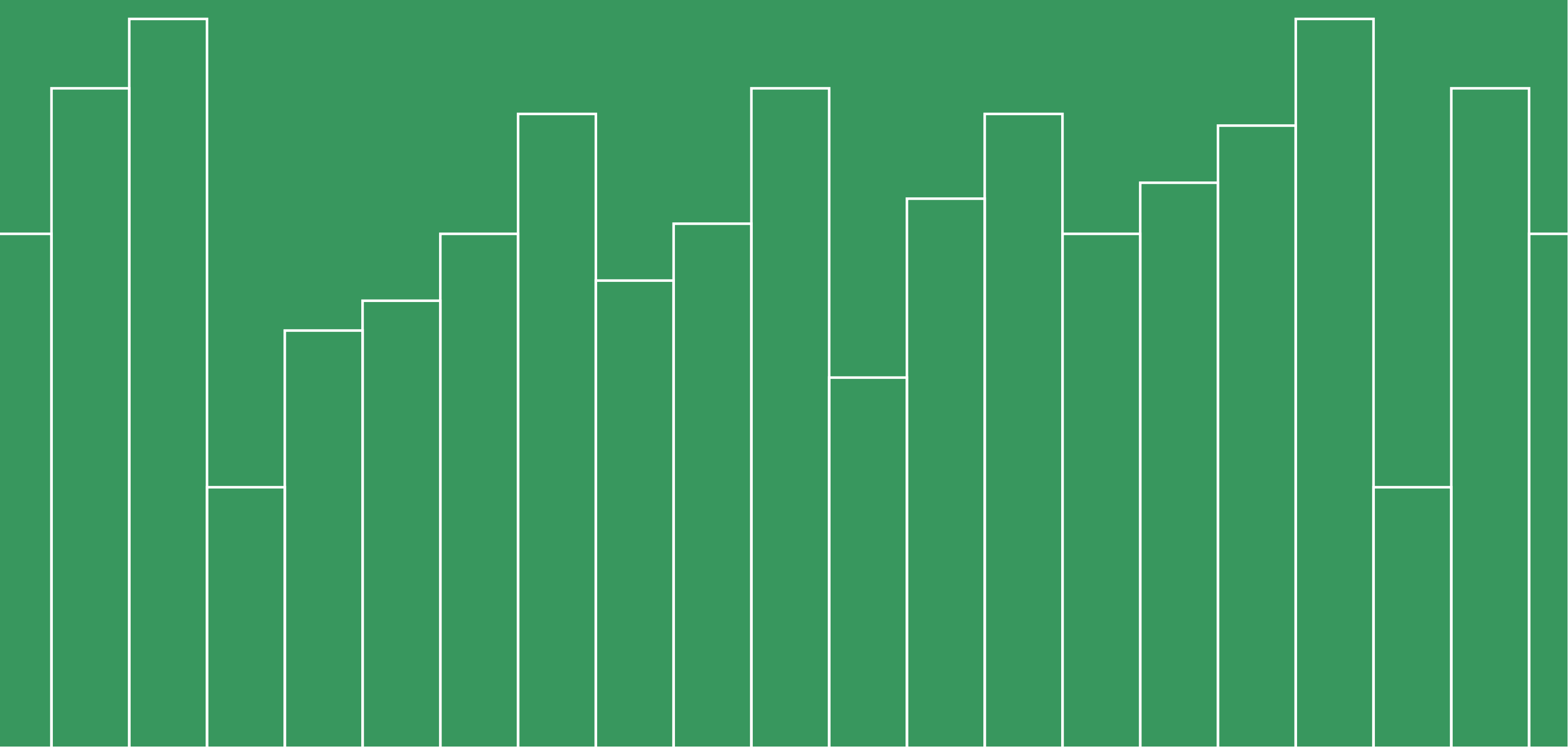
CLP 2

INTEGRAL CALCULUS

FELDMAN RECHNITZER YEAGER



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Preface

This text is a merger of the CLP Integral Calculus textbook and problembook. It is, at the time that we write this, still a work in progress; some bits and pieces around the edges still need polish. Consequently we recommend to the student that they still consult text webpage for links to the errata — especially if they think there might be a typo or error. We also request that you send us an email at clp@ugrad.math.ubc.ca

Additionally, if you are not a student at UBC and using these texts please send us an email (again using the feedback button) — we'd love to hear from you.

Joel Feldman, Andrew Rechnitzer and Elyse Yeager

To our students.

And to the many generations of scholars who have freely shared all this knowledge with us.

Acknowledgements

Elyse would like to thank her husband Seçkin Demirbaş for his endless patience, tireless support, and insightful feedback.

Andrew would like to thank Zoe for her patience and support while he hacked and also Locke for his insightful subediting. He should also thank the various coauthors for not being too impatient while other manuscripts were neglected.

Joel would like to thank a range of people for their support and encouragement over many years.

Thanks to

- Rob Beezer and David Farmer for their help converting this book from LaTeX to this online PreTeXt format.
- Nick Loewen for designing the cover art, help with figures, colours, spelling and many discussions.
- The many people who have collaborated over the last couple of decades making exams and tests for first year calculus courses at UBC Mathematics. A great many of the exercises in the text come from questions in those tests and exams.

Finally, we'd like to thank those students who reported typos and errors they found in the text. Many of these students did so through our "bug bounty" program which was supported by the Department of Mathematics, Skylight and the Loafe Cafe all at UBC.

Using the exercises in this book

Each problem in this book is split into four parts: Question, Hint, Answer, and Solution. As you are working problems, resist the temptation to prematurely peek at the hint or to click through to the answers and solutions in the appendix! It's important to allow yourself to struggle for a time with the material. Even professional mathematicians don't always know right away how to solve a problem. The art is in gathering your thoughts and figuring out a strategy to use what you know to find out what you don't.

If you find yourself at a real impasse, go ahead and look at the linked hint. Think about it for a while, and don't be afraid to read back in the notes to look for a key idea that will help you proceed. If you still can't solve the problem, well, we included the Solutions section for a reason! As you're reading the solutions, try hard to understand why we took the steps we did, instead of memorizing step-by-step how to solve that one particular problem.

If you struggled with a question quite a lot, it's probably a good idea to return to it in a few days. That might have been enough time for you to internalize the necessary ideas, and you might find it easily conquerable. Pat yourself on the back — sometimes math makes you feel good! If you're still having troubles, read over the solution again, with an emphasis on understanding why each step makes sense.

One of the reasons so many students are required to study calculus is the hope that it will improve their problem-solving skills. In this class, you will learn lots of concepts, and be asked to apply them in a variety of situations. Often, this will involve answering one really big problem by breaking it up into manageable chunks, solving those chunks, then putting the pieces back together. When you see a particularly long question, remain calm and look for a way to break it into pieces you can handle.

- ***Working with Friends***

  Study buddies are fantastic! If you don't already have friends in your class, you can ask your neighbours in lecture to form a group. Often, a question that you might bang your head against for an hour can be easily cleared up by a friend who sees what you've missed. Regular study times make sure you don't procrastinate too much, and friends help you maintain a positive attitude when you might otherwise succumb to frustration. Struggle in mathematics is desirable, but suffering is not.

When working in a group, make sure you try out problems on your own before coming together to discuss with others. Learning is a process, and getting answers to questions that you haven't considered on your own can rob you of the practice you need to master skills and concepts, and the tenacity you need to develop to become a competent problem-solver.

- ***Types of Questions***

  Questions outlined by a blue box make up the *representative question set.* This set of questions is intended to cover the most essential ideas in each section. These questions are usually highly typical of what you'd see on an exam, although some of them are atypical but carry an important moral. If you find yourself unconfident with the idea behind one of these, it's probably a good idea to practice similar questions.

  This representative question set is our suggestion for a minimal selection of questions to work on. You are highly encouraged to work on more.

  In addition to original problems, this book contains problems pulled from quizzes and exams given at UBC for Math 101 (first-semester calculus) and Math 121 (honours first-semester calculus). These problems are marked by "(*)". The authors would like to acknowledge the contributions of the many people who collaborated to produce these exams over the years.

  Finally, the questions are organized into three types: ***Stage 1***, ***Stage 2*** and ***Stage 3***.

  - Exercises — Stage 1

    The first category is meant to test and improve your understanding of basic underlying concepts. These often do not involve much calculation. They range in difficulty from very basic reviews of definitions to questions that require you to be thoughtful about the concepts covered in the section.

  - Exercises — Stage 2

    Questions in this category are for practicing skills. It's not enough to understand the philosophical grounding of an idea: you have to be able to apply it in appropriate situations. This takes practice!

  - Exercises — Stage 3

    The last questions in each section go a little farther than "***Stage 2***". Often they will combine more than one idea, incorporate review material, or ask you to apply your understanding of a concept to a new situation.

In exams, as in life, you will encounter questions of varying difficulty. A good skill to practice is recognizing the level of difficulty a problem poses. Exams will have some easy question, some standard questions, and some harder questions.

FEEDBACK ABOUT THE TEXT

The CLP-2 Integral Calculus text is still undergoing testing and changes. Because of this we request that if you find a problem or error in the text then:

1. Please check the errata list that can be found at the text webpage.
2. Is the problem in the online version or the PDF version or both?
3. Note the URL of the online version and the page number in the PDF
4. Send an email to clp@ugrad.math.ubc.ca. Please be sure to include
   - a description of the error
   - the URL of the page, if found in the online edition
   - and if the problem also exists in the PDF, then the page number in the PDF and the compile date on the front page of PDF.

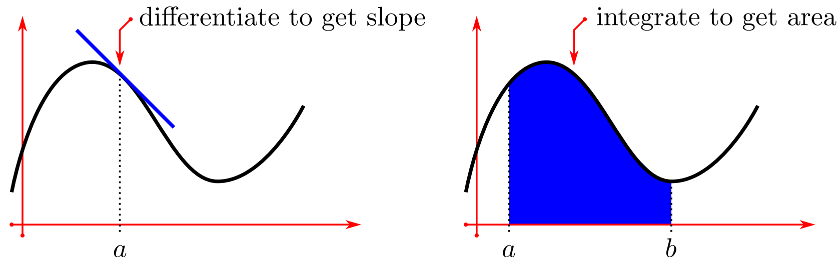
Contents

Chapter 1

Integration

Calculus is built on two operations — differentiation and integration.

- Differentiation — as we saw last term, differentiation allows us to compute and study the instantaneous rate of change of quantities. At its most basic it allows us to compute tangent lines and velocities, but it also led us to quite sophisticated applications including approximation of functions through Taylor polynomials and optimisation of quantities by studying critical and singular points.
- Integration — at its most basic, allows us to analyse the area under a curve. Of course, its application and importance extend far beyond areas and it plays a central role in solving differential equations.



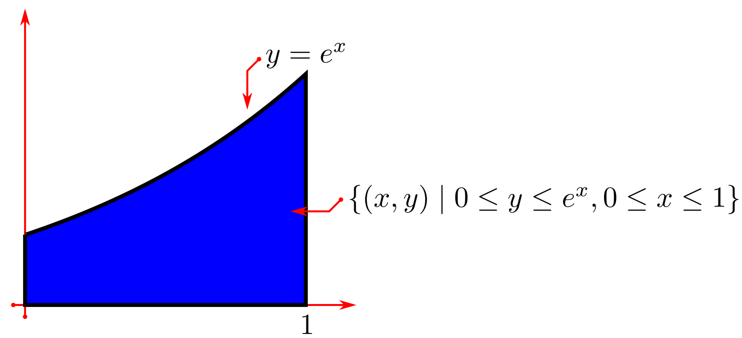
It is not immediately obvious that these two topics are related to each other. However, as we shall see, they are indeed intimately linked.

1.1 ▲ Definition of the Integral

Arguably the easiest way to introduce integration is by considering the area between the graph of a given function and the x-axis, between two specific vertical lines — such as is shown in the figure above. We'll follow this route by starting with a motivating example.

1.1.1 ➡ A Motivating Example

Let us find the area under the curve $y = e^x$ (and above the x-axis) for $0 \le x \le 1$. That is, the area of $\left\{ (x, y) \;\middle|\; 0 \le y \le e^x,\, 0 \le x \le 1 \right\}$.



This area is equal to the "definite integral"

$$\text{Area} = \int_0^1 e^x \,\mathrm{d}x$$

Do not worry about this notation or terminology just yet. We discuss it at length below. In different applications this quantity will have different interpretations — not just area. For example, if x is time and e^x is your velocity at time x, then we'll see later (in Example 1.1.18) that the specified area is the net distance travelled between time 0 and time 1. After we finish with the example, we'll mimic it to give a general definition of the integral $\int_a^b f(x)\mathrm{d}x$.

Example 1.1.1 Computing an area with vertical strips.

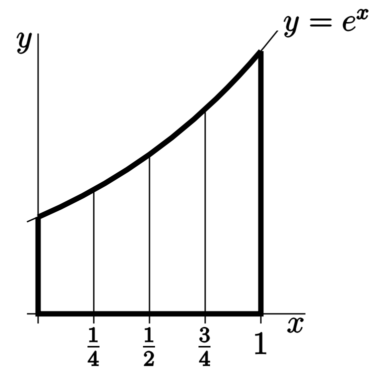
We wish to compute the area of $\left\{ (x, y) \;\middle|\; 0 \le y \le e^x,\, 0 \le x \le 1 \right\}$. We know, from our experience with e^x in differential calculus, that the curve $y = e^x$ is not easily written in terms of other simpler functions, so it is very unlikely that we would be able to write the area as a combination of simpler geometric objects such as triangles, rectangles or circles.

So rather than trying to write down the area exactly, our strategy is to approximate the area and then make our approximation more and more precise [a]. We choose [b] to approximate the area as a union of a large number of tall thin (vertical) rectangles. As we take more and more rectangles we get better and better approximations. Taking the limit as the number of rectangles goes to infinity gives the exact area [c].

As a warm up exercise, we'll now just use four rectangles. In Example 1.1.2, below, we'll consider an arbitrary number of rectangles and then take the limit as the number of rectangles goes to infinity. So

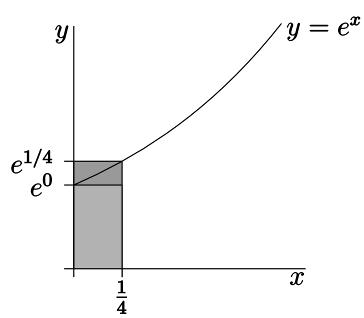
- subdivide the interval $0 \le x \le 1$ into 4 equal subintervals each of width $\frac{1}{4}$, and
- subdivide the area of interest into four corresponding vertical strips, as in the figure below.

The area we want is exactly the sum of the areas of all four strips.



Each of these strips is almost, but not quite, a rectangle. While the bottom and sides are fine (the sides are at right-angles to the base), the top of the strip is not horizontal. This is where we must start to approximate. We can replace each strip by a rectangle by just levelling off the top. But now we have to make a choice — at what height do we level off the top?

Consider, for example, the leftmost strip. On this strip, x runs from 0 to $\frac{1}{4}$. As x runs from 0 to $\frac{1}{4}$, the height y runs from e^0 to $e^{\frac{1}{4}}$. It would be reasonable to choose the height of the approximating rectangle to be somewhere between e^0 and $e^{\frac{1}{4}}$. Which



height should we choose? Well, actually it doesn't matter. When we eventually take the limit of infinitely many approximating rectangles all of those different choices give exactly the same final answer. We'll say more about this later.

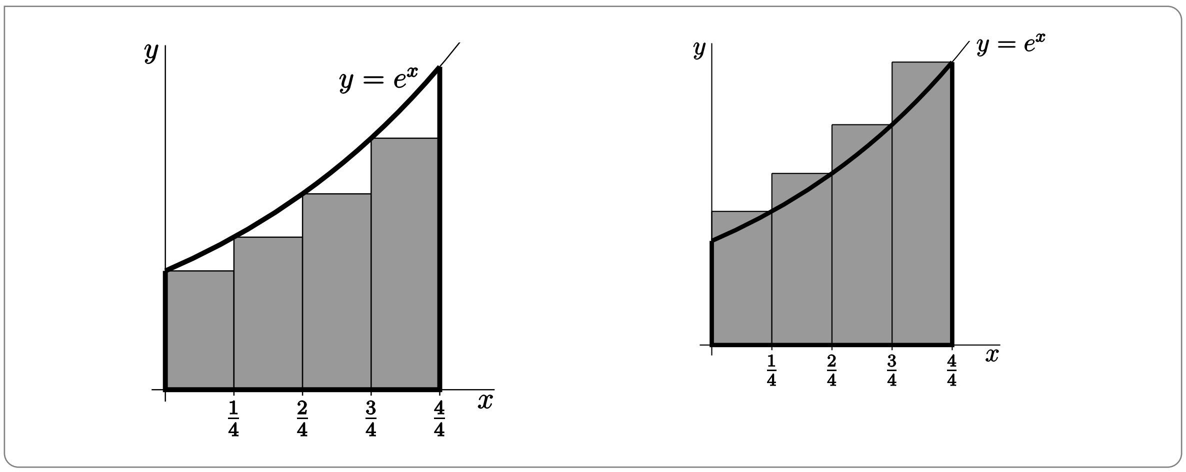
In this example we'll do two sample computations.

- For the first computation we approximate each slice by a rectangle whose height is the height of the *left* hand side of the slice.
  - On the first slice, x runs from 0 to $\frac{1}{4}$, and the height y runs from e^0, on the left hand side, to $e^{\frac{1}{4}}$, on the right hand side.
  - So we approximate the first slice by the rectangle of height e^0 and width $\frac{1}{4}$, and hence of area $\frac{1}{4}e^0 = \frac{1}{4}$.
  - On the second slice, x runs from $\frac{1}{4}$ to $\frac{1}{2}$, and the height y runs from $e^{\frac{1}{4}}$ and $e^{\frac{1}{2}}$.

- So we approximate the second slice by the rectangle of height $e^{\frac{1}{4}}$ and width $\frac{1}{4}$, and hence of area $\frac{1}{4}\,e^{\frac{1}{4}}$.
- And so on.
- All together, we approximate the area of interest by the sum of the areas of the four approximating rectangles, which is

$$\big[1 + e^{\frac{1}{4}} + e^{\frac{1}{2}} + e^{\frac{3}{4}}\big]\frac{1}{4} = 1.5124$$

- This particular approximation is called the "left Riemann sum approximation to $\int_0^1 e^x \mathrm{d}x$ with 4 subintervals". We'll explain this terminology later.
- This particular approximation represents the shaded area in the figure on the left below. Note that, because e^x *increases* as x increases, this approximation is definitely smaller than the true area.



- For the second computation we approximate each slice by a rectangle whose height is the height of the *right* hand side of the slice.
  - On the first slice, x runs from 0 to $\frac{1}{4}$, and the height y runs from e^0, on the left hand side, to $e^{\frac{1}{4}}$, on the right hand side.
  - So we approximate the first slice by the rectangle of height $e^{\frac{1}{4}}$ and width $\frac{1}{4}$, and hence of area $\frac{1}{4}\,e^{\frac{1}{4}}$.
  - On the second slice, x runs from $\frac{1}{4}$ to $\frac{1}{2}$, and the height y runs from $e^{\frac{1}{4}}$ and $e^{\frac{1}{2}}$.
  - So we approximate the second slice by the rectangle of height $e^{\frac{1}{2}}$ and width $\frac{1}{4}$, and hence of area $\frac{1}{4}\,e^{\frac{1}{2}}$.
  - And so on.

- All together, we approximate the area of interest by the sum of the areas of the four approximating rectangles, which is

$$\left[e^{\frac{1}{4}} + e^{\frac{1}{2}} + e^{\frac{3}{4}} + e^{1}\right]\frac{1}{4} = 1.9420$$

- This particular approximation is called the "right Riemann sum approximation to $\int_0^1 e^x \mathrm{d}x$ with 4 subintervals".
- This particular approximation represents the shaded area in the figure on the right above. Note that, because e^x *increases* as x increases, this approximation is definitely larger than the true area.

a This should remind the reader of the approach taken to compute the slope of a tangent line way way back at the start of differential calculus.

b Approximating the area in this way leads to a definition of integration that is called Riemann integration. This is the most commonly used approach to integration. However we could also approximate the area by using long thin horizontal strips. This leads to a definition of integration that is called Lebesgue integration. We will not be covering Lebesgue integration in these notes.

c If we want to be more careful here, we should construct two approximations, one that is always a little smaller than the desired area and one that is a little larger. We can then take a limit using the Squeeze Theorem and arrive at the exact area. More on this later.

Example 1.1.1

Now for the full computation that gives the exact area.

Example 1.1.2 Computing an area exactly.

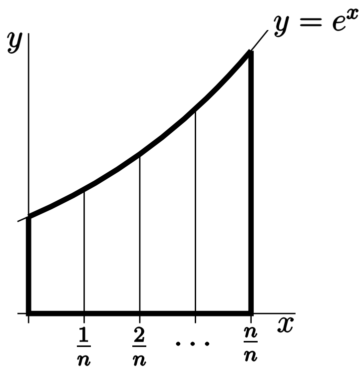
Recall that we wish to compute the area of

$$\left\{ (x,y) \;\middle|\; 0 \le y \le e^x,\ 0 \le x \le 1 \right\}$$

and that our strategy is to approximate this area by the area of a union of a large number of very thin rectangles, and then take the limit as the number of rectangles goes to infinity. In Example 1.1.1, we used just four rectangles. Now we'll consider a general number of rectangles, that we'll call n. Then we'll take the limit $n \to \infty$. So

- pick a natural number n and
- subdivide the interval $0 \le x \le 1$ into n equal subintervals each of width $\frac{1}{n}$, and
- subdivide the area of interest into corresponding thin strips, as in the figure below.

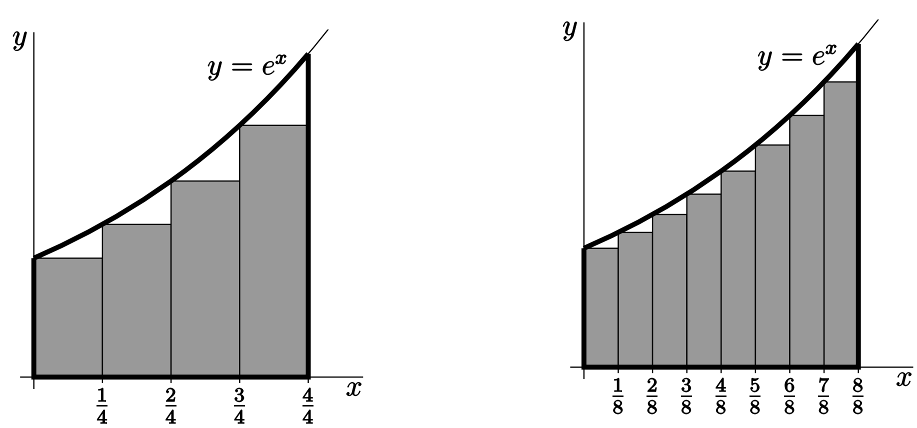
The area we want is exactly the sum of the areas of all of the thin strips.



Each of these strips is almost, but not quite, a rectangle. As in Example 1.1.1, the only problem is that the top is not horizontal. So we approximate each strip by a rectangle, just by levelling off the top. Again, we have to make a choice — at what height do we level off the top?

Consider, for example, the leftmost strip. On this strip, x runs from 0 to $\frac{1}{n}$. As x runs from 0 to $\frac{1}{n}$, the height y runs from e^0 to $e^{\frac{1}{n}}$. It would be reasonable to choose the height of the approximating rectangle to be somewhere between e^0 and $e^{\frac{1}{n}}$. Which height should we choose?

Well, as we said in Example 1.1.1, it doesn't matter. We shall shortly take the limit $n \to \infty$ and, in that limit, all of those different choices give exactly the same final answer. We won't justify that statement in this example, but there will be an (optional) section shortly that provides the justification. For this example we just, arbitrarily, choose the height of each rectangle to be the height of the graph $y = e^x$ at the smallest value of x in the corresponding strip [a]. The figure on the left below shows the approximating rectangles when $n = 4$ and the figure on the right shows the approximating rectangles when $n = 8$.



Now we compute the approximating area when there are n strips.

- We approximate the leftmost strip by a rectangle of height e^0. All of the rectangles have width $\frac{1}{n}$. So the leftmost rectangle has area $\frac{1}{n}e^0$.

- On strip number 2, x runs from $\frac{1}{n}$ to $\frac{2}{n}$. So the smallest value of x on strip number 2 is $\frac{1}{n}$, and we approximate strip number 2 by a rectangle of height $e^{\frac{1}{n}}$ and hence of area $\frac{1}{n}e^{\frac{1}{n}}$.
- And so on.
- On the last strip, x runs from $\frac{n-1}{n}$ to $\frac{n}{n} = 1$. So the smallest value of x on the last strip is $\frac{n-1}{n}$, and we approximate the last strip by a rectangle of height $e^{\frac{(n-1)}{n}}$ and hence of area $\frac{1}{n}e^{\frac{(n-1)}{n}}$.

The total area of all of the approximating rectangles is

$$\begin{aligned} \text{Total approximating area} &= \frac{1}{n}e^0 + \frac{1}{n}e^{\frac{1}{n}} + \frac{1}{n}e^{\frac{2}{n}} + \frac{1}{n}e^{\frac{3}{n}} + \cdots + \frac{1}{n}e^{\frac{(n-1)}{n}} \\ &= \frac{1}{n}\Big(1 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + e^{\frac{3}{n}} + \cdots + e^{\frac{(n-1)}{n}}\Big) \end{aligned}$$

Now the sum in the brackets might look a little intimidating because of all the exponentials, but it actually has a pretty simple structure that can be easily seen if we rename $e^{\frac{1}{n}} = r$. Then

- the first term is $1 = r^0$ and
- the second term is $e^{\frac{1}{n}} = r^1$ and
- the third term is $e^{\frac{2}{n}} = r^2$ and
- the fourth term is $e^{\frac{3}{n}} = r^3$ and
- and so on and
- the last term is $e^{\frac{(n-1)}{n}} = r^{n-1}$.

So

$$\text{Total approximating area} = \frac{1}{n}\left(1 + r + r^2 + \cdots + r^{n-1}\right)$$

The sum in brackets is known as a geometric sum and satisfies a nice simple formula:

Equation 1.1.3 Geometric sum.

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{r^n - 1}{r - 1} \qquad \text{provided } r \neq 1$$

The derivation of the above formula is not too difficult. So let's derive it in a little aside.

Geometric sum.

Denote the sum as

$$S = 1 + r + r^2 + \cdots + r^{n-1}$$

Notice that if we multiply the whole sum by r we get back almost the same thing:

$$\begin{aligned} rS & \\ &= r\left(1 + r + \cdots + r^{n-1}\right) \\ &= r + r^2 + r^3 + \cdots + r^n \end{aligned}$$

This right hand side differs from the original sum S only in that

- the right hand side, which starts with "$r+$", is missing the "$1+$" that S starts with, and
- the right hand side has an extra "$+r^n$" at the end that does not appear in S.

That is

$$rS = S - 1 + r^n$$

Moving this around a little gives

$$\begin{aligned} (r-1)S &= (r^n - 1) \\ S &= \frac{r^n - 1}{r - 1} \end{aligned}$$

as required. Notice that the last step in the manipulations only works providing $r \neq 1$ (otherwise we are dividing by zero).

Now we can go back to our area approximation armed with the above result about geometric sums.

$$\begin{aligned} \text{Total approximating area} &= \frac{1}{n}\left(1 + r + r^2 + \cdots + r^{n-1}\right) \\ &= \frac{1}{n}\frac{r^n - 1}{r - 1} \qquad\qquad \text{remember that } r = e^{1/n} \\ &= \frac{1}{n}\frac{e^{n/n} - 1}{e^{1/n} - 1} \\ &= \frac{1}{n}\frac{e - 1}{e^{1/n} - 1} \end{aligned}$$

To get the exact area [b] all we need to do is make the approximation better and better by taking the limit $n \to \infty$. The limit will look more familiar if we rename $\frac{1}{n}$ to X. As

n tends to infinity, X tends to 0, so

$$\begin{aligned}
\text{Area} &= \lim_{n\to\infty} \frac{1}{n} \frac{e-1}{e^{1/n}-1} \\
&= (e-1) \lim_{n\to\infty} \frac{1/n}{e^{1/n}-1} \\
&= (e-1) \lim_{X\to 0} \frac{X}{e^X-1} && (\text{with } X = \frac{1}{n})
\end{aligned}$$

Examining this limit we see that both numerator and denominator tend to zero as $X \to 0$, and so we cannot evaluate this limit by computing the limits of the numerator and denominator separately and then dividing the results. Despite this, the limit is not too hard to evaluate; here we give two ways:

- Perhaps the easiest way to compute the limit is by using l'Hôpital's rule [c]. Since both numerator and denominator go to zero, this is a $\frac{0}{0}$ indeterminate form. Thus

$$\lim_{X\to 0} \frac{X}{e^X-1} = \lim_{X\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}X} X}{\frac{\mathrm{d}}{\mathrm{d}X}(e^X-1)} = \lim_{X\to 0} \frac{1}{e^X} = 1$$

- Another way [d] to evaluate the same limit is to observe that it can be massaged into the form of the limit definition of the derivative. First notice that

$$\lim_{X\to 0} \frac{X}{e^X-1} = \left[\lim_{X\to 0} \frac{e^X-1}{X}\right]^{-1}$$

  provided this second limit exists and is nonzero [e]. This second limit should look a little familiar:

$$\lim_{X\to 0} \frac{e^X-1}{X} = \lim_{X\to 0} \frac{e^X-e^0}{X-0}$$

  which is just the definition of the derivative of e^x at $x=0$. Hence we have

$$\begin{aligned}
\lim_{X\to 0} \frac{X}{e^X-1} &= \left[\lim_{X\to 0} \frac{e^X-e^0}{X-0}\right]^{-1} \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}X} e^X \Big|_{X=0}\right]^{-1} \\
&= \left[e^X\big|_{X=0}\right]^{-1} \\
&= 1
\end{aligned}$$

So, after this short aside into limits, we may now conclude that

$$\text{Area} = (e-1) \lim_{X\to 0} \frac{X}{e^X-1}$$

$$= e - 1$$

a Notice that since e^x is an increasing function, this choice of heights means that each of our rectangles is smaller than the strip it came from.

b We haven't proved that this will give us the exact area, but it should be clear that taking this limit will give us a lower bound on the area. To complete things rigorously we also need an upper bound and the squeeze theorem. We do this in the next optional subsection.

c If you do not recall L'Hôpital's rule and indeterminate forms then we recommend you skim over your differential calculus notes on the topic.

d Say if you don't recall l'Hôpital's rule and have not had time to revise it.

e To hyphenate or not to hypenate: "non-zero" or "nonzero"? The authors took our lead from here and also here.

Example 1.1.3

1.1.2 ➧ Optional — A more rigorous area computation

In Example 1.1.1 above we considered the area of the region $\left\{ (x,y) \;\middle|\; 0 \le y \le e^x, 0 \le x \le 1 \right\}$. We approximated that area by the area of a union of n thin rectangles. We then claimed that upon taking the number of rectangles to infinity, the approximation of the area became the exact area. However we did not justify the claim. The purpose of this optional section is to make that calculation rigorous.

The broad set-up is the same. We divide the region up into n vertical strips, each of width $\frac{1}{n}$ and we then approximate those strips by rectangles. However rather than an uncontrolled approximation, we construct two sets of rectangles — one set always smaller than the original area and one always larger. This then gives us lower and upper bounds on the area of the region. Finally we make use of the squeeze theorem [1] to establish the result.

- To find our upper and lower bounds we make use of the fact that e^x is an increasing function. We know this because the derivative $\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$ is always positive. Consequently, the smallest and largest values of e^x on the interval $a \le x \le b$ are e^a and e^b, respectively.
- In particular, for $0 \le x \le \frac{1}{n}$, e^x takes values only between e^0 and $e^{\frac{1}{n}}$. As a result, the first strip

$$\left\{ (x,y) \;\middle|\; 0 \le x \le \frac{1}{n},\ 0 \le y \le e^x \right\}$$

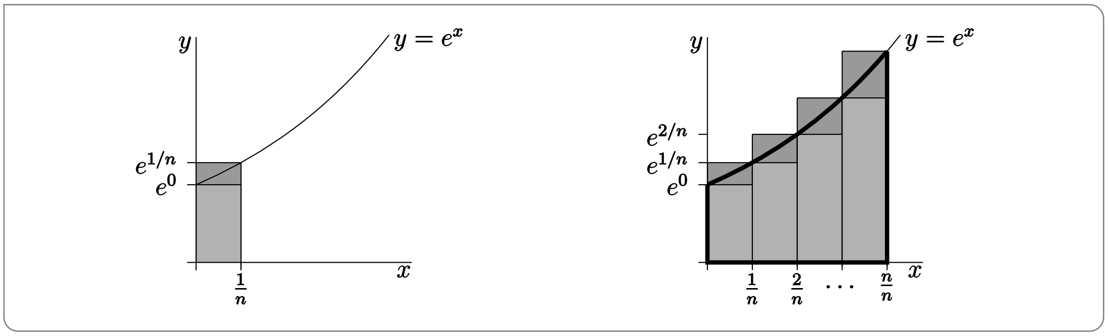
  - contains the rectangle of $0 \le x \le \frac{1}{n}$, $0 \le y \le e^0$ (the lighter rectangle in the figure on the left below) and

1 Recall that if we have 3 functions $f(x), g(x), h(x)$ that satisfy $f(x) \le g(x) \le h(x)$ and we know that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ exists and is finite, then the squeeze theorem tells us that $\lim_{x\to a} g(x) = L$.

- ◦ is contained in the rectangle $0 \le x \le \frac{1}{n}$, $0 \le y \le e^{\frac{1}{n}}$ (the largest rectangle in the figure on the left below).

Hence

$$\frac{1}{n}e^0 \le \text{Area}\{\ (x,y)\ \big|\ 0 \le x \le \frac{1}{n},\ 0 \le y \le e^x\ \} \le \frac{1}{n}e^{\frac{1}{n}}$$



- Similarly, for the second, third, ..., last strips, as in the figure on the right above,

$$\begin{aligned}
\frac{1}{n}e^{\frac{1}{n}} &\le \text{Area}\{\ (x,y)\ \big|\ \frac{1}{n} \le x \le \frac{2}{n},\ 0 \le y \le e^x\ \} &\le \frac{1}{n}e^{\frac{2}{n}}\\
\frac{1}{n}e^{\frac{2}{n}} &\le \text{Area}\{\ (x,y)\ \big|\ \frac{2}{n} \le x \le \frac{3}{n},\ 0 \le y \le e^x\ \} &\le \frac{1}{n}e^{\frac{3}{n}}\\
&\vdots\\
\frac{1}{n}e^{\frac{(n-1)}{n}} &\le \text{Area}\{\ (x,y)\ \big|\ \frac{(n-1)}{n} \le x \le \frac{n}{n},\ 0 \le y \le e^x\ \} &\le \frac{1}{n}e^{\frac{n}{n}}
\end{aligned}$$

- Adding these n inequalities together gives

$$\begin{aligned}
&\frac{1}{n}\left(1 + e^{\frac{1}{n}} + \cdots + e^{\frac{(n-1)}{n}}\right)\\
&\le \text{Area}\{\ (x,y)\ \big|\ 0 \le x \le 1,\ 0 \le y \le e^x\ \}\\
&\le \frac{1}{n}\left(e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n}{n}}\right)
\end{aligned}$$

- We can then recycle equation 1.1.3 with $r = e^{\frac{1}{n}}$, so that $r^n = \left(e^{\frac{1}{n}}\right)^n = e$. Thus we have

$$\frac{1}{n}\frac{e-1}{e^{\frac{1}{n}}-1} \le \text{Area}\{\ (x,y)\ \big|\ 0 \le x \le 1,\ 0 \le y \le e^x\ \} \le \frac{1}{n}e^{\frac{1}{n}}\frac{e-1}{e^{\frac{1}{n}}-1}$$

where we have used the fact that the upper bound is a simple multiple of the lower bound:

$$\left(e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n}{n}}\right) = e^{\frac{1}{n}}\left(1 + e^{\frac{1}{n}} + \cdots + e^{\frac{(n-1)}{n}}\right).$$

- We now apply the squeeze theorem to the above inequalities. In particular, the limits of the lower and upper bounds are $\lim_{n\to\infty} \frac{1}{n}\frac{e-1}{e^{\frac{1}{n}}-1}$ and $\lim_{n\to\infty} \frac{1}{n}e^{\frac{1}{n}}\frac{e-1}{e^{\frac{1}{n}}-1}$, respectively. As we did near the end of Example 1.1.2, we make these limits look more familiar by renaming $\frac{1}{n}$ to X. As n tends to infinity, X tends to 0, so the limits of the lower and upper bounds are

$$\lim_{n\to\infty} \frac{1}{n}\frac{e-1}{e^{\frac{1}{n}}-1} = (e-1)\lim_{X=\frac{1}{n}\to 0} \frac{X}{e^X-1} = e-1$$

(by l'Hôpital's rule) and

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}e^{\frac{1}{n}}\frac{e-1}{e^{\frac{1}{n}}-1} &= (e-1)\lim_{X=\frac{1}{n}\to 0} \cdot\frac{Xe^X}{e^X-1} \\ &= (e-1)\lim_{X\to 0} e^X \cdot \lim_{X=\to 0}\frac{X}{e^X-1} \\ &= (e-1)\cdot 1\cdot 1\end{aligned}$$

Thus, since the exact area is trapped between the lower and upper bounds, the squeeze theorem then implies that

$$\text{Exact area} = e-1.$$

1.1.3 ►► Summation notation

As you can see from the above example (and the more careful rigorous computation), our discussion of integration will involve a fair bit of work with sums of quantities. To this end, we make a quick aside into summation notation. While one can work through the material below without this notation, proper summation notation is well worth learning, so we advise the reader to persevere.

Writing out the summands explicitly can become quite impractical — for example, say we need the sum of the first 11 squares:

$$1+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2+10^2+11^2$$

This becomes tedious. Where the pattern is clear, we will often skip the middle few terms and instead write

$$1+2^2+\cdots+11^2.$$

A far more precise way to write this is using Σ (capital-sigma) notation. For example, we can write the above sum as

$$\sum_{k=1}^{11} k^2$$

This is read as

The sum from k equals 1 to 11 of k^2.

More generally

Definition 1.1.4

Let $m \leq n$ be integers and let $f(x)$ be a function defined on the integers. Then we write

$$\sum_{k=m}^{n} f(k)$$

to mean the sum of $f(k)$ for k from m to n:

$$f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n).$$

Similarly we write

$$\sum_{i=m}^{n} a_i$$

to mean

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

for some set of coefficients $\{a_m, \ldots, a_n\}$.

Consider the example

$$\sum_{k=3}^{7} \frac{1}{k^2} = \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}$$

It is important to note that the right hand side of this expression evaluates to a number [2]; it does not contain "k". The summation index k is just a "dummy" variable and it does not have to be called k. For example

$$\sum_{k=3}^{7} \frac{1}{k^2} = \sum_{i=3}^{7} \frac{1}{i^2} = \sum_{j=3}^{7} \frac{1}{j^2} = \sum_{\ell=3}^{7} \frac{1}{\ell^2}$$

Also the summation index has no meaning outside the sum. For example

$$k \sum_{k=3}^{7} \frac{1}{k^2}$$

has no mathematical meaning; it is gibberish.

A sum can be represented using summation notation in many different ways. If you are unsure as to whether or not two summation notations represent the same sum, just write out the first few terms and the last couple of terms. For example,

$$\sum_{m=3}^{15} \frac{1}{m^2} = \overbrace{\frac{1}{3^2}}^{m=3} + \overbrace{\frac{1}{4^2}}^{m=4} + \overbrace{\frac{1}{5^2}}^{m=5} + \cdots + \overbrace{\frac{1}{14^2}}^{m=14} + \overbrace{\frac{1}{15^2}}^{m=15}$$

2 Some careful addition shows it is $\frac{46181}{176400}$.

$$\sum_{m=4}^{16} \frac{1}{(m-1)^2} = \overbrace{\frac{1}{3^2}}^{m=4} + \overbrace{\frac{1}{4^2}}^{m=5} + \overbrace{\frac{1}{5^2}}^{m=6} + \cdots + \overbrace{\frac{1}{14^2}}^{m=15} + \overbrace{\frac{1}{15^2}}^{m=16}$$

are equal.

Here is a theorem that gives a few rules for manipulating summation notation.

Theorem 1.1.5 Arithmetic of Summation Notation.

Let $n \geq m$ be integers. Then for all real numbers c and a_i, b_i, $m \leq i \leq n$.

a $\sum_{i=m}^{n} ca_i = c\left(\sum_{i=m}^{n} a_i\right)$

b $\sum_{i=m}^{n} (a_i + b_i) = \left(\sum_{i=m}^{n} a_i\right) + \left(\sum_{i=m}^{n} b_i\right)$

c $\sum_{i=m}^{n} (a_i - b_i) = \left(\sum_{i=m}^{n} a_i\right) - \left(\sum_{i=m}^{n} b_i\right)$

Proof. We can prove this theorem by just writing out both sides of each equation, and observing that they are equal, by the usual laws of arithmetic [a]. For example, for the first equation, the left and right hand sides are

$$\sum_{i=m}^{n} ca_i = ca_m + ca_{m+1} + \cdots + ca_n$$

$$\text{and} \quad c\left(\sum_{i=m}^{n} a_i\right) = c(a_m + a_{m+1} + \cdots + a_n)$$

They are equal by the usual distributive law. The "distributive law" is the fancy name for $c(a+b) = ca + cb$.

□

a Since all the sums are finite, this isn't too hard. More care must be taken when the sums involve an infinite number of terms. We will examine this in Chapter 3.

Not many sums can be computed exactly [3]. Here are some that can. The first few are used a lot.

3 Of course, any finite sum can be computed exactly — just sum together the terms. What we mean by "computed exactly" in this context, is that we can rewrite the sum as a simple, and easily evaluated, formula involving the terminals of the sum. For example $\sum_{k=m}^{n} r^k = \frac{r^{n+1}-r^m}{r-1}$ provided $r \neq 1$. No matter what finite integers we choose for m and n, we can quickly compute the sum in just a few arithmetic operations. On the other hand, the sums, $\sum_{k=m}^{n} \frac{1}{k}$ and $\sum_{k=m}^{n} \frac{1}{k^2}$, cannot be expressed in such clean formulas (though you can rewrite them quite cleanly using integrals).

Theorem 1.1.6

a $\sum_{i=0}^{n} ar^i = a\frac{1-r^{n+1}}{1-r}$, for all real numbers a and $r \neq 1$ and all integers $n \geq 0$.

b $\sum_{i=1}^{n} 1 = n$, for all integers $n \geq 1$.

c $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, for all integers $n \geq 1$.

d $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$, for all integers $n \geq 1$.

e $\sum_{i=1}^{n} i^3 = \left[\frac{1}{2}n(n+1)\right]^2$, for all integers $n \geq 1$.

1.1.3.1 ⯮ Proof of Theorem 1.1.6 (Optional)

Proof.

a The first sum is

$$\sum_{i=0}^{n} ar^i = ar^0 + ar^1 + ar^2 + \cdots + ar^n$$

which is just the left hand side of equation 1.1.3, with n replaced by $n+1$ and then multiplied by a.

b The second sum is just n copies of 1 added together, so of course the sum is n.

c The third and fourth sums are discussed in the appendix of the CLP-1 text. In that discussion certain "tricks" are used to compute the sums with only simple arithmetic. Those tricks do not easily generalise to the fifth sum.

d Instead of repeating that appendix, we'll derive the third sum using a trick that generalises to the fourth and fifth sums (and also to higher powers). The trick uses the generating function [a] $S(x)$:

To explain more clearly we would need to go into a more detailed and careful discussion that is beyond the scope of this course.

Equation 1.1.7 Finite geometric sum.

$$S(x) = 1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

Notice that this is just the geometric sum given by equation 1.1.3 with n replaced by $n + 1$.

Now, consider the limit

$$\begin{aligned}
\lim_{x\to 1} S(x) &= \lim_{x\to 1} \left(1 + x + x^2 + \cdots + x^n\right) = n + 1 && \text{but also} \\
&= \lim_{x\to 1} \frac{x^{n+1} - 1}{x - 1} && \text{now use l'Hôpital's rule} \\
&= \lim_{x\to 1} \frac{(n+1)x^n}{1} = n + 1.
\end{aligned}$$

This is not so hard (or useful). But now consider the derivative of $S(x)$:

$$\begin{aligned}
S'(x) &= 1 + 2x + 3x^2 + \cdots + nx^{n-1} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^{n+1} - 1}{x - 1}\right] && \text{use the quotient rule} \\
&= \frac{(x-1)\cdot(n+1)x^n - (x^{n+1} - 1)\cdot 1}{(x-1)^2} && \text{now clean it up} \\
&= \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}.
\end{aligned}$$

Hence if we take the limit of the above expression as $x \to 1$ we recover

$$\begin{aligned}
\lim_{x\to 1} S'(x) &= 1 + 2 + 3 + \cdots + n \\
&= \lim_{x\to 1} \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2} && \text{now use l'Hôpital's rule} \\
&= \lim_{x\to 1} \frac{n(n+1)x^n - n(n+1)x^{n-1}}{2(x-1)} && \text{l'Hôpital's rule again} \\
&= \lim_{x\to 1} \frac{n^2(n+1)x^{n-1} - n(n+1)(n-1)x^{n-2}}{2} \\
&= \frac{n^2(n+1) - n(n-1)(n+1)}{2} = \frac{n(n+1)}{2}
\end{aligned}$$

as required. This computation can be done without l'Hôpital's rule, but the manipulations required are a fair bit messier.

e The derivation of the fourth and fifth sums is similar to, but even more tedious than, that of the third sum. One takes two or three derivatives of the generating functional.

□

a Generating functions are frequently used in mathematics to analyse sequences and series, but are beyond the scope of the course. The interested reader should take a look at "Generatingfunctionology" by Herb Wilf. It is an excellent book and is also free to download.

1.1.4 ⏩ The Definition of the Definite Integral

In this section we give a definition of the definite integral $\displaystyle\int_a^b f(x)\mathrm{d}x$ generalising the machinery we used in Example 1.1.1. But first some terminology and a couple of remarks to better motivate the definition.

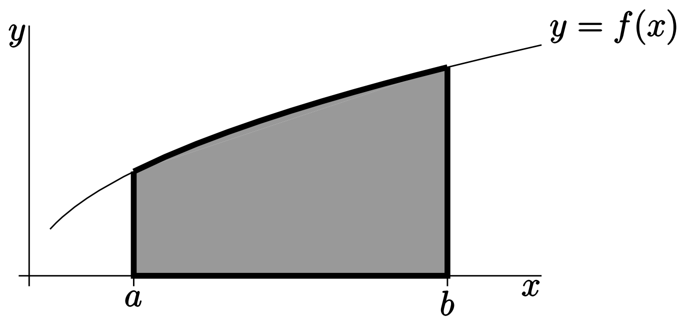
Definition 1.1.8

The symbol $\displaystyle\int_a^b f(x)\mathrm{d}x$ is read "the definite integral of the function $f(x)$ from a to b". The function $f(x)$ is called the integrand of $\int_a^b f(x)\mathrm{d}x$ and a and b are called [a] the limits of integration. The interval $a \le x \le b$ is called the interval of integration and is also called the domain of integration.

a a and b are also called the bounds of integration.

Before we explain more precisely what the definite integral actually is, a few remarks (actually — a few interpretations) are in order.

- If $f(x) \ge 0$ and $a \le b$, one interpretation of the symbol $\displaystyle\int_a^b f(x)\mathrm{d}x$ is "the area of the region $\left\{ (x,y) \;\middle|\; a \le x \le b,\ 0 \le y \le f(x) \right\}$".



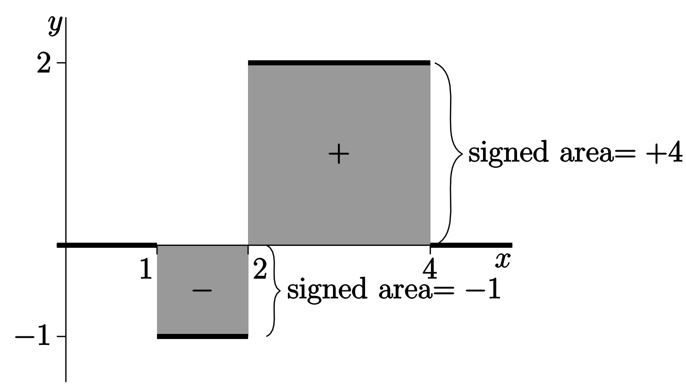
In this way we can rewrite the area in Example 1.1.1 as the definite integral $\int_0^1 e^x \mathrm{d}x$.

- This interpretation breaks down when either $a > b$ or $f(x)$ is not always positive, but it can be repaired by considering "signed areas".
- If $a \le b$, but $f(x)$ is not always positive, one interpretation of $\int_a^b f(x)\mathrm{d}x$ is "the signed area between $y = f(x)$ and the x-axis for $a \le x \le b$". For "signed area" (which is also called the "net area"), areas above the x-axis count as positive while areas below the x-axis count as negative. In the example below, we have the graph of the function

$$f(x) = \begin{cases} -1 & \text{if } 1 \le x \le 2 \\ 2 & \text{if } 2 < x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

The 2×2 shaded square above the x-axis has signed area $+2 \times 2 = +4$. The 1×1 shaded square below the x-axis has signed area $-1 \times 1 = -1$. So, for this $f(x)$,

$$\int_0^5 f(x)\mathrm{d}x = +4 - 1 = 3$$



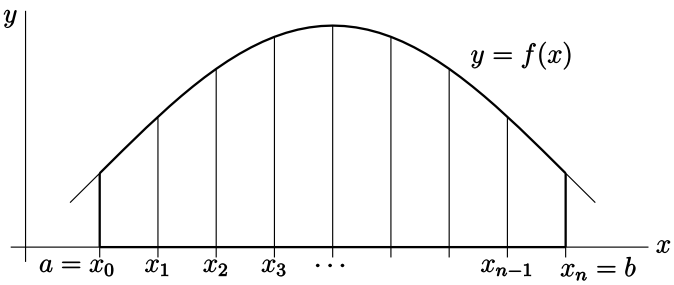
- We'll come back to the case $b < a$ later.

We're now ready to define $\int_a^b f(x)\mathrm{d}x$. The definition is a little involved, but essentially mimics what we did in Example 1.1.1 (which is why we did the example before the definition). The main differences are that we replace the function e^x by a generic function $f(x)$ and we replace the interval from 0 to 1 by the generic interval [4] from a to b.

- We start by selecting any natural number n and subdividing the interval from a to b into n equal subintervals. Each subinterval has width $\frac{b-a}{n}$.

4 We'll eventually allow a and b to be any two real numbers, not even requiring $a < b$. But it is easier to start off assuming $a < b$, and that's what we'll do.

- Just as was the case in Example 1.1.1 we will eventually take the limit as $n \to \infty$, which squeezes the width of each subinterval down to zero.
- For each integer $0 \le i \le n$, define $x_i = a + i \cdot \frac{b-a}{n}$. Note that this means that $x_0 = a$ and $x_n = b$. It is worth keeping in mind that these numbers x_i do depend on n even though our choice of notation hides this dependence.
- Subinterval number i is $x_{i-1} \le x \le x_i$. In particular, on the first subinterval, x runs from $x_0 = a$ to $x_1 = a + \frac{b-a}{n}$. On the second subinterval, x runs from x_1 to $x_2 = a + 2\frac{b-a}{n}$.



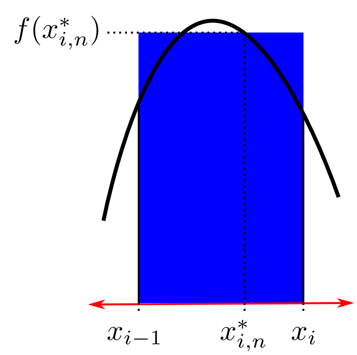
- On each subinterval we now pick $x_{i,n}^*$ between x_{i-1} and x_i. We then approximate $f(x)$ on the i^{th} subinterval by the constant function $y = f(x_{i,n}^*)$. We include n in the subscript to remind ourselves that these numbers depend on n.

  Geometrically, we're approximating the region

  $$\left\{ (x,y) \;\middle|\; x \text{ is between } x_{i-1} \text{ and } x_i, \text{ and } y \text{ is between } 0 \text{ and } f(x) \right\}$$

  by the rectangle

  $$\left\{ (x,y) \;\middle|\; x \text{ is between } x_{i-1} \text{ and } x_i, \text{ and } y \text{ is between } 0 \text{ and } f(x_{i,n}^*) \right\}$$



  In Example 1.1.1 we chose $x_{i,n}^* = x_{i-1}$ and so we approximated the function e^x on each subinterval by the value it took at the leftmost point in that subinterval.

- So, when there are n subintervals our approximation to the signed area between the curve $y = f(x)$ and the x-axis, with x running from a to b, is

$$\sum_{i=1}^{n} f(x_{i,n}^*) \cdot \frac{b-a}{n}$$

We interpret this as the signed area since the summands $f(x_{i,n}^*) \cdot \frac{b-a}{n}$ need not be positive.

- Finally we define the definite integral by taking the limit of this sum as $n \to \infty$.

Oof! This is quite an involved process, but we can now write down the definition we need.

Definition 1.1.9

Let a and b be two real numbers and let $f(x)$ be a function that is defined for all x between a and b. Then we define

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i,n}^*) \cdot \frac{b-a}{n}$$

when the limit exists and takes the same value for all choices of the $x_{i,n}^*$'s. In this case, we say that f is integrable on the interval from a to b.

Of course, it is not immediately obvious when this limit should exist. Thankfully it is easier for a function to be "integrable" than it is for it to be "differentiable".

Theorem 1.1.10

Let $f(x)$ be a function on the interval $[a, b]$. If

- $f(x)$ is continuous on $[a, b]$, or
- $f(x)$ has a finite number of jump discontinuities on $[a, b]$ (and is otherwise continuous)

then $f(x)$ is integrable on $[a, b]$.

We will not justify this theorem. But a slightly weaker statement is proved in (the optional) Section 1.1.7. Of course this does not tell us how to actually evaluate any definite integrals — but we will get to that in time.

Some comments:

- Note that, in Definition 1.1.9, we allow a and b to be any two real numbers. We do not require that $a < b$. That is, even when $a > b$, the symbol $\int_a^b f(x)\mathrm{d}x$ is still defined by the formula of Definition 1.1.9. We'll get an interpretation for $\int_a^b f(x)\mathrm{d}x$, when $a > b$, later.

- It is important to note that the definite integral $\int_a^b f(x)\mathrm{d}x$ represents a number, not a function of x. The integration variable x is another "dummy" variable, just like the summation index i in $\sum_{i=m}^{n} a_i$ (see Section 1.1.3). The integration variable does not have to be called x. For example

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(t)\mathrm{d}t = \int_a^b f(u)\mathrm{d}u$$

Just as with summation variables, the integration variable x has no meaning outside of $f(x)\mathrm{d}x$. For example

$$x\int_0^1 e^x \mathrm{d}x \qquad \text{and} \qquad \int_0^x e^x \mathrm{d}x$$

are both gibberish.

The sum inside definition 1.1.9 is named after Bernhard Riemann [5] who made the first rigorous definition of the definite integral and so placed integral calculus on rigorous footings.

Definition 1.1.11

The sum inside definition 1.1.9

$$\sum_{i=1}^{n} f(x_{i,n}^*)\,\frac{b-a}{n}$$

is called a Riemann sum. It is also often written as

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where $\Delta x = \frac{b-a}{n}$.

- If we choose each $x_{i,n}^* = x_{i-1} = a + (i-1)\frac{b-a}{n}$ to be the left hand end point of the i^{th} interval, $[x_{i-1}, x_i]$, we get the approximation

$$\sum_{i=1}^{n} f\left(a + (i-1)\frac{b-a}{n}\right)\,\frac{b-a}{n}$$

  which is called the "left Riemann sum approximation to $\int_a^b f(x)\mathrm{d}x$ with n subintervals". This is the approximation used in Example 1.1.1.

- In the same way, if we choose $x_{i,n}^* = x_i = a + i\frac{b-a}{n}$ we obtain the approximation

$$\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)\,\frac{b-a}{n}$$

  which is called the "right Riemann sum approximation to $\int_a^b f(x)\mathrm{d}x$ with n subintervals". The word "right" signifies that, on each subinterval $[x_{i-1}, x_i]$ we approximate f by its value at the right-hand end-point, $x_i = a + i\frac{b-a}{n}$, of the subinterval.

5 Bernhard Riemann was a 19th century German mathematician who made extremely important contributions to many different areas of mathematics — far too many to list here. Arguably two of the most important (after Riemann sums) are now called Riemann surfaces and the Riemann hypothesis (he didn't name them after himself).

- A third commonly used approximation is

$$\sum_{i=1}^{n} f\left(a+(i-\frac{1}{2})\frac{b-a}{n}\right)\frac{b-a}{n}$$

which is called the "midpoint Riemann sum approximation to $\int_a^b f(x)\mathrm{d}x$ with n subintervals". The word "midpoint" signifies that, on each subinterval $[x_{i-1}, x_i]$ we approximate f by its value at the midpoint, $\frac{x_{i-1}+x_i}{2} = a + (i - \frac{1}{2})\frac{b-a}{n}$, of the subinterval.

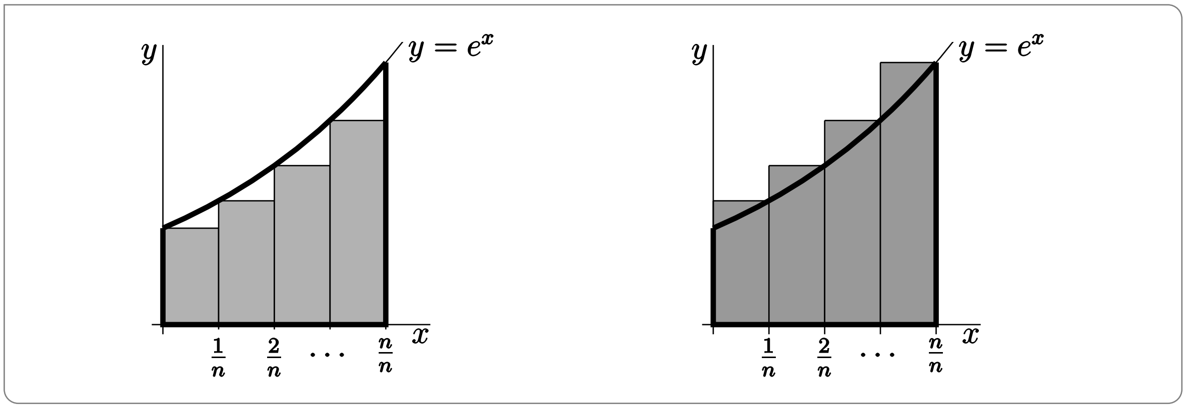
In order to compute a definite integral using Riemann sums we need to be able to compute the limit of the sum as the number of summands goes to infinity. This approach is not always feasible and we will soon arrive at other means of computing definite integrals based on antiderivatives. However, Riemann sums also provide us with a good means of approximating definite integrals — if we take n to be a large, but finite, integer, then the corresponding Riemann sum can be a good approximation of the definite integral. Under certain circumstances this can be strengthened to give rigorous bounds on the integral. Let us revisit Example 1.1.1.

Example 1.1.12 Upper and lower bounds on area.

Let's say we are again interested in the integral $\int_0^1 e^x \mathrm{d}x$. We can follow the same procedure as we used previously to construct Riemann sum approximations. However since the integrand $f(x) = e^x$ is an increasing function, we can make our approximations into upper and lower bounds without much extra work.
More precisely, we approximate $f(x)$ on each subinterval $x_{i-1} \le x \le x_i$

- by its smallest value on the subinterval, namely $f(x_{i-1})$, when we compute the left Riemann sum approximation and
- by its largest value on the subinterval, namely $f(x_i)$, when we compute the right Riemann sum approximation.

This is illustrated in the two figures below. The shaded region in the left hand figure is the left Riemann sum approximation and the shaded region in the right hand figure is the right Riemann sum approximation.



We can see that exactly because $f(x)$ is increasing, the left Riemann sum describes an area smaller than the definite integral while the right Riemann sum gives an area larger [a] than the integral.

When we approximate the integral $\int_0^1 e^x \mathrm{d}x$ using n subintervals, then, on interval number i,

- x runs from $\frac{i-1}{n}$ to $\frac{i}{n}$ and
- $y = e^x$ runs from $e^{\frac{(i-1)}{n}}$, when x is at the left hand end point of the interval, to $e^{\frac{i}{n}}$, when x is at the right hand end point of the interval.

Consequently, the left Riemann sum approximation to $\int_0^1 e^x \mathrm{d}x$ is $\sum_{i=1}^n e^{\frac{(i-1)}{n}} \frac{1}{n}$ and the right Riemann sum approximation is $\sum_{i=1}^n e^{\frac{i}{n}} \cdot \frac{1}{n}$. So

$$\sum_{i=1}^n e^{\frac{(i-1)}{n}} \frac{1}{n} \le \int_0^1 e^x \mathrm{d}x \le \sum_{i=1}^n e^{\frac{i}{n}} \cdot \frac{1}{n}$$

Thus $L_n = \sum_{i=1}^n e^{\frac{(i-1)}{n}} \frac{1}{n}$, which for any n can be evaluated by computer, is a lower bound on the exact value of $\int_0^1 e^x \mathrm{d}x$ and $R_n = \sum_{i=1}^n e^{\frac{i}{n}} \frac{1}{n}$, which for any n can also be evaluated by computer, is an upper bound on the exact value of $\int_0^1 e^x \mathrm{d}x$. For example, when $n = 1000$, $L_n = 1.7174$ and $R_n = 1.7191$ (both to four decimal places) so that, again to four decimal places,

$$1.7174 \le \int_0^1 e^x \mathrm{d}x \le 1.7191$$

Recall that the exact value is $e - 1 = 1.718281828\ldots$.

a When a function is decreasing the situation is reversed — the left Riemann sum is always larger than the integral while the right Riemann sum is smaller than the integral. For more general functions that both increase and decrease it is perhaps easiest to study each increasing (or decreasing) interval separately.

Example 1.1.12

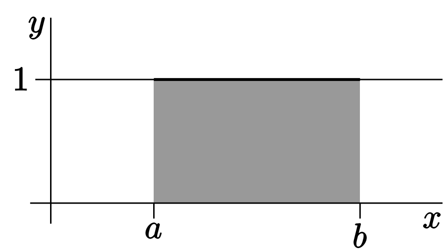
1.1.5 ▸▸ Using Known Areas to Evaluate Integrals

One of the main aims of this course is to build up general machinery for computing definite integrals (as well as interpreting and applying them). We shall start on this soon, but not quite yet. We have already seen one concrete, if laborious, method for computing definite integrals — taking limits of Riemann sums as we did in Example 1.1.1. A second method, which will work for some special integrands, works by interpreting the definite integral as "signed area". This approach will work nicely when the area under the curve decomposes into simple geometric shapes like triangles, rectangles and circles. Here are some examples of this second method.

Example 1.1.13 A very simple integral and a very simple area.

The integral $\int_a^b 1\mathrm{d}x$ (which is also written as just $\int_a^b \mathrm{d}x$) is the area of the shaded rectangle (of width $b-a$ and height 1) in the figure on the right below. So

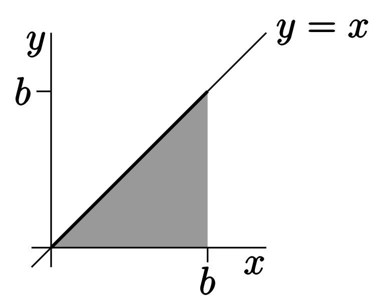
$\int_a^b \mathrm{d}x = (b-a)\times(1) = b-a$



Example 1.1.14 Another simple integral.

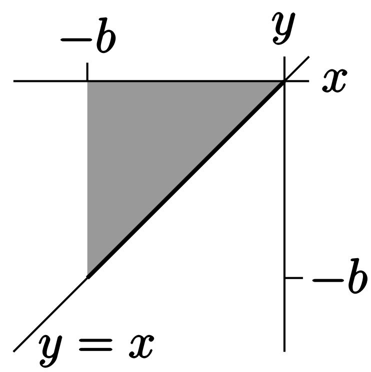
Let $b > 0$. The integral $\int_0^b x\mathrm{d}x$ is the area of the shaded triangle (of base b and of height b) in the figure on the right below. So

$\int_0^b x\mathrm{d}x = \frac{1}{2}b \times b = \frac{b^2}{2}$



The integral $\int_{-b}^0 x\mathrm{d}x$ is the signed area of the shaded triangle (again of base b and of height b) in the figure on the right below. So

$\int_{-b}^{0} x\mathrm{d}x = -\frac{b^2}{2}$



Example 1.1.14

Notice that it is very easy to extend this example to the integral $\int_0^b cx\mathrm{d}x$ for any real numbers $b, c > 0$ and find

$$\int_0^b cx\mathrm{d}x = \frac{c}{2}b^2.$$

Example 1.1.15 Evaluating $\int_{-1}^{1} (1 - |x|)\,\mathrm{d}x$.

In this example, we shall evaluate $\int_{-1}^{1} (1 - |x|)\,\mathrm{d}x$. Recall that

$$|x| = \begin{cases} -x & \text{if } x \le 0 \\ x & \text{if } x \ge 0 \end{cases}$$

so that

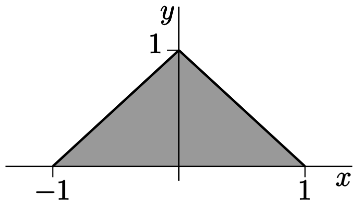
$$1 - |x| = \begin{cases} 1 + x & \text{if } x \le 0 \\ 1 - x & \text{if } x \ge 0 \end{cases}$$

To picture the geometric figure whose area the integral represents observe that

- at the left hand end of the domain of integration $x = -1$ and the integrand $1 - |x| = 1 - |-1| = 1 - 1 = 0$ and
- as x increases from -1 towards 0, the integrand $1 - |x| = 1 + x$ increases linearly, until
- when x hits 0 the integrand hits $1 - |x| = 1 - |0| = 1$ and then
- as x increases from 0, the integrand $1 - |x| = 1 - x$ decreases linearly, until
- when x hits $+1$, the right hand end of the domain of integration, the integrand hits $1 - |x| = 1 - |1| = 0$.

So the integral $\int_{-1}^{1}(1-|x|)\,\mathrm{d}x$ is the area of the shaded triangle (of base 2 and of height 1) in the figure on the right below and

$\int_{-1}^{1}(1-|x|)\,\mathrm{d}x = \frac{1}{2}\times 2\times 1 = 1$



Example 1.1.15

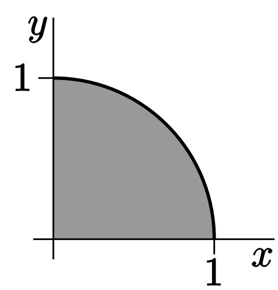
Example 1.1.16 Evaluating $\int_0^1 \sqrt{1-x^2}\mathrm{d}x$.

The integral $\int_0^1 \sqrt{1-x^2}\mathrm{d}x$ has integrand $f(x)=\sqrt{1-x^2}$. So it represents the area under $y=\sqrt{1-x^2}$ with x running from 0 to 1. But we may rewrite

$$y=\sqrt{1-x^2} \qquad \text{as} \qquad x^2+y^2=1, y\geq 0$$

But this is the (implicit) equation for a circle — the extra condition that $y \geq 0$ makes it the equation for the semi-circle centred at the origin with radius 1 lying on and above the x-axis. Thus the integral represents the area of the quarter circle of radius 1, as shown in the figure on the right below. So

$\int_0^1 \sqrt{1-x^2}\mathrm{d}x = \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$



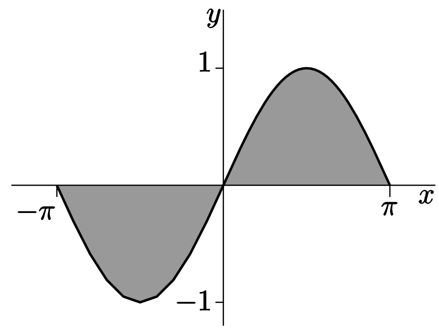
This next one is a little trickier and relies on us knowing the symmetries of the sine function.

Example 1.1.17 Integrating sine.

The integral $\int_{-\pi}^{\pi}\sin x\mathrm{d}x$ is the signed area of the shaded region in the figure on the right below. It naturally splits into two regions, one on either side of the y-axis. We don't know the formula for the area of either of these regions (yet), however the two regions are very nearly the same. In fact, the part of the shaded region below the x-axis is exactly the reflection, in the x-axis, of the part of the shaded region above the x-axis.

So the signed area of part of the shaded region below the x-axis is the negative of the signed area of part of the shaded region above the x-axis and

$\int_{-\pi}^{\pi} \sin x \mathrm{d}x = 0$



Example 1.1.17

1.1.6 ►► Another Interpretation for Definite Integrals

So far, we have only a single interpretation [6] for definite integrals — namely areas under graphs. In the following example, we develop a second interpretation.

Example 1.1.18 A moving particle.

Suppose that a particle is moving along the x-axis and suppose that at time t its velocity is $v(t)$ (with $v(t) > 0$ indicating rightward motion and $v(t) < 0$ indicating leftward motion). What is the change in its x-coordinate between time a and time $b > a$?

We'll work this out using a procedure similar to our definition of the integral. First pick a natural number n and divide the time interval from a to b into n equal subintervals, each of width $\frac{b-a}{n}$. We are working our way towards a Riemann sum (as we have done several times above) and so we will eventually take the limit $n \to \infty$.

- The first time interval runs from a to $a + \frac{b-a}{n}$. If we think of n as some large number, the width of this interval, $\frac{b-a}{n}$ is very small and over this time interval, the velocity does not change very much. Hence we can approximate the velocity over the first subinterval as being essentially constant at its value at the start of the time interval — $v(a)$. Over the subinterval the x-coordinate changes by velocity times time, namely $v(a) \cdot \frac{b-a}{n}$.
- Similarly, the second interval runs from time $a + \frac{b-a}{n}$ to time $a + 2\frac{b-a}{n}$. Again, we can assume that the velocity does not change very much and so we can approximate the velocity as being essentially constant at its value at the start of the

6 If this were the only interpretation then integrals would be a nice mathematical curiousity and unlikely to be the core topic of a large first year mathematics course.

subinterval — namely $v\left(a+\frac{b-a}{n}\right)$. So during the second subinterval the particle's x-coordinate changes by approximately $v\left(a+\frac{b-a}{n}\right)\frac{b-a}{n}$.

- In general, time subinterval number i runs from $a+(i-1)\frac{b-a}{n}$ to $a+i\frac{b-a}{n}$ and during this subinterval the particle's x-coordinate changes, essentially, by

$$,v\left(a+(i-1)\frac{b-a}{n}\right)\frac{b-a}{n}.$$

So the net change in x-coordinate from time a to time b is approximately

$$\begin{aligned}&v(a)\,\frac{b-a}{n}+v\Big(a+\frac{b-a}{n}\Big)\,\frac{b-a}{n}+\cdots+v\Big(a+(i-1)\frac{b-a}{n}\Big)\,\frac{b-a}{n}+\cdots\\&+v\Big(a+(n-1)\frac{b-a}{n}\Big)\,\frac{b-a}{n}\\&=\sum_{i=1}^{n}v\Big(a+(i-1)\frac{b-a}{n}\Big)\,\frac{b-a}{n}\end{aligned}$$

This exactly the left Riemann sum approximation to the integral of v from a to b with n subintervals. The limit as $n\to\infty$ is exactly the definite integral $\int_a^b v(t)\mathrm{d}t$. Following tradition, we have called the (dummy) integration variable t rather than x to remind us that it is time that is running from a to b.
The conclusion of the above discussion is that if a particle is moving along the x-axis and its x-coordinate and velocity at time t are $x(t)$ and $v(t)$, respectively, then, for all $b>a$,

$$x(b)-x(a)=\int_a^b v(t)\mathrm{d}t.$$

Example 1.1.18

1.1.7 ▸▸ Optional — careful definition of the integral

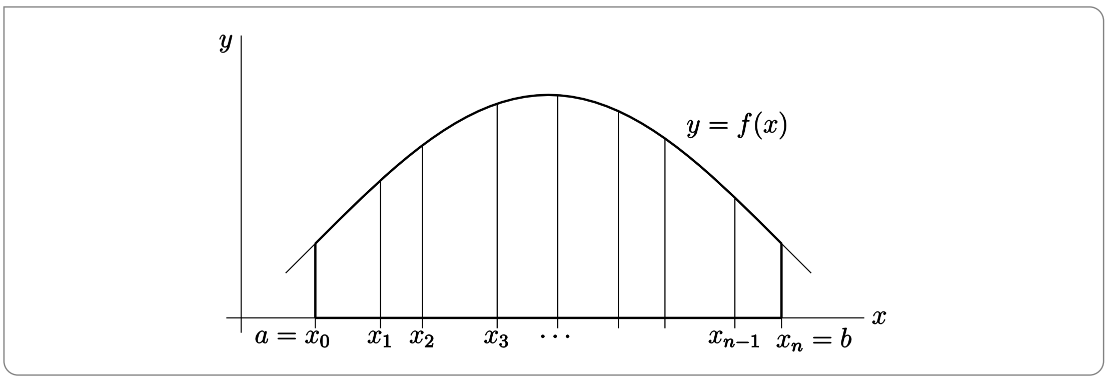
In this optional section we give a more mathematically rigorous definition of the definite integral $\displaystyle\int_a^b f(x)\mathrm{d}x$. Some textbooks use a sneakier, but equivalent, definition. The integral will be defined as the limit of a family of approximations to the area between the graph of $y=f(x)$ and the x-axis, with x running from a to b. We will then show conditions under which this limit is guaranteed to exist. We should state up front that these conditions are more restrictive than is strictly necessary — this is done so as to keep the proof accessible.

The family of approximations needed is slightly more general than that used to define Riemann sums in the previous sections, though it is quite similar. The main difference is that we do not require that all the subintervals have the same size.

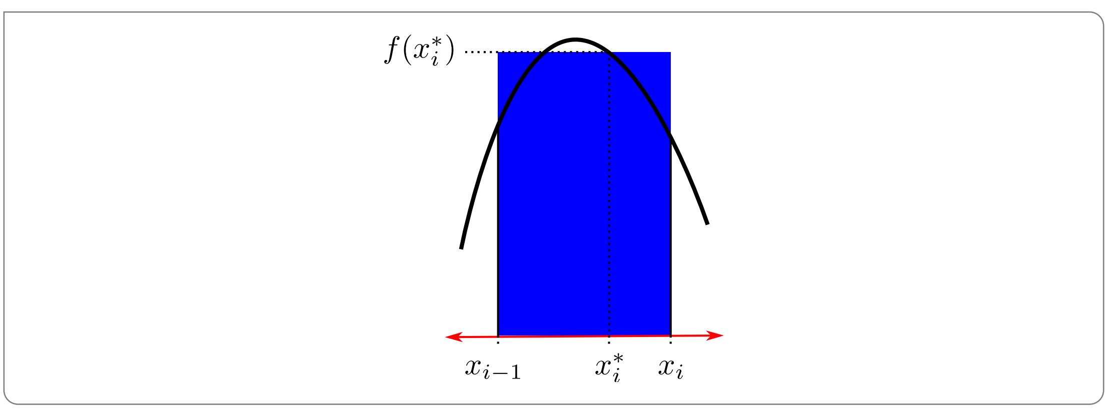
- We start by selecting a positive integer n. As was the case previously, this will be the number of subintervals used in the approximation and eventually we will take the limit as $n \to \infty$.
- Now subdivide the interval from a to b into n subintervals by selecting $n+1$ values of x that obey

  $$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

  The subinterval number i runs from x_{i-1} to x_i. This formulation does not require the subintervals to have the same size. However we will eventually require that the widths of the subintervals shrink towards zero as $n \to \infty$.
- Then for each subinterval we select a value of x in that interval. That is, for $i = 1, 2, \ldots, n$, choose x_i^* satisfying $x_{i-1} \le x_i^* \le x_i$. We will use these values of x to help approximate $f(x)$ on each subinterval.
- The area between the graph of $y = f(x)$ and the x-axis, with x running

  from x_{i-1} to x_i, i.e. the contribution, $\int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x$, from interval number i to the integral, is approximated by the area of a rectangle. The rectangle has width $x_i - x_{i-1}$ and height $f(x_i^*)$.

- Thus the approximation to the integral, using all n subintervals, is

  $$\int_a^b f(x)\mathrm{d}x \approx f(x_1^*)[x_1 - x_0] + f(x_2^*)[x_2 - x_1] + \cdots + f(x_n^*)[x_n - x_{n-1}]$$

- Of course every different choice of n and $x_1, x_2, \cdots, x_{n-1}$ and $x_1^*, x_2^*, \cdots, x_n^*$ gives a different approximation. So to simplify the discussion that follows, let us denote a particular choice of all these numbers by $\mathbb{P}$:

$$\mathbb{P} = (n, x_1, x_2, \cdots, x_{n-1}, x_1^*, x_2^*, \cdots, x_n^*) .$$

  Similarly let us denote the resulting approximation by $\mathcal{I}(\mathbb{P})$:

$$\mathcal{I}(\mathbb{P}) = f(x_1^*)[x_1 - x_0] + f(x_2^*)[x_2 - x_1] + \cdots + f(x_n^*)[x_n - x_{n-1}]$$

- We claim that, for any reasonable [7] function $f(x)$, if you take any reasonable [8] sequence of these approximations you always get the exactly the same limiting value. We define $\int_a^b f(x)\mathrm{d}x$ to be this limiting value.

- Let's be more precise. We can take the limit of these approximations in two equivalent ways. Above we did this by taking the number of subintervals n to infinity. When we did this, the width of all the subintervals went to zero. With the formulation we are now using, simply taking the number of subintervals to be very large does not imply that they will all shrink in size. We could have one very large subinterval and a large number of tiny ones. Thus we take the limit we need by taking the width of the subintervals to zero. So for any choice $\mathbb{P}$, we define

$$M(\mathbb{P}) = \max \left\{ x_1 - x_0 \ , \ x_2 - x_1 \ , \ \cdots \ , \ x_n - x_{n-1} \right\}$$

  that is the maximum width of the subintervals used in the approximation determined by $\mathbb{P}$. By forcing the maximum width to go to zero, the widths of all the subintervals go to zero.

- We then define the definite integral as the limit

$$\int_a^b f(x)\mathrm{d}x = \lim_{M(\mathbb{P})\to 0} \mathcal{I}(\mathbb{P}).$$

Of course, one is now left with the question of determining when the above limit exists. A proof of the very general conditions which guarantee existence of this limit is beyond the scope of this course, so we instead give a weaker result (with stronger conditions) which is far easier to prove.

For the rest of this section, assume

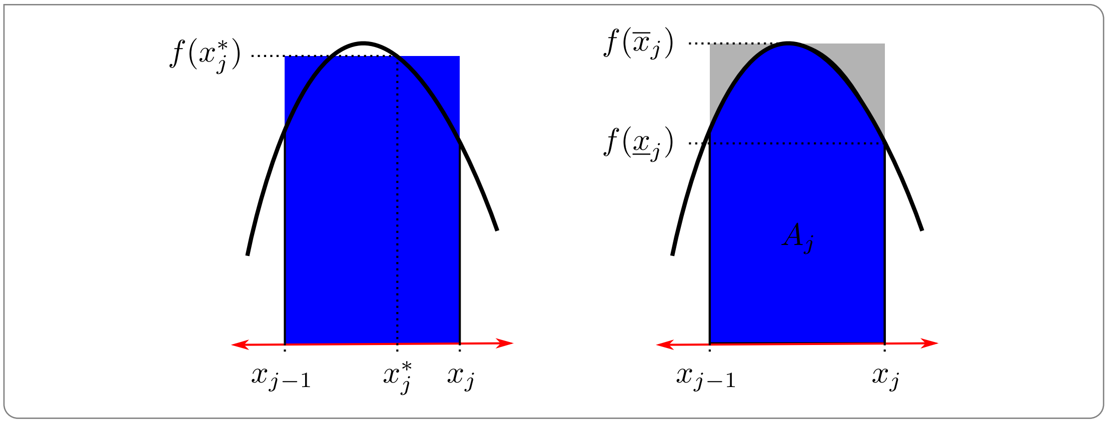
- that $f(x)$ is continuous for $a \le x \le b$,
- that $f(x)$ is differentiable for $a < x < b$, and
- that $f'(x)$ is bounded — ie $|f'(x)| \le F$ for some constant F.

7 We'll be more precise about what "reasonable" means shortly.
8 Again, we'll explain this "reasonable" shortly

We will now show that, under these hypotheses, as $M(\mathbb{P})$ approaches zero, $\mathcal{I}(\mathbb{P})$ always approaches the area, A, between the graph of $y = f(x)$ and the x-axis, with x running from a to b.

These assumptions are chosen to make the argument particularly transparent. With a little more work one can weaken the hypotheses considerably. We are cheating a little by implicitly assuming that the area A exists. In fact, one can adjust the argument below to remove this implicit assumption.

- Consider A_j, the part of the area coming from $x_{j-1} \le x \le x_j$.



  We have approximated this area by $f(x_j^*)[x_j - x_{j-1}]$ (see figure left).

- Let $f(\overline{x}_j)$ and $f(\underline{x}_j)$ be the largest and smallest values [9] of $f(x)$ for $x_{j-1} \le x \le x_j$. Then the true area is bounded by

$$f(\underline{x}_j)[x_j - x_{j-1}] \le A_j \le f(\overline{x}_j)[x_j - x_{j-1}].$$

  (see figure right).

- Now since $f(\underline{x}_j) \le f(x_j^*) \le f(\overline{x}_j)$, we also know that

$$f(\underline{x}_j)[x_j - x_{j-1}] \le f(x_j^*)[x_j - x_{j-1}] \le f(\overline{x}_j)[x_j - x_{j-1}].$$

- So both the true area, A_j, and our approximation of that area $f(x_j^*)[x_j - x_{j-1}]$ have to lie between $f(\overline{x}_j)[x_j - x_{j-1}]$ and $f(\underline{x}_j)[x_j - x_{j-1}]$. Combining these bounds we have that the difference between the true area and our approximation of that area is bounded by

$$\big|A_j - f(x_j^*)[x_j - x_{j-1}]\big| \le [f(\overline{x}_j) - f(\underline{x}_j)] \cdot [x_j - x_{j-1}].$$

  (To see this think about the smallest the true area can be and the largest our approximation can be and vice versa.)

9 Here we are using the extreme value theorem — its proof is beyond the scope of this course. The theorem says that any continuous function on a closed interval must attain a minimum and maximum at least once. In this situation this implies that for any continuous function $f(x)$, there are $x_{j-1} \le \overline{x}_j, \underline{x}_j \le x_j$ such that $f(\underline{x}_j) \le f(x) \le f(\overline{x}_j)$ for all $x_{j-1} \le x \le x_j$.

- Now since our function, $f(x)$ is differentiable we can apply one of the main theorems we learned in CLP-1 — the Mean Value Theorem [10]. The MVT implies that there exists a c between $\underline{x}_j$ and $\overline{x}_j$ such that

$$f(\overline{x}_j) - f(\underline{x}_j) = f'(c) \cdot [\overline{x}_j - \underline{x}_j]$$

- By the assumption that $|f'(x)| \leq F$ for all x and the fact that $\underline{x}_j$ and $\overline{x}_j$ must both be between x_{j-1} and x_j

$$\left|f(\overline{x}_j) - f(\underline{x}_j)\right| \leq F \cdot \left|\overline{x}_j - \underline{x}_j\right| \leq F \cdot [x_j - x_{j-1}]$$

Hence the error in this part of our approximation obeys

$$\left|A_j - f(x_j^*)[x_j - x_{j-1}]\right| \leq F \cdot [x_j - x_{j-1}]^2.$$

- That was just the error in approximating A_j. Now we bound the total error by combining the errors from approximating on all the subintervals. This gives

$$\begin{aligned}
|A - \mathcal{I}(\mathbb{P})| &= \left|\sum_{j=1}^{n} A_j - \sum_{j=1}^{n} f(x_j^*)[x_j - x_{j-1}]\right| \\
&= \left|\sum_{j=1}^{n} \big(A_j - f(x_j^*)[x_j - x_{j-1}]\big)\right| && \text{triangle inequality} \\
&\leq \sum_{j=1}^{n} \left|A_j - f(x_j^*)[x_j - x_{j-1}]\right| \\
&\leq \sum_{j=1}^{n} F \cdot [x_j - x_{j-1}]^2 && \text{from above}
\end{aligned}$$

Now do something a little sneaky. Replace one of these factors of $[x_j - x_{j-1}]$ (which is just the width of the j^{th} subinterval) by the maximum width of the subintervals:

$$\begin{aligned}
&\leq \sum_{j=1}^{n} F \cdot M(\mathbb{P}) \cdot [x_j - x_{j-1}] && F \text{ and } M(\mathbb{P}) \text{ are constant} \\
&\leq F \cdot M(\mathbb{P}) \cdot \sum_{j=1}^{n} [x_j - x_{j-1}] && \text{sum is total width} \\
&= F \cdot M(\mathbb{P}) \cdot (b - a).
\end{aligned}$$

- Since a, b and F are fixed, this tends to zero as the maximum rectangle width $M(\mathbb{P})$ tends to zero.

Thus, we have proven

10 Recall that the mean value theorem states that for a function continuous on $[a, b]$ and differentiable on (a, b), there exists a number c between a and b so that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

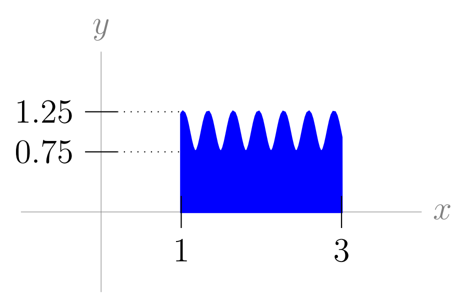
Theorem 1.1.19

Assume that $f(x)$ is continuous for $a \leq x \leq b$, and is differentiable for all $a < x < b$ with $|f'(x)| \leq F$, for some constant F. Then, as the maximum rectangle width $M(\mathbb{P})$ tends to zero, $\mathcal{I}(\mathbb{P})$ always converges to A, the area between the graph of $y = f(x)$ and the x-axis, with x running from a to b.

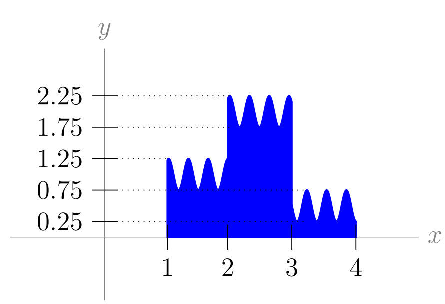
1.1.8 ▸▸ Exercises

Exercises — Stage 1 For Questions 1 through 5, we want you to develop an understanding of the model we are using to define an integral: we approximate the area under a curve by bounding it between rectangles. Later, we will learn more sophisticated methods of integration, but they are all based on this simple concept.In Questions 6 through 10, we practice using sigma notation. There are many ways to write a given sum in sigma notation. You can practice finding several, and deciding which looks the clearest.Questions 11 through 15 are meant to give you practice interpreting the formulas in Definition 1.1.11. The formulas might look complicated at first, but if you understand what each piece means, they are easy to learn.

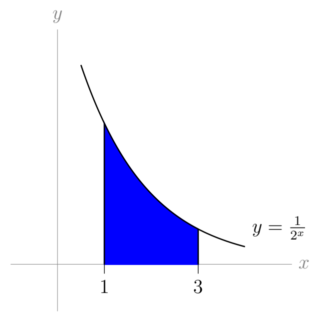
1. Give a range of possible values for the shaded area in the picture below.



2. Give a range of possible values for the shaded area in the picture below.



3. Using rectangles, find a lower and upper bound for $\displaystyle\int_1^3 \frac{1}{2^x}\mathrm{d}x$ that differ by at most 0.2 square units.



4. Let $f(x)$ be a function that is *decreasing* from $x = 0$ to $x = 5$. Which Riemann sum approximation of $\displaystyle\int_0^5 f(x)\mathrm{d}x$ is the largest–left, right, or midpoint?

5. Give an example of a function $f(x)$, an interval $[a, b]$, and a number n such that the midpoint Riemann sum of $f(x)$ over $[a, b]$ using n intervals is *larger than* both the left and right Riemann sums of $f(x)$ over $[a, b]$ using n intervals.

6. Express the following sums in sigma notation:

a $3 + 4 + 5 + 6 + 7$

b $6 + 8 + 10 + 12 + 14$

c $7 + 9 + 11 + 13 + 15$

d $1 + 3 + 5 + 7 + 9 + 11 + 13 + 15$

7. Express the following sums in sigma notation:

a $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}$

b $\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81}$

c $-\frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \frac{2}{81}$

d $\frac{2}{3} - \frac{2}{9} + \frac{2}{27} - \frac{2}{81}$

8. Express the following sums in sigma notation:

a $\frac{1}{3} + \frac{1}{3} + \frac{5}{27} + \frac{7}{81} + \frac{9}{243}$

b $\frac{1}{5} + \frac{1}{11} + \frac{1}{29} + \frac{1}{83} + \frac{1}{245}$

c $1000 + 200 + 30 + 4 + \frac{1}{2} + \frac{3}{50} + \frac{7}{1000}$

9. Evaluate the following sums. You might want to use the formulas from Theorems 5 and 6.

a $\displaystyle\sum_{i=0}^{100}\left(\frac{3}{5}\right)^i$

b $\displaystyle\sum_{i=50}^{100}\left(\frac{3}{5}\right)^i$

c $\displaystyle\sum_{i=1}^{10}\left(i^2-3i+5\right)$

d $\displaystyle\sum_{n=1}^{b}\left[\left(\frac{1}{e}\right)^n+en^3\right]$, where b is some integer greater than 1.

10. Evaluate the following sums. You might want to use the formulas from Theorem 1.1.6.

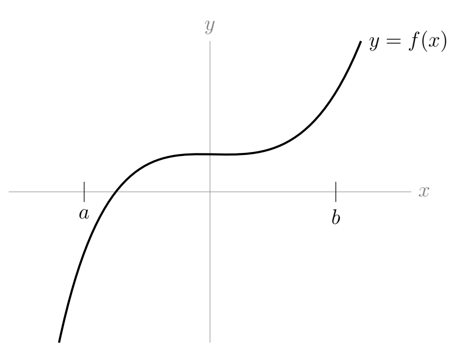
a $\displaystyle\sum_{i=50}^{100}(i-50)+\sum_{i=0}^{50}i$

b $\displaystyle\sum_{i=10}^{100}(i-5)^3$

c $\displaystyle\sum_{n=1}^{11}(-1)^n$

d $\displaystyle\sum_{n=2}^{11}(-1)^{2n+1}$

11. In the picture below, draw in the rectangles whose (signed) area is being computed by the midpoint Riemann sum $\displaystyle\sum_{i=1}^{4}\frac{b-a}{4}\cdot f\left(a+\left(i-\frac{1}{2}\right)\frac{b-a}{4}\right)$.



12. $*$. $\sum_{k=1}^{4} f(1+k)\cdot 1$ is a left Riemann sum for a function $f(x)$ on the interval $[a,b]$ with n subintervals. Find the values of a, b and n.

13. Draw a picture illustrating the area given by the following Riemann sum.

$$\sum_{i=1}^{3} 2\cdot(5+2i)^2$$

14. Draw a picture illustrating the area given by the following Riemann sum.

$$\sum_{i=1}^{5} \frac{\pi}{20}\cdot\tan\left(\frac{\pi(i-1)}{20}\right)$$

15. $*$. Fill in the blanks with right, left, or midpoint; an interval; and a value of n.

a $\sum_{k=0}^{3} f(1.5+k)\cdot 1$ is a ________ Riemann sum for f on the interval [_____ , _____] with $n=$ ____.

16. Evaluate the following integral by interpreting it as a signed area, and using geometry:

$$\int_0^5 x\,\mathrm{d}x$$

17. Evaluate the following integral by interpreting it as a signed area, and using geometry:

$$\int_{-2}^{5} x\,\mathrm{d}x$$

Exercises — Stage 2 Questions 26 and 27 use the formula for a geometric sum, Equation 1.1.3Remember that a definite integral is a signed area between a curve and the x-axis. We'll spend a lot of time learning strategies for evaluating definite integrals, but we already know lots of ways to find area of geometric shapes. In Questions 28 through 33, use your knowledge of geometry to find the signed areas described by the integrals given.

18. $*$. Use sigma notation to write the midpoint Riemann sum for $f(x)=x^8$ on $[5,15]$ with $n=50$. Do not evaluate the Riemann sum.

19. $*$. Estimate $\int_{-1}^{5} x^3\,\mathrm{d}x$ using three approximating rectangles and left hand end points.

20. $*$. Let f be a function on the whole real line. Express $\displaystyle\int_{-1}^{7} f(x)\,\mathrm{d}x$ as a limit of Riemann sums, using the right endpoints.

21. $*$. The value of the following limit is equal to the area below a graph of $y = f(x)$, integrated over the interval $[0, b]$:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\left[\sin\left(2 + \frac{4i}{n}\right)\right]^2$$

Find $f(x)$ and b.

22. $*$. For a certain function $f(x)$, the following equation holds:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2}\sqrt{1 - \frac{k^2}{n^2}} = \int_0^1 f(x)\ \mathrm{d}x$$

Find $f(x)$.

23. $*$. Express $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} e^{-i/n} \cos\left(\frac{3i}{n}\right)$ as a definite integral.

24. $*$. Let $R_n = \displaystyle\sum_{i=1}^{n} \frac{ie^{i/n}}{n^2}$. Express $\displaystyle\lim_{n\to\infty} R_n$ as a definite integral. *Do not evaluate this integral.*

25. $*$. Express $\displaystyle\lim_{n\to\infty} \left(\sum_{i=1}^{n} e^{-1-2i/n} \cdot \frac{2}{n}\right)$ as an integral in three different ways.

26. Evaluate the sum $1 + r^3 + r^6 + r^9 + \cdots + r^{3n}$.

27. Evaluate the sum $r^5 + r^6 + r^7 + \cdots + r^{100}$.

28. $*$. Evaluate $\displaystyle\int_{-1}^{2} |2x|\ \mathrm{d}x$.

29. Evaluate the following integral by interpreting it as a signed area, and using geometry:

$$\int_{-3}^{5} |t - 1|\,\mathrm{d}t$$

30. Evaluate the following integral by interpreting it as a signed area, and using geometry:

$$\int_a^b x \, \mathrm{d}x$$

where $0 \le a \le b$.

31. Evaluate the following integral by interpreting it as a signed area, and using geometry:

$$\int_a^b x \, \mathrm{d}x$$

where $a \le b \le 0$.

32. Evaluate the following integral by interpreting it as a signed area, and using geometry:

$$\int_0^4 \sqrt{16 - x^2} \mathrm{d}x$$

33. $*$. Use elementary geometry to calculate $\displaystyle\int_0^3 f(x) \, \mathrm{d}x$, where

$$f(x) = \begin{cases} x, & \text{if } x \le 1, \\ 1, & \text{if } x > 1. \end{cases}$$

34. $*$. A car's gas pedal is applied at $t = 0$ seconds and the car accelerates continuously until $t = 2$ seconds. The car's speed at half-second intervals is given in the table below. Find the best possible ***upper*** estimate for the distance that the car traveled during these two seconds.

t (s)	0	0.5	1.0	1.5	2
v (m/s)	0	14	22	30	40

35. True or false: the answer you gave for Question 34 is definitely greater than or equal to the distance the car travelled during the two seconds in question.

36. An airplane's speed at one-hour intervals is given in the table below. Approximate the distance travelled by the airplane from noon to 4pm using a midpoint Riemann sum.

time	12:00 pm	1:00 pm	2:00 pm	3:00 pm	4:00 pm
speed (km/hr)	800	700	850	900	750

Exercises — Stage 3

37. $*$. (a) Express

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\sqrt{4-\left(-2+\frac{2i}{n}\right)^2}$$

as a definite integal.
(b) Evaluate the integral of part (a).

38. $*$. Consider the integral:

$$\int_0^3 (7+x^3)\,\mathrm{d}x. \qquad (*)$$

a Approximate this integral using the left Riemann sum with $n = 3$ intervals.

b Write down the expression for the right Riemann sum with n intervals and calculate the sum. Now take the limit $n \to \infty$ in your expression for the Riemann sum, to evaluate the integral $(*)$ exactly.

You may use the identity

$$\sum_{i=1}^{n} i^3 = \frac{n^4+2n^3+n^2}{4}$$

39. $*$. Using a limit of right–endpoint Riemann sums, evaluate $\displaystyle\int_2^4 x^2\,\mathrm{d}x$. You may use the formulas $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

40. $*$. Find $\displaystyle\int_0^2 (x^3+x)\,\mathrm{d}x$ using the definition of the definite integral. You may use the summation formulas $\sum_{i=1}^{n} i^3 = \frac{n^4+2n^3+n^2}{4}$ and $\sum_{i=1}^{n} i = \frac{n^2+n}{2}$.

41. $*$. Using a limit of right-endpoint Riemann sums, evaluate $\displaystyle\int_1^4 (2x-1)\,\mathrm{d}x$. Do not use anti-differentiation, except to check your answer. [a] You may use the formula $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

a You'll learn about this method starting in Section 1.3. You can also check this answer using geometry.

42. Give a function $f(x)$ that has the following expression as a right Riemann sum when $n = 10$, $\Delta(x) = 10$ and $a = -5$:

$$\sum_{i=1}^{10} 3(7+2i)^2 \sin(4i)\,.$$

43. Using the method of Example 1.1.2, evaluate

$$\int_0^1 2^x \mathrm{d}x$$

44.

a Using the method of Example 1.1.2, evaluate

$$\int_a^b 10^x \mathrm{d}x$$

b

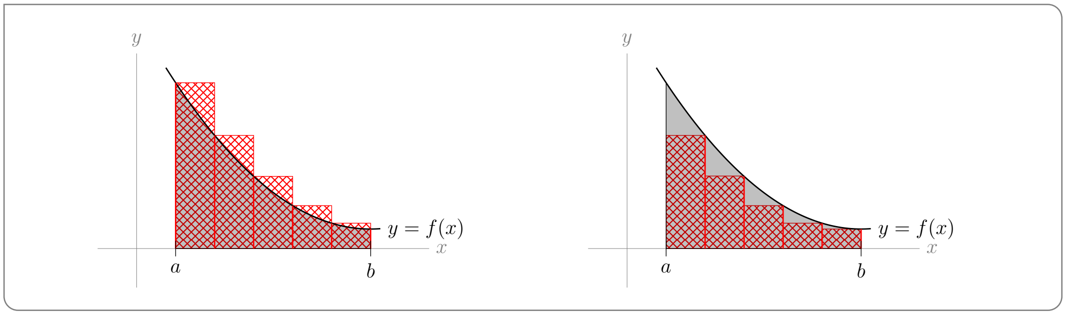
Using your answer from above, make a guess for

$$\int_a^b c^x \mathrm{d}x$$

where c is a positive constant. Does this agree with Question 43?

45. Evaluate $\int_0^a \sqrt{1-x^2}\mathrm{d}x$ using geometry, if $0 \le a \le 1$.

46. Suppose $f(x)$ is a positive, decreasing function from $x = a$ to $x = b$. You give an upper and lower bound on the area under the curve $y = f(x)$ using n rectangles and a left and right Riemann sum, respectively, as in the picture below.



a What is the difference between the lower bound and the upper bound? (That is, if we subtract the smaller estimate from the larger estimate, what do we get?) Give your answer in terms of f, a, b, and n.

b If you want to approximate the area under the curve to within 0.01 square units using this method, how many rectangles should you use? That is, what should n be?

47. Let $f(x)$ be a linear function, let $a < b$ be integers, and let n be a whole number. True or false: if we average the left and right Riemann sums for $\int_a^b f(x)\mathrm{d}x$ using n rectangles, we get the same value as the midpoint Riemann sum using n rectangles.

1.2▲ Basic properties of the definite integral

When we studied limits and derivatives, we developed methods for taking limits or derivatives of "complicated functions" like $f(x) = x^2 + \sin(x)$ by understanding how limits and derivatives interact with basic arithmetic operations like addition and subtraction. This allowed us to reduce the problem into one of of computing derivatives of simpler functions like x^2 and $\sin(x)$. Along the way we established simple rules such as

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}(f(x)+g(x)) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x}$$

Some of these rules have very natural analogues for integrals and we discuss them below. Unfortunately the analogous rules for integrals of products of functions or integrals of compositions of functions are more complicated than those for limits or derivatives. We discuss those rules at length in subsequent sections. For now let us consider some of the simpler rules of the arithmetic of integrals.

Theorem 1.2.1 Arithmetic of Integration.

Let a, b and A, B, C be real numbers. Let the functions $f(x)$ and $g(x)$ be integrable on an interval that contains a and b. Then

(a) $$\int_a^b (f(x)+g(x))\,\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x$$

(b) $$\int_a^b (f(x)-g(x))\,\mathrm{d}x = \int_a^b f(x)\mathrm{d}x - \int_a^b g(x)\mathrm{d}x$$

(c) $$\int_a^b Cf(x)\mathrm{d}x = C\cdot\int_a^b f(x)\mathrm{d}x$$

Combining these three rules we have

(d) $$\int_a^b (Af(x)+Bg(x))\,\mathrm{d}x = A\int_a^b f(x)\mathrm{d}x + B\int_a^b g(x)\mathrm{d}x$$

That is, integrals depend linearly on the integrand.

(e) $$\int_a^b \mathrm{d}x = \int_a^b 1\cdot\mathrm{d}x = b-a$$

It is not too hard to prove this result from the definition of the definite integral. Additionally we only really need to prove (d) and (e) since

- (a) follows from (d) by setting $A = B = 1$,
- (b) follows from (d) by setting $A = 1, B = -1$, and
- (c) follows from (d) by setting $A = C, B = 0$.

Proof. As noted above, it suffices for us to prove (d) and (e). Since (e) is easier, we will start with that. It is also a good warm-up for (d).

- The definite integral in (e), $\int_a^b 1\mathrm{d}x$, can be interpreted geometrically as the area of the rectangle with height 1 running from $x = a$ to $x = b$; this area is clearly $b - a$. We can also prove this formula from the definition of the integral (Definition 1.1.9):

$$\begin{aligned}
\int_a^b \mathrm{d}x &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i,n}^*)\,\frac{b-a}{n} && \text{by definition} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} 1\,\frac{b-a}{n} && \text{since } f(x) = 1 \\
&= \lim_{n\to\infty} (b-a) \sum_{i=1}^{n} \frac{1}{n} && \text{since } a, b \text{ are constants} \\
&= \lim_{n\to\infty} (b-a) \\
&= b - a
\end{aligned}$$

  as required.

- To prove (d) let us start by defining $h(x) = Af(x)+Bg(x)$ and then we need to express the integral of $h(x)$ in terms of those of $f(x)$ and $g(x)$. We use Definition 1.1.9 and some algebraic manipulations [a] to arrive at the result.

$$\begin{aligned}
\int_a^b h(x)\mathrm{d}x &= \sum_{i=1}^{n} h(x_{i,n}^*) \cdot \frac{b-a}{n} \\
&\qquad \text{by Definition 1.1.9} \\
&= \sum_{i=1}^{n} \big(Af(x_{i,n}^*) + Bg(x_{i,n}^*)\big) \cdot \frac{b-a}{n} \\
&= \sum_{i=1}^{n} \left(Af(x_{i,n}^*) \cdot \frac{b-a}{n} + Bg(x_{i,n}^*) \cdot \frac{b-a}{n}\right) \\
&= \left(\sum_{i=1}^{n} Af(x_{i,n}^*) \cdot \frac{b-a}{n}\right) + \left(\sum_{i=1}^{n} Bg(x_{i,n}^*) \cdot \frac{b-a}{n}\right) \\
&\qquad \text{by Theorem 1.1.5(b)}
\end{aligned}$$

$$= A\left(\sum_{i=1}^{n} f(x_{i,n}^*)\cdot\frac{b-a}{n}\right) + B\left(\sum_{i=1}^{n} g(x_{i,n}^*)\cdot\frac{b-a}{n}\right)$$

by Theorem 1.1.5(a)

$$= A\int_a^b f(x)\mathrm{d}x + B\int_a^b g(x)\mathrm{d}x$$

by Definition 1.1.9

as required.

□

a Now is a good time to look back at Theorem 1.1.5.

Using this Theorem we can integrate sums, differences and constant multiples of functions we know how to integrate. For example:

Example 1.2.2 The integral of a sum.

In Example 1.1.1 we saw that $\int_0^1 e^x\mathrm{d}x = e-1$. So

$$\int_0^1 \left(e^x+7\right)\mathrm{d}x = \int_0^1 e^x\mathrm{d}x + 7\int_0^1 1\mathrm{d}x$$

by Theorem 1.2.1(d) with $A=1, f(x)=e^x, B=7, g(x)=1$

$$= (e-1) + 7\times(1-0)$$

by Example 1.1.1 and Theorem 1.2.1(e)

$$= e+6$$

When we gave the formal definition of $\int_a^b f(x)\mathrm{d}x$ in Definition 1.1.9 we explained that the integral could be interpreted as the signed area between the curve $y=f(x)$ and the x-axis on the interval $[a,b]$. In order for this interpretation to make sense we required that $a<b$, and though we remarked that the integral makes sense when $a>b$ we did not explain any further. Thankfully there is an easy way to express the integral $\int_a^b f(x)\mathrm{d}x$ in terms of $\int_b^a f(x)\mathrm{d}x$ — making it always possible to write an integral so the lower limit of integration is less than the upper limit of integration. Theorem 1.2.3, below, tell us that, for example, $\int_7^3 e^x\mathrm{d}x = -\int_3^7 e^x\mathrm{d}x$. The same theorem also provides us with two other simple manipulations of the limits of integration.

Theorem 1.2.3 Arithmetic for the Domain of Integration.

Let a, b, c be real numbers. Let the function $f(x)$ be integrable on an interval that contains a, b and c. Then

(a) $$\int_a^a f(x)\mathrm{d}x = 0$$

(b) $$\int_b^a f(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x$$

(c) $$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$$

The proof of this statement is not too difficult.

Proof. Let us prove the statements in order.

- Consider the definition of the definite integral

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i,n}^*) \cdot \frac{b-a}{n}$$

If we now substitute $b = a$ in this expression we have

$$\begin{aligned}\int_a^a f(x)\mathrm{d}x &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i,n}^*) \cdot \underbrace{\frac{a-a}{n}}_{=0} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(x_{i,n}^*) \cdot 0}_{=0} \\ &= \lim_{n\to\infty} 0 \\ &= 0\end{aligned}$$

as required.

- Consider now the definite integral $\int_a^b f(x)\mathrm{d}x$. We will sneak up on the proof by first examining Riemann sum approximations to both this and $\int_b^a f(x)\mathrm{d}x$. The midpoint Riemann sum approximation to $\int_a^b f(x)\mathrm{d}x$ with 4 subintervals (so that each subinterval has width $\frac{b-a}{4}$) is

$$\begin{aligned}&\left\{f\left(a+\tfrac{1}{2}\tfrac{b-a}{4}\right) + f\left(a+\tfrac{3}{2}\tfrac{b-a}{4}\right) + f\left(a+\tfrac{5}{2}\tfrac{b-a}{4}\right) + f\left(a+\tfrac{7}{2}\tfrac{b-a}{4}\right)\right\} \cdot \tfrac{b-a}{4} \\ &= \left\{f\left(\tfrac{7}{8}a+\tfrac{1}{8}b\right) + f\left(\tfrac{5}{8}a+\tfrac{3}{8}b\right) + f\left(\tfrac{3}{8}a+\tfrac{5}{8}b\right) + f\left(\tfrac{1}{8}a+\tfrac{7}{8}b\right)\right\} \cdot \tfrac{b-a}{4}\end{aligned}$$

Now we do the same for $\int_b^a f(x)\mathrm{d}x$ with 4 subintervals. Note that b is now the lower limit on the integral and a is now the upper limit on the integral.

This is likely to cause confusion when we write out the Riemann sum, so we'll temporarily rename b to A and a to B. The midpoint Riemann sum approximation to $\int_A^B f(x)\mathrm{d}x$ with 4 subintervals is

$$\begin{aligned}&\Big\{f\big(A+\tfrac{1}{2}\tfrac{B-A}{4}\big)+f\big(A+\tfrac{3}{2}\tfrac{B-A}{4}\big)+f\big(A+\tfrac{5}{2}\tfrac{B-A}{4}\big)\\&\qquad\qquad+f\big(A+\tfrac{7}{2}\tfrac{B-A}{4}\big)\Big\}\cdot\tfrac{B-A}{4}\\&=\Big\{f\big(\tfrac{7}{8}A+\tfrac{1}{8}B\big)+f\big(\tfrac{5}{8}A+\tfrac{3}{8}B\big)+f\big(\tfrac{3}{8}A+\tfrac{5}{8}B\big)\\&\qquad\qquad+f\big(\tfrac{1}{8}A+\tfrac{7}{8}B\big)\Big\}\cdot\tfrac{B-A}{4}\end{aligned}$$

Now recalling that $A=b$ and $B=a$, we have that the midpoint Riemann sum approximation to $\int_b^a f(x)\mathrm{d}x$ with 4 subintervals is

$$\big\{f\big(\tfrac{7}{8}b+\tfrac{1}{8}a\big)+f\big(\tfrac{5}{8}b+\tfrac{3}{8}a\big)+f\big(\tfrac{3}{8}b+\tfrac{5}{8}a\big)+f\big(\tfrac{1}{8}b+\tfrac{7}{8}a\big)\big\}\cdot\tfrac{a-b}{4}$$

Thus we see that the Riemann sums for the two integrals are nearly identical — the only difference being the factor of $\frac{b-a}{4}$ versus $\frac{a-b}{4}$. Hence the two Riemann sums are negatives of each other.

The same computation with n subintervals shows that the midpoint Riemann sum approximations to $\int_b^a f(x)\mathrm{d}x$ and $\int_a^b f(x)\mathrm{d}x$ with n subintervals are negatives of each other. Taking the limit $n\to\infty$ gives $\int_b^a f(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x$.

- Finally consider (c) — we will not give a formal proof of this, but instead will interpret it geometrically. Indeed one can also interpret (a) geometrically. In both cases these become statements about areas:

$$\int_a^a f(x)\mathrm{d}x=0 \qquad \text{and} \qquad \int_a^b f(x)\mathrm{d}x=\int_a^c f(x)\mathrm{d}x+\int_c^b f(x)\mathrm{d}x$$

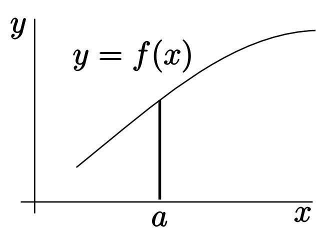
are

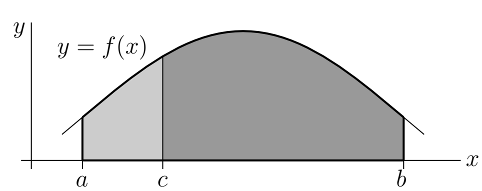
$$\text{Area}\big\{\ (x,y)\ \big|\ a\le x\le a,\ 0\le y\le f(x)\ \big\}=0$$

and

$$\begin{aligned}&\text{Area}\big\{\ (x,y)\ \big|\ a\le x\le b,\ 0\le y\le f(x)\ \big\}\\&\quad=\text{Area}\big\{\ (x,y)\ \big|\ a\le x\le c,\ 0\le y\le f(x)\ \big\}\\&\qquad+\text{Area}\big\{\ (x,y)\ \big|\ c\le x\le b,\ 0\le y\le f(x)\ \big\}\end{aligned}$$

respectively. Both of these geometric statements are intuitively obvious. See the figures below.





Note that we have assumed that $a \le c \le b$ and that $f(x) \ge 0$. One can remove these restrictions and also make the proof more formal, but it becomes quite tedious and less intuitive.

$\square$

Remark 1.2.4 For notational simplicity, let's assume that $a \le c \le b$ and $f(x) \ge 0$ for all $a \le x \le b$. The geometric interpretations of the identities

$$\int_a^a f(x)\mathrm{d}x = 0 \quad \text{and} \quad \int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$$

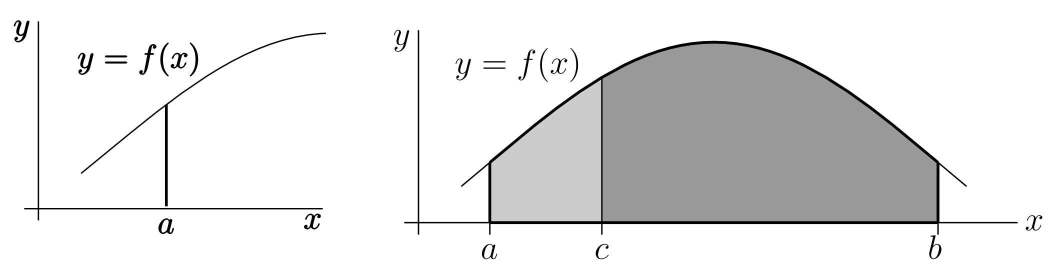
are

$$\text{Area}\big\{\ (x,y)\ \big|\ a \le x \le a,\ 0 \le y \le f(x)\ \big\} = 0$$

and

$$\begin{aligned} &\text{Area}\big\{\ (x,y)\ \big|\ a \le x \le b,\ 0 \le y \le f(x)\ \big\} \\ &\quad = \text{Area}\big\{\ (x,y)\ \big|\ a \le x \le c,\ 0 \le y \le f(x)\ \big\} \\ &\qquad + \text{Area}\big\{\ (x,y)\ \big|\ c \le x \le b,\ 0 \le y \le f(x)\ \big\} \end{aligned}$$

respectively. Both of these geometric statements are intuitively obvious. See the figures below. We won't give a formal proof.



So we concentrate on the formula $\int_b^a f(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x$. The midpoint Riemann sum approximation to $\int_a^b f(x)\mathrm{d}x$ with 4 subintervals (so that each subinterval has width $\frac{b-a}{4}$) is

$$\begin{aligned} &\Big\{ f\big(a + \tfrac{1}{2}\tfrac{b-a}{4}\big) + f\big(a + \tfrac{3}{2}\tfrac{b-a}{4}\big) + f\big(a + \tfrac{5}{2}\tfrac{b-a}{4}\big) + f\big(a + \tfrac{7}{2}\tfrac{b-a}{4}\big) \Big\}\ \tfrac{b-a}{4} \\ &= \Big\{ f\big(\tfrac{7}{8}a + \tfrac{1}{8}b\big) + f\big(\tfrac{5}{8}a + \tfrac{3}{8}b\big) + f\big(\tfrac{3}{8}a + \tfrac{5}{8}b\big) + f\big(\tfrac{1}{8}a + \tfrac{7}{8}b\big) \Big\}\ \tfrac{b-a}{4} \end{aligned} \tag{$\star$}$$

We're now going to write out the midpoint Riemann sum approximation to $\int_b^a f(x)\mathrm{d}x$ with 4 subintervals. Note that b is now the lower limit on the integral and a is now the upper limit on the integral. This is likely to cause confusion when we write out the Riemann sum, so we'll temporarily rename b to A and a to B. The midpoint Riemann sum approximation to $\int_A^B f(x)\mathrm{d}x$ with 4 subintervals is

$$\Big\{ f\big(A + \tfrac{1}{2}\tfrac{B-A}{4}\big) + f\big(A + \tfrac{3}{2}\tfrac{B-A}{4}\big) + f\big(A + \tfrac{5}{2}\tfrac{B-A}{4}\big) + f\big(A + \tfrac{7}{2}\tfrac{B-A}{4}\big) \Big\}\ \tfrac{B-A}{4}$$

$$= \left\{f\big(\tfrac{7}{8}A + \tfrac{1}{8}B\big) + f\big(\tfrac{5}{8}A + \tfrac{3}{8}B\big) + f\big(\tfrac{3}{8}A + \tfrac{5}{8}B\big) + f\big(\tfrac{1}{8}A + \tfrac{7}{8}B\big)\right\} \tfrac{B-A}{4}$$

Now recalling that $A = b$ and $B = a$, we have that the midpoint Riemann sum approximation to $\int_b^a f(x)\mathrm{d}x$ with 4 subintervals is

$$\left\{f\big(\frac{7}{8}b + \frac{1}{8}a\big) + f\big(\tfrac{5}{8}b + \tfrac{3}{8}a\big) + f\big(\tfrac{3}{8}b + \tfrac{5}{8}a\big) + f\big(\tfrac{1}{8}b + \tfrac{7}{8}a\big)\right\} \tfrac{a-b}{4} \qquad (\star\star)$$

The curly brackets in $(\star)$ and $(\star\star)$ are equal to each other — the terms are just in the reverse order. The factors multiplying the curly brackets in $(\star)$ and $(\star\star)$, namely $\frac{b-a}{4}$ and $\frac{a-b}{4}$, are negatives of each other, so $(\star\star) = -(\star)$. The same computation with n subintervals shows that the midpoint Riemann sum approximations to $\int_b^a f(x)\mathrm{d}x$ and $\int_a^b f(x)\mathrm{d}x$ with n subintervals are negatives of each other. Taking the limit $n \to \infty$ gives $\int_b^a f(x)\mathrm{d}x = -\int_a^b f(x)\mathrm{d}x$.

Example 1.2.5 Revisiting Example 1.1.14.

Back in Example 1.1.14 we saw that when $b > 0$ $\int_0^b x\mathrm{d}x = \frac{b^2}{2}$. We'll now verify that $\int_0^b x\mathrm{d}x = \frac{b^2}{2}$ is still true when $b = 0$ and also when $b < 0$.

- First consider $b = 0$. Then the statement $\int_0^b x\mathrm{d}x = \frac{b^2}{2}$ becomes

$$\int_0^0 x\mathrm{d}x = 0$$

  This is an immediate consequence of Theorem 1.2.3(a).

- Now consider $b < 0$. Let us write $B = -b$, so that $B > 0$. In Example 1.1.14 we saw that

$$\int_{-B}^0 x\mathrm{d}x = -\frac{B^2}{2}.$$

  So we have

$$\begin{aligned}
\int_0^b x\mathrm{d}x = \int_0^{-B} x\mathrm{d}x &= -\int_{-B}^0 x\mathrm{d}x && \text{by Theorem 1.2.3(b)}\\
&= -\left(-\frac{B^2}{2}\right) && \text{by Example 1.1.14}\\
&= \frac{B^2}{2} = \frac{b^2}{2}
\end{aligned}$$

We have now shown that

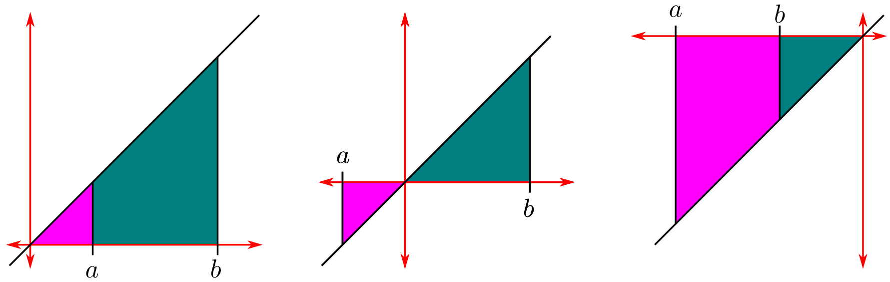
$$\int_0^b x\mathrm{d}x = \frac{b^2}{2} \qquad \text{for all real numbers } b$$

Example 1.2.6 $\int_a^b x\,\mathrm{d}x$.

Applying Theorem 1.2.3 yet again, we have, for all real numbers a and b,

$$\begin{aligned}
\int_a^b x\,\mathrm{d}x &= \int_a^0 x\,\mathrm{d}x + \int_0^b x\,\mathrm{d}x && \text{by Theorem 1.2.3}(c) \text{ with } c=0\\
&= \int_0^b x\,\mathrm{d}x - \int_0^a x\,\mathrm{d}x && \text{by Theorem 1.2.3(b)}\\
&= \frac{b^2-a^2}{2} && \text{by Example 1.2.5, twice}
\end{aligned}$$

We can also understand this result geometrically.



- (left) When $0 < a < b$, the integral represents the area in green which is the difference of two right-angle triangles — the larger with area $b^2/2$ and the smaller with area $a^2/2$.
- (centre) When $a < 0 < b$, the integral represents the signed area of the two displayed triangles. The one above the axis has area $b^2/2$ while the one below has area $-a^2/2$ (since it is below the axis).
- (right) When $a < b < 0$, the integral represents the signed area in purple of the difference between the two triangles — the larger with area $-a^2/2$ and the smaller with area $-b^2/2$.

Theorem 1.2.3(c) shows us how we can split an integral over a larger interval into one over two (or more) smaller intervals. This is particularly useful for dealing with piece-wise functions, like $|x|$.

Example 1.2.7 Integrals involving $|x|$.

Using Theorem 1.2.3, we can readily evaluate integrals involving $|x|$. First, recall that

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Now consider (for example) $\int_{-2}^{3} |x| \mathrm{d}x$. Since the integrand changes at $x = 0$, it makes sense to split the interval of integration at that point:

$$\begin{aligned}
\int_{-2}^{3} |x| \mathrm{d}x &= \int_{-2}^{0} |x| \mathrm{d}x + \int_{0}^{3} |x| \mathrm{d}x && \text{by Theorem 1.2.3} \\
&= \int_{-2}^{0} (-x) \mathrm{d}x + \int_{0}^{3} x \mathrm{d}x && \text{by definition of } |x| \\
&= -\int_{-2}^{0} x \mathrm{d}x + \int_{0}^{3} x \mathrm{d}x && \text{by Theorem 1.2.1(c)} \\
&= -(-2^2/2) + (3^2/2) = (4+9)/2 \\
&= 13/2
\end{aligned}$$

We can go further still — given a function $f(x)$ we can rewrite the integral of $f(|x|)$ in terms of the integral of $f(x)$ and $f(-x)$.

$$\begin{aligned}
\int_{-1}^{1} f\big(|x|\big) \mathrm{d}x &= \int_{-1}^{0} f\big(|x|\big) \mathrm{d}x + \int_{0}^{1} f\big(|x|\big) \mathrm{d}x \\
&= \int_{-1}^{0} f(-x) \mathrm{d}x + \int_{0}^{1} f(x) \mathrm{d}x
\end{aligned}$$

Here is a more concrete example.

Example 1.2.8 Revisiting Example 1.1.15.

Let us compute $\int_{-1}^{1} \big(1 - |x|\big) \mathrm{d}x$ again. In Example 1.1.15 we evaluated this integral by interpreting it as the area of a triangle. This time we are going to use *only* the properties given in Theorems 1.2.1 and 1.2.3 and the facts that

$$\int_{a}^{b} \mathrm{d}x = b - a \qquad \text{and} \qquad \int_{a}^{b} x \mathrm{d}x = \frac{b^2 - a^2}{2}$$

That $\int_{a}^{b} \mathrm{d}x = b - a$ is part (e) of Theorem 1.2.1. We saw that $\int_{a}^{b} x \mathrm{d}x = \frac{b^2-a^2}{2}$ in Example 1.2.6.

First we are going to get rid of the absolute value signs by splitting the interval over which we integrate. Recalling that $|x| = x$ whenever $x \geq 0$ and $|x| = -x$ whenever

$x \leq 0$, we split the interval by Theorem 1.2.3(c),

$$\begin{aligned}\int_{-1}^{1}(1-|x|)\mathrm{d}x &= \int_{-1}^{0}(1-|x|)\mathrm{d}x + \int_{0}^{1}(1-|x|)\mathrm{d}x \\ &= \int_{-1}^{0}(1-(-x))\mathrm{d}x + \int_{0}^{1}(1-x)\mathrm{d}x \\ &= \int_{-1}^{0}(1+x)\mathrm{d}x + \int_{0}^{1}(1-x)\mathrm{d}x\end{aligned}$$

Now we apply parts (a) and (b) of Theorem 1.2.1, and then

$$\begin{aligned}\int_{-1}^{1}[1-|x|]\mathrm{d}x &= \int_{-1}^{0}1\mathrm{d}x + \int_{-1}^{0}x\mathrm{d}x + \int_{0}^{1}1\mathrm{d}x - \int_{0}^{1}x\mathrm{d}x \\ &= [0-(-1)] + \frac{0^2-(-1)^2}{2} + [1-0] - \frac{1^2-0^2}{2} \\ &= 1\end{aligned}$$

Example 1.2.8

1.2.1 ▸▸ More properties of integration: even and odd functions

Recall [1] the following definition

Definition 1.2.9

Let $f(x)$ be a function. Then,

- we say that $f(x)$ is even when $f(x) = f(-x)$ for all x, and
- we say that $f(x)$ is odd when $f(x) = -f(-x)$ for all x.

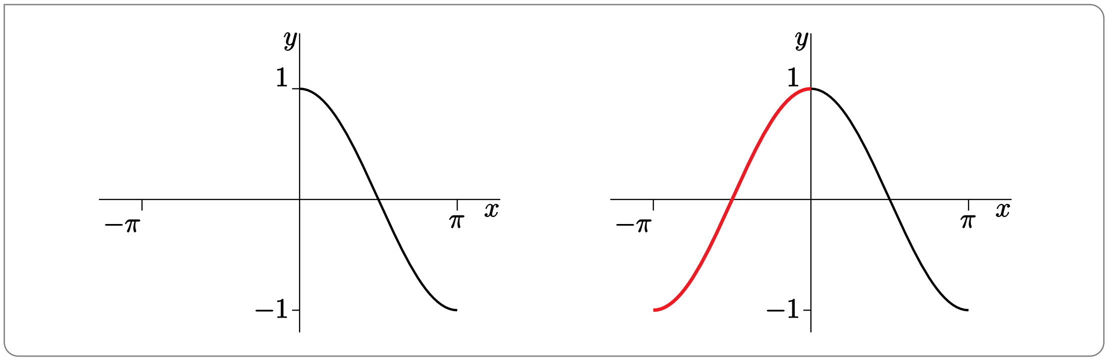
Of course most functions are neither even nor odd, but many of the standard functions you know are.

Example 1.2.10 Even functions.

- Three examples of even functions are $f(x) = |x|$, $f(x) = \cos x$ and $f(x) = x^2$. In fact, if $f(x)$ is any even power of x, then $f(x)$ is an even function.
- The part of the graph $y = f(x)$ with $x \leq 0$, may be constructed by drawing the part of the graph with $x \geq 0$ (as in the figure on the left below) and then

1 We haven't done this in this course, but you should have seen it in your differential calculus course or perhaps even earlier.

reflecting it in the y-axis (as in the figure on the right below).



- In particular, if $f(x)$ is an even function and $a > 0$, then the two sets

$$\left\{ (x,y) \mid 0 \le x \le a \text{ and } y \text{ is between } 0 \text{ and } f(x) \right\}$$
$$\left\{ (x,y) \mid -a \le x \le 0 \text{ and } y \text{ is between } 0 \text{ and } f(x) \right\}$$

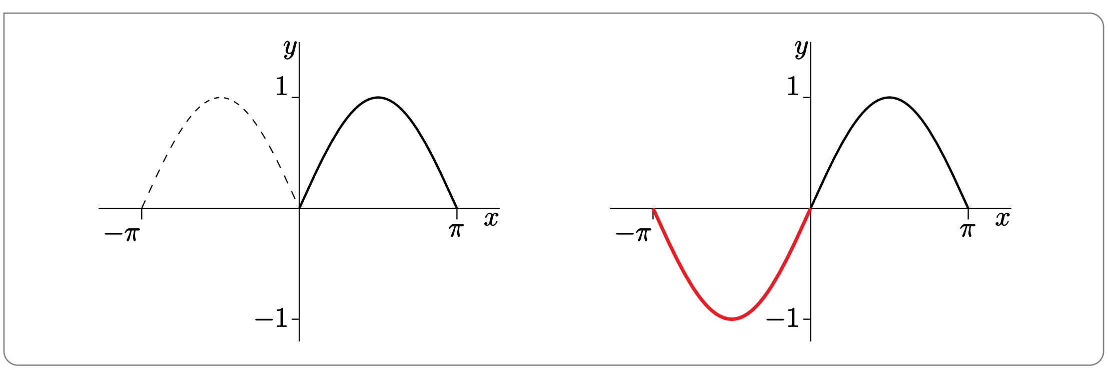
are reflections of each other in the y-axis and so have the same signed area. That is

$$\int_0^a f(x)\mathrm{d}x = \int_{-a}^0 f(x)\mathrm{d}x$$

Example 1.2.10

Example 1.2.11 Odd functions.

- Three examples of odd functions are $f(x) = \sin x$, $f(x) = \tan x$ and $f(x) = x^3$. In fact, if $f(x)$ is any odd power of x, then $f(x)$ is an odd function.
- The part of the graph $y = f(x)$ with $x \le 0$, may be constructed by drawing the part of the graph with $x \ge 0$ (like the solid line in the figure on the left below) and then reflecting it in the y-axis (like the dashed line in the figure on the left below) and then reflecting the result in the x-axis (i.e. flipping it upside down, like in the figure on the right, below).



- In particular, if $f(x)$ is an odd function and $a > 0$, then the signed areas of the two sets

$$\left\{\ (x,y)\ \middle|\ 0 \le x \le a \text{ and } y \text{ is between } 0 \text{ and } f(x)\ \right\}$$
$$\left\{\ (x,y)\ \middle|\ -a \le x \le 0 \text{ and } y \text{ is between } 0 \text{ and } f(x)\ \right\}$$

are negatives of each other — to get from the first set to the second set, you flip it upside down, in addition to reflecting it in the y-axis. That is

$$\int_0^a f(x)\mathrm{d}x = -\int_{-a}^0 f(x)\mathrm{d}x$$

Example 1.2.11

We can exploit the symmetries noted in the examples above, namely

$$\int_0^a f(x)\mathrm{d}x = \int_{-a}^0 f(x)\mathrm{d}x \qquad \text{for } f \text{ even}$$
$$\int_0^a f(x)\mathrm{d}x = -\int_{-a}^0 f(x)\mathrm{d}x \qquad \text{for } f \text{ odd}$$

together with Theorem 1.2.3 Theorem 1.2.3

$$\int_{-a}^a f(x)\mathrm{d}x = \int_{-a}^0 f(x)\mathrm{d}x + \int_0^a f(x)\mathrm{d}x$$

in order to simplify the integration of even and odd functions over intervals of the form $[-a, a]$.

Theorem 1.2.12 Even and Odd.

Let $a > 0$.

a If $f(x)$ is an even function, then

$$\int_{-a}^a f(x)\mathrm{d}x = 2\int_0^a f(x)\mathrm{d}x$$

b If $f(x)$ is an odd function, then

$$\int_{-a}^a f(x)\mathrm{d}x = 0$$

Proof. For any function

$$\int_{-a}^{a} f(x)\mathrm{d}x = \int_{0}^{a} f(x)\mathrm{d}x + \int_{-a}^{0} f(x)\mathrm{d}x$$

When f is even, the two terms on the right hand side are equal. When f is odd, the two terms on the right hand side are negatives of each other.

□

1.2.2 ⏩ Optional — More properties of integration: inequalities for integrals

We are still unable to integrate many functions, however with a little work we can infer bounds on integrals from bounds on their integrands.

Theorem 1.2.13 Inequalities for Integrals.

Let $a \le b$ be real numbers and let the functions $f(x)$ and $g(x)$ be integrable on the interval $a \le x \le b$.

a If $f(x) \ge 0$ for all $a \le x \le b$, then

$$\int_a^b f(x)\,\mathrm{d}x \ge 0$$

b If $f(x) \le g(x)$ for all $a \le x \le b$, then

$$\int_a^b f(x)\,\mathrm{d}x \le \int_a^b g(x)\,\mathrm{d}x$$

c If there are constants m and M such that $m \le f(x) \le M$ for all $a \le x \le b$, then

$$m(b-a) \le \int_a^b f(x)\,\mathrm{d}x \le M(b-a)$$

d We have

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \le \int_a^b |f(x)|\,\mathrm{d}x$$

Proof.

a By interpreting the integral as the signed area, this statement simply says that if the curve $y = f(x)$ lies above the x-axis and $a \le b$, then the signed area of $\left\{ (x,y) \;\middle|\; a \le x \le b,\ 0 \le y \le f(x) \right\}$ is at least zero. This is quite clear. Alternatively, we could argue more algebraically from Definition 1.1.9. We observe that when we define $\int_a^b f(x)\mathrm{d}x$ via Riemann sums, every summand, $f(x_{i,n}^*)\,\frac{b-a}{n} \ge 0$. Thus the whole sum is nonnegative and consequently, so is the limit, and thus so is the integral.

b We are assuming that $g(x) - f(x) \ge 0$, so part (a) gives

$$\begin{aligned}\int_a^b \big[g(x) - f(x)\big]\,\mathrm{d}x \ge 0 &\implies \int_a^b g(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x \ge 0\\ &\implies \int_a^b f(x)\,\mathrm{d}x \le \int_a^b g(x)\,\mathrm{d}x\end{aligned}$$

c Applying part (b) with $g(x) = M$ for all $a \le x \le b$ gives

$$\int_a^b f(x)\,\mathrm{d}x \le \int_a^b M\,\mathrm{d}x = M(b-a)$$

Similarly, viewing m as a (constant) function, and applying part (b) gives

$$m \le f(x) \implies \overbrace{\int_a^b m\,\mathrm{d}x}^{=m(b-a)} \le \int_a^b f(x)\,\mathrm{d}x$$

d For any x, $|f(x)|$ is either $f(x)$ or $-f(x)$ (depending on whether $f(x)$ is positive or negative), so we certainly have

$$f(x) \le |f(x)| \qquad \text{and} \qquad -f(x) \le |f(x)|$$

Applying part (c) to each of those inequalities gives

$$\int_a^b f(x)\mathrm{d}x \le \int_a^b |f(x)|\mathrm{d}x \qquad \text{and} \qquad -\int_a^b f(x)\mathrm{d}x \le \int_a^b |f(x)|\mathrm{d}x$$

Now $\left|\int_a^b f(x)\mathrm{d}x\right|$ is either equal to $\int_a^b f(x)\mathrm{d}x$ or $-\int_a^b f(x)\mathrm{d}x$ (depending on whether the integral is positive or negative). In either case we can apply the above two inequalities to get the same result, namely

$$\left|\int_a^b f(x)\mathrm{d}x\right| \le \int_a^b |f(x)|\mathrm{d}x.$$

□

Example 1.2.14 $\int_0^{\frac{\pi}{3}} \sqrt{\cos x}\mathrm{d}x$.

Consider the integral

$$\int_0^{\frac{\pi}{3}} \sqrt{\cos x}\mathrm{d}x$$

This is not so easy to compute exactly [a], but we can bound it quite quickly.
For x between 0 and $\frac{\pi}{3}$, the function $\cos x$ takes values [b] between 1 and $\frac{1}{2}$. Thus the function $\sqrt{\cos x}$ takes values between 1 and $\frac{1}{\sqrt{2}}$. That is

$$\frac{1}{\sqrt{2}} \le \sqrt{\cos x} \le 1 \qquad \text{for } 0 \le x \le \frac{\pi}{3}.$$

Consequently, by Theorem 1.2.13(b) with $a = 0$, $b = \frac{\pi}{3}$, $m = \frac{1}{\sqrt{2}}$ and $M = 1$,

$$\frac{\pi}{3\sqrt{2}} \le \int_0^{\frac{\pi}{3}} \sqrt{\cos x}\mathrm{d}x \le \frac{\pi}{3}$$

Plugging these expressions into a calculator gives us

$$0.7404804898 \le \int_0^{\frac{\pi}{3}} \sqrt{\cos x}\mathrm{d}x \le 1.047197551$$

a It is not too hard to use Riemann sums and a computer to evaluate it numerically: $0.948025319\ldots$.
b You know the graphs of sine and cosine, so you should be able to work this out without too much difficulty.

1.2.3 ▸▸ Exercises

Exercises — Stage 1

1. For each of the following properties of definite integrals, draw a picture illustrating the concept, interpreting definite integrals as areas under a curve.
For simplicity, you may assume that $a \le c \le b$, and that $f(x), g(x)$ give positive values.

a $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0$, (Theorem 1.2.3, part (a))

b $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$, (Theorem 1.2.3, part (c)) c $\int_a^b (f(x) + g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$, (Theorem 1.2.1, part (a))

2. If $\int_0^b \cos x \mathrm{d}x = \sin b$, then what is $\int_a^b \cos x \mathrm{d}x$?

3. $*$. Decide whether each of the following statements is true or false. If false, provide a counterexample. If true, provide a brief justification. (Assume that $f(x)$ and $g(x)$ are continuous functions.)

a $\int_{-3}^{-2} f(x)\mathrm{d}x = -\int_3^2 f(x)\mathrm{d}x$.

b If $f(x)$ is an odd function, then $\int_{-3}^{-2} f(x)\,\mathrm{d}x = \int_2^3 f(x)\,\mathrm{d}x$.

c $\int_0^1 f(x) \cdot g(x)\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x \cdot \int_0^1 g(x)\mathrm{d}x$.

4. Suppose we want to make a right Riemann sum with 100 intervals to approximate $\int_5^0 f(x)\mathrm{d}x$, where $f(x)$ is a function that gives only positive values.

a What is Δx?

b Are the heights of our rectangles positive or negative?

c Is our Riemann sum positive or negative?

d Is the signed area under the curve $y = f(x)$ from $x = 0$ to $x = 5$ positive or negative?

Exercises — Stage 2

5. $*$. Suppose $\int_2^3 f(x)\,\mathrm{d}x = -1$ and $\int_2^3 g(x)\,\mathrm{d}x = 5$. Evaluate $\int_2^3 \big(6f(x) - 3g(x)\big)\,\mathrm{d}x$.

6. $*$. If $\int_0^2 f(x)\,\mathrm{d}x = 3$ and $\int_0^2 g(x)\,\mathrm{d}x = -4$, calculate $\int_0^2 \big(2f(x) + 3g(x)\big)\,\mathrm{d}x$.

7. $*$. The functions $f(x)$ and $g(x)$ obey

$$\int_0^{-1} f(x)\,\mathrm{d}x = 1 \qquad \int_0^2 f(x)\,\mathrm{d}x = 2$$

$$\int_{-1}^{0} g(x)\,\mathrm{d}x = 3 \qquad\qquad \int_{0}^{2} g(x)\,\mathrm{d}x = 4$$

Find $\int_{-1}^{2} \big[3g(x) - f(x)\big]\,\mathrm{d}x$.

8. In Question 1.1.8.45, Section 1.1, we found that

$$\int_0^a \sqrt{1-x^2}\mathrm{d}x = \frac{\pi}{4} - \frac{1}{2}\arccos(a) + \frac{1}{2}a\sqrt{1-a^2}$$

when $0 \le a \le 1$.
Using this fact, evaluate the following:

a $\displaystyle\int_a^0 \sqrt{1-x^2}\mathrm{d}x$, where $-1 \le a \le 0$

b $\displaystyle\int_a^1 \sqrt{1-x^2}\mathrm{d}x$, where $0 \le a \le 1$

9. $*$. Evaluate $\displaystyle\int_{-1}^{2} |2x|\mathrm{d}x$.

You may use the result from Example 1.2.6 that $\int\limits_a^b x\mathrm{d}x = \frac{b^2-a^2}{2}$.

10. Evaluate $\displaystyle\int_{-5}^{5} x|x|\mathrm{d}x$.

11. Suppose $f(x)$ is an even function and $\displaystyle\int_{-2}^{2} f(x)\mathrm{d}x = 10$. What is $\displaystyle\int_{-2}^{0} f(x)\mathrm{d}x$?

Exercises — Stage 3

12. $*$. Evaluate $\displaystyle\int_{-2}^{2} \left(5+\sqrt{4-x^2}\right)\mathrm{d}x$.

13. $*$. Evaluate $\displaystyle\int_{-2012}^{+2012} \frac{\sin x}{\log(3+x^2)}\mathrm{d}x$.

14. $*$. Evaluate $\displaystyle\int_{-2012}^{+2012} x^{1/3}\cos x\,\mathrm{d}x$.

15. Evaluate $\displaystyle\int_0^6 (x-3)^3\,\mathrm{d}x$.

16. We want to compute the area of an ellipse, $(ax)^2 + (by)^2 = 1$ for some (let's say positive) constants a and b.

a Solve the equation for the upper half of the ellipse. It should have the form "$y = \cdots$"

b Write an integral for the area of the upper half of the ellipse. Using properties of integrals, make the integrand look like the upper half of a circle.

c Using geometry and your answer to part (b), find the area of the ellipse.

17. Fill in the following table: the product of an (even/odd) function with an (even/odd) function is an (even/odd) function. You may assume that both functions are defined for all real numbers.

$\times$	even	odd
even		
odd		

18. Suppose $f(x)$ is an odd function and $g(x)$ is an even function, both defined at $x = 0$. What are the possible values of $f(0)$ and $g(0)$?

19. Suppose $f(x)$ is a function defined on all real numbers that is both even and odd. What could $f(x)$ be?

20. Is the derivative of an even function even or odd? Is the derivative of an odd function even or odd?

1.3 ▲ The Fundamental Theorem of Calculus

1.3.1 ▸▸ The Fundamental Theorem of Calculus

We have spent quite a few pages (and lectures) talking about definite integrals, what they are (Definition 1.1.9), when they exist (Theorem 1.1.10), how to compute some special cases (Section 1.1.5), some ways to manipulate them (Theorem 1.2.1 and 1.2.3) and how to bound them (Theorem 1.2.13). Conspicuously missing from all of this has been a discussion of how to compute them in general. It is high time we rectified that.

The single most important tool used to evaluate integrals is called "the fundamental theorem of calculus". Its grand name is justified — it links the two branches of calculus by connecting derivatives to integrals. In so doing it also tells us how to compute integrals. Very roughly speaking the derivative of an integral is the original function. This fact allows us to compute integrals using antiderivatives [1]. Of course "very rough" is not enough — let's be precise.

Theorem 1.3.1 Fundamental Theorem of Calculus.

Let $a < b$ and let $f(x)$ be a function which is defined and continuous on $[a, b]$.

- ***Part 1***. Let $F(x) = \displaystyle\int_a^x f(t)\mathrm{d}t$ for any $x \in [a, b]$. Then the function $F(x)$ is differentiable and further

$$F'(x) = f(x)$$

- ***Part 2***. Let $G(x)$ be any function which is defined and continuous on $[a, b]$. Further let $G(x)$ be differentiable with $G'(x) = f(x)$ for all $a < x < b$. Then

$$\int_a^b f(x)\mathrm{d}x = G(b) - G(a) \quad \text{or equivalently} \quad \int_a^b G'(x)\mathrm{d}x = G(b) - G(a)$$

Before we prove this theorem and look at a bunch of examples of its application, it is important that we recall one definition from differential calculus — antiderivatives. If $F'(x) = f(x)$ on some interval, then $F(x)$ is called an antiderivative of $f(x)$ on that interval. So Part 2 of the the fundamental theorem of calculus tells us how to evaluate the definite integral of $f(x)$ in terms of any of its antiderivatives — if $G(x)$ is any antiderivative of $f(x)$ then

$$\int_a^b f(x)\mathrm{d}x = G(b) - G(a)$$

The form $\int_a^b G'(x)\,\mathrm{d}x = G(b) - G(a)$ of the fundamental theorem relates the rate of change of $G(x)$ over the interval $a \le x \le b$ to the net change of G between $x = a$ and $x = b$. For that reason, it is sometimes called the "net change theorem".

We'll start with a simple example. Then we'll see why the fundamental theorem is true and then we'll do many more, and more involved, examples.

Example 1.3.2 A first example.

Consider the integral $\int_a^b x\mathrm{d}x$ which we have explored previously in Example 1.2.6.

- The integrand is $f(x) = x$.
- We can readily verify that $G(x) = \frac{x^2}{2}$ satisfies $G'(x) = f(x)$ and so is an antiderivative of the integrand.

1 You learned these near the end of your differential calculus course. Now is a good time to revise — but we'll go over them here since they are so important in what follows.

- Part 2 of Theorem 1.3.1 then tells us that

$$\int_a^b f(x)\mathrm{d}x = G(b) - G(a)$$
$$\int_a^b x\mathrm{d}x = \frac{b^2}{2} - \frac{a^2}{2}$$

which is precisely the result we obtained (with more work) in Example 1.2.6.

Example 1.3.2

We do not give completely rigorous proofs of the two parts of the theorem — that is not really needed for this course. We just give the main ideas of the proofs so that you can understand why the theorem is true.

Part 1. We wish to show that if

$$F(x) = \int_a^x f(t)\mathrm{d}t \qquad \text{then} \qquad F'(x) = f(x)$$

- Assume that F is the above integral and then consider $F'(x)$. By definition

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

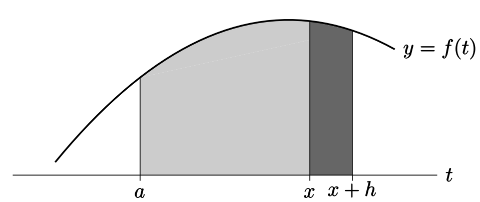
- To understand this limit, we interpret the terms $F(x), F(x+h)$ as signed areas. To simplify this further, let's only consider the case that f is always nonnegative and that $h > 0$. These restrictions are not hard to remove, but the proof ideas are a bit cleaner if we keep them in place. Then we have

$$F(x+h) = \text{the area of the region } \left\{ (t,y) \;\middle|\; a \le t \le x+h,\ 0 \le y \le f(t) \right\}$$
$$F(x) = \text{the area of the region } \left\{ (t,y) \;\middle|\; a \le t \le x, \quad 0 \le y \le f(t) \right\}$$

- Then the numerator

$$F(x+h) - F(x) = \text{the area of } \left\{ (t,y) \;\middle|\; x \le t \le x+h,\ 0 \le y \le f(t) \right\}$$

This is just the more darkly shaded region in the figure



- We will be taking the limit $h \to 0$. So suppose that h is very small. Then, as t runs from x to $x+h$, $f(t)$ runs only over a very narrow range of values[a], all close to $f(x)$.

- So the darkly shaded region is almost a rectangle of width h and height $f(x)$ and so has an area which is very close to $f(x)h$. Thus $\frac{F(x+h)-F(x)}{h}$ is very close to $f(x)$.
- In the limit $h \to 0$, $\frac{F(x+h)-F(x)}{h}$ becomes exactly $f(x)$, which is precisely what we want.

$\square$

a Notice that if f were discontinuous, then this might be false.

We can make the above more rigorous using the Mean Value Theorem [2].

Part 2. We want to show that $\int_a^b f(t)\mathrm{d}t = G(b)-G(a)$. To do this we exploit the fact that the derivative of a constant is zero.

- Let
$$H(x) = \int_a^x f(t)\mathrm{d}t - G(x) + G(a)$$
Then the result we wish to prove is that $H(b) = 0$. We will do this by showing that $H(x) = 0$ for all x between a and b.
- We first show that $H(x)$ is constant by computing its derivative:
$$H'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t - \frac{\mathrm{d}}{\mathrm{d}x}(G(x)) + \frac{\mathrm{d}}{\mathrm{d}x}(G(a))$$
Since $G(a)$ is a constant, its derivative is 0 and by assumption the derivative of $G(x)$ is just $f(x)$, so
$$= \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t - f(x)$$
Now Part 1 of the theorem tells us that this derivative is just $f(x)$, so
$$= f(x) - f(x) = 0$$
Hence H is constant.
- To determine which constant we just compute $H(a)$:
$$H(a) = \int_a^a f(t)\mathrm{d}t - G(a) + G(a)$$

2 The MVT tells us that there is a number c between x and $x+h$ so that $F'(c) = \frac{F(x+h)-F(x)}{(x+h)-x} = \frac{F(x+h)-F(x)}{h}$. But since $F'(x) = f(x)$, this tells us that $\frac{F(x+h)-F(x)}{h} = f(c)$ where c is trapped between $x+h$ and x. Now when we take the limit as $h \to 0$ we have that this number c is squeezed to x and the result follows.

$$= \int_a^a f(t)\mathrm{d}t \qquad \text{by Theorem 1.2.3(a)}$$
$$= 0$$

as required.

□

The simple example we did above (Example 1.3.2), demonstrates the application of part 2 of the fundamental theorem of calculus. Before we do more examples (and there will be many more over the coming sections) we should do some examples illustrating the use of part 1 of the fundamental theorem of calculus. Then we'll move on to part 2.

Example 1.3.3 $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x t\mathrm{d}t$.

Consider the integral $\int_0^x t\,\mathrm{d}t$. We know how to evaluate this — it is just Example 1.3.2 with $a = 0$, $b = x$. So we have two ways to compute the derivative. We can evaluate the integral and then take the derivative, or we can apply Part 1 of the fundamental theorem. We'll do both, and check that the two answers are the same.
First, Example 1.3.2 gives

$$F(x) = \int_0^x t\,\mathrm{d}t = \frac{x^2}{2}$$

So of course $F'(x) = x$. Second, Part 1 of the fundamental theorem of calculus tells us that the derivative of $F(x)$ is just the integrand. That is, Part 1 of the fundamental theorem of calculus also gives $F'(x) = x$.

In the previous example we were able to evaluate the integral explicitly, so we did not need the fundamental theorem to determine its derivative. Here is an example that really does require the use of the fundamental theorem.

Example 1.3.4 $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x e^{-t^2}\mathrm{d}t$.

We would like to find $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x e^{-t^2}\mathrm{d}t$. In the previous example, we were able to compute the corresponding derivative in two ways — we could explicitly compute the integral and then differentiate the result, or we could apply part 1 of the fundamental theorem of calculus. In this example we do not know the integral explicitly. Indeed it is not possible to express [a] the integral $\int_0^x e^{-t^2}\mathrm{d}t$ as a finite combination of standard functions such as polynomials, exponentials, trigonometric functions and so on.
Despite this, we can find its derivative by just applying the first part of the fundamental theorem of calculus with $f(t) = e^{-t^2}$ and $a = 0$. That gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x e^{-t^2}\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f(t)\mathrm{d}t$$

$$= f(x) = e^{-x^2}$$

a The integral $\int_0^x e^{-t^2}\mathrm{d}t$ is closely related to the "error function" which is an extremely important function in mathematics. While we cannot express this integral (or the error function) as a *finite* combination of polynomials, exponentials etc, we can express it as an infinite series $\int_0^x e^{-t^2}\mathrm{d}t = x - \frac{x^3}{3\cdot 1} + \frac{x^5}{5\cdot 2} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)\cdot k!} + \cdots$. But more on this in Chapter 3.

Example 1.3.4

Let us ratchet up the complexity of the previous example — we can make the limits of the integral more complicated functions. So consider the previous example with the upper limit x replaced by x^2:

Example 1.3.5 $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} e^{-t^2}\mathrm{d}t$.

Consider the integral $\int_0^{x^2} e^{-t^2}\mathrm{d}t$. We would like to compute its derivative with respect to x using part 1 of the fundamental theorem of calculus.

The fundamental theorem tells us how to compute the derivative of functions of the form $\int_a^x f(t)\mathrm{d}t$ but the integral at hand is *not* of the specified form because the upper limit we have is x^2, rather than x, — so more care is required. Thankfully we can deal with this obstacle with only a little extra work. The trick is to define an auxiliary function by simply changing the upper limit to x. That is, define

$$E(x) = \int_0^x e^{-t^2}\mathrm{d}t$$

Then the integral we want to work with is

$$E(x^2) = \int_0^{x^2} e^{-t^2}\mathrm{d}t$$

The derivative $E'(x)$ can be found via part 1 of the fundamental theorem of calculus (as we did in Example 1.3.4) and is $E'(x) = e^{-x^2}$. We can then use this fact with the chain rule to compute the derivative we need:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} e^{-t^2}\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}E(x^2) && \text{use the chain rule}\\
&= 2xE'(x^2)\\
&= 2xe^{-x^4}
\end{aligned}$$

What if both limits of integration are functions of x? We can still make this work, but we have to split the integral using Theorem 1.2.3.

Example 1.3.6 $\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x^2} e^{-t^2}\mathrm{d}t$.

Consider the integral

$$\int_x^{x^2} e^{-t^2}\mathrm{d}t$$

As was the case in the previous example, we have to do a little pre-processing before we can apply the fundamental theorem.

This time (by design), not only is the upper limit of integration x^2 rather than x, but the lower limit of integration also depends on x — this is different from the integral $\int_a^x f(t)\mathrm{d}t$ in the fundamental theorem where the *lower* limit of integration is a constant.

Fortunately we can use the basic properties of integrals (Theorem 1.2.3(b) and (c)) to split $\int_x^{x^2} e^{-t^2}\mathrm{d}t$ into pieces whose derivatives we already know.

$$\begin{aligned}
\int_x^{x^2} e^{-t^2}\mathrm{d}t &= \int_x^{0} e^{-t^2}\mathrm{d}t + \int_0^{x^2} e^{-t^2}\mathrm{d}t && \text{by Theorem 1.2.3(c)}\\
&= -\int_0^{x} e^{-t^2}\mathrm{d}t + \int_0^{x^2} e^{-t^2}\mathrm{d}t && \text{by Theorem 1.2.3(b)}
\end{aligned}$$

With this pre-processing, both integrals are of the right form. Using what we have learned in the the previous two examples,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x^2} e^{-t^2}\mathrm{d}t &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-\int_0^{x} e^{-t^2}\mathrm{d}t + \int_0^{x^2} e^{-t^2}\mathrm{d}t\right)\\
&= -\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x} e^{-t^2}\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} e^{-t^2}\mathrm{d}t\\
&= -e^{-x^2} + 2xe^{-x^4}
\end{aligned}$$

Before we start to work with part 2 of the fundamental theorem, we need a little terminology and notation. First some terminology — you may have seen this definition in your differential calculus course.

Definition 1.3.7 Antiderivatives.

Let $f(x)$ and $F(x)$ be functions. If $F'(x) = f(x)$ on an interval, then we say that $F(x)$ is an antiderivative of $f(x)$ on that interval.

As we saw above, an antiderivative of $f(x) = x$ is $F(x) = x^2/2$ — we can easily verify this by differentiation. Notice that $x^2/2 + 3$ is also an antiderivative of x, as is $x^2/2 + C$ for any constant C. This observation gives us the following simple lemma.

Lemma 1.3.8

Let $f(x)$ be a function and let $F(x)$ be an antiderivative of $f(x)$. Then $F(x)+C$ is also an antiderivative for any constant C. Further, every antiderivative of $f(x)$ must be of this form.

Proof. There are two parts to the lemma and we prove each in turn.

- Let $F(x)$ be an antiderivative of $f(x)$ and let C be some constant. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(F(x)+C\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(F(x)\right)+\frac{\mathrm{d}}{\mathrm{d}x}\left(C\right)\\ &= f(x)+0\end{aligned}$$

  since the derivative of a constant is zero, and by definition the derivative of $F(x)$ is just $f(x)$. Thus $F(x)+C$ is also an antiderivative of $f(x)$.

- Now let $F(x)$ and $G(x)$ both be antiderivatives of $f(x)$ — we will show that $G(x) = F(x)+C$ for some constant C. To do this let $H(x) = G(x) - F(x)$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}H(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(G(x)-F(x)\right) = \frac{\mathrm{d}}{\mathrm{d}x}G(x) - \frac{\mathrm{d}}{\mathrm{d}x}F(x)\\ &= f(x)-f(x) = 0\end{aligned}$$

  Since the derivative of $H(x)$ is zero, $H(x)$ must be a constant function [a]. Thus $H(x) = G(x) - F(x) = C$ for some constant C and the result follows.

$\square$

[a] This follows from the Mean Value Theorem. Say $H(x)$ were not constant, then there would be two numbers $a < b$ so that $H(a) \neq H(b)$. Then the MVT tells us that there is a number c between a and b so that $H'(c) = \frac{H(b)-H(a)}{b-a}$. Since both numerator and denominator are nonzero, we know the derivative at c is nonzero. But this would contradict the assumption that derivative of H is zero. Hence we cannot have $a < b$ with $H(a) \neq H(b)$ and so $H(x)$ must be constant.

Based on the above lemma we have the following definition.

Definition 1.3.9

The "indefinite integral of $f(x)$" is denoted by $\int f(x)\mathrm{d}x$ and should be regarded as the general antiderivative of $f(x)$. In particular, if $F(x)$ is an antiderivative of

$f(x)$ then

$$\int f(x)\mathrm{d}x = F(x) + C$$

where the C is an arbitrary constant. In this context, the constant C is also often called a "constant of integration".

Now we just need a tiny bit more notation.

Definition 1.3.10

The symbol

$$\int f(x)\mathrm{d}x\bigg|_a^b$$

denotes the change in an antiderivative of $f(x)$ from $x = a$ to $x = b$. More precisely, let $F(x)$ be any antiderivative of $f(x)$. Then

$$\int f(x)\mathrm{d}x\bigg|_a^b = F(x)\big|_a^b = F(b) - F(a)$$

Notice that this notation allows us to write part 2 of the fundamental theorem as

$$\begin{aligned}\int_a^b f(x)\mathrm{d}x &= \int f(x)\mathrm{d}x\bigg|_a^b \\ &= F(x)\big|_a^b = F(b) - F(a)\end{aligned}$$

Some texts also use an equivalent notation using square brackets:

$$\int_a^b f(x)\mathrm{d}x = \Big[F(x)\Big]_a^b = F(b) - F(a).$$

You should be familiar with both notations.

We'll soon develop some strategies for computing more complicated integrals. But for now, we'll try a few integrals that are simple enough that we can just guess the answer. Of course, any antiderivative that we can guess we can also check — simply differentiate the guess and verify you get back to the original function:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int f(x)\mathrm{d}x = f(x).$$

We do these examples in some detail to help us become comfortable finding indefinite integrals.

Example 1.3.11 Compute the definite integral $\int_1^2 x\mathrm{d}x$.

Compute the definite integral $\int_1^2 x\mathrm{d}x$.

Solution: We have already seen, in Example 1.2.6, that $\int_1^2 x\mathrm{d}x = \frac{2^2-1^2}{2} = \frac{3}{2}$. We shall now rederive that result using the fundamental theorem of calculus.

- The main difficulty in this approach is finding the indefinite integral (an antiderivative) of x. That is, we need to find a function $F(x)$ whose derivative is x. So think back to all the derivatives you computed last term [a] and try to remember a function whose derivative was something like x.
- This shouldn't be too hard — we recall that the derivatives of polynomials are polynomials. More precisely, we know that

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$$

So if we want to end up with just $x = x^1$, we need to take $n = 2$. However this gives us

$$\frac{\mathrm{d}}{\mathrm{d}x}x^2 = 2x$$

- This is pretty close to what we want except for the factor of 2. Since this is a constant we can just divide both sides by 2 to obtain:

$$\begin{aligned} \frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}x^2 &= \frac{1}{2}\cdot 2x && \text{which becomes}\\ \cdot\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2}{2} &= x \end{aligned}$$

which is exactly what we need. It tells us that $x^2/2$ is an antiderivative of x.
- Once one has an antiderivative, it is easy to compute the indefinite integral

$$\int x\mathrm{d}x = \frac{1}{2}x^2 + C$$

as well as the definite integral:

$$\begin{aligned} \int_1^2 x\mathrm{d}x &= \frac{1}{2}x^2\bigg|_1^2 && \text{since } x^2/2 \text{ is an antiderivative of } x\\ &= \frac{1}{2}2^2 - \frac{1}{2}1^2 = \frac{3}{2} \end{aligned}$$

a Of course, this assumes that you did your differential calculus course last term. If you did that course at a different time then please think back to that point in time. If it is long enough ago that you don't quite remember when it was, then you should probably do some revision of derivatives of simple functions before proceeding further.

While the previous example could be computed using signed areas, the following example would be very difficult to compute without using the fundamental theorem of calculus.

Example 1.3.12 Compute $\int_0^{\frac{\pi}{2}} \sin x \mathrm{d}x$.

Compute $\int_0^{\frac{\pi}{2}} \sin x \mathrm{d}x$.

Solution:

- Once again, the crux of the solution is guessing the antiderivative of $\sin x$ — that is finding a function whose derivative is $\sin x$.
- The standard derivative that comes closest to $\sin x$ is

$$\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x$$

which is the derivative we want, multiplied by a factor of -1.

- Just as we did in the previous example, we multiply this equation by a constant to remove this unwanted factor:

$$\begin{aligned} (-1) \cdot \frac{\mathrm{d}}{\mathrm{d}x} \cos x &= (-1) \cdot (-\sin x) && \text{giving us} \\ \frac{\mathrm{d}}{\mathrm{d}x}\big(-\cos x\big) &= \sin x \end{aligned}$$

This tells us that $-\cos x$ is an antiderivative of $\sin x$.

- Now it is straightforward to compute the integral:

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \sin x \mathrm{d}x &= -\cos x\Big|_0^{\frac{\pi}{2}} && \text{since } -\cos x \text{ is an antiderivative of } \sin x \\ &= -\cos\frac{\pi}{2} + \cos 0 \\ &= 0 + 1 = 1 \end{aligned}$$

Example 1.3.13 Compute $\int_1^2 \frac{1}{x} \mathrm{d}x$.

Find $\int_1^2 \frac{1}{x} \mathrm{d}x$.

Solution:

- Once again, the crux of the solution is guessing a function whose derivative is $\frac{1}{x}$. Our standard way to differentiate powers of x, namely

$$\frac{\mathrm{d}}{\mathrm{d}x} x^n = nx^{n-1},$$

doesn't work in this case — since it would require us to pick $n = 0$ and this would give

$$\frac{\mathrm{d}}{\mathrm{d}x}x^0 = \frac{\mathrm{d}}{\mathrm{d}x}1 = 0.$$

- Fortunately, we also know [a] that

$$\frac{\mathrm{d}}{\mathrm{d}x}\log x = \frac{1}{x}$$

which is exactly the derivative we want.

- We're now ready to compute the prescribed integral.

$$\begin{aligned}\int_1^2 \frac{1}{x}\mathrm{d}x &= \log x\big|_1^2 && \text{since } \log x \text{ is an antiderivative of } 1/x\\ &= \log 2 - \log 1 && \text{since } \log 1 = 0\\ &= \log 2\end{aligned}$$

a Recall that in most mathematics courses (especially this one) we use $\log x$ without any indicated base to denote the natural logarithm — the logarithm base e. Many widely used computer languages, like Java, C, Python, MATLAB, $\cdots$, use $\log(x)$ to denote the logarithm base e too. But many texts also use $\ln x$ to denote the natural logarithm $\log x = \log_e x = \ln x$. The reader should be comfortable with all three notations for this function. They should also be aware that in different contexts — such as in chemistry or physics — it is common to use $\log x$ to denote the logarithm base 10, while in computer science often $\log x$ denotes the logarithm base 2. Context is key.

Example 1.3.13

Example 1.3.14 $\int_{-2}^{-1} \frac{1}{x}\mathrm{d}x$.

Find $\int_{-2}^{-1} \frac{1}{x}\mathrm{d}x$.
Solution:

- As we saw in the last example,

$$\frac{\mathrm{d}}{\mathrm{d}x}\log x = \frac{1}{x}$$

and if we naively use this here, then we will obtain

$$\int_{-2}^{-1} \frac{1}{x}\mathrm{d}x = \log(-1) - \log(-2)$$

which makes no sense since the logarithm is only defined for positive numbers [a].

- We can work around this problem using a slight variation of the logarithm — $\log|x|$.
  - When $x > 0$, we know that $|x| = x$ and so we have
    $$\begin{aligned} \log|x| &= \log x && \text{differentiating gives us} \\ \frac{\mathrm{d}}{\mathrm{d}x}\log|x| &= \frac{\mathrm{d}}{\mathrm{d}x}\log x = \frac{1}{x}. \end{aligned}$$
  - When $x < 0$ we have that $|x| = -x$ and so
    $$\begin{aligned} \log|x| &= \log(-x) && \text{differentiating with the chain rule gives} \\ \frac{\mathrm{d}}{\mathrm{d}x}\log|x| &= \frac{\mathrm{d}}{\mathrm{d}x}\log(-x) \\ &= \frac{1}{(-x)}\cdot(-1) = \frac{1}{x} \end{aligned}$$
  - Indeed, more generally we should write the indefinite integral of $1/x$ as
    $$\int \frac{1}{x}\mathrm{d}x = \log|x| + C$$
    which is valid for all positive and negative x. It is, however, undefined at $x = 0$.
- We're now ready to compute the prescribed integral.
  $$\begin{aligned} \int_{-2}^{-1} \frac{1}{x}\mathrm{d}x &= \log|x|\Big|_{-2}^{-1} && \text{since } \log|x| \text{ is an antiderivative of } 1/x \\ &= \log|-1| - \log|-2| = \log 1 - \log 2 \\ &= -\log 2 = \log\frac{1}{2}. \end{aligned}$$

a This is not entirely true — one can extend the definition of the logarithm to negative numbers, but to do so one needs to understand complex numbers which is a topic beyond the scope of this course.

Example 1.3.14

This next example raises a nasty issue that requires a little care. We know that the function $1/x$ is not defined at $x = 0$ — so can we integrate over an interval that contains $x = 0$ and still obtain an answer that makes sense? More generally can we integrate a function over an interval on which that function has discontinuities?

Example 1.3.15 $\int_{-1}^{1} \frac{1}{x^2}\mathrm{d}x$.

Find $\int_{-1}^{1} \frac{1}{x^2}\mathrm{d}x$.

Solution: Beware that this is a particularly nasty example, which illustrates a booby trap hidden in the fundamental theorem of calculus. The booby trap explodes when the theorem is applied sloppily.

- The sloppy solution starts, as our previous examples have, by finding an antiderivative of the integrand. In this case we know that

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x} = -\frac{1}{x^2}$$

which means that $-x^{-1}$ is an antiderivative of x^{-2}.

- This suggests (if we proceed naively) that

$$\begin{aligned} \int_{-1}^{1} x^{-2}\mathrm{d}x &= -\frac{1}{x}\bigg|_{-1}^{1} && \text{since } -1/x \text{ is an antiderivative of } 1/x^2 \\ &= -\frac{1}{1} - \Big(-\frac{1}{-1}\Big) \\ &= -2 \end{aligned}$$

Unfortunately,

- At this point we should really start to be concerned. This answer cannot be correct. Our integrand, being a square, is positive everywhere. So our integral represents the area of a region above the x-axis and must be positive.

- So what has gone wrong? The flaw in the computation is that the fundamental theorem of calculus, which says that

$$\text{if } F'(x) = f(x) \text{ then } \int_a^b f(x)\mathrm{d}x = F(b) - F(a),$$

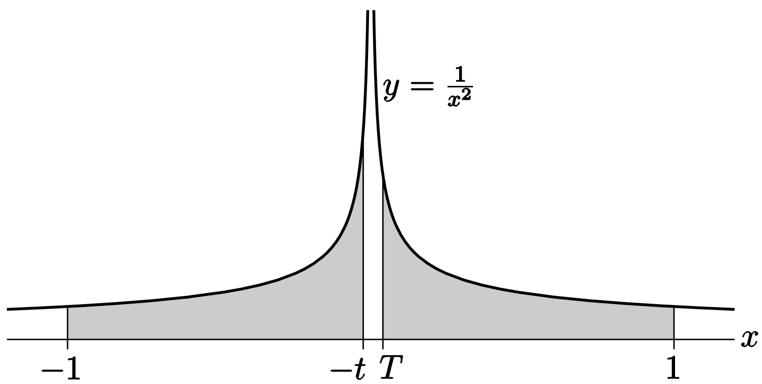
is *only* applicable when $F'(x)$ exists and equals $f(x)$ for *all* x between a and b.

- In this case $F'(x) = \frac{1}{x^2}$ does not exist for $x = 0$. So we cannot apply the fundamental theorem of calculus as we tried to above.

An integral, like $\int_{-1}^{1} \frac{1}{x^2}\mathrm{d}x$, whose integrand is undefined somewhere in the domain of integration is called improper. We'll give a more thorough treatment of improper integrals later in the text. For now, we'll just say that the correct way to define (and evaluate) improper integrals is as a limit of well-defined approximating integrals. We shall later see that, not only is $\int_{-1}^{1} \frac{1}{x^2}\mathrm{d}x$ not negative, it is infinite.

Remark 1.3.16 For completeness we'll show how to evaluate this integral by sneaking up on the point of discontinuity in the interval of integration. As noted above, we will give a fuller explanation of such integrals later in the text.

- Rather than evaluating the integral directly, we will approximate the integral using definite integrals on intervals that avoid the discontinuity. In the current example, the original domain of integration is $-1 \leq x \leq 1$. The domains of integration of the approximating integrals exclude from $[-1, 1]$ small intervals around $x = 0$.
- The shaded area in the figure below illustrates a typical approximating integral, whose domain of integration consists of the original domain of integration, $[-1, 1]$, but with the interval $[-t, T]$ excluded.



The full domain of integration is only recovered in the limit $t, T \to 0$.

- For this example, the correct computation is

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{x^2} \mathrm{d}x &= \lim_{t \to 0^+} \int_{-1}^{-t} \frac{1}{x^2} \mathrm{d}x \ + \lim_{T \to 0^+} \int_{T}^{1} \frac{1}{x^2} \mathrm{d}x \\
&= \lim_{t \to 0^+} \left[-\frac{1}{x} \right]_{-1}^{-t} + \lim_{T \to 0^+} \left[-\frac{1}{x} \right]_{T}^{1} \\
&= \lim_{t \to 0^+} \left[\left(-\frac{1}{-t} \right) - \left(-\frac{1}{-1} \right) \right] + \lim_{T \to 0^+} \left[\left(-\frac{1}{1} \right) - \left(-\frac{1}{T} \right) \right] \\
&= \lim_{t \to 0^+} \frac{1}{t} + \lim_{T \to 0^+} \frac{1}{T} - 2 \\
&= +\infty
\end{aligned}$$

- We can interpret this to mean that the signed area under the curve x^{-2} between $x = -1$ and $x = 1$ is infinite.

Example 1.3.16

The above examples have illustrated how we can use the fundamental theorem of calculus to convert knowledge of derivatives into knowledge of integrals. We are now in a position to easily build a table of integrals. Here is a short table of the most important derivatives that we know.

$F(x)$	1	x^n	$\sin x$	$\cos x$	$\tan x$	e^x	$\log_e \lvert x\rvert$	$\arcsin x$	$\arctan x$
$f(x) = F'(x)$	0	nx^{n-1}	$\cos x$	$-\sin x$	$\sec^2 x$	e^x	$\frac{1}{x}$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$

Of course we know other derivatives, such as those of $\sec x$ and $\cot x$, however the ones listed above are arguably the most important ones. From this table (with a very little massaging) we can write down a short table of indefinite integrals.

Theorem 1.3.17 Important indefinite integrals.

$f(x)$	$F(x) = \int f(x)\mathrm{d}x$
1	$x + C$
x^n	$\frac{1}{n+1}x^{n+1} + C$ provided that $n \neq -1$
$\dfrac{1}{x}$	$\log_e \lvert x\rvert + C$
e^x	$e^x + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$
$\dfrac{1}{1+x^2}$	$\arctan x + C$

Example 1.3.18 Using Theorem 1.3.17 to compute some integrals.

Find the following integrals

i $\int_2^7 e^x \mathrm{d}x$

ii $\int_{-2}^2 \frac{1}{1+x^2}\mathrm{d}x$

iii $\int_0^3 (2x^3 + 7x - 2)\mathrm{d}x$

Solution: We can proceed with each of these as before — find the antiderivative and then apply the fundamental theorem. The third integral is a little more complicated, but we can split it up into monomials using Theorem 1.2.1 and do each separately.

i An antiderivative of e^x is just e^x, so

$$\int_2^7 e^x \mathrm{d}x = e^x \Big|_2^7$$

$$= e^7 - e^2 = e^2(e^5 - 1).$$

ii An antiderivative of $\frac{1}{1+x^2}$ is $\arctan(x)$, so

$$\begin{aligned}\int_{-2}^{2} \frac{1}{1+x^2}\mathrm{d}x &= \arctan(x)\Big|_{-2}^{2} \\ &= \arctan(2) - \arctan(-2)\end{aligned}$$

We can simplify this a little further by noting that $\arctan(x)$ is an odd function, so $\arctan(-2) = -\arctan(2)$ and thus our integral is

$$= 2\arctan(2)$$

iii We can proceed by splitting the integral using Theorem 1.2.1(d)

$$\begin{aligned}\int_0^3 (2x^3 + 7x - 2)\mathrm{d}x &= \int_0^3 2x^3\mathrm{d}x + \int_0^3 7x\mathrm{d}x - \int_0^3 2\mathrm{d}x \\ &= 2\int_0^3 x^3\mathrm{d}x + 7\int_0^3 x\mathrm{d}x - 2\int_0^3 \mathrm{d}x\end{aligned}$$

and because we know that $x^4/4, x^2/2, x$ are antiderivatives of $x^3, x, 1$ respectively, this becomes

$$\begin{aligned}&= \left[\frac{x^4}{2}\right]_0^3 + \left[\frac{7x^2}{2}\right]_0^3 - [2x]_0^3 \\ &= \frac{81}{2} + \frac{7\cdot 9}{2} - 6 \\ &= \frac{81 + 63 - 12}{2} = \frac{132}{2} = 66.\end{aligned}$$

We can also just find the antiderivative of the whole polynomial by finding the antiderivatives of each term of the polynomial and then recombining them. This is equivalent to what we have done above, but perhaps a little neater:

$$\begin{aligned}\int_0^3 (2x^3 + 7x - 2)\mathrm{d}x &= \left[\frac{x^4}{2} + \frac{7x^2}{2} - 2x\right]_0^3 \\ &= \frac{81}{2} + \frac{7\cdot 9}{2} - 6 = 66.\end{aligned}$$

Example 1.3.18

1.3.2 ▸▸ Exercises

Exercises — Stage 1 Questions 11 through 14 are meant to help reinforce key ideas in the Fundamental Theorem of Calculus and its proof.So far, we have been able to guess many antiderivatives. Often, however, antiderivatives are very difficult to guess. In Questions 16 through 19, we will find some antiderivatives that might appear in a table of integrals. Coming up with the antiderivative might be quite difficult (strategies to do just that will form a large part of this semester), but *verifying* that your antiderivative is correct is as simple as differentiating.

1. $*$. Suppose that $f(x)$ is a function and $F(x) = e^{(x^2-3)} + 1$ is an antiderivative of $f(x)$. Evaluate the definite integral $\displaystyle\int_1^{\sqrt{5}} f(x)\,\mathrm{d}x$.

2. $*$. For the function $f(x) = x^3 - \sin 2x$, find its antiderivative $F(x)$ that satisfies $F(0) = 1$.

3. $*$. Decide whether each of the following statements is true or false. Provide a brief justification.

a If $f(x)$ is continuous on $[1, \pi]$ and differentiable on $(1, \pi)$, then $\displaystyle\int_1^{\pi} f'(x)\,\mathrm{d}x = f(\pi) - f(1)$.

b $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,\mathrm{d}x = 0$.

c If f is continuous on $[a, b]$ then $\displaystyle\int_a^b x f(x)\,\mathrm{d}x = x \int_a^b f(x)\,\mathrm{d}x$.

4. True or false: an antiderivative of $\dfrac{1}{x^2}$ is $\log(x^2)$ (where by $\log x$ we mean logarithm base e).

5. True or false: an antiderivative of $\cos(e^x)$ is $\frac{\sin(e^x)}{e^x}$.

6. Suppose $F(x) = \displaystyle\int_7^x \sin(t^2)\mathrm{d}t$. What is the instantaneous rate of change of $F(x)$ with respect to x?

7. Suppose $F(x) = \displaystyle\int_2^x e^{1/t}\mathrm{d}t$. What is the slope of the tangent line to $y = F(x)$ when $x = 3$?

8. Suppose $F'(x) = f(x)$. Give two different antiderivatives of $f(x)$.

9. In Question 1.1.8.45, Section 1.1, we found that

$$\int_0^a \sqrt{1-x^2}\mathrm{d}x = \frac{\pi}{4} - \frac{1}{2}\arccos(a) + \frac{1}{2}a\sqrt{1-a^2}.$$

a Verify that $\dfrac{\mathrm{d}}{\mathrm{d}a}\left\{\dfrac{\pi}{4} - \dfrac{1}{2}\arccos(a) + \dfrac{1}{2}a\sqrt{1-a^2}\right\} = \sqrt{1-a^2}$.

b Find a function $F(x)$ that satisfies $F'(x) = \sqrt{1-x^2}$ and $F(0) = \pi$.

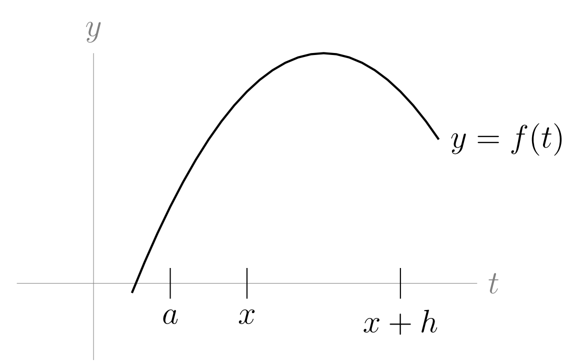
10. Evaluate the following integrals using the Fundamental Theorem of Calculus Part 2, or explain why it does not apply.

a $\displaystyle\int_{-\pi}^{\pi} \cos x\mathrm{d}x$.

b $\displaystyle\int_{-\pi}^{\pi} \sec^2 x\mathrm{d}x$.

c $\displaystyle\int_{-2}^{0} \frac{1}{x+1}\mathrm{d}x$.

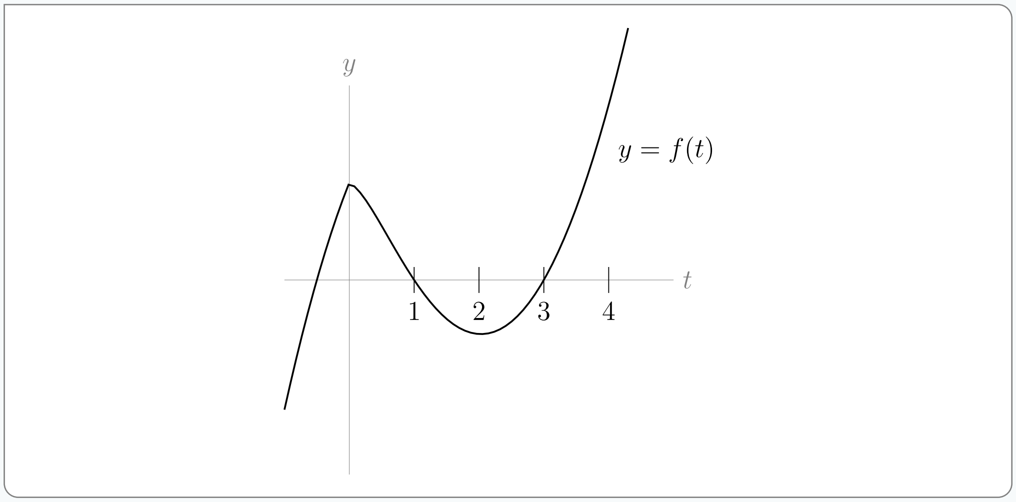
11. As in the proof of the Fundamental Theorem of Calculus, let $F(x) = \int_a^x f(t)\mathrm{d}t$. In the diagram below, shade the area corresponding to $F(x+h) - F(x)$.



12. Let $F(x) = \displaystyle\int_0^x f(t)\mathrm{d}t$, where $f(t)$ is shown in the graph below, and $0 \leq x \leq 4$.

a Is $F(0)$ positive, negative, or zero?

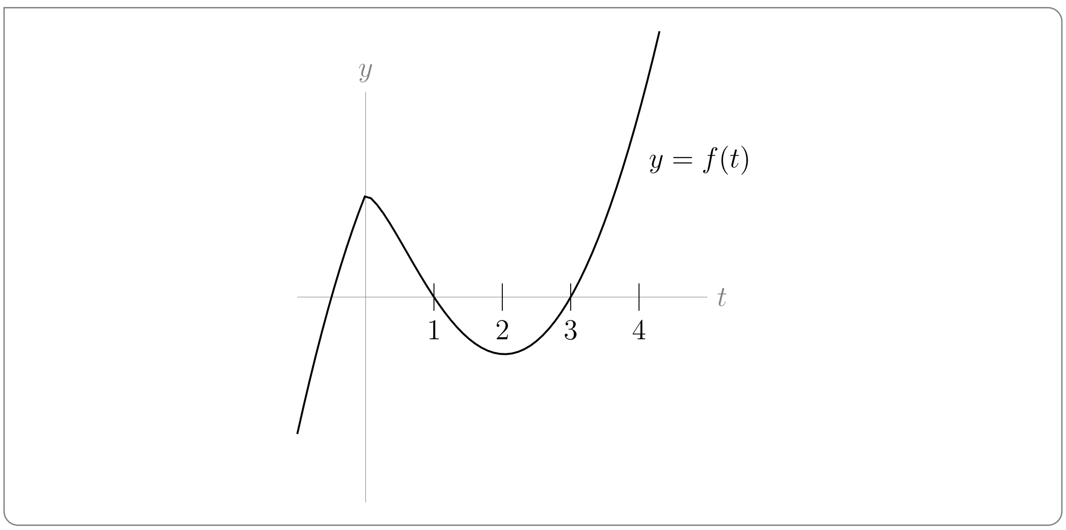
b Where is $F(x)$ increasing and where is it decreasing?



13. Let $G(x) = \int_x^0 f(t)\mathrm{d}t$, where $f(t)$ is shown in the graph below, and $0 \le x \le 4$.

a Is $G(0)$ positive, negative, or zero?

b Where is $G(x)$ increasing and where is it decreasing?



14. Let $F(x) = \int_a^x t\mathrm{d}t$. Using the definition of the derivative, find $F'(x)$.

15. Give a continuous function $f(x)$ so that $F(x) = \int_0^x f(t)\mathrm{d}t$ is a constant.

16. Evaluate and simplify $\frac{\mathrm{d}}{\mathrm{d}x}\{x\log(ax) - x\}$, where a is some constant and $\log(x)$ is the logarithm base e. What *antiderivative* does this tell you?

17. Evaluate and simplify $\frac{\mathrm{d}}{\mathrm{d}x}\{e^x\left(x^3 - 3x^2 + 6x - 6\right)\}$. What *antiderivative* does this tell you?

18. Evaluate and simplify $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\log\left|x+\sqrt{x^2+a^2}\right|\right\}$, where a is some constant. What *antiderivative* does this tell you?

19. Evaluate and simplify $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\sqrt{x(a+x)} - a\log\left(\sqrt{x}+\sqrt{a+x}\right)\right\}$, where a is some constant. What *antiderivative* does this tell you?

Exercises — Stage 2

20. $*$. Evaluate $\displaystyle\int_0^2 \left(x^3+\sin x\right)\mathrm{d}x$.

21. $*$. Evaluate $\displaystyle\int_1^2 \frac{x^2+2}{x^2}\,\mathrm{d}x$.

22. Evaluate $\displaystyle\int \frac{1}{1+25x^2}\mathrm{d}x$.

23. Evaluate $\displaystyle\int \frac{1}{\sqrt{2-x^2}}\mathrm{d}x$.

24. Evaluate $\displaystyle\int \tan^2 x\mathrm{d}x$.

25. Evaluate $\displaystyle\int 3\sin x\cos x\mathrm{d}x$.

26. Evaluate $\displaystyle\int \cos^2 x\mathrm{d}x$.

27. $*$. If

$$F(x) = \int_0^x \log(2+\sin t)\,\mathrm{d}t \quad \text{and} \quad G(y) = \int_y^0 \log(2+\sin t)\,\mathrm{d}t$$

find $F'\left(\frac{\pi}{2}\right)$ and $G'\left(\frac{\pi}{2}\right)$.

28. $*$. Let $f(x) = \displaystyle\int_1^x 100(t^2-3t+2)e^{-t^2}\mathrm{d}t$. Find the interval(s) on which f is increasing.

29. $*$. If $F(x) = \displaystyle\int_0^{\cos x} \frac{1}{t^3+6}\,\mathrm{d}t$, find $F'(x)$.

30. $*$. Compute $f'(x)$ where $f(x) = \displaystyle\int_0^{1+x^4} e^{t^2}\mathrm{d}t$.

31. $*$. Evaluate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\displaystyle\int_0^{\sin x}(t^6+8)\mathrm{d}t\right\}$.

32. $*$. Let $F(x) = \displaystyle\int_0^{x^3} e^{-t} \sin\left(\frac{\pi t}{2}\right) \mathrm{d}t$. Calculate $F'(1)$.

33. $*$. Find $\displaystyle\frac{\mathrm{d}}{\mathrm{d}u}\left\{\int_{\cos u}^{0} \frac{\mathrm{d}t}{1+t^3}\right\}$.

34. $*$. Find $f(x)$ if $x^2 = 1 + \displaystyle\int_1^x f(t)\mathrm{d}t$.

35. $*$. If $x \sin(\pi x) = \displaystyle\int_0^x f(t)\,\mathrm{d}t$ where f is a continuous function, find $f(4)$.

36. $*$. Consider the function $F(x) = \displaystyle\int_0^{x^2} e^{-t}\,\mathrm{d}t + \int_{-x}^{0} e^{-t^2}\,\mathrm{d}t$.

a Find $F'(x)$.

b Find the value of x for which $F(x)$ takes its minimum value.

37. $*$. If $F(x)$ is defined by $F(x) = \displaystyle\int_{x^4-x^3}^{x} e^{\sin t}\,\mathrm{d}t$, find $F'(x)$.

38. $*$. Evaluate $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_{x^5}^{-x^2} \cos\left(e^t\right)\mathrm{d}t\right\}$.

39. $*$. Differentiate $\displaystyle\int_x^{e^x} \sqrt{\sin t}\,\mathrm{d}t$ for $0 < x < \log \pi$.

40. $*$. Evaluate $\displaystyle\int_1^5 f(x)\,\mathrm{d}x$, where $f(x) = \begin{cases} 3 & \text{if } x \le 3 \\ x & \text{if } x \ge 3 \end{cases}$.

Exercises — Stage 3

41. $*$. If $f'(1) = 2$ and $f'(2) = 3$, find $\displaystyle\int_1^2 f'(x)f''(x)\,\mathrm{d}x$.

42. $*$. A car traveling at 30 m/s applies its brakes at time $t = 0$, its velocity (in m/s) decreasing according to the formula $v(t) = 30 - 10t$. How far does the car go before it stops?

43. $*$. Compute $f'(x)$ where $f(x) = \displaystyle\int_0^{2x-x^2} \log\left(1+e^t\right)\mathrm{d}t$. Does $f(x)$ have an absolute maximum? Explain.

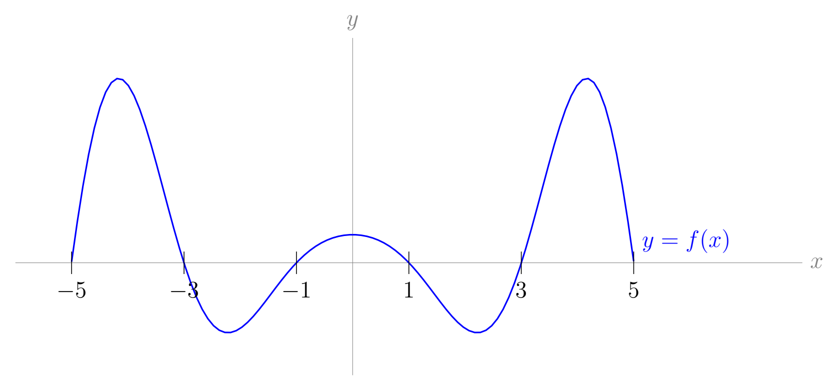
44. $*$. Find the minimum value of $\displaystyle\int_0^{x^2-2x} \frac{\mathrm{d}t}{1+t^4}$. Express your answer as an integral.

45. $*$. Define the function $F(x) = \displaystyle\int_0^{x^2} \sin(\sqrt{t})\,\mathrm{d}t$ on the interval $0 < x < 4$. On this interval, where does $F(x)$ have a maximum?

46. $*$. Evaluate $\displaystyle\lim_{n\to\infty} \frac{\pi}{n}\sum_{j=1}^{n} \sin\left(\frac{j\pi}{n}\right)$ by interpreting it as a limit of Riemann sums.

47. $*$. Use Riemann sums to evaluate the limit $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \frac{1}{1+\frac{j}{n}}$.

48. Below is the graph of $y = f(t)$, $-5 \le t \le 5$. Define $F(x) = \displaystyle\int_0^x f(t)\mathrm{d}t$ for any x in $[-5, 5]$. Sketch $F(x)$.



49. $*$. Define $f(x) = x^3 \displaystyle\int_0^{x^3+1} e^{t^3}\,\mathrm{d}t$.

a Find a formula for the derivative $f'(x)$. (Your formula may include an integral sign.)

b Find the equation of the tangent line to the graph of $y = f(x)$ at $x = -1$.

50. Two students calculate $\int f(x)\mathrm{d}x$ for some function $f(x)$.

- Student A calculates $\int f(x)\mathrm{d}x = \tan^2 x + x + C$
- Student B calculates $\int f(x)\mathrm{d}x = \sec^2 x + x + C$
- It is a fact that $\frac{\mathrm{d}}{\mathrm{d}x}\{\tan^2 x\} = f(x) - 1$

Who ended up with the correct answer?

51. Let $F(x) = \int_0^x x^3 \sin(t)\mathrm{d}t$.

a Evaluate $F(3)$.

b What is $F'(x)$?

52. Let $f(x)$ be an even function, defined everywhere, and let $F(x)$ be an antiderivative of $f(x)$. Is $F(x)$ even, odd, or not necessarily either one? (You may use your answer from Section 1.2, Question 1.2.3.20.)

1.4 ▲ Substitution

1.4.1 ▶▶ Substitution

In the previous section we explored the fundamental theorem of calculus and the link it provides between definite integrals and antiderivatives. Indeed, integrals with simple integrands are usually evaluated via this link. In this section we start to explore methods for integrating more complicated integrals. We have already seen — via Theorem 1.2.1 — that integrals interact very nicely with addition, subtraction and multiplication by constants:

$$\int_a^b \left(Af(x) + Bg(x)\right)\mathrm{d}x = A\int_a^b f(x)\mathrm{d}x + B\int_a^b g(x)\mathrm{d}x$$

for A, B constants. By combining this with the list of indefinite integrals in Theorem 1.3.17, we can compute integrals of linear combinations of simple functions. For example

$$\begin{aligned}\int_1^4 \left(e^x - 2\sin x + 3x^2\right)\mathrm{d}x &= \int_1^4 e^x\mathrm{d}x - 2\int_1^4 \sin x\mathrm{d}x + 3\int_1^4 x^2\mathrm{d}x \\ &= \left(e^x + (-2)\cdot(-\cos x) + 3\frac{x^3}{3}\right)\Bigg|_1^4 \qquad \text{and so on}\end{aligned}$$

Of course there are a great many functions that can be approached in this way, however there are some very simple examples that cannot.

$$\int \sin(\pi x)\mathrm{d}x \qquad \int xe^x\mathrm{d}x \qquad \int \frac{x}{x^2 - 5x + 6}\mathrm{d}x$$

In each case the integrands are not linear combinations of simpler functions; in order to compute them we need to understand how integrals (and antiderivatives) interact with compositions, products and quotients. We reached a very similar point in our differential calculus course where we understood the linearity of the derivative,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(Af(x)+Bg(x)\right)=A\frac{\mathrm{d}f}{\mathrm{d}x}+B\frac{\mathrm{d}g}{\mathrm{d}x},$$

but had not yet seen the chain, product and quotient rules [1]. While we will develop tools to find the second and third integrals in later sections, we should really start with how to integrate compositions of functions.

It is important to state up front, that in general one cannot write down the integral of the composition of two functions — even if those functions are simple. This is not because the integral does not exist. Rather it is because the integral cannot be written down as a finite combination of the standard functions we know. A very good example of this, which we encountered in Example 1.3.4, is the composition of e^x and $-x^2$. Even though we know

$$\int e^x \mathrm{d}x = e^x + C \qquad \text{and} \qquad \int -x^2 \mathrm{d}x = -\frac{1}{3}x^3 + C$$

there is no simple function that is equal to the indefinite integral

$$\int e^{-x^2}\mathrm{d}x.$$

even though the indefinite integral exists. In this way integration is very different from differentiation.

With that caveat out of the way, we can introduce the substitution rule. The substitution rule is obtained by antidifferentiating the chain rule. In some sense it is the chain rule in reverse. For completeness, let us restate the chain rule:

Theorem 1.4.1 The chain rule.

Let $F(u)$ and $u(x)$ be differentiable functions and form their composition $F(u(x))$. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}F\big(u(x)\big)=F'\big(u(x)\big)\cdot u'(x)$$

Equivalently, if $y(x)=F(u(x))$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}F}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Consider a function $f(u)$, which has antiderivative $F(u)$. Then we know that

$$\int f(u)\mathrm{d}u=\int F'(u)\mathrm{d}u=F(u)+C$$

1 If your memory of these rules is a little hazy then you really should go back and revise them before proceeding. You will definitely need a good grasp of the chain rule for what follows in this section.

Now take the above equation and substitute into it $u = u(x)$ — i.e. replace the variable u with any (differentiable) function of x to get

$$\int f(u)\mathrm{d}u\bigg|_{u=u(x)} = F(u(x)) + C$$

But now the right-hand side is a function of x, so we can differentiate it with respect to x to get

$$\frac{\mathrm{d}}{\mathrm{d}x}F(u(x)) = F'(u(x)) \cdot u'(x)$$

This tells us that $F(u(x))$ is an antiderivative of the function $F'(u(x)) \cdot u'(x) = f(u(x))u'(x)$. Thus we know

$$\int f\big(u(x)\big) \cdot u'(x)\,\mathrm{d}x = F\big(u(x)\big) + C = \int f(u)\,\mathrm{d}u\bigg|_{u=u(x)}$$

This is the substitution rule for indefinite integrals.

Theorem 1.4.2 The substitution rule — indefinite integral version.

For any differentiable function $u(x)$:

$$\int f(u(x))u'(x)\mathrm{d}x = \int f(u)\mathrm{d}u\bigg|_{u=u(x)}$$

In order to apply the substitution rule successfully we will have to write the integrand in the form $f(u(x)) \cdot u'(x)$. To do this we need to make a good choice of the function $u(x)$; after that it is not hard to then find $f(u)$ and $u'(x)$. Unfortunately there is no one strategy for choosing $u(x)$. This can make applying the substitution rule more art than science [2]. Here we suggest two possible strategies for picking $u(x)$:

1 Factor the integrand and choose one of the factors to be $u'(x)$. For this to work, you must be able to easily find the antiderivative of the chosen factor. The antiderivative will be $u(x)$.

2 Look for a factor in the integrand that is a function with an argument that is more complicated than just "x". That factor will play the role of $f\big(u(x)\big)$ Choose $u(x)$ to be the complicated argument.

Here are two examples which illustrate each of those strategies in turn.

2 Thankfully this does become easier with experience and we recommend that the reader read some examples and then practice a LOT.

Example 1.4.3 $\int 9\sin^8(x)\cos(x)\mathrm{d}x.$

Consider the integral

$$\int 9\sin^8(x)\cos(x)\mathrm{d}x$$

We want to massage this into the form of the integrand in the substitution rule — namely $f(u(x)) \cdot u'(x)$. Our integrand can be written as the product of the two factors

$$\underbrace{9\sin^8(x)}_{\text{first factor}} \cdot \underbrace{\cos(x)}_{\text{second factor}}$$

and we start by determining (or guessing) which factor plays the role of $u'(x)$. We can choose $u'(x) = 9\sin^8(x)$ or $u'(x) = \cos(x)$.

- If we choose $u'(x) = 9\sin^8(x)$, then antidifferentiating this to find $u(x)$ is really not very easy. So it is perhaps better to investigate the other choice before proceeding further with this one.
- If we choose $u'(x) = \cos(x)$, then we know (Theorem 1.3.17) that $u(x) = \sin(x)$. This also works nicely because it makes the other factor simplify quite a bit $9\sin^8(x) = 9u^8$. This looks like the right way to go.

So we go with the second choice. Set $u'(x) = \cos(x), u(x) = \sin(x)$, then

$$\begin{aligned}\int 9\sin^8(x)\cos(x)\mathrm{d}x &= \int 9u(x)^8 \cdot u'(x)\mathrm{d}x \\ &= \int 9u^8\mathrm{d}u\bigg|_{u=\sin(x)} && \text{by the substitution rule}\end{aligned}$$

We are now left with the problem of antidifferentiating a monomial; this we can do with Theorem 1.3.17.

$$\begin{aligned}&= \left(u^9 + C\right)\bigg|_{u=\sin(x)} \\ &= \sin^9(x) + C\end{aligned}$$

Note that $9\sin^8(x)\cos(x)$ is a function of x. So our answer, which is the indefinite integral of $9\sin^8(x)\cos(x)$, must also be a function of x. This is why we have substituted $u = \sin(x)$ in the last step of our solution — it makes our solution a function of x.

Example 1.4.4 $\int 3x^2\cos(x^3)\mathrm{d}x$.

Evaluate the integral

$$\int 3x^2\cos(x^3)\mathrm{d}x$$

Solution: Again we are going to use the substitution rule and helpfully our integrand is a product of two factors

$$\underbrace{3x^2}_{\text{first factor}} \cdot \underbrace{\cos(x^3)}_{\text{second factor}}$$

The second factor, $\cos\left(x^3\right)$ is a function, namely cos, with a complicated argument, namely x^3. So we try $u(x) = x^3$. Then $u'(x) = 3x^2$, which is the other factor in the integrand. So the integral becomes

$$\begin{aligned}
\int 3x^2\cos(x^3)\mathrm{d}x &= \int u'(x)\cos\big(u(x)\big)\mathrm{d}x && \text{just swap order of factors}\\
&= \int \cos\big(u(x)\big)u'(x)\mathrm{d}x && \text{by the substitution rule}\\
&= \int \cos(u)\mathrm{d}u\bigg|_{u=x^3}\\
&= \left(\sin(u)+C\right)\bigg|_{u=x^3} && \text{using Theorem 1.3.17)}\\
&= \sin(x^3)+C
\end{aligned}$$

One more — we'll use this to show how to use the substitution rule with definite integrals.

Example 1.4.5 $\int_0^1 e^x\sin(e^x)\mathrm{d}x$.

Compute

$$\int_0^1 e^x\sin\left(e^x\right)\mathrm{d}x.$$

Solution: Again we use the substitution rule.

- The integrand is again the product of two factors and we can choose $u'(x) = e^x$ or $u'(x) = \sin(e^x)$.
- If we choose $u'(x) = e^x$ then $u(x) = e^x$ and the other factor becomes $\sin(u)$ — this looks promising. Notice that if we applied the other strategy of looking for a complicated argument then we would arrive at the same choice.

- So we try $u'(x) = e^x$ and $u(x) = e^x$. This gives (if we ignore the limits of integration for a moment)

$$\begin{aligned}
\int e^x \sin\big(e^x\big)\mathrm{d}x &= \int \sin\big(u(x)\big)u'(x)\mathrm{d}x && \text{apply the substitution rule} \\
&= \int \sin(u)\mathrm{d}u \Big|_{u=e^x} \\
&= (-\cos(u) + C)\Big|_{u=e^x} \\
&= -\cos\big(e^x\big) + C
\end{aligned}$$

- But what happened to the limits of integration? We can incorporate them now. We have just shown that the indefinite integral is $-\cos(e^x)$, so by the fundamental theorem of calculus

$$\begin{aligned}
\int_0^1 e^x \sin\big(e^x\big)\mathrm{d}x &= \big[-\cos\big(e^x\big)\big]_0^1 \\
&= -\cos(e^1) - (-\cos(e^0)) \\
&= -\cos(e) + \cos(1)
\end{aligned}$$

Example 1.4.5

Theorem 1.4.2, the substitution rule for indefinite integrals, tells us that if $F(u)$ is any antiderivative for $f(u)$, then $F\big(u(x)\big)$ is an antiderivative for $f\big(u(x)\big)u'(x)$. So the fundamental theorem of calculus gives us

$$\begin{aligned}
\int_a^b f\big(u(x)\big)u'(x)\,\mathrm{d}x &= F\big(u(x)\big)\Big|_{x=a}^{x=b} \\
&= F\big(u(b)\big) - F\big(u(a)\big) \\
&= \int_{u(a)}^{u(b)} f(u)\,\mathrm{d}u && \text{since } F(u) \text{ is an antiderivative for } f(u)
\end{aligned}$$

and we have just found

Theorem 1.4.6 The substitution rule — definite integral version.

For any differentiable function $u(x)$:

$$\int_a^b f(u(x))u'(x)\mathrm{d}x = \int_{u(a)}^{u(b)} f(u)\mathrm{d}u$$

Notice that to get from the integral on the left hand side to the integral on the right hand side you

- substitute [3] $u(x) \to u$ and $u'(x)\mathrm{d}x \to \mathrm{d}u$,
- set the lower limit for the u integral to the value of u (namely $u(a)$) that corresponds to the lower limit of the x integral (namely $x = a$), and
- set the upper limit for the u integral to the value of u (namely $u(b)$) that corresponds to the upper limit of the x integral (namely $x = b$).

Also note that we now have two ways to evaluate definite integrals of the form $\int_a^b f\big(u(x)\big)u'(x)\,\mathrm{d}x$.

- We can find the indefinite integral $\int f\big(u(x)\big)u'(x)\,\mathrm{d}x$, using Theorem 1.4.2, and then evaluate the result between $x = a$ and $x = b$. This is what was done in Example 1.4.5.
- Or we can apply Theorem 1.4.2. This entails finding the indefinite integral $\int f(u)\,\mathrm{d}u$ and evaluating the result between $u = u(a)$ and $u = u(b)$. This is what we will do in the following example.

Example 1.4.7 $\int_0^1 x^2\sin(x^3+1)\mathrm{d}x$.

Compute

$$\int_0^1 x^2 \sin\big(x^3+1\big)\mathrm{d}x$$

Solution:

- In this example the integrand is already neatly factored into two pieces. While we could deploy either of our two strategies, it is perhaps easier in this case to choose $u(x)$ by looking for a complicated argument.
- The second factor of the integrand is $\sin\big(x^3+1\big)$, which is the function sin evaluated at x^3+1. So set $u(x) = x^3+1$, giving $u'(x) = 3x^2$ and $f(u) = \sin(u)$
- The first factor of the integrand is x^2 which is not quite $u'(x)$, however we can easily massage the integrand into the required form by multiplying and dividing by 3:

$$x^2\sin\big(x^3+1\big) = \frac{1}{3}\cdot 3x^2\cdot\sin\big(x^3+1\big).$$

- We want this in the form of the substitution rule, so we do a little massaging:

$$\int_0^1 x^2\sin\big(x^3+1\big)\mathrm{d}x = \int_0^1 \frac{1}{3}\cdot 3x^2\cdot\sin\big(x^3+1\big)\mathrm{d}x$$

3 A good way to remember this last step is that we replace $\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$ by just $\mathrm{d}u$ — which looks like we cancelled out the $\mathrm{d}x$ terms: $\frac{\mathrm{d}u}{\cancel{\mathrm{d}x}}\cancel{\mathrm{d}x} = \mathrm{d}u$. While using "cancel the $\mathrm{d}x$" is a good mnemonic (memory aid), you should not think of the derivative $\frac{\mathrm{d}u}{\mathrm{d}x}$ as a fraction — you are not dividing $\mathrm{d}u$ by $\mathrm{d}x$.

$$= \frac{1}{3}\int_0^1 \sin\left(x^3+1\right)\cdot 3x^2 \mathrm{d}x$$

by Theorem 1.2.1(c)

- Now we are ready for the substitution rule:

$$\frac{1}{3}\int_0^1 \sin\left(x^3+1\right)\cdot 3x^2\mathrm{d}x = \frac{1}{3}\int_0^1 \underbrace{\sin\left(x^3+1\right)}_{=f(u(x))}\cdot \underbrace{3x^2}_{=u'(x)}\mathrm{d}x$$

Now set $u(x) = x^3+1$ and $f(u) = \sin(u)$

$$\begin{aligned}
&= \frac{1}{3}\int_0^1 f(u(x))u'(x)\mathrm{d}x \\
&= \frac{1}{3}\int_{u(0)}^{u(1)} f(u)\mathrm{d}u && \text{by the substitution rule} \\
&= \frac{1}{3}\int_1^2 \sin(u)\mathrm{d}u && \text{since } u(0)=1 \text{ and } u(1)=2 \\
&= \frac{1}{3}\big[-\cos(u)\big]_1^2 \\
&= \frac{1}{3}\big(-\cos(2)-(-\cos(1))\big) \\
&= \frac{\cos(1)-\cos(2)}{3}.
\end{aligned}$$

Example 1.4.7

There is another, and perhaps easier, way to view the manipulations in the previous example. Once you have chosen $u(x)$ you

- make the substitution $u(x) \to u$,
- replace $\mathrm{d}x \to \dfrac{1}{u'(x)}\mathrm{d}u$.

In so doing, we take the integral

$$\begin{aligned}
\int_a^b f(u(x))\cdot u'(x)\mathrm{d}x &= \int_{u(a)}^{u(b)} f(u)\cdot u'(x)\cdot \frac{1}{u'(x)}\mathrm{d}u \\
&= \int_{u(a)}^{u(b)} f(u)\mathrm{d}u && \text{exactly the substitution rule}
\end{aligned}$$

but we do not have to manipulate the integrand so as to make $u'(x)$ explicit. Let us redo the previous example by this approach.

Example 1.4.8 Example 1.4.7 revisited.

Compute the integral

$$\int_0^1 x^2 \sin\left(x^3+1\right)\mathrm{d}x$$

Solution:

- We have already observed that one factor of the integrand is $\sin\left(x^3+1\right)$, which is sin evaluated at x^3+1. Thus we try setting $u(x) = x^3+1$.
- This makes $u'(x) = 3x^2$, and we replace $u(x) = x^3+1 \to u$ and $\mathrm{d}x \to \frac{1}{u'(x)}\mathrm{d}u = \frac{1}{3x^2}\mathrm{d}u$:

$$\begin{aligned}\int_0^1 x^2 \sin\left(x^3+1\right)\mathrm{d}x &= \int_{u(0)}^{u(1)} x^2 \underbrace{\sin\left(x^3+1\right)}_{=\sin(u)} \frac{1}{3x^2}\mathrm{d}u \\ &= \int_1^2 \sin(u)\frac{x^2}{3x^2}\mathrm{d}u \\ &= \int_1^2 \frac{1}{3}\sin(u)\mathrm{d}u \\ &= \frac{1}{3}\int_1^2 \sin(u)\mathrm{d}u\end{aligned}$$

  which is precisely the integral we found in Example 1.4.7.

Example 1.4.9 Some more substitutions.

Compute the indefinite integrals

$$\int \sqrt{2x+1}\mathrm{d}x \qquad \text{and} \qquad \int e^{3x-2}\mathrm{d}x$$

Solution:

- Starting with the first integral, we see that it is not too hard to spot the complicated argument. If we set $u(x) = 2x+1$ then the integrand is just $\sqrt{u}$.
- Hence we substitute $2x+1 \to u$ and $\mathrm{d}x \to \frac{1}{u'(x)}\mathrm{d}u = \frac{1}{2}\mathrm{d}u$:

$$\begin{aligned}\int \sqrt{2x+1}\mathrm{d}x &= \int \sqrt{u}\frac{1}{2}\mathrm{d}u \\ &= \int u^{1/2}\frac{1}{2}\mathrm{d}u\end{aligned}$$

$$= \left(\frac{2}{3}u^{3/2} \cdot \frac{1}{2} + C\right)\Big|_{u=2x+1}$$
$$= \frac{1}{3}(2x+1)^{3/2} + C$$

- We can evaluate the second integral in much the same way. Set $u(x) = 3x - 2$ and replace $\mathrm{d}x$ by $\frac{1}{u'(x)}\mathrm{d}u = \frac{1}{3}\mathrm{d}u$:

$$\begin{aligned}\int e^{3x-2}\mathrm{d}x &= \int e^u \frac{1}{3}\mathrm{d}u \\ &= \left(\frac{1}{3}e^u + C\right)\Big|_{u=3x-2} \\ &= \frac{1}{3}e^{3x-2} + C\end{aligned}$$

Example 1.4.9

This last example illustrates that substitution can be used to easily deal with arguments of the form $ax + b$, i.e. that are linear functions of x, and suggests the following theorem.

Theorem 1.4.10

Let $F(u)$ be an antiderivative of $f(u)$ and let a, b be constants. Then

$$\int f(ax+b)\mathrm{d}x = \frac{1}{a}F(ax+b) + C$$

Proof. We can show this using the substitution rule. Let $u(x) = ax + b$ so $u'(x) = a$, then

$$\begin{aligned}\int f(ax+b)\mathrm{d}x &= \int f(u) \cdot \frac{1}{u'(x)}\mathrm{d}u \\ &= \int \frac{1}{a}f(u)\mathrm{d}u \\ &= \frac{1}{a}\int f(u)\mathrm{d}u && \text{since } a \text{ is a constant} \\ &= \frac{1}{a}F(u)\Big|_{u=ax+b} + C && \text{since } F(u) \text{ is an antiderivative of } f(u) \\ &= \frac{1}{a}F(ax+b) + C.\end{aligned}$$

□

Now we can do the following example using the substitution rule or the above theorem:

Example 1.4.11 $\int_0^{\frac{\pi}{2}} \cos(3x)\mathrm{d}x$.

Compute $\int_0^{\frac{\pi}{2}} \cos(3x)\mathrm{d}x$.

- In this example we should set $u = 3x$, and substitute $\mathrm{d}x \to \frac{1}{u'(x)}\mathrm{d}u = \frac{1}{3}\mathrm{d}u$. When we do this we also have to convert the limits of the integral: $u(0) = 0$ and $u(\pi/2) = 3\pi/2$. This gives

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos(3x)\mathrm{d}x &= \int_0^{\frac{3\pi}{2}} \cos(u)\frac{1}{3}\mathrm{d}u \\ &= \left[\frac{1}{3}\sin(u)\right]_0^{\frac{3\pi}{2}} \\ &= \frac{\sin(3\pi/2) - \sin(0)}{3} \\ &= \frac{-1-0}{3} = -\frac{1}{3}.\end{aligned}$$

- We can also do this example more directly using the above theorem. Since $\sin(x)$ is an antiderivative of $\cos(x)$, Theorem 1.4.10 tells us that $\frac{\sin(3x)}{3}$ is an antiderivative of $\cos(3x)$. Hence

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos(3x)\mathrm{d}x &= \left[\frac{\sin(3x)}{3}\right]_0^{\frac{\pi}{2}} \\ &= \frac{\sin(3\pi/2) - \sin(0)}{3} \\ &= -\frac{1}{3}.\end{aligned}$$

The rest of this section is just more examples of the substitution rule. We recommend that you after reading these that you practice many examples by yourself under exam conditions.

Example 1.4.12 $\int_0^1 x^2 \sin(1-x^3)\mathrm{d}x$.

This integral looks a lot like that of Example 1.4.7. It makes sense to try $u(x) = 1-x^3$ since it is the argument of $\sin(1-x^3)$. We

- substitute $u = 1 - x^3$ and
- replace $\mathrm{d}x$ with $\frac{1}{u'(x)}\mathrm{d}u = \frac{1}{-3x^2}\mathrm{d}u$,

- when $x = 0$, we have $u = 1 - 0^3 = 1$ and
- when $x = 1$, we have $u = 1 - 1^3 = 0$.

So

$$\begin{aligned}\int_0^1 x^2 \sin\left(1 - x^3\right) \cdot \mathrm{d}x &= \int_1^0 x^2 \sin(u) \cdot \frac{1}{-3x^2}\mathrm{d}u \\ &= \int_1^0 -\frac{1}{3}\sin(u)\mathrm{d}u.\end{aligned}$$

Note that the lower limit of the u-integral, namely 1, is larger than the upper limit, which is 0. There is absolutely nothing wrong with that. We can simply evaluate the u-integral in the normal way. Since $-\cos(u)$ is an antiderivative of $\sin(u)$:

$$\begin{aligned}&= \left[\frac{\cos(u)}{3}\right]_1^0 \\ &= \frac{\cos(0) - \cos(1)}{3} \\ &= \frac{1 - \cos(1)}{3}.\end{aligned}$$

Example 1.4.12

Example 1.4.13 $\int_0^1 \frac{1}{(2x+1)^3}\mathrm{d}x$.

Compute $\int_0^1 \frac{1}{(2x+1)^3}\mathrm{d}x$.
We could do this one using Theorem 1.4.10, but its not too hard to do without. We can think of the integrand as the function "one over a cube" with the argument $2x + 1$. So it makes sense to substitute $u = 2x + 1$. That is

- set $u = 2x + 1$ and
- replace $\mathrm{d}x \to \frac{1}{u'(x)}\mathrm{d}u = \frac{1}{2}\mathrm{d}u$.
- When $x = 0$, we have $u = 2 \times 0 + 1 = 1$ and
- when $x = 1$, we have $u = 2 \times 1 + 1 = 3$.

So

$$\begin{aligned}\int_0^1 \frac{1}{(2x+1)^3}\mathrm{d}x &= \int_1^3 \frac{1}{u^3} \cdot \frac{1}{2}\mathrm{d}u \\ &= \frac{1}{2}\int_1^3 u^{-3}\mathrm{d}u\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\left[\frac{u^{-2}}{-2}\right]_1^3 \\
&= \frac{1}{2}\left(\frac{1}{-2}\cdot\frac{1}{9} - \frac{1}{-2}\cdot\frac{1}{1}\right) \\
&= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{18}\right) = \frac{1}{2}\cdot\frac{8}{18} \\
&= \frac{2}{9}
\end{aligned}$$

Example 1.4.13

Example 1.4.14 $\int_0^1 \frac{x}{1+x^2}\mathrm{d}x$.

Evaluate $\int_0^1 \frac{x}{1+x^2}\mathrm{d}x$.
Solution:

- The integrand can be rewritten as $x \cdot \frac{1}{1+x^2}$. This second factor suggests that we should try setting $u = 1 + x^2$ — and so we interpret the second factor as the function "one over" evaluated at argument $1 + x^2$.
- With this choice we
  - set $u = 1 + x^2$,
  - substitute $\mathrm{d}x \to \frac{1}{2x}\mathrm{d}u$, and
  - translate the limits of integration: when $x = 0$, we have $u = 1 + 0^2 = 1$ and when $x = 1$, we have $u = 1 + 1^2 = 2$.
- The integral then becomes

$$\begin{aligned}
\int_0^1 \frac{x}{1+x^2}\mathrm{d}x &= \int_1^2 \frac{x}{u}\frac{1}{2x}\mathrm{d}u \\
&= \int_1^2 \frac{1}{2u}\mathrm{d}u \\
&= \frac{1}{2}\big[\log|u|\big]_1^2 \\
&= \frac{\log 2 - \log 1}{2} = \frac{\log 2}{2}.
\end{aligned}$$

Remember that we are using the notation "log" for the natural logarithm, i.e. the logarithm with base e. You might also see it written as "$\ln x$", or with the base made explicit as "$\log_e x$".

Example 1.4.15 $\int x^3 \cos\left(x^4+2\right)\mathrm{d}x$.

Compute the integral $\int x^3 \cos\left(x^4+2\right)\mathrm{d}x$.

Solution:

- The integrand is the product of cos evaluated at the argument x^4+2 times x^3, which aside from a factor of 4, is the derivative of the argument x^4+2.
- Hence we set $u = x^4+2$ and then substitute $\mathrm{d}x \to \frac{1}{u'(x)}\mathrm{d}u = \frac{1}{4x^3}\mathrm{d}u$.
- Before proceeding further, we should note that this is an indefinite integral so we don't have to worry about the limits of integration. However we do need to make sure our answer is a function of x — we cannot leave it as a function of u.
- With this choice of u, the integral then becomes

$$\begin{aligned}\int x^3 \cos\left(x^4+2\right)\mathrm{d}x &= \int x^3 \cos(u)\frac{1}{4x^3}\mathrm{d}u\Big|_{u=x^4+2}\\ &= \int \frac{1}{4}\cos(u)\mathrm{d}u\Big|_{u=x^4+2}\\ &= \left(\frac{1}{4}\sin(u)+C\right)\Big|_{u=x^4+2}\\ &= \frac{1}{4}\sin(x^4+2)+C.\end{aligned}$$

The next two examples are more involved and require more careful thinking.

Example 1.4.16 $\int \sqrt{1+x^2}\,x^3\mathrm{d}x$.

Compute $\int \sqrt{1+x^2}\,x^3\mathrm{d}x$.

- An obvious choice of u is the argument inside the square root. So substitute $u = 1+x^2$ and $\mathrm{d}x \to \frac{1}{2x}\mathrm{d}u$.
- When we do this we obtain

$$\begin{aligned}\int \sqrt{1+x^2}\cdot x^3\mathrm{d}x &= \int \sqrt{u}\cdot x^3\cdot\frac{1}{2x}\mathrm{d}u\\ &= \int \frac{1}{2}\sqrt{u}\cdot x^2\mathrm{d}u\end{aligned}$$

  Unlike all our previous examples, we have not cancelled out all of the x's from the integrand. However before we do the integral with respect to u, the integrand must be expressed solely in terms of u — no x's are allowed. (Look that integrand on the right hand side of Theorem 1.4.2.)

- But all is not lost. We can rewrite the factor x^2 in terms of the variable u. We know that $u = 1+x^2$, so this means $x^2 = u-1$. Substituting this into our integral gives

$$\begin{aligned}\int \sqrt{1+x^2}\cdot x^3 \mathrm{d}x &= \int \frac{1}{2}\sqrt{u}\cdot x^2 \mathrm{d}u \\ &= \int \frac{1}{2}\sqrt{u}\cdot (u-1)\mathrm{d}u \\ &= \frac{1}{2}\int \left(u^{3/2} - u^{1/2}\right)\mathrm{d}u \\ &= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\Bigg|_{u=x^2+1} + C \\ &= \left(\frac{1}{5}u^{5/2} - \frac{1}{3}u^{3/2}\right)\Bigg|_{u=x^2+1} + C \\ &= \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C.\end{aligned}$$

Oof!

- Don't forget that you can always check the answer by differentiating:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{5}(x^2+1)^{5/2}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3}(x^2+1)^{3/2}\right) \\ &= \frac{1}{5}\cdot 2x\cdot\frac{5}{2}\cdot(x^2+1)^{3/2} - \frac{1}{3}\cdot 2x\cdot\frac{3}{2}\cdot(x^2+1)^{1/2} \\ &= x(x^2+1)^{3/2} - x(x^2+1)^{1/2} \\ &= x\big[(x^2+1)-1\big]\cdot\sqrt{x^2+1} \\ &= x^3\sqrt{x^2+1}.\end{aligned}$$

which is the original integrand ✓.

Example 1.4.16

Example 1.4.17 $\int \tan x \mathrm{d}x$.

Evaluate the indefinite integral $\int \tan(x)\mathrm{d}x$.
Solution:

- At first glance there is nothing to manipulate here and so very little to go on. However we can rewrite $\tan x$ as $\frac{\sin x}{\cos x}$, making the integral $\int \frac{\sin x}{\cos x}\mathrm{d}x$. This gives us more to work with.

- Now think of the integrand as being the product $\frac{1}{\cos x} \cdot \sin x$. This suggests that we set $u = \cos x$ and that we interpret the first factor as the function "one over" evaluated at $u = \cos x$.
- Substitute $u = \cos x$ and $\mathrm{d}x \to \frac{1}{-\sin x}\mathrm{d}u$ to give:

$$\begin{aligned}\int \frac{\sin x}{\cos x}\mathrm{d}x &= \int \frac{\sin x}{u}\frac{1}{-\sin x}\mathrm{d}u\Big|_{u=\cos x}\\ &= \int -\frac{1}{u}\mathrm{d}u\Big|_{u=\cos x}\\ &= -\log|\cos x| + C \qquad \text{and if we want to go further}\\ &= \log\left|\frac{1}{\cos x}\right| + C\\ &= \log|\sec x| + C.\end{aligned}$$

Example 1.4.17

In all of the above substitution examples we expressed the new integration variable, u, as a function, $u(x)$, of the old integration variable x. It is also possible to express the old integration variable, x, as a function, $x(u)$, of the new integration variable u. We shall see examples of this in Section 1.9.

1.4.2 ▸▸ Exercises

Recall that we are using $\log x$ to denote the logarithm of x with base e. In other courses it is often denoted $\ln x$.

Exercises — Stage 1

1.

a True or False: $\displaystyle\int \sin(e^x) \cdot e^x \mathrm{d}x = \int \sin(u)\mathrm{d}u\Big|_{u=e^x} = -\cos(e^x) + C$

b True or False: $\displaystyle\int_0^1 \sin(e^x) \cdot e^x \mathrm{d}x = \int_0^1 \sin(u)\mathrm{d}u = 1 - \cos(1)$

2. Is the following reasoning sound? If not, fix it.

Problem: Evaluate $\displaystyle\int (2x+1)^2 \mathrm{d}x$.

Work: We use the substitution $u = 2x + 1$. Then:

$$\int (2x+1)^2 \mathrm{d}x = \int u^2 \mathrm{d}u$$

$$
\begin{aligned}
&= \frac{1}{3}u^3 + C \\
&= \frac{1}{3}(2x+1)^3 + C
\end{aligned}
$$

3. Is the following reasoning sound? If not, fix it.

Problem: Evaluate $\displaystyle\int_1^\pi \frac{\cos(\log t)}{t}\mathrm{d}t$.

Work: We use the substitution $u = \log t$, so $\mathrm{d}u = \frac{1}{t}\mathrm{d}t$. Then:

$$
\begin{aligned}
\int_1^\pi \frac{\cos(\log t)}{t}\mathrm{d}t &= \int_1^\pi \cos(u)\mathrm{d}u \\
&= \sin(\pi) - \sin(1) = -\sin(1)\,.
\end{aligned}
$$

4. Is the following reasoning sound? If not, fix it.

Problem: Evaluate $\displaystyle\int_0^{\pi/4} x\tan(x^2)\mathrm{d}x$.

Work: We begin with the substitution $u = x^2$, $\mathrm{d}u = 2x\mathrm{d}x$:

$$
\begin{aligned}
\int_0^{\pi/4} x\tan(x^2)\mathrm{d}x &= \int_0^{\pi/4} \frac{1}{2}\tan(x^2)\cdot 2x\mathrm{d}x \\
&= \int_0^{\pi^2/16} \frac{1}{2}\tan u\mathrm{d}u \\
&= \frac{1}{2}\int_0^{\pi^2/16} \frac{\sin u}{\cos u}\mathrm{d}u
\end{aligned}
$$

Now we use the substitution $v = \cos u$, $\mathrm{d}v = -\sin u\mathrm{d}u$:

$$
\begin{aligned}
&= \frac{1}{2}\int_{\cos 0}^{\cos(\pi^2/16)} -\frac{1}{v}\mathrm{d}v \\
&= -\frac{1}{2}\int_1^{\cos(\pi^2/16)} \frac{1}{v}\mathrm{d}v \\
&= -\frac{1}{2}\left[\log|v|\right]_1^{\cos(\pi^2/16)} \\
&= -\frac{1}{2}\Big(\log\big(\cos(\pi^2/16)\big) - \log(1)\Big) \\
&= -\frac{1}{2}\log\big(\cos(\pi^2/16)\big)
\end{aligned}
$$

5. $*$. What is the integral that results when the substitution $u = \sin x$ is applied to the integral $\displaystyle\int_0^{\pi/2} f(\sin x)\,\mathrm{d}x$?

6. Let f and g be functions that are continuous and differentiable everywhere. Simplify

$$\int f'(g(x))g'(x)\mathrm{d}x - f(g(x)).$$

Exercises — Stage 2

7. $*$. Use substitution to evaluate $\displaystyle\int_0^1 xe^{x^2}\cos(e^{x^2})\,\mathrm{d}x$.

8. $*$. Let $f(t)$ be any function for which $\displaystyle\int_1^8 f(t)\,\mathrm{d}t = 1$. Calculate the integral $\displaystyle\int_1^2 x^2 f(x^3)\,\mathrm{d}x$.

9. $*$. Evaluate $\displaystyle\int \frac{x^2}{(x^3+1)^{101}}\mathrm{d}x$.

10. $*$. Evaluate $\displaystyle\int_e^{e^4} \frac{\mathrm{d}x}{x\log x}$.

11. $*$. Evaluate $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1+\sin x}\,\mathrm{d}x$.

12. $*$. Evaluate $\displaystyle\int_0^{\pi/2} \cos x\cdot(1+\sin^2 x)\,\mathrm{d}x$.

13. $*$. Evaluate $\displaystyle\int_1^3 (2x-1)e^{x^2-x}\mathrm{d}x$.

14. $*$. Evaluate $\displaystyle\int \frac{(x^2-4)x}{\sqrt{4-x^2}}\,\mathrm{d}x$.

15. Evaluate $\displaystyle\int \frac{e^{\sqrt{\log x}}}{2x\sqrt{\log x}}\mathrm{d}x$.

Exercises — Stage 3 Questions 18 through 22 can be solved by substitution, but it may not be obvious which substitution will work. In general, when evaluating integrals, it is not always immediately clear which methods are appropriate. If this happens to you, don't despair, and definitely don't give up! Just guess a method and try it. Even if it fails, you'll probably learn something that you can use to make a better guess. [4]

16. $*$. Calculate $\displaystyle\int_{-2}^2 xe^{x^2}\,\mathrm{d}x$.

4 This is also pretty decent life advice.

17. $*$. Calculate $\lim_{n\to\infty}\sum_{j=1}^{n}\frac{j}{n^2}\sin\left(1+\frac{j^2}{n^2}\right)$.

18. Evaluate $\int_0^1 \frac{u^3}{u^2+1}\mathrm{d}u$.

19. Evaluate $\int \tan^3\theta\ \mathrm{d}\theta$.

20. Evaluate $\int \frac{1}{e^x+e^{-x}}\mathrm{d}x$

21. Evaluate $\int_0^1 (1-2x)\sqrt{1-x^2}\mathrm{d}x$

22. Evaluate $\int \tan x\cdot\log(\cos x)\,\mathrm{d}x$

23. $*$. Evaluate $\lim_{n\to\infty}\sum_{j=1}^{n}\frac{j}{n^2}\cos\left(\frac{j^2}{n^2}\right)$.

24. $*$. Calculate $\lim_{n\to\infty}\sum_{j=1}^{n}\frac{j}{n^2}\sqrt{1+\frac{j^2}{n^2}}$.

25. Using Riemann sums, prove that

$$\int_a^b 2f(2x)\mathrm{d}x = \int_{2a}^{2b} f(x)\mathrm{d}x$$

1.5▲ Area between curves

1.5.1 ⯮ Area between curves

Before we continue our exploration of different methods for integrating functions, we have now have sufficient tools to examine some simple applications of definite integrals. One of the motivations for our definition of "integral" was the problem of finding the area between some curve and the x-axis for x running between two specified values.

More precisely

$$\int_a^b f(x)\mathrm{d}x$$

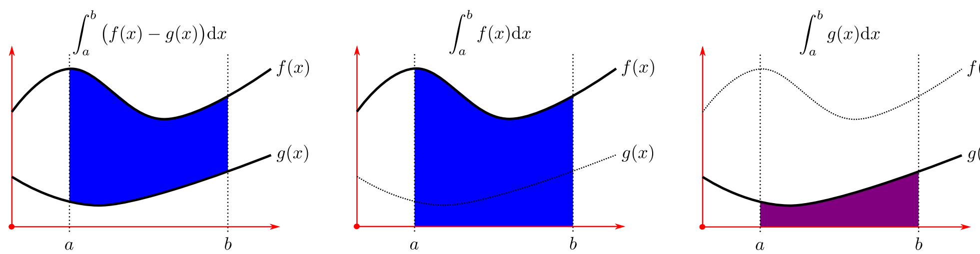
is equal to the signed area between the the curve $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$.

We found the area of this region by approximating it by the union of tall thin rectangles, and then found the exact area by taking the limit as the width of the approximating rectangles went to zero. We can use the same strategy to find areas of more complicated regions in the xy-plane.

As a preview of the material to come, let $f(x) > g(x) > 0$ and $a < b$ and suppose that we are interested in the area of the region

$$S_1 = \left\{ \ (x, y) \ \middle| \ a \le x \le b\,,\ g(x) \le y \le f(x) \ \right\}$$

that is sketched in the left hand figure below.



We already know that $\int_a^b f(x)\,\mathrm{d}x$ is the area of the region

$$S_2 = \left\{ \ (x, y) \ \middle| \ a \le x \le b\,,\ 0 \le y \le f(x) \ \right\}$$

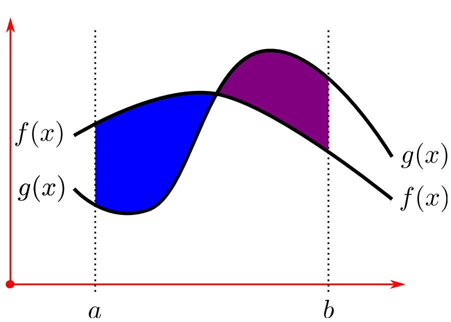
sketched in the middle figure above and that $\int_a^b g(x)\,\mathrm{d}x$ is the area of the region

$$S_3 = \left\{ \ (x, y) \ \middle| \ a \le x \le b\,,\ 0 \le y \le g(x) \ \right\}$$

sketched in the right hand figure above. Now the region S_1 of the left hand figure can be constructed by taking the region S_2 of center figure and removing from it the region S_3 of the right hand figure. So the area of S_1 is exactly

$$\int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x = \int_a^b \big(f(x) - g(x)\big)\,\mathrm{d}x$$

This computation depended on the assumption that $f(x) > g(x)$ and, in particular, that the curves $y = g(x)$ and $y = f(x)$ did not cross. If they do cross, as in this figure



then we have to be a lot more careful. The idea is to separate the domain of integration depending on where $f(x) - g(x)$ changes sign — i.e. where the curves intersect. We will illustrate this in Example 1.5.5 below.

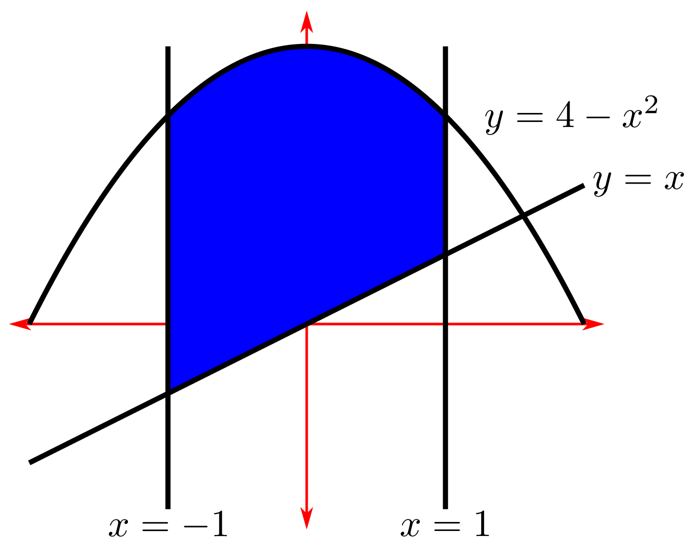
Let us start with an example that makes the link to Riemann sums and definite integrals quite explicit.

Example 1.5.1 The area between $y = 4 - x^2$ and $y = x$.

Find the area bounded by the curves $y = 4 - x^2$, $y = x$, $x = -1$ and $x = 1$.

Solution:

- Before we do any calculus, it is a very good idea to make a sketch of the area in question. The curves $y = x$, $x = -1$ and $x = 1$ are all straight lines, while the curve $y = 4 - x^2$ is a parabola whose apex is at $(0, 4)$ and then curves down (because of the minus sign in $-x^2$) with x-intercepts at $(\pm 2, 0)$. Putting these together gives



Notice that the curves $y = 4 - x^2$ and $y = x$ intersect when $4 - x^2 = x$, namely when $x = \frac{1}{2}\left(-1 \pm \sqrt{17}\right) \approx 1.56, -2.56$. Hence the curve $y = 4 - x^2$ lies above the line $y = x$ for all $-1 \le x \le 1$.

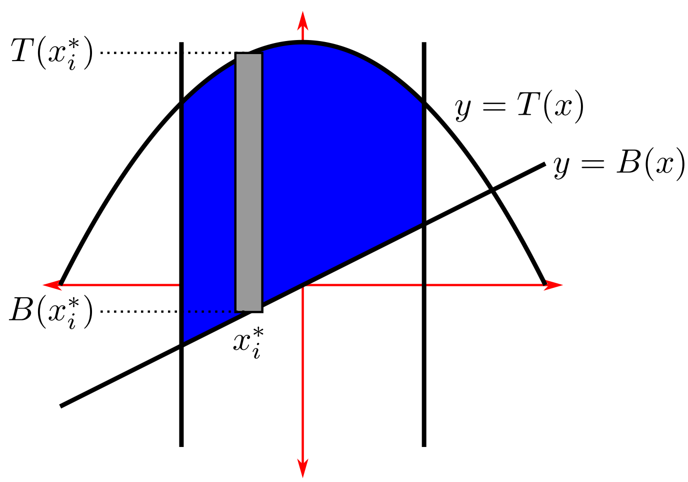
- We are to find the area of the shaded region. Each point (x, y) in this shaded region has $-1 \le x \le 1$ and $x \le y \le 4 - x^2$. When we were defining the integral (way back in Definition 1.1.9) we used a and b to denote the smallest and largest allowed values of x; let's do that here too. Let's also use $B(x)$ to denote the bottom curve (i.e. to denote the smallest allowed value of y for a given x) and use $T(x)$ to denote the top curve (i.e. to denote the largest allowed value of y for a given x). So in this example

$$a = -1 \qquad b = 1 \qquad B(x) = x \qquad T(x) = 4 - x^2$$

and the shaded region is

$$\left\{ (x,y) \;\middle|\; a \le x \le b,\ B(x) \le y \le T(x) \right\}$$

- We use the same strategy as we used when defining the integral in Section 1.1.4:
  - Pick a natural number n (that we will later send to infinity), then
  - subdivide the region into n narrow slices, each of width $\Delta x = \frac{b-a}{n}$.
  - For each $i = 1, 2, \cdots, n$, slice number i runs from $x = x_{i-1}$ to $x = x_i$, and we approximate its area by the area of a rectangle. We pick a number x_i^* between x_{i-1} and x_i and approximate the slice by a rectangle whose top is at $y = T(x_i^*)$ and whose bottom is at $y = B(x_i^*)$.
  - Thus the area of slice i is approximately $\big[T(x_i^*) - B(x_i^*)\big]\Delta x$ (as shown in the figure below).



- So the Riemann sum approximation of the area is

$$\text{Area} \approx \sum_{i=1}^{n} \big[T(x_i^*) - B(x_i^*)\big]\Delta x$$

- By taking the limit as $n \to \infty$ (i.e. taking the limit as the width of the rectangles goes to zero), we convert the Riemann sum into a definite integral (see Definition 1.1.9) and at the same time our approximation of the area becomes the exact area:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \big[T(x_i^*) - B(x_i^*)\big]\Delta x = \int_a^b \big[T(x) - B(x)\big]\,\mathrm{d}x$$

Riemann sum $\to$ integral

$$
\begin{aligned}
&= \int_{-1}^{1} \big[(4-x^2)-x\big]\mathrm{d}x \\
&= \int_{-1}^{1} \big[4-x-x^2\big]\mathrm{d}x \\
&= \left[4x-\frac{x^2}{2}-\frac{x^3}{3}\right]_{-1}^{1} \\
&= \left(4-\frac{1}{2}-\frac{1}{3}\right)-\left(-4-\frac{1}{2}+\frac{1}{3}\right) \\
&= \frac{24-3-2}{6}-\frac{-24-3+2}{6} \\
&= \frac{19}{6}+\frac{25}{6} \\
&= \frac{44}{6}=\frac{22}{3}.
\end{aligned}
$$

Example 1.5.1

Oof! Thankfully we generally do not need to go through the Riemann sum steps to get to the answer. Usually, provided we are careful to check where curves intersect and which curve lies above which, we can just jump straight to the integral

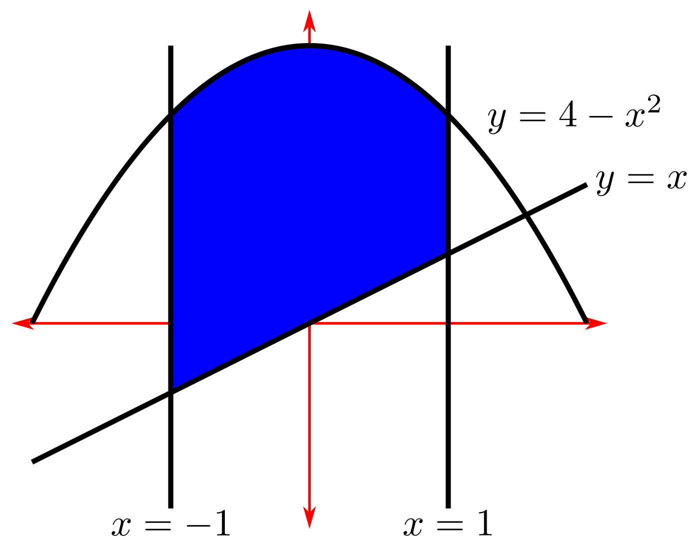
$$\text{Area} = \int_a^b \big[T(x)-B(x)\big]\mathrm{d}x.$$

So let us redo the above example.

Example 1.5.2 Example 1.5.1 revisited.

Find the area bounded by the curves $y=4-x^2$, $y=x$, $x=-1$ and $x=1$.
Solution:

- We first sketch the region



and verify [a] that $y = T(x) = 4 - x^2$ lies above the curve $y = B(x) = x$ on the region $-1 \leq x \leq 1$.

- The area between the curves is then

$$\begin{aligned}\text{Area} &= \int_a^b \big[T(x) - B(x)\big] \mathrm{d}x \\ &= \int_{-1}^{1} \big[4 - x - x^2\big] \mathrm{d}x \\ &= \left[4x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{1} \\ &= \frac{19}{6} + \frac{25}{6} = \frac{44}{6} = \frac{22}{3}.\end{aligned}$$

a We should do this by checking where the curves intersect; that is by solving $T(x) = B(x)$ and seeing if any of the solutions lie in the range $-1 \leq x \leq 1$.

Example 1.5.2

Example 1.5.3 The area between $y = x^2$ and $y = 6x - 2x^2$.

Find the area of the finite region bounded by $y = x^2$ and $y = 6x - 2x^2$.
Solution: This is a little different from the previous question, since we are not given bounding lines $x = a$ and $x = b$ — instead we have to determine the minimum and maximum allowed values of x by determining where the curves intersect. Hence our very first task is to get a good idea of what the region looks like by sketching it.

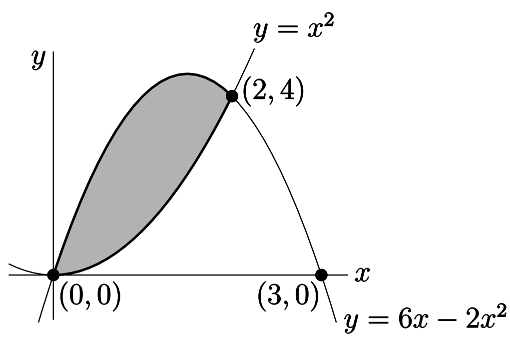
- Start by sketching the region:

- The curve $y = x^2$ is a parabola. The point on this parabola with the smallest y-coordinate is $(0,0)$. As $|x|$ increases, y increases so the parabola opens upward.
- The curve $y = 6x - 2x^2 = -2(x^2 - 3x) = -2(x - \frac{3}{2})^2 + \frac{9}{2}$ is also a parabola. The point on this parabola with the largest value of y has $x = \frac{3}{2}$ (so that the negative term in $-2(x - \frac{3}{2})^2 + \frac{9}{2}$ is zero). So the point with the largest value of y is is $(\frac{3}{2}, \frac{9}{2})$. As x moves away from $\frac{3}{2}$, either to the right or to the left, y decreases. So the parabola opens downward. The parabola crosses the x-axis when $0 = 6x - 2x^2 = 2x(3-x)$. That is, when $x = 0$ and $x = 3$.
- The two parabolas intersect when $x^2 = 6x - 2x^2$, or

$$\begin{aligned} 3x^2 - 6x &= 0 \\ 3x(x-2) &= 0 \end{aligned}$$

  So there are two points of intersection, one being $x = 0$, $y = 0^2 = 0$ and the other being $x = 2$, $y = 2^2 = 4$.
- The finite region between the curves lies between these two points of intersection.

This leads us to the sketch



- So on this region we have $0 \le x \le 2$, the top curve is $T(x) = 6x - x^2$ and the bottom curve is $B(x) = x^2$. Hence the area is given by

$$\begin{aligned} \text{Area} &= \int_a^b \big[T(x) - B(x)\big]\,\mathrm{d}x \\ &= \int_0^2 \big[(6x - 2x^2) - (x^2)\big]\,\mathrm{d}x \\ &= \int_0^2 \big[6x - 3x^2\big]\,\mathrm{d}x \\ &= \left[6\frac{x^2}{2} - 3\frac{x^3}{3}\right]_0^2 \\ &= 3(2)^2 - 2^3 = 4 \end{aligned}$$

Example 1.5.3

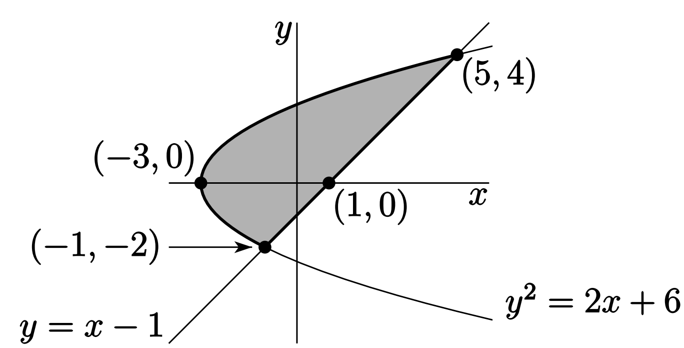
Example 1.5.4 The area between $y^2 = 2x+6$ and $y = x-1$.

Find the area of the finite region bounded by $y^2 = 2x+6$ and $y = x-1$.
Solution: We show two different solutions to this problem. The first takes the approach we have in Example 1.5.3 but leads to messy algebra. The second requires a little bit of thinking at the beginning but then is quite straightforward. Before we get to that we should start by by sketching the region.

- The curve $y^2 = 2x+6$, or equivalently $x = \frac{1}{2}y^2 - 3$ is a parabola. The point on this parabola with the smallest x-coordinate has $y = 0$ (so that the positive term in $\frac{1}{2}y^2 - 3$ is zero). So the point on this parabola with the smallest x-coordinate is $(-3, 0)$. As $|y|$ increases, x increases so the parabola opens to the right.
- The curve $y = x-1$ is a straight line of slope 1 that passes through $x = 1$, $y = 0$.
- The two curves intersect when $\frac{y^2}{2} - 3 = y+1$, or

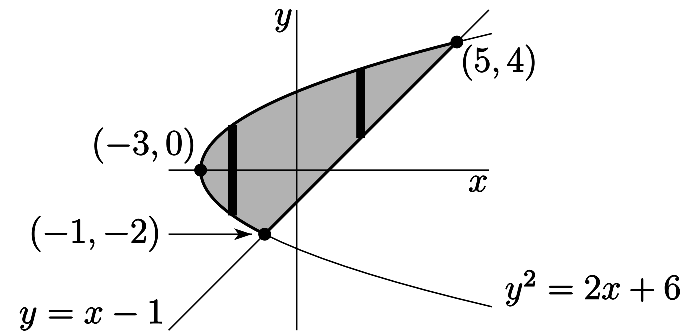
$$\begin{aligned} y^2 - 6 &= 2y+2 \\ y^2 - 2y - 8 &= 0 \\ (y+2)(y-4) &= 0 \end{aligned}$$

  So there are two points of intersection, one being $y = 4$, $x = 4+1 = 5$ and the other being $y = -2$, $x = -2+1 = -1$.
- Putting this all together gives us the sketch



As noted above, we can find the area of this region by approximating it by a union of narrow vertical rectangles, as we did in Example 1.5.3 — though it is a little harder. The easy way is to approximate it by a union of narrow horizontal rectangles. Just for practice, here is the hard solution. The easy solution is after it.
Harder solution:

- As we have done previously, we approximate the region by a union of narrow vertical rectangles, each of width Δx. Two of those rectangles are illustrated in the sketch



- In this region, x runs from $a = -3$ to $b = 5$. The curve at the top of the region is

$$y = T(x) = \sqrt{2x+6}$$

The curve at the bottom of the region is more complicated. To the left of $(-1,-2)$ the lower half of the parabola gives the bottom of the region while to the right of $(-1,-2)$ the straight line gives the bottom of the region. So

$$B(x) = \begin{cases} -\sqrt{2x+6} & \text{if } -3 \le x \le -1 \\ x-1 & \text{if } -1 \le x \le 5 \end{cases}$$

- Just as before, the area is still given by the formula $\int_a^b \big[T(x) - B(x)\big]\mathrm{d}x$, but to accommodate our $B(x)$, we have to split up the domain of integration when we evaluate the integral.

$$\begin{aligned}
&\int_a^b \big[T(x) - B(x)\big]\mathrm{d}x \\
&= \int_{-3}^{-1} \big[T(x) - B(x)\big]\mathrm{d}x + \int_{-1}^{5} \big[T(x) - B(x)\big]\mathrm{d}x \\
&= \int_{-3}^{-1} \big[\sqrt{2x+6} - (-\sqrt{2x+6})\big]\mathrm{d}x + \int_{-1}^{5} \big[\sqrt{2x+6} - (x-1)\big]\mathrm{d}x \\
&= 2\int_{-3}^{-1} \sqrt{2x+6}\mathrm{d}x + \int_{-1}^{5} \sqrt{2x+6} - \int_{-1}^{5} (x-1)\mathrm{d}x
\end{aligned}$$

- The third integral is straightforward, while we evaluate the first two via the substitution rule. In particular, set $u = 2x+6$ and replace $\mathrm{d}x \to \frac{1}{2}\mathrm{d}u$. Also $u(-3) = 0, u(-1) = 4, u(5) = 16$. Hence

$$\begin{aligned}
\text{Area} &= 2\int_0^4 \sqrt{u}\,\frac{\mathrm{d}u}{2} + \int_4^{16} \sqrt{u}\,\frac{\mathrm{d}u}{2} - \int_{-1}^{5} (x-1)\mathrm{d}x \\
&= 2\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\frac{1}{2}\right]_0^4 + \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\frac{1}{2}\right]_4^{16} - \left[\frac{x^2}{2} - x\right]_{-1}^{5}
\end{aligned}$$

$$\begin{aligned}
&= \frac{2}{3}[8-0] + \frac{1}{3}[64-8] - \Big[\Big(\frac{25}{2}-5\Big) - \Big(\frac{1}{2}+1\Big)\Big] \\
&= \frac{72}{3} - \frac{24}{2} + 6 \\
&= 18
\end{aligned}$$

Oof!

Easier solution: The easy way to determine the area of our region is to approximate by narrow horizontal rectangles, rather than narrow vertical rectangles. (Really we are just swapping the roles of x and y in this problem)

- Look at our sketch of the region again — each point (x, y) in our region has $-2 \le y \le 4$ and $\frac{1}{2}(y^2 - 6) \le x \le y + 1$.
- Let's use
  - c to denote the smallest allowed value of y,
  - d to denote the largest allowed value of y
  - $L(y)$ ("L" stands for "left") to denote the smallest allowed value of x, when the y-coordinate is y, and
  - $R(y)$ ("R" stands for "right") to denote the largest allowed value of x, when the y-coordinate is y.

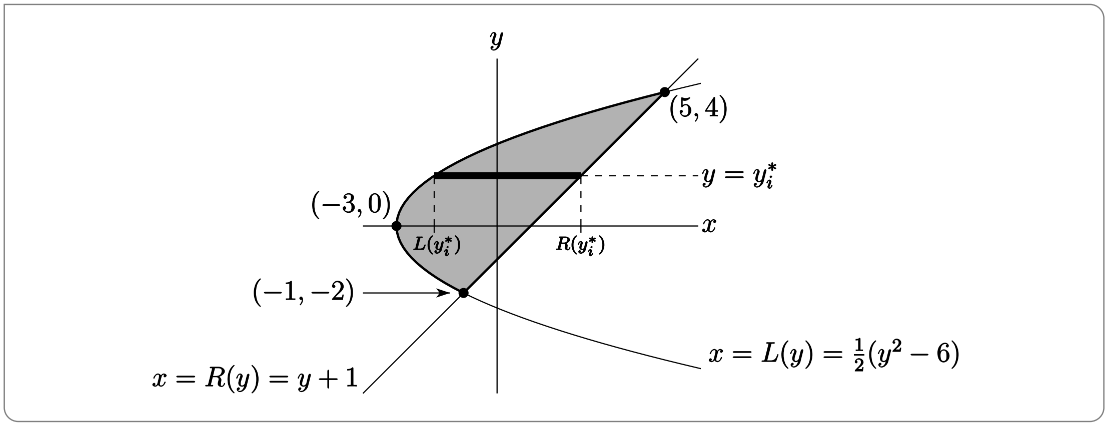
  So, in this example,

  $$c = -2 \qquad d = 4 \qquad L(y) = \frac{1}{2}(y^2 - 6) \qquad R(y) = y + 1$$

  and the shaded region is

  $$\Big\{\ (x, y)\ \Big|\ c \le y \le d,\ L(y) \le x \le R(y)\ \Big\}$$

- Our strategy is now nearly the same as that used in Example 1.5.1:
  - Pick a natural number n (that we will later send to infinity), then
  - subdivide the interval $c \le y \le d$ into n narrow subintervals, each of width $\Delta y = \frac{d-c}{n}$. Each subinterval cuts a thin horizontal slice from the region (see the figure below).
  - We approximate the area of slice number i by the area of a thin horizontal rectangle (indicated by the dark rectangle in the figure below). On this slice, the y-coordinate runs over a very narrow range. We pick a number y_i^*, somewhere in that range. We approximate slice i by a rectangle whose left side is at $x = L(y_i^*)$ and whose right side is at $x = R(y_i^*)$.
  - Thus the area of slice i is approximately $\big[R(x_i^*) - L(x_i^*)\big]\Delta y$.



- The desired area is

$$\lim_{n\to\infty}\sum_{i=1}^{n}\big[R(y_i^*) - L(y_i^*)\big]\Delta y = \int_c^d \big[R(y) - L(y)\big]\,\mathrm{d}y$$

Riemann sum → integral

$$\begin{aligned}
&= \int_{-2}^{4}\Big[(y+1) - \tfrac{1}{2}\big(y^2-6\big)\Big]\mathrm{d}y \\
&= \int_{-2}^{4}\Big[-\tfrac{1}{2}y^2 + y + 4\Big]\mathrm{d}y \\
&= \Big[-\tfrac{1}{6}y^3 + \tfrac{1}{2}y^2 + 4y\Big]_{-2}^{4} \\
&= -\tfrac{1}{6}\big(64-(-8)\big) + \tfrac{1}{2}(16-4) + 4(4+2) \\
&= -12 + 6 + 24 \\
&= 18
\end{aligned}$$

Example 1.5.4

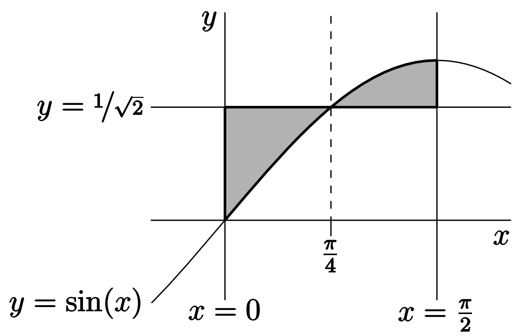
One last example.

Example 1.5.5 Another area.

Find the area between the curves $y = \dfrac{1}{\sqrt{2}}$ and $y = \sin(x)$ with x running from 0 to $\frac{\pi}{2}$.

Solution: This one is a little trickier since (as we shall see) the region is split into two pieces and we need to treat them separately.

- Again we start by sketching the region.



We want the shaded area.

- Unlike our previous examples, the bounding curves $y = \frac{1}{\sqrt{2}}$ and $y = \sin(x)$ cross in the middle of the region of interest. They cross when $y = \frac{1}{\sqrt{2}}$ and $\sin(x) = y = \frac{1}{\sqrt{2}}$, i.e. when $x = \frac{\pi}{4}$. So
  - to the left of $x = \frac{\pi}{4}$, the top boundary is part of the straight line $y = \frac{1}{\sqrt{2}}$ and the bottom boundary is part of the curve $y = \sin(x)$
  - while to the right of $x = \frac{\pi}{4}$, the top boundary is part of the curve $y = \sin(x)$ and the bottom boundary is part of the straight line $y = \frac{1}{\sqrt{2}}$.
- Thus the formulae for the top and bottom boundaries are

$$T(x) = \begin{Bmatrix} \frac{1}{\sqrt{2}} & \text{if } 0 \le x \le \frac{\pi}{4} \\ \sin(x) & \text{if } \frac{\pi}{4} \le x \le \frac{\pi}{2} \end{Bmatrix}$$

$$B(x) = \begin{Bmatrix} \sin(x) & \text{if } 0 \le x \le \frac{\pi}{4} \\ \frac{1}{\sqrt{2}} & \text{if } \frac{\pi}{4} \le x \le \frac{\pi}{2} \end{Bmatrix}$$

  We may compute the area of interest using our canned formula

$$\text{Area} = \int_a^b \big[T(x) - B(x)\big]\,\mathrm{d}x$$

  but since the formulas for $T(x)$ and $B(x)$ change at the point $x = \frac{\pi}{4}$, we must split the domain of the integral in two at that point [a].

- Our integral over the domain $0 \le x \le \frac{\pi}{2}$ is split into an integral over $0 \le x \le \frac{\pi}{4}$ and one over $\frac{\pi}{4} \le x \le \frac{\pi}{2}$:

$$\begin{aligned} \text{Area} &= \int_0^{\frac{\pi}{2}} \big[T(x) - B(x)\big]\,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{4}} \big[T(x) - B(x)\big]\,\mathrm{d}x + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \big[T(x) - B(x)\big]\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\frac{\pi}{4}} \left[\frac{1}{\sqrt{2}} - \sin(x)\right] \mathrm{d}x + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\sin(x) - \frac{1}{\sqrt{2}}\right] \mathrm{d}x \\
&= \left[\frac{x}{\sqrt{2}} + \cos(x)\right]_0^{\frac{\pi}{4}} + \left[-\cos(x) - \frac{x}{\sqrt{2}}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= \left[\frac{1}{\sqrt{2}}\frac{\pi}{4} + \frac{1}{\sqrt{2}} - 1\right] + \left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{\pi}{4}\right] \\
&= \frac{2}{\sqrt{2}} - 1 \\
&= \sqrt{2} - 1
\end{aligned}
$$

a We are effectively computing the area of the region by computing the area of the two disjoint pieces separately. Alternatively, if we set $f(x) = \sin(x)$ and $g(x) = \frac{1}{\sqrt{2}}$, we can rewrite the integral $\int_a^b [T(x) - B(x)]\,\mathrm{d}x$ as $\int_a^b |f(x) - g(x)|\,\mathrm{d}x$. To see that the two integrals are the same, split the domain of integration where $f(x) - g(x)$ changes sign.

Example 1.5.5

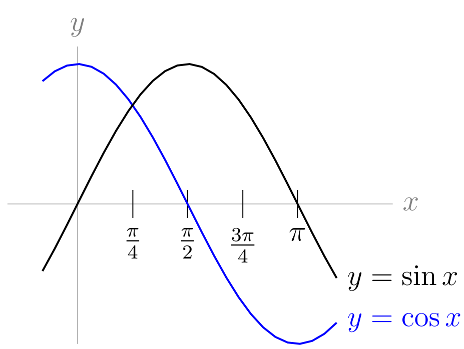
1.5.2 ▸▸ Exercises

Exercises — Stage 1

1. We want to approximate the area between the graphs of $y = \cos x$ and $y = \sin x$ from $x = 0$ to $x = \pi$ using a left Riemann sum with $n = 4$ rectangles.

a On the graph below, sketch the four rectangles.

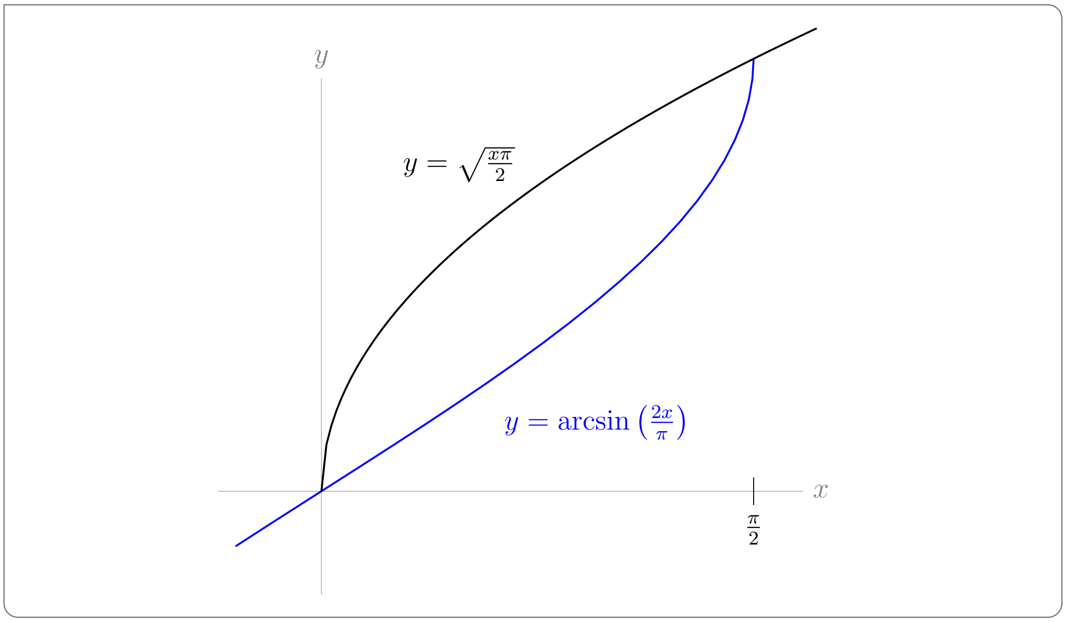
b Calculate the Riemann approximation.



2. We want to approximate the bounded area between the curves $y = \arcsin\left(\frac{2x}{\pi}\right)$ and $y = \sqrt{\frac{\pi x}{2}}$ using $n = 5$ rectangles.

a Draw the five (vertical) rectangles on the picture below corresponding to a right Riemann sum.

b Draw five rectangles on the picture below we might use if we were using horizontal rectangles.



3. $*$. Write down a definite integral that represents the finite area bounded by the curves $y = x^3 - x$ and $y = x$ for $x \geq 0$. *Do not evaluate the integral explicitly.*

4. $*$. Write down a definite integral that represents the area of the region bounded by the line $y = -\dfrac{x}{2}$ and the parabola $y^2 = 6 - \dfrac{5x}{4}$. *Do not evaluate the integral explicitly.*

5. $*$. Write down a definite integral that represents the area of the finite plane region bounded by $y^2 = 4ax$ and $x^2 = 4ay$, where $a > 0$ is a constant. *Do not evaluate the integral explicitly.*

6. $*$. Write down a definite integral that represents the area of the region bounded between the line $x + 12y + 5 = 0$ and the curve $x = 4y^2$. *Do not evaluate the integral explicitly.*

Exercises — Stage 2

7. $*$. Find the area of the region bounded by the graph of $f(x) = \dfrac{1}{(2x-4)^2}$ and the x–axis between $x = 0$ and $x = 1$.

8. $*$. Find the area between the curves $y = x$ and $y = 3x - x^2$, by first identifying the points of intersection and then integrating.

9. $*$. Calculate the area of the region enclosed by $y = 2^x$ and $y = \sqrt{x} + 1$.

10. $*$. Find the area of the finite region bounded between the two curves $y = \sqrt{2}\cos(\pi x/4)$ and $y = |x|$.

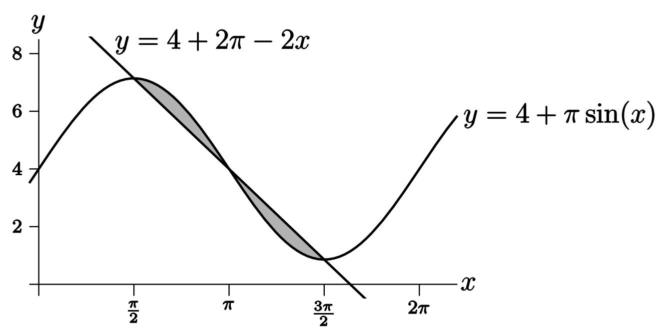
11. $*$. Find the area of the finite region that is bounded by the graphs of $f(x) = x^2\sqrt{x^3 + 1}$ and $g(x) = 3x^2$.

12. $*$. Find the area to the left of the y–axis and to the right of the curve $x = y^2 + y$.

13. Find the area of the finite region below $y = \sqrt{9 - x^2}$ and above both $y = |x|$ and $y = \sqrt{1 - x^2}$.

Exercises — Stage 3

14. $*$. The graph below shows the region between $y = 4 + \pi \sin x$ and $y = 4 + 2\pi - 2x$.



Find the area of this region.

15. $*$. Compute the area of the finite region bounded by the curves $x = 0$, $x = 3$, $y = x + 2$ and $y = x^2$.

16. $*$. Find the total area between the curves $y = x\sqrt{25 - x^2}$ and $y = 3x$, on the interval $0 \le x \le 4$.

17. Find the area of the finite region below $y = \sqrt{9 - x^2}$ and $y = x$, and above $y = \sqrt{1 - (x - 1)^2}$.

18. Find the area of the finite region bounded by the curve $y = x(x^2 - 4)$ and the line $y = x - 2$.

1.6▲ Volumes

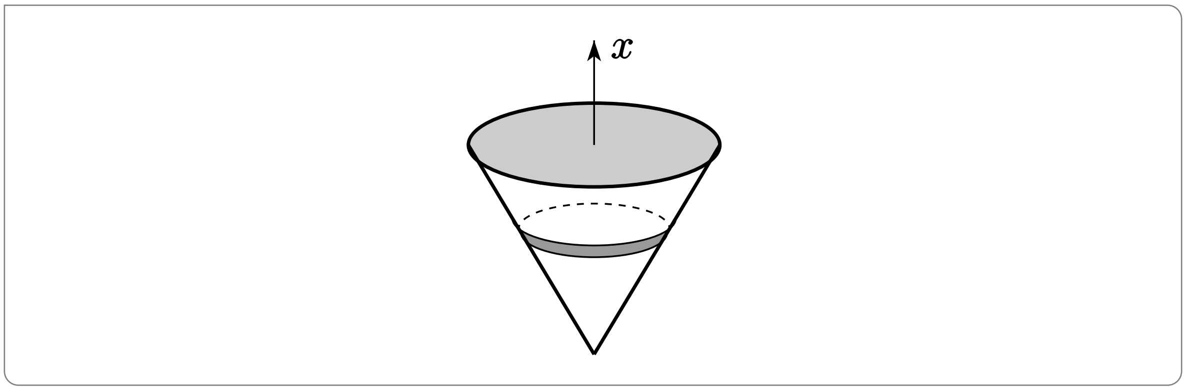
Another simple [1] application of integration is computing volumes. We use the same strategy as we used to express areas of regions in two dimensions as integrals — approximate the region by a union of small, simple pieces whose volume we can compute and then then take the limit as the "piece size" tends to zero.

In many cases this will lead to "multivariable integrals" that are beyond our present scope [2]. But there are some special cases in which this leads to integrals that we can handle. Here are some examples.

Example 1.6.1 Cone.

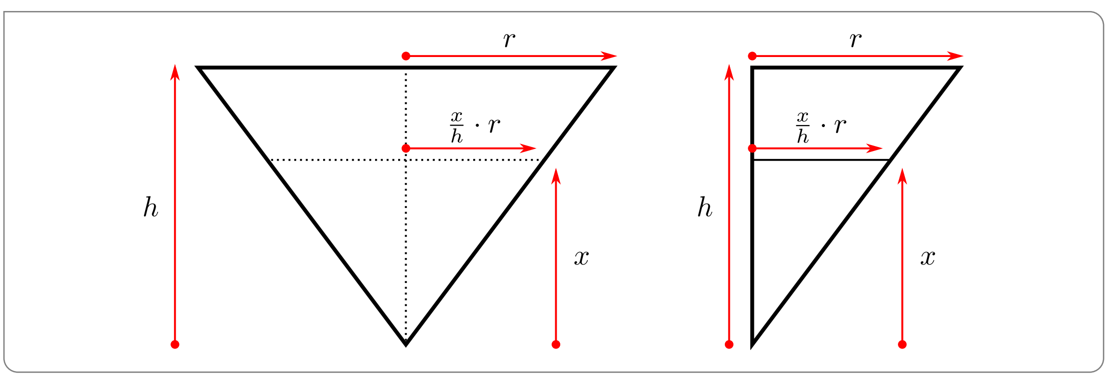
Find the volume of the circular cone of height h and radius r.

Solution: Here is a sketch of the cone.



We have called the vertical axis x, just so that we end up with a "dx" integral.

- In what follows we will slice the cone into thin horizontal "pancakes". In order to approximate the volume of those slices, we need to know the radius of the cone at a height x above its point. Consider the cross sections shown in the following figure.

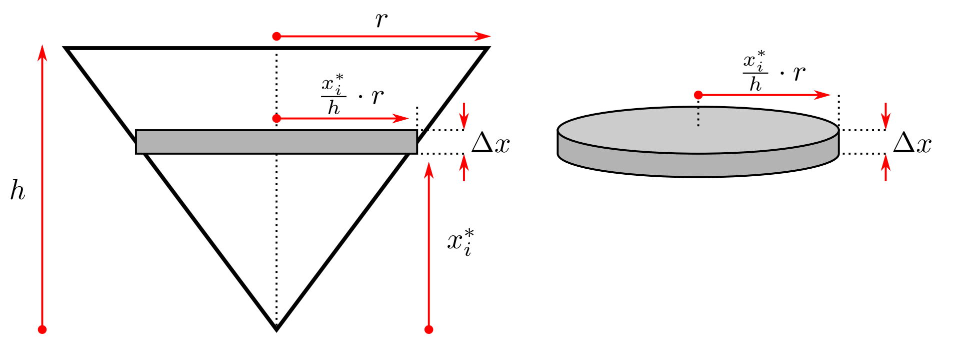


1 Well — arguably the idea isn't too complicated and is a continuation of the idea used to compute areas in the previous section. In practice this can be quite tricky as we shall see.

2 Typically such integrals (and more) are covered in a third calculus course.

At full height h, the cone has radius r. If we cut the cone at height x, then by similar triangles (see the figure on the right) the radius will be $\frac{x}{h} \cdot r$.

- Now think of cutting the cone into n thin horizontal "pancakes". Each such pancake is approximately a squat cylinder of height $\Delta x = \frac{h}{n}$. This is very similar to how we approximated the area under a curve by n tall thin rectangles. Just as we approximated the area under the curve by summing these rectangles, we can approximate the volume of the cone by summing the volumes of these cylinders. Here is a side view of the cone and one of the cylinders.



- We follow the method we used in Example 1.5.1, except that our slices are now pancakes instead of rectangles.
  - Pick a natural number n (that we will later send to infinity), then
  - subdivide the cone into n thin pancakes, each of width $\Delta x = \frac{h}{n}$.
  - For each $i = 1, 2, \cdots, n$, pancake number i runs from $x = x_{i-1} = (i-1) \cdot \Delta x$ to $x = x_i = i \cdot \Delta x$, and we approximate its volume by the volume of a squat cone. We pick a number x_i^* between x_{i-1} and x_i and approximate the pancake by a cylinder of height Δx and radius $\frac{x_i^*}{h} r$.
  - Thus the volume of pancake i is approximately $\pi \left(\frac{x_i^*}{h} r\right)^2 \Delta x$ (as shown in the figure above).
- So the Riemann sum approximation of the volume is

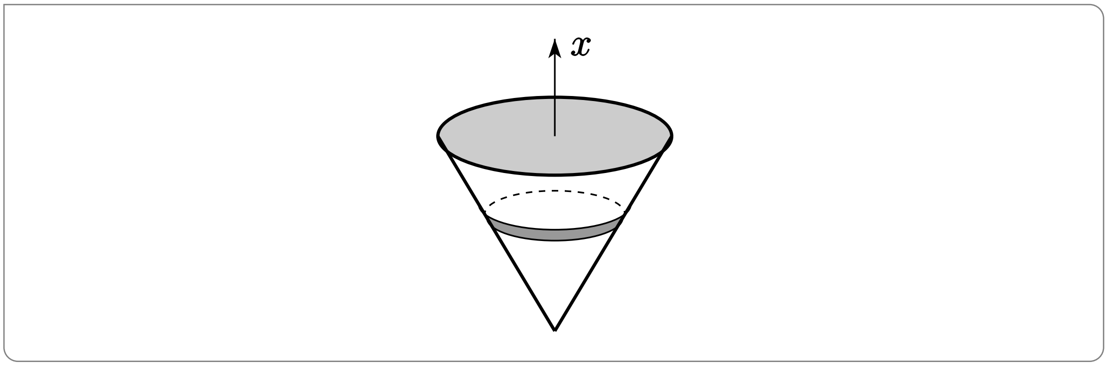
$$\text{Volume} \approx \sum_{i=1}^{n} \pi \left(\frac{x_i^*}{h} r\right)^2 \Delta x$$

- By taking the limit as $n \to \infty$ (i.e. taking the limit as the thickness of the pancakes goes to zero), we convert the Riemann sum into a definite integral (see Definition 1.1.9) and at the same time our approximation of the volume becomes the exact volume:

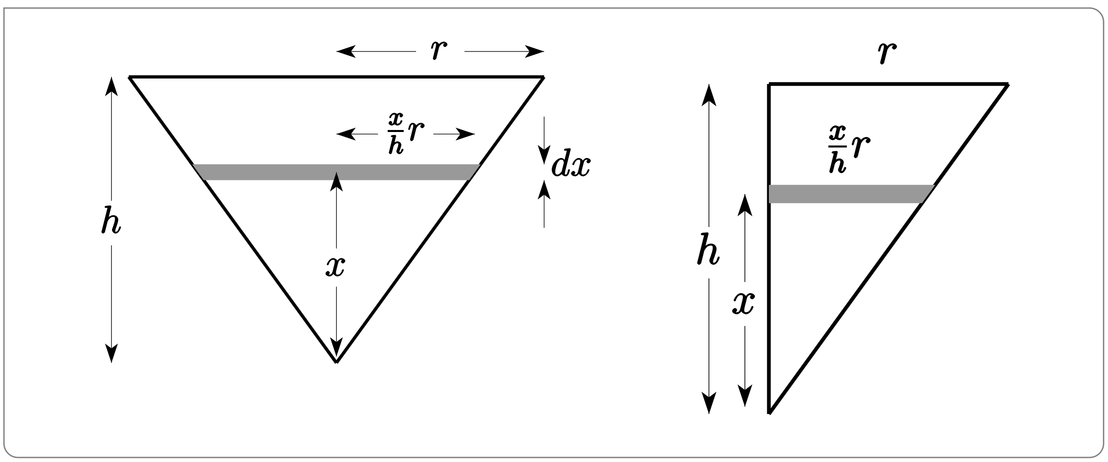
$$\int_0^h \pi \left(\frac{x}{h} r\right)^2 \mathrm{d}x$$

Our life [a] would be easier if we could avoid all this formal work with Riemann sums every time we encounter a new volume. So before we compute the above integral, let us redo the above calculation in a less formal manner.

- Start again from the picture of the cone



and think of slicing it into thin pancakes, each of width $\mathrm{d}x$.



- The pancake at height x above the point of the cone (which is the fraction $\frac{x}{h}$ of the total height of the cone) has
  - radius $\frac{x}{h} \cdot r$ (the fraction $\frac{x}{h}$ of the full radius, r) and so
  - cross-sectional area $\pi\left(\frac{x}{h}r\right)^2$,
  - thickness $\mathrm{d}x$ — we have done something a little sneaky here, see the discussion below.
  - volume $\pi\left(\frac{x}{h}r\right)^2\mathrm{d}x$

As x runs from 0 to h, the total volume is

$$\int_0^h \pi\left(\frac{x}{h}r\right)^2\mathrm{d}x = \frac{\pi r^2}{h^2}\int_0^h x^2\mathrm{d}x$$

$$= \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h$$
$$= \frac{1}{3}\pi r^2 h$$

In this second computation we are using a time-saving trick. As we saw in the formal computation above, what we really need to do is pick a natural number n, slice the cone into n pancakes each of thickness $\Delta x = \frac{h}{n}$ and then take the limit as $n \to \infty$. This led to the Riemann sum

$$\sum_{i=1}^{n} \pi \left(\frac{x_i^*}{h} r\right)^2 \Delta x \qquad \text{which becomes} \int_0^h \pi \left(\frac{x}{h} r\right)^2 \mathrm{d}x$$

So knowing that we will replace

$$\sum_{i=1}^{n} \longrightarrow \int_0^h$$
$$x_i^* \longrightarrow x$$
$$\Delta x \longrightarrow \mathrm{d}x$$

when we take the limit, we have just skipped the intermediate steps. While this is not entirely rigorous, it can be made so, and does save us a lot of algebra.

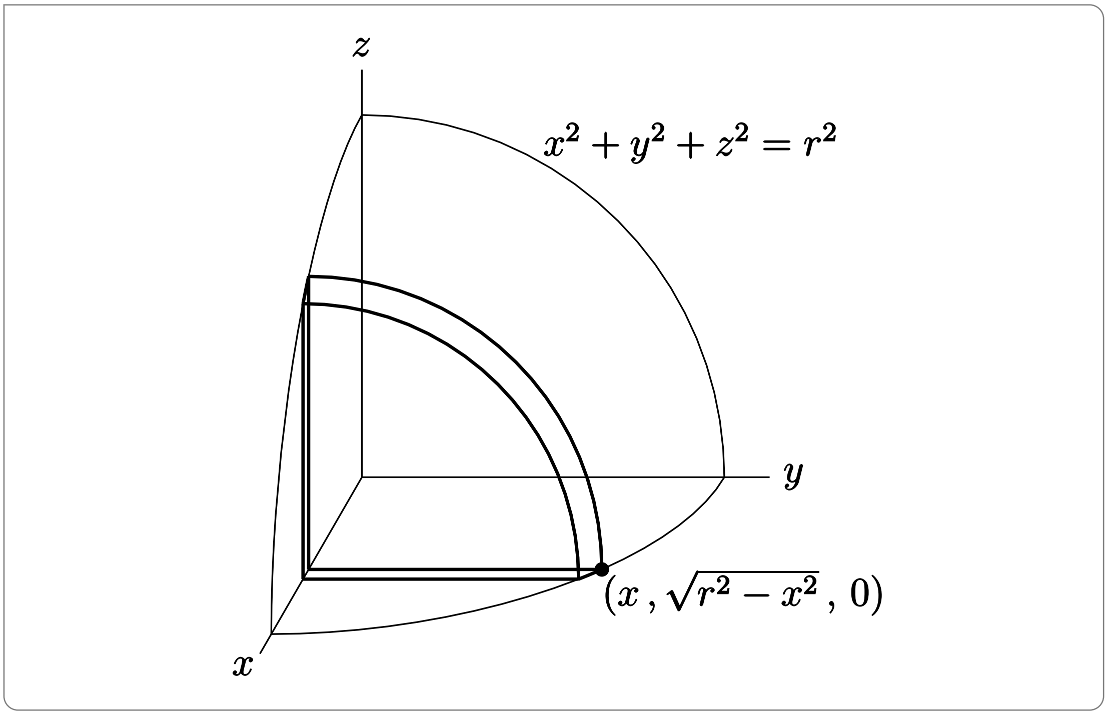
a At least the bits of it involving integrals.

Example 1.6.1

Example 1.6.2 Sphere.

Find the volume of the sphere of radius r.
Solution: We'll find the volume of the part of the sphere in the first octant [a], sketched below. Then we'll multiply by 8.

- To compute the volume,



we slice it up into thin vertical "pancakes" (just as we did in the previous example).

- Each pancake is one quarter of a thin circular disk. The pancake a distance x from the yz-plane is shown in the sketch above. The radius of that pancake is the distance from the dot shown in the figure to the x-axis, i.e. the y-coordinate of the dot. To get the coordinates of the dot, observe that
  - it lies the xy-plane, and so has z-coordinate zero, and that
  - it also lies on the sphere, so that its coordinates obey $x^2 + y^2 + z^2 = r^2$. Since $z = 0$ and $y > 0$, $y = \sqrt{r^2 - x^2}$.
- So the pancake at distance x from the yz-plane has
  - thickness [b] $\mathrm{d}x$ and
  - radius $\sqrt{r^2 - x^2}$
  - cross-sectional area $\frac{1}{4}\pi\big(\sqrt{r^2 - x^2}\,\big)^2$ and hence
  - volume $\frac{\pi}{4}\big(r^2 - x^2\big)\mathrm{d}x$
- As x runs from 0 to r, the total volume of the part of the sphere in the first octant is

$$\int_0^r \frac{\pi}{4}\big(r^2 - x^2\big)\mathrm{d}x = \frac{\pi}{4}\left[r^2 x - \frac{x^3}{3}\right]_0^r = \frac{1}{6}\pi r^3$$

and the total volume of the whole sphere is eight times that, which is $\frac{4}{3}\pi r^3$, as expected.

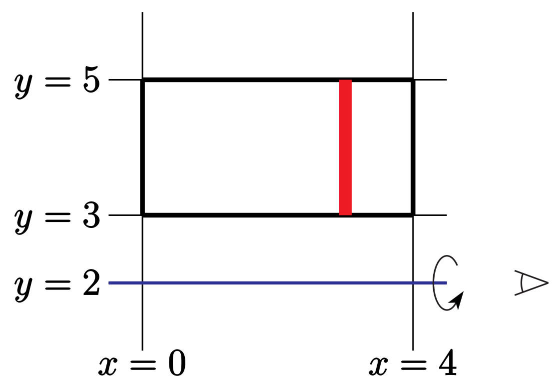
a The first octant is the set of all points (x, y, z) with $x \geq 0$, $y \geq 0$ and $z \geq 0$.
b Yet again what we really do is pick a natural number n, slice the octant of the sphere into n pancakes each of thickness $\Delta x = \frac{r}{n}$ and then take the limit $n \to \infty$. In the integral Δx is replaced by $\mathrm{d}x$. Knowing that this is what is going to happen, we again just skip a few steps.

Example 1.6.2

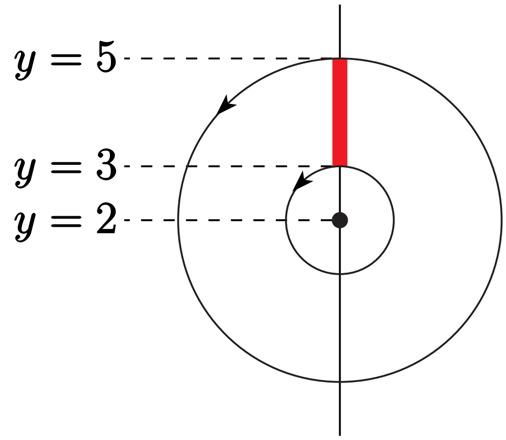
Example 1.6.3 Revolving a region.

The region between the lines $y = 3$, $y = 5$, $x = 0$ and $x = 4$ is rotated around the line $y = 2$. Find the volume of the region swept out.

Solution: As with most of these problems, we should start by sketching the problem.



- Consider the region and slice it into thin vertical strips of width $\mathrm{d}x$.
- Now we are to rotate this region about the line $y = 2$. Imagine looking straight down the axis of rotation, $y = 2$, end on. The symbol in the figure above just to the right of the end the line $y = 2$ is supposed to represent your eye [a]. Here is what you see as the rotation takes place.



- Upon rotation about the line $y = 2$ our strip sweeps out a "washer"
  - whose cross-section is a disk of radius $5 - 2 = 3$ from which a disk of radius $3 - 2 = 1$ has been removed so that it has a
  - cross-sectional area of $\pi 3^2 - \pi 1^2 = 8\pi$ and a
  - thickness $\mathrm{d}x$ and hence a
  - volume $8\pi\, \mathrm{d}x$.
- As our leftmost strip is at $x = 0$ and our rightmost strip is at $x = 4$, the total

$$\text{Volume} = \int_0^4 8\pi\, \mathrm{d}x = (8\pi)(4) = 32\pi$$

Notice that we could also reach this answer by writing the volume as the difference of two cylinders.

- The outer cylinder has radius $(5 - 2)$ and length 4. This has volume

$$V_{outer} = \pi r^2 \ell = \pi \cdot 3^2 \cdot 4 = 36\pi.$$

- The inner cylinder has radius $(3 - 2)$ and length 4. This has volume

$$V_{inner} = \pi r^2 \ell = \pi \cdot 1^2 \cdot 4 = 4\pi.$$

- The volume we want is the difference of these two, namely

$$V = V_{outer} - V_{inner} = 32\pi.$$

a Okay okay... We missed the pupil. I'm sure there is a pun in there somewhere.

Example 1.6.3

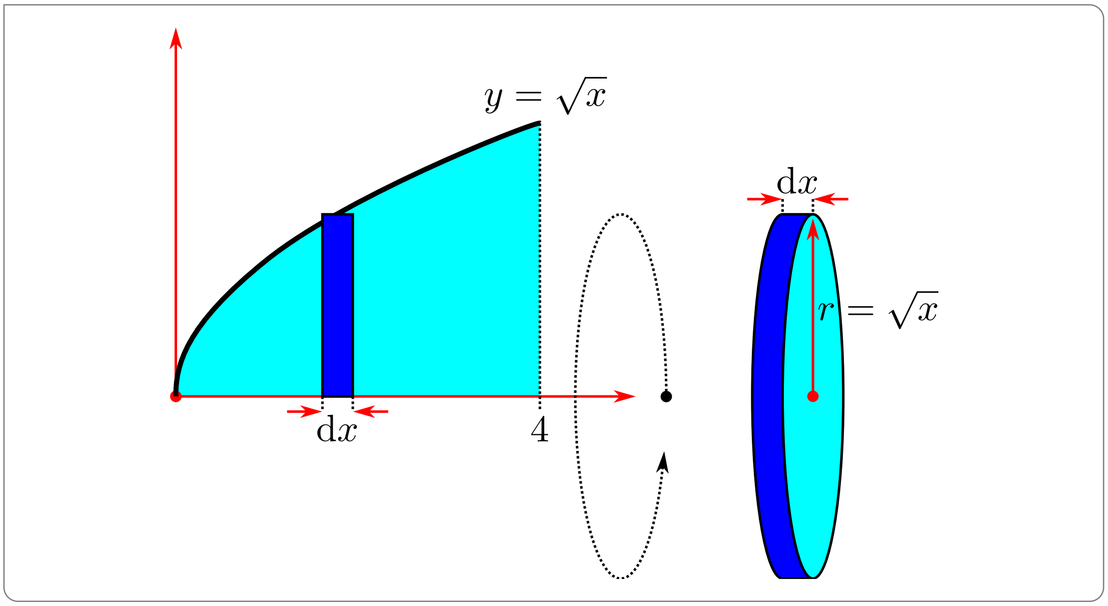
Let us turn up the difficulty a little on this last example.

Example 1.6.4 Revolving again.

The region between the curve $y = \sqrt{x}$, and the lines $y = 0$, $x = 0$ and $x = 4$ is rotated around the line $y = 0$. Find the volume of the region swept out.

Solution: We can approach this in much the same way as the previous example.

- Consider the region and cut it into thin vertical strips of width $\mathrm{d}x$.



- When we rotate the region about the line $y = 0$, each strip sweeps out a thin pancake
    - whose cross-section is a disk of radius $\sqrt{x}$ with a
    - cross-sectional area of $\pi(\sqrt{x})^2 = \pi x$ and a
    - thickness $\mathrm{d}x$ and hence a
    - volume $\pi x \mathrm{d}x$.
- As our leftmost strip is at $x = 0$ and our rightmost strip is at $x = 4$, the total

$$\text{Volume} = \int_0^4 \pi x \mathrm{d}x = \left[\frac{\pi}{2}x^2\right]_0^4 = 8\pi$$

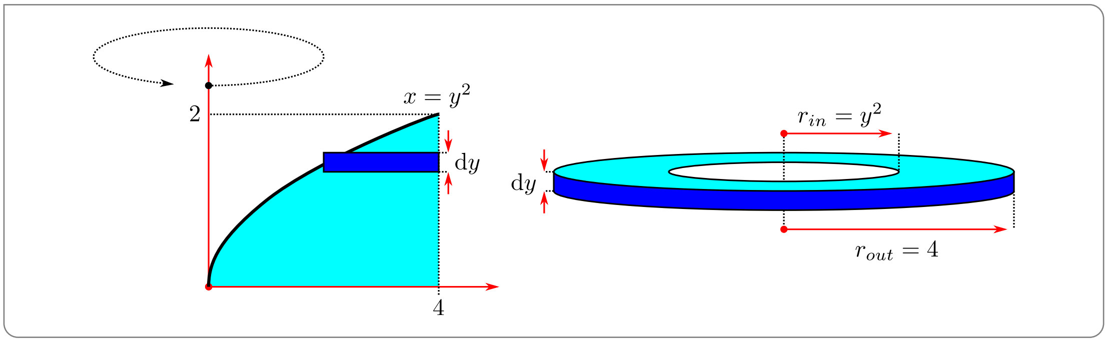
In the last example we considered rotating a region around the x-axis. Let us do the same but rotating around the y-axis.

Example 1.6.5 Revolving yet again.

The region between the curve $y = \sqrt{x}$, and the lines $y = 0$, $x = 0$ and $x = 4$ is rotated around the line $x = 0$. Find the volume of the region swept out.

Solution:

- We will cut the region into horizontal slices, so we should write x as a function of y. That is, the region is bounded by $x = y^2$, $x = 4$, $y = 0$ and $y = 2$.
- Now slice the region into thin horizontal strips of width $\mathrm{d}y$.



- When we rotate the region about the line $x = 0$, each strip sweeps out a thin washer
  - whose inner radius is y^2 and outer radius is 4, and
  - thickness is $\mathrm{d}y$ and hence
  - has volume $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}y = \pi(16 - y^4)\mathrm{d}y$.
- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = 2$, the total

$$\text{Volume} = \int_0^2 \pi(16 - y^4)\mathrm{d}y = \left[16\pi y - \frac{\pi}{5}y^5\right]_0^2 = 32\pi - \frac{32\pi}{5} = \frac{128\pi}{5}.$$

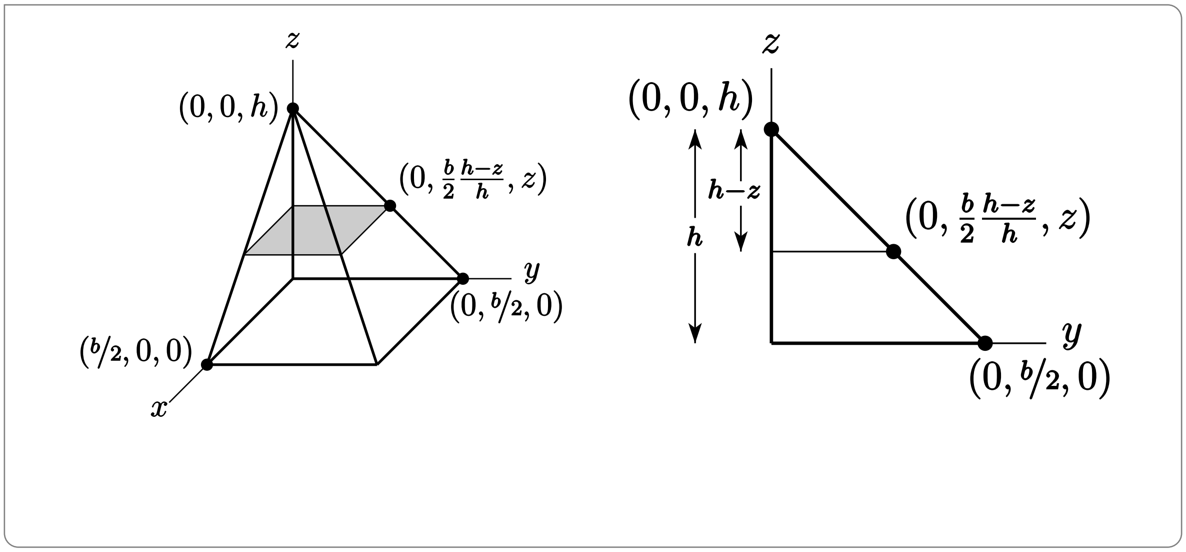
There is another way [3] to do this one which we show at the end of this section.

Example 1.6.6 Pyramid.

Find the volume of the pyramid which has height h and whose base is a square of side b.

Solution: Here is a sketch of the part of the pyramid that is in the first octant; we display only this portion to make the diagrams simpler.

3 The method is not a core part of the course and should be considered optional.



Note that this diagram shows only 1 quarter of the whole pyramid.

- To compute its volume, we slice it up into thin horizontal "square pancakes". A typical pancake also appears in the sketch above.
  - The pancake at height z is the fraction $\frac{h-z}{h}$ of the distance from the peak of the pyramid to its base.
  - So the *full* pancake [a] at height z is a square of side $\frac{h-z}{h}b$. As a check, note that when $z = h$ the pancake has side $\frac{h-h}{h}b = 0$, and when $z = 0$ the pancake has side $\frac{h-0}{h}b = b$.
  - So the pancake has cross-sectional area $\left(\frac{h-z}{h}b\right)^2$ and thickness [b] $\mathrm{d}z$ and hence
  - volume $\left(\frac{h-z}{h}b\right)^2\mathrm{d}z$.
- The volume of the whole pyramid (not just the part of the pyramid in the first octant) is

$$\int_0^h \left(\frac{h-z}{h}b\right)^2\mathrm{d}z = \frac{b^2}{h^2}\int_0^h (h-z)^2\mathrm{d}z$$

Now use the substitution rule with $t = (h-z), \mathrm{d}z \to -\mathrm{d}t$

$$\begin{aligned}
&= \frac{b^2}{h^2}\int_h^0 -t^2\mathrm{d}t \\
&= -\frac{b^2}{h^2}\left[\frac{t^3}{3}\right]_h^0 \\
&= -\frac{b^2}{h^2}\left[-\frac{h^3}{3}\right] \\
&= \frac{1}{3}b^2h
\end{aligned}$$

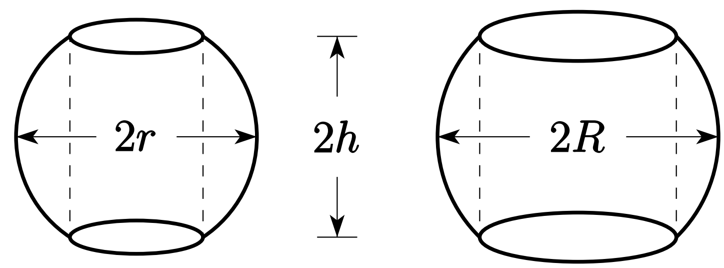
a Note that this is the full pancake, not just the part in the first octant.
b We are again using our Riemann sum avoiding trick.

Example 1.6.6

Let's ramp up the difficulty a little.

Example 1.6.7 Napkin Ring.

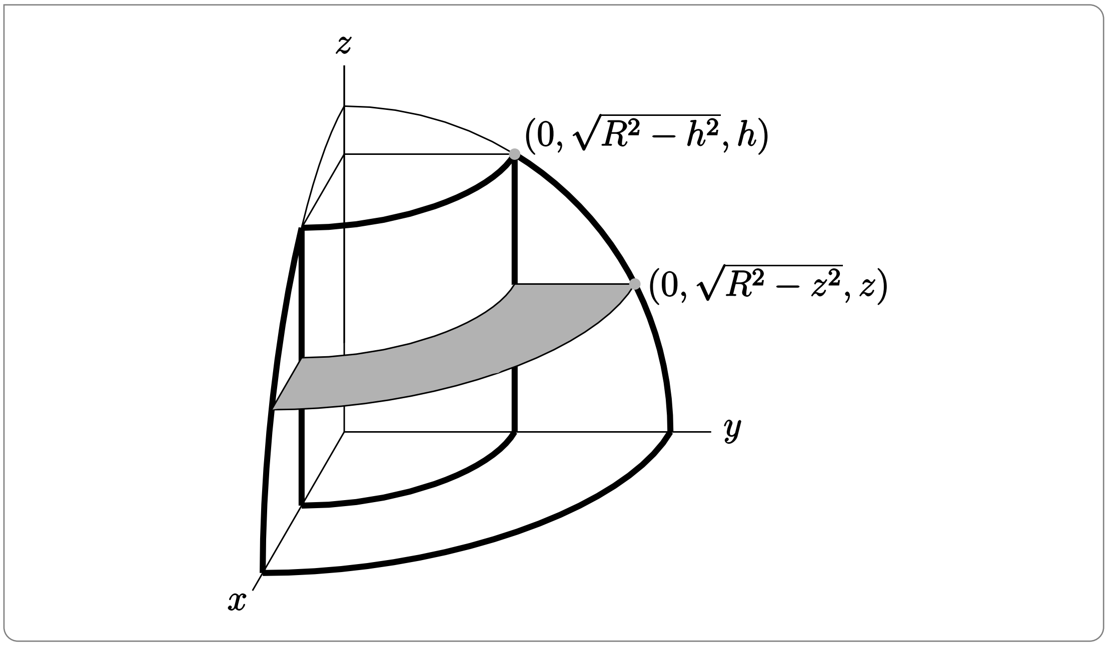
Suppose you make two napkin rings [a] by drilling holes with different diameters through two wooden balls. One ball has radius r and the other radius R with $r < R$. You choose the diameter of the holes so that both napkin rings have the same height, $2h$. See the figure below.



Which [b] ring has more wood in it?

Solution: We'll compute the volume of the napkin ring with radius R. We can then obtain the volume of the napkin ring of radius r, by just replacing $R \mapsto r$ in the result.

- To compute the volume of the napkin ring of radius R, we slice it up into thin horizontal "pancakes". Here is a sketch of the part of the napkin ring in the first octant showing a typical pancake.



- The coordinates of the two points marked in the yz-plane of that figure are found by remembering that
  - the equation of the sphere is $x^2 + y^2 + z^2 = R^2$.
  - The two points have $y > 0$ and are in the yz-plane, so that $x = 0$ for them. So $y = \sqrt{R^2 - z^2}$.
  - In particular, at the top of the napkin ring $z = h$, so that $y = \sqrt{R^2 - h^2}$.
- The pancake at height z, shown in the sketch, is a "washer" — a circular disk with a circular hole cut in its center.
  - The outer radius of the washer is $\sqrt{R^2 - z^2}$ and
  - the inner radius of the washer is $\sqrt{R^2 - h^2}$. So the
  - cross-sectional area of the washer is

$$\pi\left(\sqrt{R^2 - z^2}\,\right)^2 - \pi\left(\sqrt{R^2 - h^2}\,\right)^2 = \pi(h^2 - z^2)$$

- The pancake at height z
  - has thickness dz and
  - cross-sectional area $\pi(h^2 - z^2)$ and hence
  - volume $\pi(h^2 - z^2)\mathrm{d}z$.
- Since z runs from $-h$ to $+h$, the total volume of wood in the napkin ring of radius R is

$$\int_{-h}^{h} \pi(h^2 - z^2)\mathrm{d}z = \pi\left[h^2 z - \frac{z^3}{3}\right]_{-h}^{h}$$

$$
\begin{aligned}
&= \pi\Big[\Big(h^3 - \frac{h^3}{3}\Big) - \Big((-h)^3 - \frac{(-h)^3}{3}\Big)\Big] \\
&= \pi\Big[\frac{2}{3}h^3 - \frac{2}{3}\big(-h\big)^3\Big] \\
&= \frac{4\pi}{3}h^3
\end{aligned}
$$

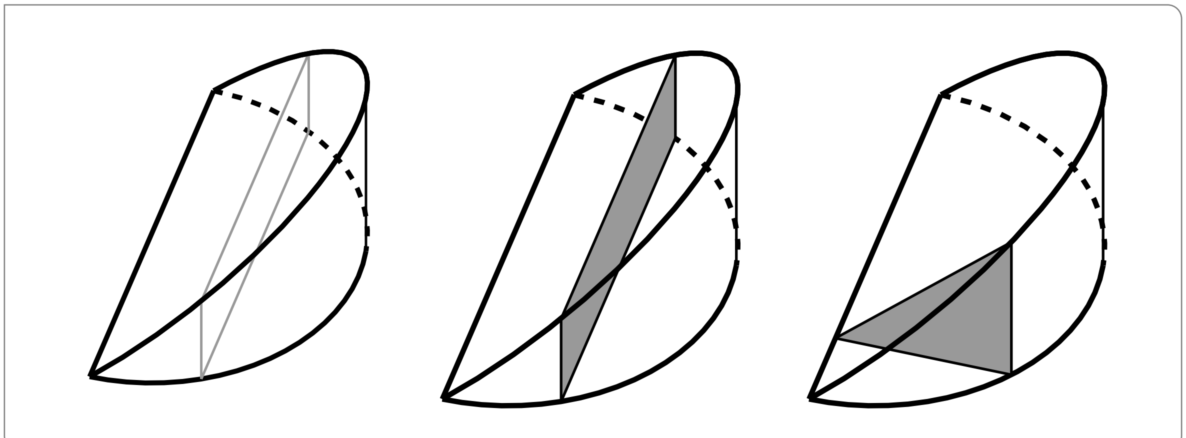
This volume is independent of R. Hence the napkin ring of radius r contains precisely the same volume of wood as the napkin ring of radius R!

a Handy things to have (when combined with cloth napkins) if your parents are coming to dinner and you want to convince them that you are "taking care of yourself".
b A good question to ask to distract your parents from the fact you are serving frozen burritos.

Example 1.6.7

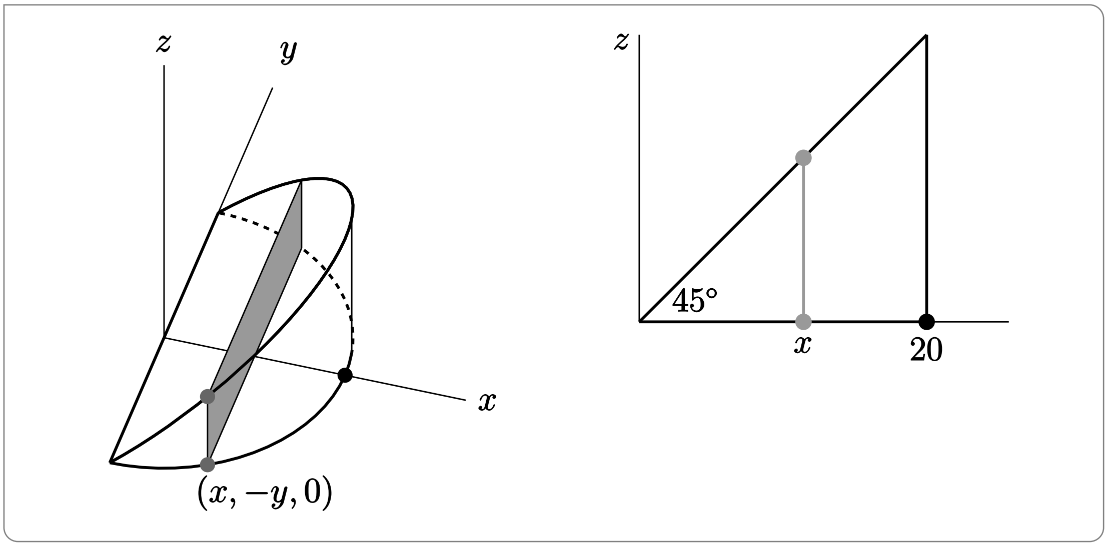
Example 1.6.8 Notch.

A 45° notch is cut to the centre of a cylindrical log having radius 20cm. One plane face of the notch is perpendicular to the axis of the log. See the sketch below. What volume of wood was removed?



Solution: We show two solutions to this problem which are of comparable difficulty. The difference lies in the shape of the pancakes we use to slice up the volume. In solution 1 we cut rectangular pancakes parallel to the yz-plane and in solution 2 we slice triangular pancakes parallel to the xz-plane.
Solution 1:

- Concentrate on the notch. Rotate it around so that the plane face lies in the xy-plane.
- Then slice the notch into vertical rectangles (parallel to the yz-plane) as in the figure on the left below.



- The cylindrical log had radius 20cm. So the circular part of the boundary of the base of the notch has equation $x^2 + y^2 = 20^2$. (We're putting the origin of the xy-plane at the centre of the circle.) If our coordinate system is such that x is constant on each slice, then
  - the base of the slice is the line segment from $(x, -y, 0)$ to $(x, +y, 0)$ where $y = \sqrt{20^2 - x^2}$ so that
  - the slice has width $2y = 2\sqrt{20^2 - x^2}$ and
  - height x (since the upper face of the notch is at 45° to the base — see the side view sketched in the figure on the right above).
  - So the slice has cross-sectional area $2x\sqrt{20^2 - x^2}$.
- On the base of the notch x runs from 0 to 20 so the volume of the notch is

$$V = \int_0^{20} 2x\sqrt{20^2 - x^2}\mathrm{d}x$$

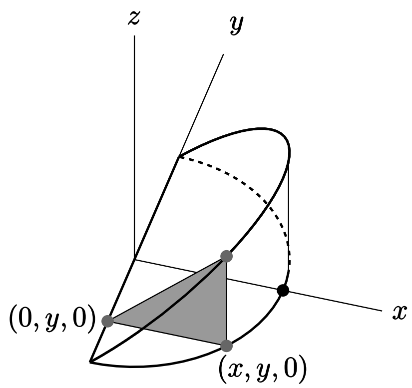
Make the change of variables $u = 20^2 - x^2$ (don't forget to change $\mathrm{d}x \to -\frac{1}{2x}\mathrm{d}u$):

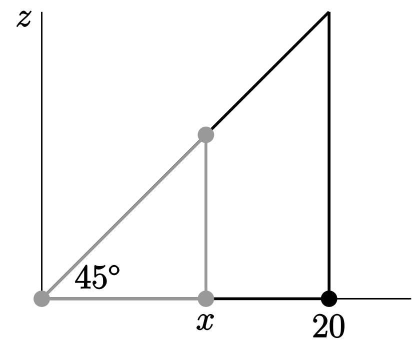
$$\begin{aligned} V &= \int_{20^2}^{0} -\sqrt{u}\,\mathrm{d}u \\ &= \left[-\frac{u^{3/2}}{3/2}\right]_{20^2}^{0} \\ &= \frac{2}{3}20^3 = \frac{16,000}{3} \end{aligned}$$

Solution 2:

- Concentrate of the notch. Rotate it around so that its base lies in the xy-plane with the skinny edge along the y-axis.

- Slice the notch into triangles parallel to the xz-plane as in the figure on the left below. In the figure below, the triangle happens to lie in a plane where y is negative.





- The cylindrical log had radius 20cm. So the circular part of the boundary of the base of the notch has equation $x^2 + y^2 = 20^2$. Our coordinate system is such that y is constant on each slice, so that
  - the base of the triangle is the line segment from $(0, y, 0)$ to $(x, y, 0)$ where $x = \sqrt{20^2 - y^2}$ so that
  - the triangle has base $x = \sqrt{20^2 - y^2}$ and
  - height $x = \sqrt{20^2 - y^2}$ (since the upper face of the notch is at 45° to the base — see the side view sketched in the figure on the right above).
  - So the slice has cross-sectional area $\frac{1}{2}\left(\sqrt{20^2 - y^2}\right)^2$.
- On the base of the notch y runs from -20 to 20, so the volume of the notch is

$$\begin{aligned} V &= \frac{1}{2}\int_{-20}^{20} (20^2 - y^2)\mathrm{d}y \\ &= \int_0^{20} (20^2 - y^2)\mathrm{d}y \\ &= \left[20^2 y - \frac{y^3}{3}\right]_0^{20} \\ &= \frac{2}{3}20^3 = \frac{16,000}{3} \end{aligned}$$

Example 1.6.8

1.6.1 ➡ Optional — Cylindrical shells

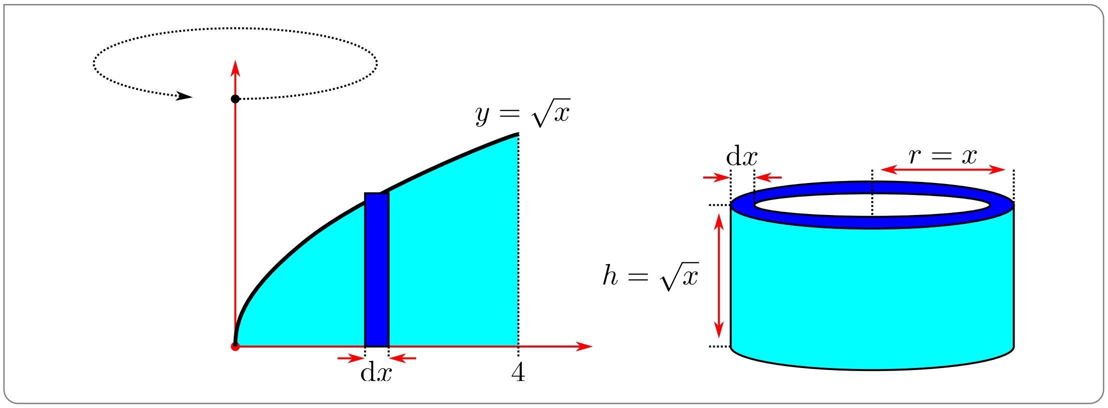
Let us return to Example 1.6.5 in which we rotate a region around the y-axis. Here we show another solution to this problem which is obtained by slicing the region into vertical strips. When rotated about the y-axis, each such strip sweeps out a thin cylindrical shell. Hence the name of this approach (and this subsection).

Example 1.6.9 Revolving yet again.

The region between the curve $y = \sqrt{x}$, and the lines $y = 0$, $x = 0$ and $x = 4$ is rotated around the line $x = 0$. Find the volume of the region swept out.

Solution:

- Consider the region and cut it into thin vertical strips of width $\mathrm{d}x$.



- When we rotate the region about the line $y = 0$, each strip sweeps out a thin cylindrical shell
  - whose radius is x,
  - height is $\sqrt{x}$, and
  - thickness is $\mathrm{d}x$ and hence
  - has volume $2\pi \times \text{radius} \times \text{height} \times \text{thickness} = 2\pi x^{3/2}\mathrm{d}x$.
- As our leftmost strip is at $x = 0$ and our rightmost strip is at $x = 4$, the total

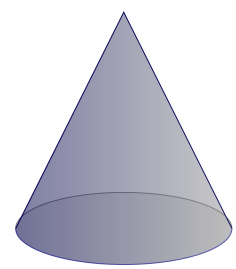
$$\text{Volume} = \int_0^4 2\pi x^{3/2}\mathrm{d}x = \left[\frac{4\pi}{5}x^{5/2}\right]_0^4 = \frac{4\pi}{5} \cdot 32 = \frac{128\pi}{5}$$

which (thankfully) agrees with our previous computation.

1.6.2 ▸▸ Exercises

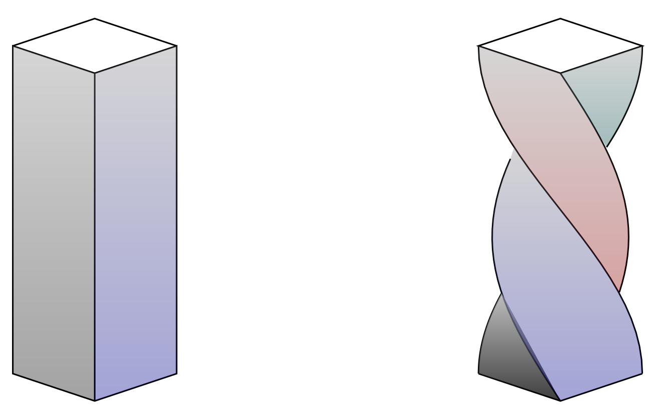
Exercises — Stage 1

1. Consider a right circular cone.



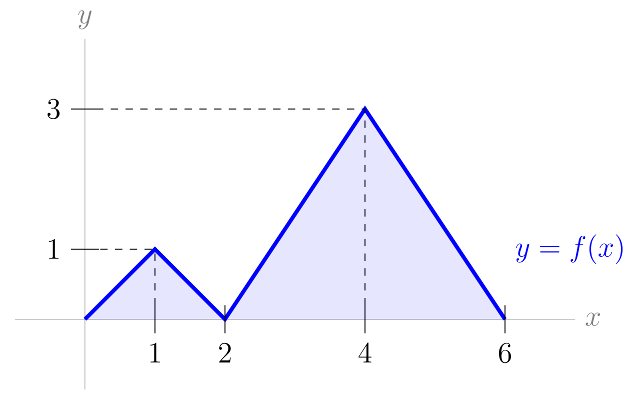
What shape are horizontal cross-sections? Are the vertical cross-sections the same?

2. Two potters start with a block of clay h units tall, and identical square cookie cutters. They form columns by pushing the square cookie cutter straight down over the clay, so that its cross-section is the same square as the cookie cutter. Potter A pushes their cookie cutter down while their clay block is sitting motionless on a table; Potter B pushes their cookie cutter down while their clay block is rotating on a potter's wheel, so their column looks twisted. Which column has greater volume?



Column A Column B

3. Let R be the region bounded above by the graph of $y = f(x)$ shown below and bounded below by the x-axis, from $x = 0$ to $x = 6$. Sketch the washers that are formed by rotating R about the y-axis. In your sketch, label all the radii in terms of y, and label the thickness.



4. $*$. Write down definite integrals that represent the following quantities. *Do not evaluate the integrals explicitly.*

a The volume of the solid obtained by rotating around the x–axis the region between the x–axis and $y = \sqrt{x}\, e^{x^2}$ for $0 \le x \le 3$.

b The volume of the solid obtained by revolving the region bounded by the curves $y = x^2$ and $y = x + 2$ about the line $x = 3$.

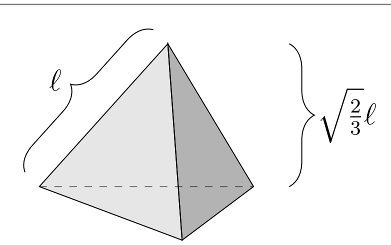
5. $*$. Write down definite integrals that represent the following quantities. *Do not evaluate the integrals explicitly.*

a The volume of the solid obtained by rotating the finite plane region bounded by the curves $y = 1 - x^2$ and $y = 4 - 4x^2$ about the line $y = -1$.

b The volume of the solid obtained by rotating the finite plane region bounded by the curve $y = x^2 - 1$ and the line $y = 0$ about the line $x = 5$.

6. $*$. Write down a definite integral that represents the volume of the solid obtained by rotating around the line $y = -1$ the region between the curves $y = x^2$ and $y = 8 - x^2$. *Do not evaluate the integrals explicitly.*

7. A tetrahedron is a three-dimensional shape with four faces, each of which is an equilateral triangle. (You might have seen this shape as a 4-sided die; think of a pyramid with a triangular base.) Using the methods from this section, calculate the volume of a tetrahedron with side-length ℓ. You may assume without proof that the height of a tetrahedron with side-length ℓ is $\sqrt{\frac{2}{3}}\ell$.



Exercises — Stage 2

8. $*$. Let $a > 0$ be a constant. Let R be the finite region bounded by the graph of $y = 1 + \sqrt{x}e^{x^2}$, the line $y = 1$, and the line $x = a$. Using vertical slices, find the volume generated when R is rotated about the line $y = 1$.

9. $*$. Find the volume of the solid generated by rotating the finite region bounded by $y = 1/x$ and $3x + 3y = 10$ about the x–axis.

10. $*$. Let R be the region inside the circle $x^2 + (y - 2)^2 = 1$. Let S be the solid obtained by rotating R about the x-axis.

a Write down an integral representing the volume of S.

b Evaluate the integral you wrote down in part (a).

11. $*$. The region R is the portion of the first quadrant which is below the parabola $y^2 = 8x$ and above the hyperbola $y^2 - x^2 = 15$.

a Sketch the region R.

b Find the volume of the solid obtained by revolving R about the x axis.

12. $*$. The region R is bounded by $y = \log x$, $y = 0$, $x = 1$ and $x = 2$. (Recall that we are using $\log x$ to denote the logarithm of x with base e. In other courses it is often denoted $\ln x$.)

a Sketch the region R.

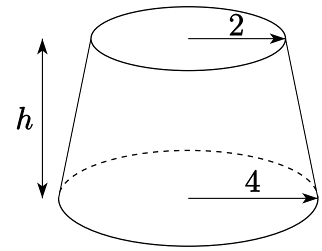
b Find the volume of the solid obtained by revolving this region about the y axis.

13. $*$. The finite region between the curves $y = \cos(\frac{x}{2})$ and $y = x^2 - \pi^2$ is rotated about the line $y = -\pi^2$. Using vertical slices (disks and/or washers), find the volume of the resulting solid.

14. $*$. The solid V is 2 meters high and has square horizontal cross sections. The length of the side of the square cross section at height x meters above the base is $\frac{2}{1+x}$ m. Find the volume of this solid.

15. $*$. Consider a solid whose base is the finite portion of the xy–plane bounded by the curves $y = x^2$ and $y = 8 - x^2$. The cross–sections perpendicular to the x–axis are squares with one side in the xy–plane. Compute the volume of this solid.

16. $*$. A frustrum of a right circular cone (as shown below) has height h. Its base is a circular disc with radius 4 and its top is a circular disc with radius 2. Calculate the volume of the frustrum.



Exercises — Stage 3

17. The shape of the earth is often approximated by an oblate spheroid, rather than a sphere. An *oblate spheroid* is formed by rotating an ellipse about its *minor axis* (its shortest diameter).

a Find the volume of the oblate spheroid obtained by rotating the upper (positive) half of the ellipse $(ax)^2 + (by)^2 = 1$ about the x-axis, where a and b are positive constants with $a \geq b$.

b Suppose [a] the earth has radius at the equator of 6378.137 km, and radius at the poles of 6356.752 km. If we model the earth as an oblate spheroid formed by rotating the upper half of the ellipse $(ax)^2 + (by)^2 = 1$ about the x-axis, what are a and b?

c What is the volume of this model of the earth? (Use a calculator.)

d Suppose we had calculated the volume of the earth by modelling it as a sphere with radius 6378.137 km. What would our absolute and relative errors be, compared to our oblate spheroid calculation?

a Earth Fact Sheet, NASA, accessed 2 July 2017

18. $*$. Let R be the bounded region that lies between the curve $y = 4 - (x-1)^2$ and the line $y = x + 1$.

a Sketch R and find its area.

b Write down a definite integral giving the volume of the region obtained by rotating R about the line $y = 5$. *Do not evaluate this integral.*

19. $*$. Let $\mathcal{R} = \big\{(x, y) \ : \ (x-1)^2 + y^2 \leq 1 \text{ and } x^2 + (y-1)^2 \leq 1 \ \big\}$.

a Sketch $\mathcal{R}$ and find its area.

b If $\mathcal{R}$ rotates around the y–axis, what volume is generated?

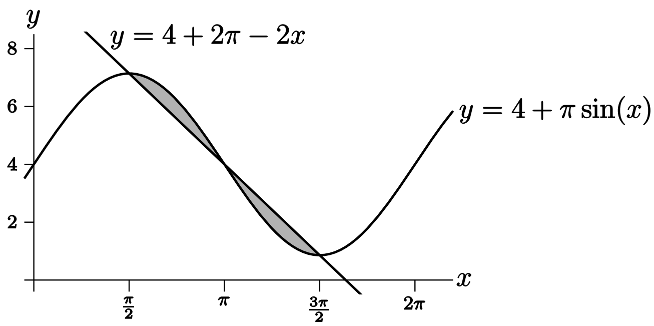
20. ***.** Let $\mathcal{R}$ be the plane region bounded by $x = 0$, $x = 1$, $y = 0$ and $y = c\sqrt{1+x^2}$, where $c \geq 0$ is a constant.

a Find the volume V_1 of the solid obtained by revolving $\mathcal{R}$ about the x–axis.

b Find the volume V_2 of the solid obtained by revolving $\mathcal{R}$ about the y–axis.

c If $V_1 = V_2$, what is the value of c?

21. ***.** The graph below shows the region between $y = 4 + \pi \sin x$ and $y = 4 + 2\pi - 2x$.



The region is rotated about the line $y = -1$. Express in terms of definite integrals the volume of the resulting solid. Do not evaluate the integrals.

22. On a particular, highly homogeneous [a] planet, we observe that the density of the atmosphere h kilometres above the surface is given by the equation $\rho(h) = c2^{-h/6}\ \frac{\text{kg}}{\text{m}^3}$, where c is the density on the planet's surface.

a What is the mass of the atmosphere contained in a vertical column with radius one metre, sixty kilometres high?

b What height should a column be to contain $\dfrac{3000c\pi}{\log 2}$ kilograms of air?

[a] This is clearly a simplified model: air density changes all the time, and depends on lots of complicated factors aside from altitude. However, the equation we're using is not so far off from an idealized model of the earth's atmosphere, taken from Pressure and the Gas Laws by H.P. Schmid, accessed 3 July 2017.

1.7 ▲ Integration by parts

1.7.1 ⏩ Integration by parts

The fundamental theorem of calculus tells us that it is very easy to integrate a derivative. In particular, we know that

$$\int \frac{\mathrm{d}}{\mathrm{d}x}\left(F(x)\right)\mathrm{d}x = F(x) + C$$

We can exploit this in order to develop another rule for integration — in particular a rule to help us integrate products of simpler function such as

$$\int xe^x \mathrm{d}x$$

In so doing we will arrive at a method called “integration by parts”.

To do this we start with the product rule and integrate. Recall that the product rule says

$$\frac{\mathrm{d}}{\mathrm{d}x}u(x)v(x) = u'(x)\,v(x) + u(x)\,v'(x)$$

Integrating this gives

$$\begin{aligned}\int \left[u'(x)\,v(x) + u(x)\,v'(x)\right]\mathrm{d}x &= \left[\text{a function whose derivative is } u'v + uv'\right] + C \\ &= u(x)v(x) + C\end{aligned}$$

Now this, by itself, is not terribly useful. In order to apply it we need to have a function whose integrand is a sum of products that is in exactly this form $u'(x)v(x)+u(x)v'(x)$. This is far too specialised.

However if we tease this apart a little:

$$\int \left[u'(x)\,v(x) + u(x)\,v'(x)\right]\mathrm{d}x = \int u'(x)\,v(x)\,\mathrm{d}x + \int u(x)\,v'(x)\,\mathrm{d}x$$

Bring one of the integrals to the left-hand side

$$u(x)v(x) - \int u'(x)\,v(x)\mathrm{d}x = \int u(x)\,v'(x)\mathrm{d}x$$

Swap left and right sides

$$\int u(x)\,v'(x)\mathrm{d}x = u(x)v(x) - \int u'(x)\,v(x)\mathrm{d}x$$

In this form we take the integral of one product and express it in terms of the integral of a different product. If we express it like that, it doesn't seem too useful. However, if the second integral is easier, then this process helps us.

Let us do a simple example before explaining this more generally.

Example 1.7.1 $\int xe^x \mathrm{d}x$.

Compute the integral $\displaystyle\int xe^x \mathrm{d}x$.

Solution:

- We start by taking the equation above
$$\int u(x)\, v'(x)\mathrm{d}x = u(x)v(x) - \int u'(x)\, v(x)\mathrm{d}x$$
- Now set $u(x) = x$ and $v'(x) = e^x$. How did we know how to make this choice? We will explain some strategies later. For now, let us just accept this choice and keep going.
- In order to use the formula we need to know $u'(x)$ and $v(x)$. In this case it is quite straightforward: $u'(x) = 1$ and $v(x) = e^x$.
- Plug everything into the formula:
$$\int xe^x \mathrm{d}x = xe^x - \int e^x \mathrm{d}x$$
So our original more difficult integral has been turned into a question of computing an easy one.
$$= xe^x - e^x + C$$
- We can check our answer by differentiating:
$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(xe^x - e^x + C\right) &= \underbrace{xe^x + 1\cdot e^x}_{\text{by product rule}} - e^x + 0 \\ &= xe^x && \text{as required.} \end{aligned}$$

The process we have used in the above example is called "integration by parts". When our integrand is a product we try to write it as $u(x)v'(x)$ — we need to choose one factor to be $u(x)$ and the other to be $v'(x)$. We then compute $u'(x)$ and $v(x)$ and then apply the following theorem:

Theorem 1.7.2 Integration by parts.

Let $u(x)$ and $v(x)$ be continuously differentiable. Then
$$\int u(x)\, v'(x)\mathrm{d}x = u(x)\, v(x) - \int v(x)\, u'(x)\mathrm{d}x$$

If we write $\mathrm{d}v$ for $v'(x)\mathrm{d}x$ and $\mathrm{d}u$ for $u'(x)\mathrm{d}x$ (as the substitution rule suggests), then the formula becomes

$$\int u\mathrm{d}v = u\,v - \int v\mathrm{d}u$$

The application of this formula is known as integration by parts.
The corresponding statement for definite integrals is

$$\int_a^b u(x)\,v'(x)\mathrm{d}x = u(b)\,v(b) - u(a)\,v(a) - \int_a^b v(x)\,u'(x)\mathrm{d}x$$

Integration by parts is not as easy to apply as the product rule for derivatives. This is because it relies on us

1 judiciously choosing $u(x)$ and $v'(x)$, then

2 computing $u'(x)$ and $v(x)$ — which requires us to antidifferentiate $v'(x)$, and finally

3 that the integral $\int u'(x)v(x)\mathrm{d}x$ is easier than the integral we started with.

Notice that any antiderivative of $v'(x)$ will do. All antiderivatives of $v'(x)$ are of the form $v(x) + A$ with A a constant. Putting this into the integration by parts formula gives

$$\begin{aligned}\int u(x)v'(x)\mathrm{d}x &= u(x)\left(v(x)+A\right) - \int u'(x)\left(v(x)+A\right)\mathrm{d}x\\ &= u(x)v(x) + Au(x) - \int u'(x)v(x)\mathrm{d}x - \underbrace{A\int u'(x)\mathrm{d}x}_{=Au(x)+C}\\ &= u(x)v(x) - \int u'(x)v(x)\mathrm{d}x + C\end{aligned}$$

So that constant A will always cancel out.

In most applications (but not all) our integrand will be a product of two factors so we have two choices for $u(x)$ and $v'(x)$. Typically one of these choices will be "good" (in that it results in a simpler integral) while the other will be "bad" (we cannot antidifferentiate our choice of $v'(x)$ or the resulting integral is harder). Let us illustrate what we mean by returning to our previous example.

Example 1.7.3 $\int xe^x\mathrm{d}x$ — again.

Our integrand is the product of two factors

$$x \qquad \text{and} \qquad e^x$$

This gives us two obvious choices of u and v':

$$u(x) = x \qquad\qquad v'(x) = e^x$$
or
$$u(x) = e^x \qquad\qquad v'(x) = x$$

We should explore both choices:

1. If take $u(x) = x$ and $v'(x) = e^x$. We then quickly compute

$$u'(x) = 1 \qquad \text{and} \qquad v(x) = e^x$$

which means we will need to integrate (in the right-hand side of the integration by parts formula)

$$\int u'(x)v(x)\mathrm{d}x = \int 1 \cdot e^x \mathrm{d}x$$

which looks straightforward. This is a good indication that this is the right choice of $u(x)$ and $v'(x)$.

2. But before we do that, we should also explore the other choice, namely $u(x) = e^x$ and $v'(x) = x$. This implies that

$$u'(x) = e^x \qquad \text{and} \qquad v(x) = \frac{1}{2}x^2$$

which means we need to integrate

$$\int u'(x)v(x)\mathrm{d}x = \int \frac{1}{2}x^2 \cdot e^x \mathrm{d}x.$$

This is at least as hard as the integral we started with. Hence we should try the first choice.

With our choice made, we integrate by parts to get

$$\begin{aligned}\int xe^x \mathrm{d}x &= xe^x - \int e^x \mathrm{d}x \\ &= xe^x - e^x + C.\end{aligned}$$

The above reasoning is a very typical workflow when using integration by parts.

Example 1.7.3

Integration by parts is often used

- to eliminate factors of x from an integrand like xe^x by using that $\frac{\mathrm{d}}{\mathrm{d}x}x = 1$ and
- to eliminate a $\log x$ from an integrand by using that $\frac{\mathrm{d}}{\mathrm{d}x}\log x = \frac{1}{x}$ and

- to eliminate inverse trig functions, like $\arctan x$, from an integrand by using that, for example, $\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$.

Example 1.7.4 $\int x \sin x \mathrm{d}x$.

Solution:

- Again we have a product of two factors giving us two possible choices.

  1 If we choose $u(x) = x$ and $v'(x) = \sin x$, then we get

  $$u'(x) = 1 \qquad \text{and} \qquad v(x) = -\cos x$$

  which is looking promising.

  2 On the other hand if we choose $u(x) = \sin x$ and $v'(x) = x$, then we have

  $$u'(x) = \cos x \qquad \text{and} \qquad v(x) = \frac{1}{2}x^2$$

  which is looking worse — we'd need to integrate $\int \frac{1}{2}x^2 \cos x \mathrm{d}x$.

- So we stick with the first choice. Plugging $u(x) = x$, $v(x) = -\cos x$ into integration by parts gives us

$$\begin{aligned}\int x \sin x \mathrm{d}x &= -x\cos x - \int 1 \cdot (-\cos x)\mathrm{d}x \\ &= -x\cos x + \sin x + C\end{aligned}$$

- Again we can check our answer by differentiating:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(-x\cos x + \sin x + C) &= -\cos x + x\sin x + \cos x + 0 \\ &= x\sin x \checkmark\end{aligned}$$

Once we have practised this a bit we do not really need to write as much. Let us solve it again, but showing only what we need to.

Solution:

- We use integration by parts to solve the integral.

- Set $u(x) = x$ and $v'(x) = \sin x$. Then $u'(x) = 1$ and $v(x) = -\cos x$, and

$$\begin{aligned}\int x \sin x \mathrm{d}x &= -x\cos x + \int \cos x \mathrm{d}x \\ &= -x\cos x + \sin x + C.\end{aligned}$$

It is pretty standard practice to reduce the notation even further in these problems. As noted above, many people write the integration by parts formula as

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u$$

where $\mathrm{d}u, \mathrm{d}v$ are shorthand for $u'(x)\mathrm{d}x, v'(x)\mathrm{d}x$. Let us write up the previous example using this notation.

Example 1.7.5 $\int x \sin x \mathrm{d}x$ yet again.

Solution: Using integration by parts, we set $u = x$ and $\mathrm{d}v = \sin x \mathrm{d}x$. This makes $\mathrm{d}u = 1\mathrm{d}x$ and $v = -\cos x$. Consequently

$$\begin{aligned}\int x \sin x \mathrm{d}x &= \int u\mathrm{d}v \\ &= uv - \int v\mathrm{d}u \\ &= -x\cos x + \int \cos x \mathrm{d}x \\ &= -x\cos x + \sin x + C\end{aligned}$$

You can see that this is a very neat way to write up these problems and we will continue using this shorthand in the examples that follow below.

We can also use integration by parts to eliminate higher powers of x. We just need to apply the method more than once.

Example 1.7.6 $\int x^2 e^x \mathrm{d}x$.

Solution:

- Let $u = x^2$ and $\mathrm{d}v = e^x\mathrm{d}x$. This then gives $\mathrm{d}u = 2x\mathrm{d}x$ and $v = e^x$, and

$$\int x^2e^x\mathrm{d}x = x^2e^x - \int 2xe^x\mathrm{d}x$$

- So we have reduced the problem of computing the original integral to one of integrating $2xe^x$. We know how to do this — just integrate by parts again:

$$\begin{aligned}\int x^2e^x\mathrm{d}x &= x^2e^x - \int 2xe^x\mathrm{d}x && \text{set } u = 2x, \mathrm{d}v = e^x\mathrm{d}x \\ &= x^2e^x - \left(2xe^x - \int 2e^x\mathrm{d}x\right) && \text{since } \mathrm{d}u = 2\mathrm{d}x, v = e^x \\ &= x^2e^x - 2xe^x + 2e^x + C\end{aligned}$$

- We can, if needed, check our answer by differentiating:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2e^x - 2xe^x + 2e^x + C\right)$$

$$= \left(x^2e^x + 2xe^x\right) - (2xe^x + 2e^x) + 2e^x + 0$$
$$= x^2e^x \checkmark$$

A similar iterated application of integration by parts will work for integrals

$$\int P(x)\left(Ae^{ax} + B\sin(bx) + C\cos(cx)\right)\mathrm{d}x$$

where $P(x)$ is a polynomial and A, B, C, a, b, c are constants.

Example 1.7.6

Now let us look at integrands containing logarithms. We don't know the antiderivative of $\log x$, but we can eliminate $\log x$ from an integrand by using integration by parts with $u = \log x$. Remember $\log x = \log_e x = \ln x$.

Example 1.7.7 $\int x \log x \mathrm{d}x$.

Solution:

- We have two choices for u and $\mathrm{d}v$.

  1 Set $u = x$ and $\mathrm{d}v = \log x\mathrm{d}x$. This gives $\mathrm{d}u = \mathrm{d}x$ but v is hard to compute — we haven't done it yet [a]. Before we go further along this path, we should look to see what happens with the other choice.

  2 Set $u = \log x$ and $\mathrm{d}v = x\mathrm{d}x$. This gives $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$ and $v = \frac{1}{2}x^2$, and we have to integrate

  $$\int v\,\mathrm{d}u = \int \frac{1}{x}\cdot\frac{1}{2}x^2\mathrm{d}x$$

  which is easy.

- So we proceed with the second choice.

$$\int x\log x\mathrm{d}x = \frac{1}{2}x^2\log x - \int \frac{1}{2}x\mathrm{d}x$$
$$= \frac{1}{2}x^2\log x - \frac{1}{4}x^2 + C$$

- We can check our answer quickly:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{2}\ln x - \frac{x^2}{4} + C\right) = x\,\ln x + \frac{x^2}{2}\frac{1}{x} - \frac{x}{2} + 0 = x\,\ln x$$

a We will soon.

Example 1.7.8 $\int \log x \mathrm{d}x$.

It is not immediately obvious that one should use integration by parts to compute the integral

$$\int \log x \mathrm{d}x$$

since the integrand is not a product. But we should persevere — indeed this is a situation where our shorter notation helps to clarify how to proceed.
Solution:

- In the previous example we saw that we could remove the factor $\log x$ by setting $u = \log x$ and using integration by parts. Let us try repeating this. When we make this choice, we are then forced to take $\mathrm{d}v = \mathrm{d}x$ — that is we choose $v'(x) = 1$. Once we have made this sneaky move everything follows quite directly.
- We then have $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$ and $v = x$, and the integration by parts formula gives us

$$\begin{aligned}\int \log x \mathrm{d}x &= x \log x - \int \frac{1}{x} \cdot x \mathrm{d}x \\ &= x \log x - \int 1 \mathrm{d}x \\ &= x \log x - x + C\end{aligned}$$

- As always, it is a good idea to check our result by verifying that the derivative of the answer really is the integrand.

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(x \ln x - x + C\big) = \ln x + x\,\frac{1}{x} - 1 + 0 = \ln x$$

The same method works almost exactly to compute the antiderivatives of $\arcsin(x)$ and $\arctan(x)$:

Example 1.7.9 $\int \arctan(x)\mathrm{d}x$ and $\int \arcsin(x)\mathrm{d}x$.

Compute the antiderivatives of the inverse sine and inverse tangent functions.
Solution:

- Again neither of these integrands are products, but that is no impediment. In both cases we set $\mathrm{d}v = \mathrm{d}x$ (ie $v'(x) = 1$) and choose $v(x) = x$.
- For inverse tan we choose $u = \arctan(x)$, so $\mathrm{d}u = \frac{1}{1+x^2}\mathrm{d}x$:

$$\int \arctan(x)\mathrm{d}x = x \arctan(x) - \int x \cdot \frac{1}{1+x^2}\mathrm{d}x$$

now use substitution rule with $w(x) = 1 + x^2, w'(x) = 2x$

$$\begin{aligned}
&= x\arctan(x) - \int \frac{w'(x)}{2} \cdot \frac{1}{w}\mathrm{d}x \\
&= x\arctan(x) - \frac{1}{2}\int \frac{1}{w}\mathrm{d}w \\
&= x\arctan(x) - \frac{1}{2}\log|w| + C \\
&= x\arctan(x) - \frac{1}{2}\log|1+x^2| + C \qquad \text{but } 1+x^2>0\text{, so} \\
&= x\arctan(x) - \frac{1}{2}\log(1+x^2) + C
\end{aligned}$$

- Similarly for inverse sine we choose $u = \arcsin(x)$ so $\mathrm{d}u = \frac{1}{\sqrt{1-x^2}}\mathrm{d}x$:

$$\int \arcsin(x)\mathrm{d}x = x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\mathrm{d}x$$

Now use substitution rule with $w(x) = 1 - x^2, w'(x) = -2x$

$$\begin{aligned}
&= x\arcsin(x) - \int \frac{-w'(x)}{2} \cdot w^{-1/2}\mathrm{d}x \\
&= x\arcsin(x) + \frac{1}{2}\int w^{-1/2}\mathrm{d}w \\
&= x\arcsin(x) + \frac{1}{2}\cdot 2w^{1/2} + C \\
&= x\arcsin(x) + \sqrt{1-x^2} + C
\end{aligned}$$

- Both can be checked quite quickly by differentiating — but we leave that as an exercise for the reader.

Example 1.7.9

There are many other examples we could do, but we'll finish with a tricky one.

Example 1.7.10 $\int e^x \sin x\mathrm{d}x$.

Solution: Let us attempt this one a little naively and then we'll come back and do it more carefully (and successfully).

- We can choose either $u = e^x, \mathrm{d}v = \sin x\mathrm{d}x$ or the other way around.

  1. Let $u = e^x, \mathrm{d}v = \sin x\mathrm{d}x$. Then $\mathrm{d}u = e^x\mathrm{d}x$ and $v = -\cos x$. This gives

$$\int e^x \sin x = -e^x\cos x + \int e^x \cos x\mathrm{d}x$$

So we are left with an integrand that is very similar to the one we started with. What about the other choice?

2. Let $u = \sin x, \mathrm{d}v = e^x \mathrm{d}x$. Then $\mathrm{d}u = \cos x \mathrm{d}x$ and $v = e^x$. This gives

$$\int e^x \sin x = e^x \sin x - \int e^x \cos x \mathrm{d}x$$

So we are again left with an integrand that is very similar to the one we started with.

- How do we proceed? — It turns out to be easier if you do both $\int e^x \sin x \mathrm{d}x$ and $\int e^x \cos x \mathrm{d}x$ simultaneously. We do so in the next example.

Example 1.7.10

Example 1.7.11 $\int_a^b e^x \sin x \mathrm{d}x$ and $\int_a^b e^x \cos x \mathrm{d}x$.

This time we're going to do the two integrals

$$I_1 = \int_a^b e^x \sin x \mathrm{d}x \qquad\qquad I_2 = \int_a^b e^x \cos x \mathrm{d}x$$

at more or less the same time.

- First

$$I_1 = \int_a^b e^x \sin x \mathrm{d}x = \int_a^b u \mathrm{d}v$$

Choose $u = e^x, \mathrm{d}v = \sin x \mathrm{d}x$, so $v = -\cos x, \mathrm{d}u = e^x \mathrm{d}x$

$$= \Big[- e^x \cos x \Big]_a^b + \int_a^b e^x \cos x \mathrm{d}x$$

We have not found I_1 but we have related it to I_2.

$$I_1 = \Big[- e^x \cos x \Big]_a^b + I_2$$

- Now start over with I_2.

$$I_2 = \int_a^b e^x \cos x \mathrm{d}x = \int_a^b u \mathrm{d}v$$

Choose $u = e^x, \mathrm{d}v = \cos x \mathrm{d}x$, so $v = \sin x, \mathrm{d}u = e^x \mathrm{d}x$

$$= \Big[e^x \sin x \Big]_a^b - \int_a^b e^x \sin x \mathrm{d}x$$

Once again, we have not found I_2 but we have related it back to I_1.

$$I_2 = \Big[e^x \sin x \Big]_a^b - I_1$$

- So summarising, we have

$$I_1 = \Big[-e^x \cos x\Big]_a^b + I_2 \qquad\qquad I_2 = \Big[e^x \sin x\Big]_a^b - I_1$$

- So now, substitute the expression for I_2 from the second equation into the first equation to get

$$\begin{aligned} I_1 &= \Big[-e^x \cos x + e^x \sin x\Big]_a^b - I_1 \\ &\text{which implies} \qquad I_1 = \frac{1}{2}\Big[e^x\big(\sin x - \cos x\big)\Big]_a^b \end{aligned}$$

If we substitute the other way around we get

$$\begin{aligned} I_2 &= \Big[e^x \sin x + e^x \cos x\Big]_a^b - I_2 \\ &\text{which implies} \qquad I_2 = \frac{1}{2}\Big[e^x\big(\sin x + \cos x\big)\Big]_a^b \end{aligned}$$

That is,

$$\begin{aligned} \int_a^b e^x \sin x \mathrm{d}x &= \frac{1}{2}\Big[e^x\big(\sin x - \cos x\big)\Big]_a^b \\ \int_a^b e^x \cos x \mathrm{d}x &= \frac{1}{2}\Big[e^x\big(\sin x + \cos x\big)\Big]_a^b \end{aligned}$$

- This also says, for example, that $\frac{1}{2}e^x\big(\sin x - \cos x\big)$ is an antiderivative of $e^x \sin x$ so that

$$\int e^x \sin x \mathrm{d}x = \frac{1}{2}e^x\big(\sin x - \cos x\big) + C$$

- Note that we can always check whether or not this is correct. It is correct if and only if the derivative of the right hand side is $e^x \sin x$. Here goes. By the product rule

$$\begin{aligned} &= \frac{\mathrm{d}}{\mathrm{d}x}\Big[\frac{1}{2}e^x\big(\sin x - \cos x\big) + C\Big] \\ &= \frac{1}{2}\Big[e^x\big(\sin x - \cos x\big) + e^x\big(\cos x + \sin x\big)\Big] = e^x \sin x \end{aligned}$$

which is the desired derivative.

- There is another way to find $\int e^x \sin x \mathrm{d}x$ and $\int e^x \cos x \mathrm{d}x$ that, in contrast to the above computations, doesn't involve any trickery. But it does require the use of complex numbers and so is beyond the scope of this course. The secret is to use that $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$ and $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, where i is the square root of -1 of the complex number system. See Example B.2.6.

Example 1.7.11

1.7.2 ⏩ Exercises

Exercises — Stage 1

1. The method of integration by substitution comes from the rule for differentiation.
The method of integration by parts comes from the rule for differentiation.

2. Suppose you want to evaluate an integral using integration by parts. You choose part of your integrand to be u, and part to be $\mathrm{d}v$. The part chosen as u will be: (differentiated, antidifferentiated). The part chosen as $\mathrm{d}v$ will be: (differentiated, antidifferentiated).

3. Let $f(x)$ and $g(x)$ be differentiable functions. Using the quotient rule for differentiation, give an equivalent expression to $\displaystyle\int \frac{f'(x)}{g(x)}\mathrm{d}x$.

4. Suppose we want to use integration by parts to evaluate $\displaystyle\int u(x)\cdot v'(x)\mathrm{d}x$ for some differentiable functions u and v. We need to find an antiderivative of $v'(x)$, but there are infinitely many choices. Show that every antiderivative of $v'(x)$ gives an equivalent final answer.

5. Suppose you want to evaluate $\displaystyle\int f(x)\mathrm{d}x$ using integration by parts. Explain why $\mathrm{d}v = f(x)\mathrm{d}x$, $u = 1$ is generally a bad choice.
Note: compare this to Example 1.7.8, where we chose $u = f(x)$, $\mathrm{d}v = 1\mathrm{d}x$.

Exercises — Stage 2

6. $*$. Evaluate $\displaystyle\int x\log x\,\mathrm{d}x$.

7. $*$. Evaluate $\displaystyle\int \frac{\log x}{x^7}\,\mathrm{d}x$.

8. $*$. Evaluate $\displaystyle\int_0^{\pi} x\sin x\,\mathrm{d}x$.

9. $*$. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} x\cos x\,\mathrm{d}x$.

10. Evaluate $\displaystyle\int x^3e^x\mathrm{d}x$.

11. Evaluate $\int x \log^3 x \mathrm{d}x$.

12. Evaluate $\int x^2 \sin x \mathrm{d}x$.

13. Evaluate $\int (3t^2 - 5t + 6) \log t \mathrm{d}t$.

14. Evaluate $\int \sqrt{s} e^{\sqrt{s}} \mathrm{d}s$.

15. Evaluate $\int \log^2 x \mathrm{d}x$.

16. Evaluate $\int 2x e^{x^2+1} \mathrm{d}x$.

17. ***.** Evaluate $\int \arccos y \, \mathrm{d}y$.

Exercises — Stage 3

18. ***.** Evaluate $\int 4y \arctan(2y) \, \mathrm{d}y$.

19. Evaluate $\int x^2 \arctan x \mathrm{d}x$.

20. Evaluate $\int e^{x/2} \cos(2x) \mathrm{d}x$.

21. Evaluate $\int \sin(\log x) \mathrm{d}x$.

22. Evaluate $\int 2^{x + \log_2 x} \mathrm{d}x$.

23. Evaluate $\int e^{\cos x} \sin(2x) \mathrm{d}x$.

24. Evaluate $\int \frac{xe^{-x}}{(1-x)^2} \mathrm{d}x$.

25. ***.** A reduction formula.

a Derive the reduction formula

$$\int \sin^n(x) \, \mathrm{d}x = -\frac{\sin^{n-1}(x)\cos(x)}{n} + \frac{n-1}{n} \int \sin^{n-2}(x) \, \mathrm{d}x.$$

b Calculate $\displaystyle\int_0^{\pi/2} \sin^8(x)\,\mathrm{d}x$.

26. $*$. Let R be the part of the first quadrant that lies below the curve $y = \arctan x$ and between the lines $x = 0$ and $x = 1$.

a Sketch the region R and determine its area.

b Find the volume of the solid obtained by rotating R about the y–axis.

27. $*$. Let R be the region between the curves $T(x) = \sqrt{x}e^{3x}$ and $B(x) = \sqrt{x}(1 + 2x)$ on the interval $0 \le x \le 3$. (It is true that $T(x) \ge B(x)$ for all $0 \le x \le 3$.) Compute the volume of the solid formed by rotating R about the x-axis.

28. $*$. Let $f(0) = 1$, $f(2) = 3$ and $f'(2) = 4$. Calculate $\displaystyle\int_0^4 f''\big(\sqrt{x}\big)\,\mathrm{d}x$.

29. Evaluate $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(\frac{2}{n}i - 1\right) e^{\frac{2}{n}i-1}$.

1.8 ▲ Trigonometric Integrals

Integrals of polynomials of the trigonometric functions $\sin x$, $\cos x$, $\tan x$ and so on, are generally evaluated by using a combination of simple substitutions and trigonometric identities. There are of course a very large number [1] of trigonometric identities, but usually we use only a handful of them. The most important three are:

Equation 1.8.1

$$\sin^2 x + \cos^2 x = 1$$

Equation 1.8.2

$$\sin(2x) = 2\sin x \cos x$$

1 The more pedantic reader could construct an infinite list of them.

Equation 1.8.3

$$\begin{aligned}\cos(2x) &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x\end{aligned}$$

Notice that the last two lines of Equation 1.8.3 follow from the first line by replacing either $\sin^2 x$ or $\cos^2 x$ using Equation 1.8.1. It is also useful to rewrite these last two lines:

Equation 1.8.4

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

Equation 1.8.5

$$\cos^2 x = \frac{1+\cos(2x)}{2}$$

These last two are particularly useful since they allow us to rewrite higher powers of sine and cosine in terms of lower powers. For example:

$$\begin{aligned}\sin^4(x) &= \left[\frac{1-\cos(2x)}{2}\right]^2 && \text{by Equation 1.8.4} \\ &= \frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\underbrace{\cos^2(2x)}_{\text{do it again}} && \text{use Equation 1.8.5} \\ &= \frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{8}(1+\cos(4x)) \\ &= \frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\end{aligned}$$

So while it was hard to integrate $\sin^4(x)$ directly, the final expression is quite straightforward (with a little substitution rule).

There are many such tricks for integrating powers of trigonometric functions. Here we concentrate on two families

$$\int \sin^m x \cos^n x \mathrm{d}x \qquad \text{and} \qquad \int \tan^m x \sec^n x \mathrm{d}x$$

for integer n, m. The details of the technique depend on the parity of n and m — that is, whether n and m are even or odd numbers.

1.8.1 ▸▸ Integrating $\int \sin^m x \cos^n x \mathrm{d}x$

1.8.1.1 ▸▸▸ One of n and m is odd

Consider the integral $\int \sin^2 x \cos x \mathrm{d}x$. We can integrate this by substituting $u = \sin x$ and $\mathrm{d}u = \cos x \mathrm{d}x$. This gives

$$\begin{aligned}\int \sin^2 x \cos x \mathrm{d}x &= \int u^2 \mathrm{d}u \\ &= \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3 x + C\end{aligned}$$

This method can be used whenever n is an odd integer.

- Substitute $u = \sin x$ and $\mathrm{d}u = \cos x \mathrm{d}x$.
- This leaves an even power of cosines — convert them using $\cos^2 x = 1 - \sin^2 x = 1 - u^2$.

Here is an example.

Example 1.8.6 $\int \sin^2 x \cos^3 x \mathrm{d}x$.

Start by factoring off one power of $\cos x$ to combine with $\mathrm{d}x$ to get $\cos x \mathrm{d}x = \mathrm{d}u$.

$$\begin{aligned}\int \sin^2 x \cos^3 x \mathrm{d}x &= \int \underbrace{\sin^2 x}_{=u^2} \underbrace{\cos^2 x}_{=1-u^2} \underbrace{\cos x \mathrm{d}x}_{=\mathrm{d}u} && \text{set } u = \sin x \\ &= \int u^2 \ (1 - u^2)\mathrm{d}u \\ &= \frac{u^3}{3} - \frac{u^5}{5} + C \\ &= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C\end{aligned}$$

Of course if m is an odd integer we can use the same strategy with the roles of $\sin x$ and $\cos x$ exchanged. That is, we substitute $u = \cos x$, $\mathrm{d}u = -\sin x \mathrm{d}x$ and $\sin^2 x = 1 - \cos^2 x = 1 - u^2$.

1.8.1.2 ⯮ Both n and m are even

If m and n are both even, the strategy is to use the trig identities 1.8.4 and 1.8.5 to get back to the m or n odd case. This is typically more laborious than the previous case we studied. Here are a couple of examples that arise quite commonly in applications.

Example 1.8.7 $\int \cos^2 x\mathrm{d}x$.

By 1.8.5

$$\int \cos^2 x\mathrm{d}x = \frac{1}{2}\int \big[1+\cos(2x)\big]\mathrm{d}x = \frac{1}{2}\Big[x+\frac{1}{2}\sin(2x)\Big]+C$$

Example 1.8.8 $\int \cos^4 x\mathrm{d}x$.

First we'll prepare the integrand $\cos^4 x$ for easy integration by applying 1.8.5 a couple times. We have already used 1.8.5 once to get

$$\cos^2 x = \frac{1}{2}\big[1+\cos(2x)\big]$$

Squaring it gives

$$\cos^4 x = \frac{1}{4}\big[1+\cos(2x)\big]^2 = \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{4}\cos^2(2x)$$

Now by 1.8.5 a second time

$$\begin{aligned}\cos^4 x &= \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{4}\,\frac{1+\cos(4x)}{2}\\ &= \frac{3}{8}+\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)\end{aligned}$$

Now it's easy to integrate

$$\begin{aligned}\int \cos^4 x\mathrm{d}x &= \frac{3}{8}\int \mathrm{d}x+\frac{1}{2}\int \cos(2x)\mathrm{d}x+\frac{1}{8}\int \cos(4x)\mathrm{d}x\\ &= \frac{3}{8}x+\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+C\end{aligned}$$

Example 1.8.9 $\int \cos^2 x\sin^2 x\mathrm{d}x$.

Here we apply both 1.8.4 and 1.8.5.

$$\int \cos^2 x\sin^2 x\mathrm{d}x = \frac{1}{4}\int \big[1+\cos(2x)\big]\big[1-\cos(2x)\big]\mathrm{d}x$$

$$= \frac{1}{4}\int \big[1 - \cos^2(2x)\big]\,\mathrm{d}x$$

We can then apply 1.8.5 again

$$\begin{aligned} &= \frac{1}{4}\int \Big[1 - \frac{1}{2}\left(1+\cos(4x)\right)\Big]\,\mathrm{d}x \\ &= \frac{1}{8}\int \big[1 - \cos(4x)\big]\,\mathrm{d}x \\ &= \frac{1}{8}x - \frac{1}{32}\sin(4x) + C \end{aligned}$$

Oof! We could also have done this one using 1.8.2 to write the integrand as $\sin^2(2x)$ and then used 1.8.4 to write it in terms of $\cos(4x)$.

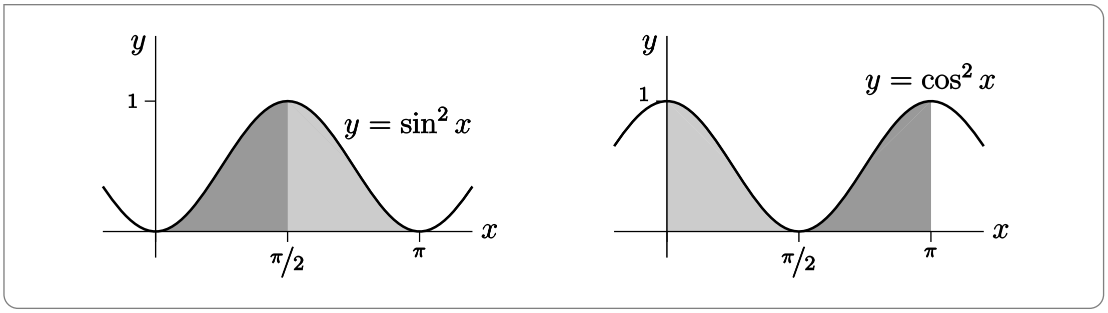
Example 1.8.9

Example 1.8.10 $\int_0^\pi \cos^2 x\,\mathrm{d}x$ and $\int_0^\pi \sin^2 x\,\mathrm{d}x$.

Of course we can compute the definite integral $\int_0^\pi \cos^2 x\,\mathrm{d}x$ by using the antiderivative for $\cos^2 x$ that we found in Example 1.8.7. But here is a trickier way to evaluate that integral, and also the integral $\int_0^\pi \sin^2 x\,\mathrm{d}x$ at the same time, very quickly without needing the antiderivative of Example 1.8.7.

Solution:

- Observe that $\int_0^\pi \cos^2 x\,\mathrm{d}x$ and $\int_0^\pi \sin^2 x\,\mathrm{d}x$ are equal because they represent the same area — look at the graphs below — the darkly shaded regions in the two graphs have the same area and the lightly shaded regions in the two graphs have the same area.



- Consequently,

$$\begin{aligned} \int_0^\pi \cos^2 x\,\mathrm{d}x = \int_0^\pi \sin^2 x\,\mathrm{d}x &= \frac{1}{2}\left[\int_0^\pi \sin^2 x\,\mathrm{d}x + \int_0^\pi \cos^2 x\,\mathrm{d}x\right] \\ &= \frac{1}{2}\int_0^\pi \big[\sin^2 x + \cos^2 x\big]\,\mathrm{d}x \end{aligned}$$

$$= \frac{1}{2}\int_0^\pi \mathrm{d}x$$
$$= \frac{\pi}{2}$$

Example 1.8.10

1.8.2 ▸▸ Integrating $\int \tan^m x \sec^n x \mathrm{d}x$

The strategy for dealing with these integrals is similar to the strategy that we used to evaluate integrals of the form $\int \sin^m x \cos^n x \mathrm{d}x$ and again depends on the parity of the exponents n and m. It uses [2]

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sec x = \sec x\,\tan x \qquad 1+\tan^2 x = \sec^2 x$$

We split the methods for integrating $\int \tan^m x \sec^n x \mathrm{d}x$ into 5 cases which we list below. These will become much more clear after an example (or two).

1 When m is odd and any n — rewrite the integrand in terms of $\sin x$ and $\cos x$:

$$\begin{aligned}\tan^m x\,\sec^n x\mathrm{d}x &= \left(\frac{\sin x}{\cos x}\right)^m \left(\frac{1}{\cos x}\right)^n \mathrm{d}x \\ &= \frac{\sin^{m-1} x}{\cos^{n+m} x}\ \sin x\mathrm{d}x\end{aligned}$$

and then substitute $u = \cos x$, $\mathrm{d}u = -\sin x\mathrm{d}x$, $\sin^2 x = 1-\cos^2 x = 1-u^2$. See Examples 1.8.11 and 1.8.12.

2 Alternatively, if m is odd and $n \geq 1$ move one factor of $\sec x\,\tan x$ to the side so that you can see $\sec x\,\tan x\mathrm{d}x$ in the integral, and substitute $u = \sec x$, $\mathrm{d}u = \sec x\,\tan x\,\mathrm{d}x$ and $\tan^2 x = \sec^2 x - 1 = u^2 - 1$. See Example 1.8.13.

3 If n is even with $n \geq 2$, move one factor of $\sec^2 x$ to the side so that you can see $\sec^2 x\mathrm{d}x$ in the integral, and substitute $u = \tan x$, $\mathrm{d}u = \sec^2 x\,\mathrm{d}x$ and $\sec^2 x = 1+\tan^2 x = 1+u^2$. See Example 1.8.14.

4 When m is even and $n = 0$ — that is the integrand is just an even power of tangent — we can still use the $u = \tan x$ substitution, after using $\tan^2 x = \sec^2 x - 1$ (possibly more than once) to create a $\sec^2 x$. See Example 1.8.16.

5 This leaves the case n odd and m even. There are strategies like those above for treating this case. But they are more complicated and also involve more tricks (that basically have to be memorized). Examples using them are provided in

2 You will need to memorise the derivatives of tangent and secant. However there is no need to memorise $1+\tan^2 x = \sec^2 x$. To derive it very quickly just divide $\sin^2 x + \cos^2 x = 1$ by $\cos^2 x$.

the optional section entitled "Integrating $\sec x$, $\csc x$, $\sec^3 x$ and $\csc^3 x$", below. A more straight forward strategy uses another technique called "partial fractions". We shall return to this strategy after we have learned about partial fractions. See Example 1.10.5 and 1.10.6 in Section 1.10.

1.8.2.1 ▶▶▶ m is odd — odd power of tangent

In this case we rewrite the integrand in terms of sine and cosine and then substitute $u = \cos x, \mathrm{d}u = -\sin x \mathrm{d}x$.

Example 1.8.11 $\int \tan x \mathrm{d}x$.

Solution:

- Write the integrand $\tan x = \frac{1}{\cos x} \sin x$.
- Now substitute $u = \cos x$, $\mathrm{d}u = -\sin x\,\mathrm{d}x$ just as we did in treating integrands of the form $\sin^m x \cos^n x$ with m odd.

$$\begin{aligned}
\int \tan x\,\mathrm{d}x &= \int \frac{1}{\cos x} \sin x\,\mathrm{d}x && \text{substitute } u = \cos x \\
&= \int \frac{1}{u} \cdot (-1)\mathrm{d}u \\
&= -\log|u| + C \\
&= -\log|\cos x| + C && \text{can also write in terms of secant} \\
&= \log|\cos x|^{-1} + C = \log|\sec x| + C
\end{aligned}$$

Example 1.8.12 $\int \tan^3 x \mathrm{d}x$.

Solution:

- Write the integrand $\tan^3 x = \frac{\sin^2 x}{\cos^3 x} \sin x$.
- Again substitute $u = \cos x$, $\mathrm{d}u = -\sin x\,\mathrm{d}x$. We rewrite the remaining even powers of $\sin x$ using $\sin^2 x = 1 - \cos^2 x = 1 - u^2$.
- Hence

$$\begin{aligned}
\int \tan^3 x\,\mathrm{d}x &= \int \frac{\sin^2 x}{\cos^3 x} \sin x\,\mathrm{d}x && \text{substitute } u = \cos x \\
&= \int \frac{1 - u^2}{u^3} (-1)\mathrm{d}u \\
&= \frac{u^{-2}}{2} + \log|u| + C
\end{aligned}$$

$$= \frac{1}{2\cos^2 x} + \log|\cos x| + C \qquad \text{rewrite in terms of secant}$$
$$= \frac{1}{2}\sec^2 x - \log|\sec x| + C$$

Example 1.8.12

1.8.2.2 m is odd and $n \geq 1$ — odd power of tangent and at least one secant

Here we collect a factor of $\tan x \sec x$ and then substitute $u = \sec x$ and $\mathrm{d}u = \sec x \tan x \mathrm{d}x$. We can then rewrite any remaining even powers of $tanx$ in terms of $\sec x$ using $\tan^2 x = \sec^2 x - 1 = u^2 - 1$.

Example 1.8.13 $\int \tan^3 x \sec^4 x \mathrm{d}x$.

Solution:

- Start by factoring off one copy of $\sec x \tan x$ and combine it with $\mathrm{d}x$ to form $\sec x \tan x \mathrm{d}x$, which will be $\mathrm{d}u$.
- Now substitute $u = \sec x$, $\mathrm{d}u = \sec x \tan x \mathrm{d}x$ and $\tan^2 x = \sec^2 x - 1 = u^2 - 1$.
- This gives

$$\int \tan^3 x \sec^4 x \mathrm{d}x = \int \underbrace{\tan^2 x}_{u^2-1} \underbrace{\sec^3 x}_{u^3} \underbrace{\sec x \tan x \mathrm{d}x}_{\mathrm{d}u}$$
$$= \int [u^2 - 1]u^3 \mathrm{d}u$$
$$= \frac{u^6}{6} - \frac{u^4}{4} + C$$
$$= \frac{1}{6}\sec^6 x - \frac{1}{4}\sec^4 x + C$$

1.8.2.3 $n \geq 2$ is even — a positive even power of secant

In the previous case we substituted $u = \sec x$, while in this case we substitute $u = \tan x$. When we do this we write $\mathrm{d}u = \sec^2 x \mathrm{d}x$ and then rewrite any remaining even powers of $\sec x$ as powers of $\tan x$ using $\sec^2 x = 1 + \tan^2 x = 1 + u^2$.

Example 1.8.14 $\int \sec^4 x \mathrm{d}x$.

Solution:

- Factor off one copy of $\sec^2 x$ and combine it with $\mathrm{d}x$ to form $\sec^2 x\mathrm{d}x$, which will be $\mathrm{d}u$.
- Then substitute $u = \tan x$, $\mathrm{d}u = \sec^2 x\mathrm{d}x$ and rewrite any remaining even powers of $\sec x$ as powers of $\tan x = u$ using $\sec^2 x = 1 + \tan^2 x = 1 + u^2$.
- This gives

$$\begin{aligned}\int \sec^4 x\mathrm{d}x &= \int \underbrace{\sec^2 x}_{1+u^2}\ \underbrace{\sec^2 x\mathrm{d}x}_{\mathrm{d}u} \\ &= \int [1 + u^2]\mathrm{d}u \\ &= u + \frac{u^3}{3} + C \\ &= \tan x + \frac{1}{3}\tan^3 x + C\end{aligned}$$

Example 1.8.15 $\int \tan^3 x \sec^4 x\mathrm{d}x$ — redux.

Solution: Let us revisit this example using this slightly different approach.

- Factor off one copy of $\sec^2 x$ and combine it with $\mathrm{d}x$ to form $\sec^2 x\mathrm{d}x$, which will be $\mathrm{d}u$.
- Then substitute $u = \tan x$, $\mathrm{d}u = \sec^2 x\mathrm{d}x$ and rewrite any remaining even powers of $\sec x$ as powers of $\tan x = u$ using $\sec^2 x = 1 + \tan^2 x = 1 + u^2$.
- This gives

$$\begin{aligned}\int \tan^3 x\sec^4 x\mathrm{d}x &= \int \underbrace{\tan^3 x}_{u^3}\underbrace{\sec^2 x}_{1+u^2}\ \underbrace{\sec^2 x\mathrm{d}x}_{\mathrm{d}u} \\ &= \int [u^3 + u^5]\mathrm{d}u \\ &= \frac{u^4}{4} + \frac{u^6}{6} + C \\ &= \frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + C\end{aligned}$$

- This is not quite the same as the answer we got above in Example 1.8.13. However we can show they are (nearly) equivalent. To do so we substitute $v = \sec x$ and

$\tan^2 x = \sec^2 x - 1 = v^2 - 1$:

$$\begin{aligned}
\frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x &= \frac{1}{6}(v^2-1)^3 + \frac{1}{4}(v^2-1)^2 \\
&= \frac{1}{6}(v^6 - 3v^4 + 3v^2 - 1) + \frac{1}{4}(v^4 - 2v^2 + 1) \\
&= \frac{v^6}{6} - \frac{v^4}{2} + \frac{v^2}{2} - \frac{1}{6} + \frac{v^4}{4} - \frac{v^2}{2} + \frac{1}{4} \\
&= \frac{v^6}{6} - \frac{v^4}{4} + 0 \cdot v^2 + \left(\frac{1}{4} - \frac{1}{6}\right) \\
&= \frac{1}{6}\sec^6 x - \frac{1}{4}\sec^4 x + \frac{1}{12}.
\end{aligned}$$

So while $\frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x \neq \frac{1}{6}\sec^6 x - \frac{1}{4}\sec^4 x$, they only differ by a constant. Hence both are valid antiderivatives of $\tan^3 x \sec^4 x$.

Example 1.8.15

1.8.2.4 ▸▸▸ m is even and $n = 0$ — even powers of tangent

We integrate this by setting $u = \tan x$. For this to work we need to pull one factor of $\sec^2 x$ to one side to form $\mathrm{d}u = \sec^2 x\mathrm{d}x$. To find this factor of $\sec^2 x$ we (perhaps repeatedly) apply the identity $\tan^2 x = \sec^2 x - 1$.

Example 1.8.16 $\int \tan^4 x\mathrm{d}x$.

Solution:

- There is no $\sec^2 x$ term present, so we try to create it from $\tan^4 x$ by using $\tan^2 x = \sec^2 x - 1$.

$$\begin{aligned}
\tan^4 x &= \tan^2 x \cdot \tan^2 x \\
&= \tan^2 x\big[\sec^2 x - 1\big] \\
&= \tan^2 x \sec^2 x - \underbrace{\tan^2 x}_{\sec^2 x - 1} \\
&= \tan^2 x \sec^2 x - \sec^2 x + 1
\end{aligned}$$

- Now we can substitute $u = \tan x$, $\mathrm{d}u = \sec^2 x\mathrm{d}x$.

$$\begin{aligned}
\int \tan^4 x\mathrm{d}x &= \int \underbrace{\tan^2 x}_{u^2}\ \underbrace{\sec^2 x\mathrm{d}x}_{\mathrm{d}u} - \int \underbrace{\sec^2 x\mathrm{d}x}_{\mathrm{d}u} + \int \mathrm{d}x \\
&= \int u^2\mathrm{d}u - \int \mathrm{d}u + \int \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \frac{u^3}{3} - u + x + C \\
&= \frac{\tan^3 x}{3} - \tan x + x + C
\end{aligned}
$$

Example 1.8.16

Example 1.8.17 $\int \tan^8 x \mathrm{d}x$.

Solution: Let us try the same approach.

- First pull out a factor of $\tan^2 x$ to create a $\sec^2 x$ factor:

$$
\begin{aligned}
\tan^8 x &= \tan^6 x \cdot \tan^2 x \\
&= \tan^6 x \cdot \left[\sec^2 x - 1\right] \\
&= \tan^6 x \sec^2 x - \tan^6 x
\end{aligned}
$$

The first term is now ready to be integrated, but we need to reapply the method to the second term:

$$
\begin{aligned}
&= \tan^6 x \sec^2 x - \tan^4 x \cdot \left[\sec^2 x - 1\right] \\
&= \tan^6 x \sec^2 x - \tan^4 x \sec^2 x + \tan^4 x \qquad \text{do it again} \\
&= \tan^6 x \sec^2 x - \tan^4 x \sec^2 x + \tan^2 x \cdot \left[\sec^2 x - 1\right] \\
&= \tan^6 x \sec^2 x - \tan^4 x \sec^2 x + \tan^2 x \sec^2 x - \tan^2 x \qquad \text{and again} \\
&= \tan^6 x \sec^2 x - \tan^4 x \sec^2 x + \tan^2 x \sec^2 x - \left[\sec^2 x - 1\right]
\end{aligned}
$$

- Hence

$$
\begin{aligned}
&\int \tan^8 x \mathrm{d}x \\
&\quad= \int \left[\tan^6 x \sec^2 x - \tan^4 x \sec^2 x + \tan^2 x \sec^2 x - \sec^2 x + 1\right] \mathrm{d}x \\
&\quad= \int \left[\tan^6 x - \tan^4 x + \tan^2 x - 1\right] \sec^2 x \mathrm{d}x + \int \mathrm{d}x \\
&\quad= \int \left[u^6 - u^4 + u^2 - 1\right] \mathrm{d}u + x + C \\
&\quad= \frac{u^7}{7} - \frac{u^5}{5} + \frac{u^3}{3} - u + x + C \\
&\quad= \frac{1}{7}\tan^7 x - \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x - \tan x + x + C
\end{aligned}
$$

Indeed this example suggests that for integer $k \geq 0$:

$$
\begin{aligned}
\int \tan^{2k} x \mathrm{d}x = \frac{1}{2k-1}\tan^{2k-1}(x) - \frac{1}{2k-3}\tan^{2k-3} x + \cdots \\
- (-1)^k \tan x + (-1)^k x + C
\end{aligned}
$$

This last example also shows how we might integrate an odd power of tangent:

Example 1.8.18 $\int \tan^7 x$.

Solution: We follow the same steps

- Pull out a factor of $\tan^2 x$ to create a factor of $\sec^2 x$:

$$\begin{aligned}
\tan^7 x &= \tan^5 x \cdot \tan^2 x \\
&= \tan^5 x \cdot \left[\sec^2 x - 1\right] \\
&= \tan^5 x \sec^2 x - \tan^5 x \qquad \text{do it again} \\
&= \tan^5 x \sec^2 x - \tan^3 x \cdot \left[\sec^2 x - 1\right] \\
&= \tan^5 x \sec^2 x - \tan^3 x \sec^2 x + \tan^3 x \qquad \text{and again} \\
&= \tan^5 x \sec^2 x - \tan^3 x \sec^2 x + \tan x\left[\sec^2 x - 1\right] \\
&= \tan^5 x \sec^2 x - \tan^3 x \sec^2 x + \tan x \sec^2 x - \tan x
\end{aligned}$$

- Now we can substitute $u = \tan x$ and $\mathrm{d}u = \sec^2 x\mathrm{d}x$ and also use the result from Example 1.8.11 to take care of the last term:

$$\begin{aligned}
\int \tan^7 x\mathrm{d}x = \int & \left[\tan^5 x \sec^2 x - \tan^3 x \sec^2 x + \tan x \sec^2 x\right]\mathrm{d}x \\
& - \int \tan x\mathrm{d}x
\end{aligned}$$

Now factor out the common $\sec^2 x$ term and integrate $\tan x$ via Example 1.8.11

$$\begin{aligned}
&= \int \left[\tan^5 x - \tan^3 x + \tan x\right] \sec x\mathrm{d}x - \log|\sec x| + C \\
&= \int \left[u^5 - u^3 + u\right]\mathrm{d}u - \log|\sec x| + C \\
&= \frac{u^6}{6} - \frac{u^4}{4} + \frac{u^2}{2} - \log|\sec x| + C \\
&= \frac{1}{6}\tan^6 x - \frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x - \log|\sec x| + C
\end{aligned}$$

This example suggests that for integer $k \geq 0$:

$$\begin{aligned}
\int \tan^{2k+1} x\mathrm{d}x = & \frac{1}{2k}\tan^{2k}(x) - \frac{1}{2k-2}\tan^{2k-2} x + \cdots \\
& - (-1)^k \frac{1}{2}\tan^2 x + (-1)^k \log|\sec x| + C
\end{aligned}$$

Of course we have not considered integrals involving powers of $\cot x$ and $\csc x$. But they can be treated in much the same way as $\tan x$ and $\sec x$ were.

1.8.3 ➤ Optional — Integrating $\sec x$, $\csc x$, $\sec^3 x$ and $\csc^3 x$

As noted above, when n is odd and m is even, one can use similar strategies as to the previous cases. However the computations are often more involved and more tricks need to be deployed. For this reason we make this section optional — the computations are definitely non-trivial. Rather than trying to construct a coherent "method" for this case, we instead give some examples to give the idea of what to expect.

Example 1.8.19 $\int \sec x \mathrm{d}x$ — by trickery.

Solution: There is a very sneaky trick to compute this integral[a].

- The standard trick for this integral is to multiply the integrand by $1 = \frac{\sec x+\tan x}{\sec x+\tan x}$

$$\sec x = \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} = \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}$$

- Notice now that the numerator of this expression is exactly the derivative its denominator. Hence we can substitute $u = \sec x + \tan x$ and $\mathrm{d}u = (\sec x \tan x + \sec^2 x)\,\mathrm{d}x$.

- Hence

$$\begin{aligned}
\int \sec x \mathrm{d}x &= \int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} \mathrm{d}x = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \mathrm{d}x \\
&= \int \frac{1}{u} \mathrm{d}u \\
&= \log |u| + C \\
&= \log |\sec x + \tan x| + C
\end{aligned}$$

- The above trick appears both totally unguessable and very hard to remember. Fortunately, there is a simple way[b] to recover the trick. Here it is.
  - The goal is to guess a function whose derivative is $\sec x$.
  - So get out a table of derivatives and look for functions whose derivatives at least contain $\sec x$. There are two:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \tan x &= \sec^2 x \\
\frac{\mathrm{d}}{\mathrm{d}x} \sec x &= \tan x \, \sec x
\end{aligned}$$

  - Notice that if we add these together we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\big(\sec x + \tan x\big) &= (\sec x + \tan x) \sec x \qquad \implies \\
\frac{\frac{\mathrm{d}}{\mathrm{d}x}\big(\sec x + \tan x\big)}{\sec x + \tan x} &= \sec x
\end{aligned}$$

- We've done it! The right hand side is $\sec x$ and the left hand side is the derivative of $\log|\sec x + \tan x|$.

[a] The integral of secant played an important role in constructing accurate Mercator projection maps of the earth. See https://en.wikipedia.org/wiki/Integral_of_the_secant_function and https://arxiv.org/pdf/2204.11187.pdf. The method for evaluating the integral that is being presented in this example was invented by the Scottish mathematician James Gregory in 1668. There are also a number of other methods. See the previous two references.

[b] We thank Serban Raianu for bringing this to our attention.

Example 1.8.19

There is another method for integrating $\int \sec x \mathrm{d}x$, that is more tedious, but more straight forward. In particular, it does not involve a memorized trick. We first use the substitution $u = \sin x$, $\mathrm{d}u = \cos x\,\mathrm{d}x$, together with $\cos^2 x = 1 - \sin^2 x = 1 - u^2$. This converts the integral into

$$\begin{aligned}\int \sec x \mathrm{d}x &= \int \frac{1}{\cos x}\mathrm{d}x = \int \frac{\cos x\ \mathrm{d}x}{\cos^2 x}\\ &= \int \frac{\mathrm{d}u}{1-u^2}\bigg|_{u=\sin x}\end{aligned}$$

The integrand $\frac{1}{1-u^2}$ is a rational function, i.e. a ratio of two polynomials. There is a procedure, called the method of partial fractions, that may be used to integrate any rational function. We shall learn about it in Section 1.10 "Partial Fractions". The detailed evaluation of the integral $\int \sec x\,\mathrm{d}x = \int \frac{\mathrm{d}u}{1-u^2}$ by the method of partial fractions is presented in Example 1.10.5 below.

In addition, there is a standard trick for evaluating $\int \frac{\mathrm{d}u}{1-u^2}$ that allows us to avoid going through the whole partial fractions algorithm.

Example 1.8.20 $\int \sec x \mathrm{d}x$ — by more trickery.

Solution: We have already seen that

$$\int \sec x \mathrm{d}x = \int \frac{\mathrm{d}u}{1-u^2}\bigg|_{u=\sin x}$$

The trick uses the obervations that

- $\frac{1}{1-u^2} = \frac{1+u-u}{1-u^2} = \frac{1}{1-u} - \frac{u}{1-u^2}$
- $\frac{1}{1-u}$ has antiderivative $-\log(1-u)$ (for $u < 1$)
- The derivative $\frac{\mathrm{d}}{\mathrm{d}u}(1-u^2) = -2u$ of the denominator of $\frac{u}{1-u^2}$ is the same, up to a factor of -2, as the numerator of $\frac{u}{1-u^2}$. So we can easily evaluate the integral of $\frac{u}{1-u^2}$ by substituting $v = 1-u^2$, $\mathrm{d}v = -2u\,\mathrm{d}u$.

$$\int \frac{u\,\mathrm{d}u}{1-u^2} = \int \frac{\frac{\mathrm{d}v}{-2}}{v}\bigg|_{v=1-u^2} = -\frac{1}{2}\log(1-u^2) + C$$

Combining these observations gives

$$\begin{aligned}\int \sec x \mathrm{d}x &= \left[\int \frac{\mathrm{d}u}{1-u^2}\right]_{u=\sin x} = \left[\int \frac{1}{1-u}\mathrm{d}u - \int \frac{u}{1-u^2}\mathrm{d}u\right]_{u=\sin x} \\ &= \left[-\log(1-u) + \frac{1}{2}\log(1-u^2) + C\right]_{u=\sin x} \\ &= -\log(1-\sin x) + \frac{1}{2}\log(1-\sin^2 x) + C \\ &= -\log(1-\sin x) + \frac{1}{2}\log(1-\sin x) + \frac{1}{2}\log(1+\sin x) + C \\ &= \frac{1}{2}\log\frac{1+\sin x}{1-\sin x} + C\end{aligned}$$

Example 1.8.20

Example 1.8.20 has given the answer

$$\int \sec x \mathrm{d}x = \frac{1}{2}\log\frac{1+\sin x}{1-\sin x} + C$$

which appears to be different than the answer in Example 1.8.19. But they are really the same since

$$\begin{aligned}&\frac{1+\sin x}{1-\sin x} = \frac{(1+\sin x)^2}{1-\sin^2 x} = \frac{(1+\sin x)^2}{\cos^2 x} \\ \Longrightarrow\ &\frac{1}{2}\log\frac{1+\sin x}{1-\sin x} = \frac{1}{2}\log\frac{(1+\sin x)^2}{\cos^2 x} = \log\left|\frac{\sin x+1}{\cos x}\right| = \log|\tan x + \sec x|\end{aligned}$$

Oof!

Example 1.8.21 $\int \csc x \mathrm{d}x$ — by the $u = \tan\frac{x}{2}$ substitution.

Solution: The integral $\int \csc x \mathrm{d}x$ may also be evaluated by both the methods above. That is either

- by multiplying the integrand by a cleverly chosen $1 = \frac{\cot x - \csc x}{\cot x - \csc x}$ and then substituting $u = \cot x - \csc x$, $\mathrm{d}u = (-\csc^2 x + \csc x \cot x)\,\mathrm{d}x$, or
- by substituting $u = \cos x$, $\mathrm{d}u = -\sin x\,\mathrm{d}x$ to give $\int \csc x \mathrm{d}x = -\int \frac{\mathrm{d}u}{1-u^2}$ and then using the method of partial fractions.

These two methods give the answers

$$\int \csc x \mathrm{d}x = \log|\cot x - \csc x| + C = -\frac{1}{2}\log\frac{1+\cos x}{1-\cos x} + C \qquad (\star)$$

In this example, we shall evaluate $\int \csc x \mathrm{d}x$ by yet a third method, which can be used to integrate rational functions [a] of $\sin x$ and $\cos x$.

- This method uses the substitution

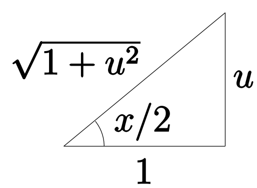
$$x = 2\arctan u \qquad \text{i.e. } u = \tan\frac{x}{2} \qquad \text{and } \mathrm{d}x = \frac{2}{1+u^2}\mathrm{d}u$$

— a half-angle substitution.

- To express $\sin x$ and $\cos x$ in terms of u, we first use the double angle trig identities (Equations 1.8.2 and 1.8.3 with $x \mapsto \frac{x}{2}$) to express $\sin x$ and $\cos x$ in terms of $\sin\frac{x}{2}$ and $\cos\frac{x}{2}$:

$$\begin{aligned} \sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2} \\ \cos x &= \cos^2\frac{x}{2} - \sin^2\frac{x}{2} \end{aligned}$$

- We then use the triangle



to express $\sin\frac{x}{2}$ and $\cos\frac{x}{2}$ in terms of u. The bottom and right hand sides of the triangle have been chosen so that $\tan\frac{x}{2} = u$. This tells us that

$$\sin\frac{x}{2} = \frac{u}{\sqrt{1+u^2}} \qquad \cos\frac{x}{2} = \frac{1}{\sqrt{1+u^2}}$$

- This in turn implies that:

$$\begin{aligned} \sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{u}{\sqrt{1+u^2}}\frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2} \\ \cos x &= \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1}{1+u^2} - \frac{u^2}{1+u^2} = \frac{1-u^2}{1+u^2} \end{aligned}$$

Oof!

- Let's use this substitution to evaluate $\int \csc x \,\mathrm{d}x$.

$$\begin{aligned} \int \csc x \mathrm{d}x &= \int \frac{1}{\sin x}\mathrm{d}x = \int \frac{1+u^2}{2u}\frac{2}{1+u^2}\mathrm{d}u = \int \frac{1}{u}\mathrm{d}u \\ &= \log|u| + C = \log\left|\tan\frac{x}{2}\right| + C \end{aligned}$$

To see that this answer is really the same as that in $(\star)$, note that

$$\cot x - \csc x = \frac{\cos x - 1}{\sin x} = \frac{-2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)} = -\tan\frac{x}{2}$$

a A rational function of $\sin x$ and $\cos x$ is a ratio with both the numerator and denominator being finite sums of terms of the form $a \sin^m x \cos^n x$, where a is a constant and m and n are positive integers.

Example 1.8.21

Example 1.8.22 $\int \sec^3 x\mathrm{d}x$ — by trickery.

Solution: The standard trick used to evaluate $\int \sec^3 x\mathrm{d}x$ is integration by parts.

- Set $u = \sec x$, $\mathrm{d}v = \sec^2 x\mathrm{d}x$. Hence $\mathrm{d}u = \sec x \tan x\mathrm{d}x$, $v = \tan x$ and

$$\int \sec^3 x\mathrm{d}x = \int \underbrace{\sec x}_{u}\ \underbrace{\sec^2 x\mathrm{d}x}_{\mathrm{d}v}$$
$$= \underbrace{\sec x}_{u}\ \underbrace{\tan x}_{v} - \int \underbrace{\tan x}_{v}\ \underbrace{\sec x \tan x\mathrm{d}x}_{\mathrm{d}u}$$

- Since $\tan^2 x + 1 = \sec^2 x$, we have $\tan^2 x = \sec^2 x - 1$ and

$$\int \sec^3 x\mathrm{d}x = \sec x\ \tan x - \int [\sec^3 x - \sec x]\mathrm{d}x$$
$$= \sec x\ \tan x + \log|\sec x + \tan x| + C - \int \sec^3 x\mathrm{d}x$$

  where we used $\int \sec x\mathrm{d}x = \log|\sec x + \tan x| + C$, which we saw in Example 1.8.19.

- Now moving the $\int \sec^3 x\mathrm{d}x$ from the right hand side to the left hand side

$$2\int \sec^3 x\mathrm{d}x = \sec x \tan x + \log|\sec x + \tan x| + C \qquad \text{and so}$$
$$\int \sec^3 x\mathrm{d}x = \frac{1}{2}\sec x \tan x + \frac{1}{2}\log|\sec x + \tan x| + C$$

  for a new arbitrary constant C (which is just one half the old one).

The integral $\int \sec^3 \mathrm{d}x$ can also be evaluated by two other methods.

- Substitute $u = \sin x$, $\mathrm{d}u = \cos x\mathrm{d}x$ to convert $\int \sec^3 x\mathrm{d}x$ into $\int \frac{\mathrm{d}u}{[1-u^2]^2}$ and evaluate the latter using the method of partial fractions. This is done in Example 1.10.6 in Section 1.10.
- Use the $u = \tan\frac{x}{2}$ substitution. We use this method to evaluate $\int \csc^3 x\mathrm{d}x$ in Example 1.8.23, below.

Example 1.8.23 $\int \csc^3 x \mathrm{d}x$ – by the $u = \tan\frac{x}{2}$ substitution.

Solution: Let us use the half-angle substitution that we introduced in Example 1.8.21.

- In this method we set

$$u = \tan\frac{x}{2} \quad \mathrm{d}x = \frac{2}{1+u^2}\mathrm{d}u \quad \sin x = \frac{2u}{1+u^2} \quad \cos x = \frac{1-u^2}{1+u^2}$$

- The integral then becomes

$$\begin{aligned}\int \csc^3 x \mathrm{d}x &= \int \frac{1}{\sin^3 x}\mathrm{d}x \\ &= \int \left(\frac{1+u^2}{2u}\right)^3 \frac{2}{1+u^2}\mathrm{d}u \\ &= \frac{1}{4}\int \frac{1+2u^2+u^4}{u^3}\mathrm{d}u \\ &= \frac{1}{4}\left\{\frac{u^{-2}}{-2} + 2\log|u| + \frac{u^2}{2}\right\} + C \\ &= \frac{1}{8}\left\{-\cot^2\frac{x}{2} + 4\log\left|\tan\frac{x}{2}\right| + \tan^2\frac{x}{2}\right\} + C\end{aligned}$$

Oof!

- This is a perfectly acceptable answer. But if you don't like the $\frac{x}{2}$'s, they may be eliminated by using

$$\begin{aligned}\tan^2\frac{x}{2} - \cot^2\frac{x}{2} &= \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} - \frac{\cos^2\frac{x}{2}}{\sin^2\frac{x}{2}} \\ &= \frac{\sin^4\frac{x}{2} - \cos^4\frac{x}{2}}{\sin^2\frac{x}{2}\cos^2\frac{x}{2}} \\ &= \frac{\left(\sin^2\frac{x}{2} - \cos^2\frac{x}{2}\right)\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right)}{\sin^2\frac{x}{2}\cos^2\frac{x}{2}} \\ &= \frac{\sin^2\frac{x}{2} - \cos^2\frac{x}{2}}{\sin^2\frac{x}{2}\cos^2\frac{x}{2}} && \text{since } \sin^2\frac{x}{2} + \cos^2\frac{x}{2} = 1 \\ &= \frac{-\cos x}{\frac{1}{4}\sin^2 x} && \text{by 1.8.2 and 1.8.3}\end{aligned}$$

and

$$\begin{aligned}\tan\frac{x}{2} &= \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{\sin^2\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}} \\ &= \frac{\frac{1}{2}[1-\cos x]}{\frac{1}{2}\sin x} && \text{by 1.8.2 and 1.8.3}\end{aligned}$$

So we may also write

$$\int \csc^3 x \mathrm{d}x = -\frac{1}{2}\cot x \csc x + \frac{1}{2}\log|\csc x - \cot x| + C$$

That last optional section was a little scary — let's get back to something a little easier.

1.8.4 ▸▸ Exercises

Recall that we are using $\log x$ to denote the logarithm of x with base e. In other courses it is often denoted $\ln x$.

Exercises — Stage 1

1. Suppose you want to evaluate $\displaystyle\int_0^{\pi/4} \sin x \cos^n x \mathrm{d}x$ using the substitution $u = \cos x$. Which of the following need to be true for your substitution to work?

a n must be even

b n must be odd

c n must be an integer

d n must be positive

e n can be any real number

2. Evaluate $\displaystyle\int \sec^n x \tan x \mathrm{d}x$, where n is a strictly positive integer.

3. Derive the identity $\tan^2 x + 1 = \sec^2 x$ from the easier-to-remember identity $\sin^2 x + \cos^2 x = 1$.

Exercises — Stage 2 Questions 4 through 10 deal with powers of sines and cosines. Review Section 1.8.1 in the notes for integration strategies.Questions 12 through 21 deal with powers of tangents and secants. Review Section 1.8.2 in the notes for strategies.

4. **∗.** Evaluate $\displaystyle\int \cos^3 x \, \mathrm{d}x$.

5. **∗.** Evaluate $\displaystyle\int_0^{\pi} \cos^2 x \, \mathrm{d}x$.

6. **∗.** Evaluate $\displaystyle\int \sin^{36} t \, \cos^3 t \, \mathrm{d}t$.

7. Evaluate $\displaystyle\int \frac{\sin^3 x}{\cos^4 x} \mathrm{d}x$.

8. Evaluate $\displaystyle\int_0^{\pi/3} \sin^4 x \mathrm{d}x$.

9. Evaluate $\displaystyle\int \sin^5 x \mathrm{d}x$.

10. Evaluate $\displaystyle\int \sin^{1.2} x \cos x \mathrm{d}x$.

11. Evaluate $\displaystyle\int \tan x \sec^2 x \mathrm{d}x$.

12. $*$. Evaluate $\displaystyle\int \tan^3 x \sec^5 x \, \mathrm{d}x$.

13. $*$. Evaluate $\displaystyle\int \sec^4 x \ \tan^{46} x \, \mathrm{d}x$.

14. Evaluate $\displaystyle\int \tan^3 x \sec^{1.5} x \mathrm{d}x$.

15. Evaluate $\displaystyle\int \tan^3 x \sec^2 x \mathrm{d}x$.

16. Evaluate $\displaystyle\int \tan^4 x \sec^2 x \mathrm{d}x$.

17. Evaluate $\displaystyle\int \tan^3 x \sec^{-0.7} x \mathrm{d}x$.

18. Evaluate $\displaystyle\int \tan^5 x \mathrm{d}x$.

19. Evaluate $\displaystyle\int_0^{\pi/6} \tan^6 x \mathrm{d}x$.

20. Evaluate $\displaystyle\int_0^{\pi/4} \tan^8 x \sec^4 x \mathrm{d}x$.

21. Evaluate $\displaystyle\int \tan x \sqrt{\sec x} \mathrm{d}x$.

22. Evaluate $\displaystyle\int \sec^8 \theta \tan^e \theta \mathrm{d}\theta$.

Exercises — Stage 3

23. $*$. A reduction formula.

a Let n be a positive integer with $n \geq 2$. Derive the reduction formula

$$\int \tan^n(x)\,\mathrm{d}x = \frac{\tan^{n-1}(x)}{n-1} - \int \tan^{n-2}(x)\,\mathrm{d}x.$$

b Calculate $\int_0^{\pi/4} \tan^6(x)\,\mathrm{d}x$.

24. Evaluate $\int \tan^5 x \cos^2 x \mathrm{d}x$.

25. Evaluate $\int \frac{1}{\cos^2\theta} \mathrm{d}\theta$.

26. Evaluate $\int \cot x \mathrm{d}x$.

27. Evaluate $\int e^x \sin(e^x)\cos(e^x)\mathrm{d}x$.

28. Evaluate $\int \sin(\cos x)\sin^3 x \mathrm{d}x$.

29. Evaluate $\int x \sin x \cos x \mathrm{d}x$.

1.9▲ Trigonometric Substitution

1.9.1 ⏩ Trigonometric Substitution

In this section we discuss substitutions that simplify integrals containing square roots of the form

$$\sqrt{a^2 - x^2} \qquad \sqrt{a^2 + x^2} \qquad \sqrt{x^2 - a^2}.$$

When the integrand contains one of these square roots, then we can use trigonometric substitutions to eliminate them. That is, we substitute

$$x = a\sin u \qquad \text{or} \qquad x = a\tan u \qquad \text{or} \qquad x = a\sec u$$

and then use trigonometric identities

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{and} \quad 1 + \tan^2\theta = \sec^2\theta$$

to simplify the result. To be more precise, we can

- eliminate $\sqrt{a^2 - x^2}$ from an integrand by substituting $x = a\sin u$ to give

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2 u} = \sqrt{a^2\cos^2 u} = |a\cos u|$$

- eliminate $\sqrt{a^2 + x^2}$ from an integrand by substituting $x = a\tan u$ to give

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2 u} = \sqrt{a^2\sec^2 u} = |a\sec u|$$

- eliminate $\sqrt{x^2 - a^2}$ from an integrand by substituting $x = a\sec u$ to give

$$\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2 u - a^2} = \sqrt{a^2\tan^2 u} = |a\tan u|$$

  Be very careful with signs and absolute values when using this substitution. See Example 1.9.6.

When we have used substitutions before, we usually gave the new integration variable, u, as a function of the old integration variable x. Here we are doing the reverse — we are giving the old integration variable, x, in terms of the new integration variable u. We may do so, as long as we may invert to get u as a function of x. For example, with $x = a\sin u$, we may take $u = \arcsin\frac{x}{a}$. This is a good time for you to review the definitions of $\arcsin\theta$, $\arctan\theta$ and $\operatorname{arcsec}\theta$. See Section 2.12, "Inverse Functions", of the CLP-1 text.

As a warm-up, consider the area of a quarter of the unit circle.

Example 1.9.1 Quarter of the unit circle.

Compute the area of the unit circle lying in the first quadrant.

Solution: We know that the answer is $\frac{\pi}{4}$, but we can also compute this as an integral — we saw this way back in Example 1.1.16:

$$\text{area} = \int_0^1 \sqrt{1 - x^2}\,\mathrm{d}x$$

- To simplify the integrand we substitute $x = \sin u$. With this choice $\frac{\mathrm{d}x}{\mathrm{d}u} = \cos u$ and so $\mathrm{d}x = \cos u\,\mathrm{d}u$.
- We also need to translate the limits of integration and it is perhaps easiest to do this by writing u as a function of x — namely $u(x) = \arcsin x$. Hence $u(0) = 0$ and $u(1) = \frac{\pi}{2}$.
- Hence the integral becomes

$$\int_0^1 \sqrt{1 - x^2}\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 u} \cdot \cos u\,\mathrm{d}u$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\cos^2 u} \cdot \cos u \mathrm{d}u$$
$$= \int_0^{\frac{\pi}{2}} \cos^2 u \mathrm{d}u$$

Notice that here we have used that the *positive* square root $\sqrt{\cos^2 u} = |\cos u| = \cos u$ because $\cos(u) \geq 0$ for $0 \leq u \leq \frac{\pi}{2}$.

- To go further we use the techniques of Section 1.8.

$$\begin{aligned}
\int_0^1 \sqrt{1-x^2}\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \cos^2 u \mathrm{d}u \qquad \text{and since } \cos^2 u = \frac{1+\cos 2u}{2} \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos(2u))\mathrm{d}u \\
&= \frac{1}{2}\left[u + \frac{1}{2}\sin(2u)\right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{2}\left(\frac{\pi}{2} - 0 + \frac{\sin \pi}{2} - \frac{\sin 0}{2}\right) \\
&= \frac{\pi}{4} \checkmark
\end{aligned}$$

Example 1.9.1

Example 1.9.2 $\int \frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x$.

Solution: We proceed much as we did in the previous example.

- To simplify the integrand we substitute $x = \sin u$. With this choice $\frac{\mathrm{d}x}{\mathrm{d}u} = \cos u$ and so $\mathrm{d}x = \cos u \mathrm{d}u$. Also note that $u = \arcsin x$.
- The integral becomes

$$\begin{aligned}
\int \frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x &= \int \frac{\sin^2 u}{\sqrt{1-\sin^2 u}} \cdot \cos u \mathrm{d}u \\
&= \int \frac{\sin^2 u}{\sqrt{\cos^2 u}} \cdot \cos u \mathrm{d}u
\end{aligned}$$

- To proceed further we need to get rid of the square-root. Since $u = \arcsin x$ has domain $-1 \leq x \leq 1$ and range $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$, it follows that $\cos u \geq 0$ (since cosine is non-negative on these inputs). Hence

$$\sqrt{\cos^2 u} = \cos u \qquad \text{when } -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$$

- So our integral now becomes

$$\begin{aligned}\int \frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x &= \int \frac{\sin^2 u}{\sqrt{\cos^2 u}} \cdot \cos u \mathrm{d}u \\ &= \int \frac{\sin^2 u}{\cos u} \cdot \cos u \mathrm{d}u \\ &= \int \sin^2 u \mathrm{d}u \\ &= \frac{1}{2}\int (1-\cos 2u)\mathrm{d}u \qquad \text{by Equation 1.8.4} \\ &= \frac{u}{2} - \frac{1}{4}\sin 2u + C \\ &= \frac{1}{2}\arcsin x - \frac{1}{4}\sin(2\arcsin x) + C\end{aligned}$$

- We can simplify this further using a double-angle identity. Recall that $u = \arcsin x$ and that $x = \sin u$. Then

$$\sin 2u = 2\sin u \cos u$$

We can replace $\cos u$ using $\cos^2 u = 1 - \sin^2 u$. Taking a square-root of this formula gives $\cos u = \pm\sqrt{1-\sin^2 u}$. We need the positive branch here since $\cos u \geq 0$ when $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$ (which is exactly the range of $\arcsin x$). Continuing along:

$$\begin{aligned}\sin 2u &= 2\sin u \cdot \sqrt{1-\sin^2 u} \\ &= 2x\sqrt{1-x^2}\end{aligned}$$

Thus our solution is

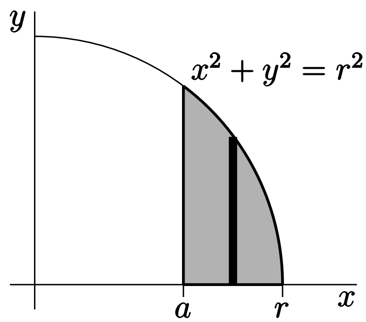
$$\begin{aligned}\int \frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x &= \frac{1}{2}\arcsin x - \frac{1}{4}\sin(2\arcsin x) + C \\ &= \frac{1}{2}\arcsin x - \frac{1}{2}x\sqrt{1-x^2} + C\end{aligned}$$

Example 1.9.2

The above two example illustrate the main steps of the approach. The next example is similar, but with more complicated limits of integration.

Example 1.9.3 $\int_a^r \sqrt{r^2 - x^2}\mathrm{d}x$.

Let's find the area of the shaded region in the sketch below.



We'll set up the integral using vertical strips. The strip in the figure has width $\mathrm{d}x$ and height $\sqrt{r^2 - x^2}$. So the area is given by the integral

$$\text{area} = \int_a^r \sqrt{r^2 - x^2}\mathrm{d}x$$

Which is very similar to the previous example.

Solution:

- To evaluate the integral we substitute

$$x = x(u) = r \sin u \qquad \mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u = r \cos u \mathrm{d}u$$

  It is also helpful to write u as a function of x — namely $u = \arcsin \frac{x}{r}$.

- The integral runs from $x = a$ to $x = r$. These correspond to

$$u(r) = \arcsin \frac{r}{r} = \arcsin 1 = \frac{\pi}{2}$$
$$u(a) = \arcsin \frac{a}{r} \quad \text{which does not simplify further}$$

- The integral then becomes

$$\begin{aligned}\int_a^r \sqrt{r^2 - x^2}\mathrm{d}x &= \int_{\arcsin(a/r)}^{\frac{\pi}{2}} \sqrt{r^2 - r^2 \sin^2 u} \cdot r \cos u \mathrm{d}u \\ &= \int_{\arcsin(a/r)}^{\frac{\pi}{2}} r^2 \sqrt{1 - \sin^2 u} \cdot \cos u \mathrm{d}u \\ &= r^2 \int_{\arcsin(a/r)}^{\frac{\pi}{2}} \sqrt{\cos^2 u} \cdot \cos u \mathrm{d}u\end{aligned}$$

  To proceed further (as we did in Examples 1.9.1 and 1.9.2) we need to think about whether $\cos u$ is positive or negative.

- Since a (as shown in the diagram) satisfies $0 \leq a \leq r$, we know that $u(a)$ lies between $\arcsin(0) = 0$ and $\arcsin(1) = \frac{\pi}{2}$. Hence the variable u lies between 0 and $\frac{\pi}{2}$, and on this range $\cos u \geq 0$. This allows us get rid of the square-root:

$$\sqrt{\cos^2 u} = |\cos u| = \cos u$$

- Putting this fact into our integral we get

$$\begin{aligned} \int_a^r \sqrt{r^2 - x^2}\mathrm{d}x &= r^2 \int_{\arcsin(a/r)}^{\frac{\pi}{2}} \sqrt{\cos^2 u} \cdot \cos u \mathrm{d}u \\ &= r^2 \int_{\arcsin(a/r)}^{\frac{\pi}{2}} \cos^2 u \mathrm{d}u \end{aligned}$$

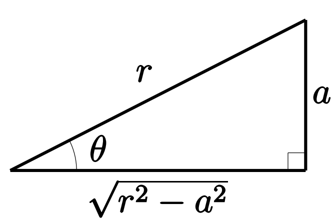
Recall the identity $\cos^2 u = \frac{1+\cos 2u}{2}$ from Section 1.8

$$\begin{aligned} &= \frac{r^2}{2} \int_{\arcsin(a/r)}^{\frac{\pi}{2}} (1 + \cos 2u)\mathrm{d}u \\ &= \frac{r^2}{2} \left[u + \frac{1}{2}\sin(2u) \right]_{\arcsin(a/r)}^{\frac{\pi}{2}} \\ &= \frac{r^2}{2} \left(\frac{\pi}{2} + \frac{1}{2}\sin\pi - \arcsin(a/r) - \frac{1}{2}\sin(2\arcsin(a/r)) \right) \\ &= \frac{r^2}{2} \left(\frac{\pi}{2} - \arcsin(a/r) - \frac{1}{2}\sin(2\arcsin(a/r)) \right) \end{aligned}$$

Oof! But there is a little further to go before we are done.

- We can again simplify the term $\sin(2\arcsin(a/r))$ using a double angle identity. Set $\theta = \arcsin(a/r)$. Then θ is the angle in the triangle on the right below. By the double angle formula for $\sin(2\theta)$ (Equation 1.8.2)

$$\begin{aligned} \sin(2\theta) &= 2\sin\theta \ \cos\theta \\ &= 2 \ \frac{a}{r} \ \frac{\sqrt{r^2 - a^2}}{r}. \end{aligned}$$



- So finally the area is

$$\text{area} = \int_a^r \sqrt{r^2 - x^2}\mathrm{d}x$$

$$= \frac{r^2}{2}\left(\frac{\pi}{2} - \arcsin(a/r) - \frac{1}{2}\sin(2\arcsin(a/r))\right)$$
$$= \frac{\pi r^2}{4} - \frac{r^2}{2}\arcsin(a/r) - \frac{a}{2}\sqrt{r^2-a^2}$$

- This is a relatively complicated formula, but we can make some "reasonableness" checks, by looking at special values of a.
  - If $a = 0$ the shaded region, in the figure at the beginning of this example, is exactly one quarter of a disk of radius r and so has area $\frac{1}{4}\pi r^2$. Substituting $a = 0$ into our answer does indeed give $\frac{1}{4}\pi r^2$.
  - At the other extreme, if $a = r$, the shaded region disappears completely and so has area 0. Subbing $a = r$ into our answer does indeed give 0, since $\arcsin 1 = \frac{\pi}{2}$.

Example 1.9.3

Example 1.9.4 $\int_a^r x\sqrt{r^2-x^2}\mathrm{d}x$.

The integral $\int_a^r x\sqrt{r^2-x^2}\mathrm{d}x$ looks a lot like the integral we just did in the previous 3 examples. It can also be evaluated using the trigonometric substitution $x = r\sin u$ — but that is unnecessarily complicated. Just because you have now learned how to use trigonometric substitution [a] doesn't mean that you should forget everything you learned before.

Solution: This integral is *much* more easily evaluated using the simple substitution $u = r^2 - x^2$.

- Set $u = r^2 - x^2$. Then $\mathrm{d}u = -2x\mathrm{d}x$, and so

$$\int_a^r x\sqrt{r^2-x^2}\mathrm{d}x = \int_{r^2-a^2}^0 \sqrt{u}\,\frac{\mathrm{d}u}{-2}$$
$$= -\frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_{r^2-a^2}^0$$
$$= \frac{1}{3}\left[r^2-a^2\right]^{3/2}$$

a To paraphrase the Law of the Instrument, possibly Mark Twain and definitely some psychologists, when you have a shiny new hammer, everything looks like a nail.

Enough sines and cosines — let us try a tangent substitution.

Example 1.9.5 $\int \frac{\mathrm{d}x}{x^2\sqrt{9+x^2}}$.

Solution: As per our guidelines at the start of this section, the presence of the square root term $\sqrt{3^2+x^2}$ tells us to substitute $x = 3\tan u$.

- Substitute

$$x = 3\tan u \qquad\qquad \mathrm{d}x = 3\sec^2 u\mathrm{d}u$$

  This allows us to remove the square root:

$$\sqrt{9+x^2} = \sqrt{9+9\tan^2 u} = 3\sqrt{1+\tan^2 u} = 3\sqrt{\sec^2 u} = 3|\sec u|$$

- Hence our integral becomes

$$\int \frac{\mathrm{d}x}{x^2\sqrt{9+x^2}} = \int \frac{3\sec^2 u}{9\tan^2 u \cdot 3|\sec u|}\mathrm{d}u$$

- To remove the absolute value we must consider the range of values of u in the integral. Since $x = 3\tan u$ we have $u = \arctan(x/3)$. The range [a] of arctangent is $-\frac{\pi}{2} \leq \arctan \leq \frac{\pi}{2}$ and so $u = \arctan(x/3)$ will always like between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$. Hence $\cos u$ will always be positive, which in turn implies that $|\sec u| = \sec u$.

- Using this fact our integral becomes:

$$\begin{aligned}\int \frac{\mathrm{d}x}{x^2\sqrt{9+x^2}} &= \int \frac{3\sec^2 u}{27\tan^2 u|\sec u|}\mathrm{d}u \\ &= \frac{1}{9}\int \frac{\sec u}{\tan^2 u}\mathrm{d}u && \text{since } \sec u > 0\end{aligned}$$

- Rewrite this in terms of sine and cosine

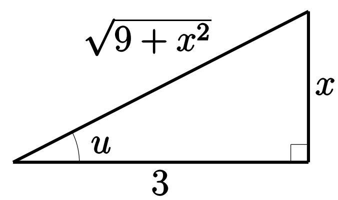
$$\begin{aligned}\int \frac{\mathrm{d}x}{x^2\sqrt{9+x^2}} &= \frac{1}{9}\int \frac{\sec u}{\tan^2 u}\mathrm{d}u \\ &= \frac{1}{9}\int \frac{1}{\cos u}\cdot\frac{\cos^2 u}{\sin^2 u}\mathrm{d}u = \frac{1}{9}\int \frac{\cos u}{\sin^2 u}\mathrm{d}u\end{aligned}$$

  Now we can use the substitution rule with $y = \sin u$ and $\mathrm{d}y = \cos u\mathrm{d}u$

$$\begin{aligned}&= \frac{1}{9}\int \frac{\mathrm{d}y}{y^2} \\ &= -\frac{1}{9y} + C \\ &= -\frac{1}{9\sin u} + C\end{aligned}$$

- The original integral was a function of x, so we still have to rewrite $\sin u$ in terms of x. Remember that $x = 3\tan u$ or $u = \arctan(x/3)$. So u is the angle shown in the triangle below and we can read off the triangle that

$$\sin u = \frac{x}{\sqrt{9+x^2}}$$
$$\implies \int \frac{\mathrm{d}x}{x^2\sqrt{9+x^2}} = -\frac{\sqrt{9+x^2}}{9x} + C$$



a To be pedantic, we mean the range of the "standard" arctangent function or its "principle value". One can define other arctangent functions with different ranges.

Example 1.9.5

Example 1.9.6 $\int \frac{x^2}{\sqrt{x^2-1}}\mathrm{d}x$.

Solution: This one requires a secant substitution, but otherwise is very similar to those above.

- Set $x = \sec u$ and $\mathrm{d}x = \sec u \tan u \mathrm{d}u$. Then

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2-1}}\mathrm{d}x &= \int \frac{\sec^2 u}{\sqrt{\sec^2 u - 1}} \sec u \tan u \mathrm{d}u \\ &= \int \sec^3 u \cdot \frac{\tan u}{\sqrt{\tan^2 u}}\mathrm{d}u \qquad \text{since } \tan^2 u = \sec^2 u - 1 \\ &= \int \sec^3 u \cdot \frac{\tan u}{|\tan u|}\mathrm{d}u\end{aligned}$$

- As before we need to consider the range of u values in order to determine the sign of $\tan u$. Notice that the integrand is only defined when either $x < -1$ or $x > 1$; thus we should treat the cases $x < -1$ and $x > 1$ separately. Let us assume that $x > 1$ and we will come back to the case $x < -1$ at the end of the example.

  When $x > 1$, our $u = \operatorname{arcsec} x$ takes values in $(0, \frac{\pi}{2})$. This follows since when $0 < u < \frac{\pi}{2}$, we have $0 < \cos u < 1$ and so $\sec u > 1$. Further, when $0 < u < \frac{\pi}{2}$, we have $\tan u > 0$. Thus $|\tan u| = \tan u$.

- Back to our integral, when $x > 1$:

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x^2-1}}\mathrm{d}x &= \int \sec^3 u \cdot \frac{\tan u}{|\tan u|}\mathrm{d}u \\
&= \int \sec^3 u \mathrm{d}u && \text{since } \tan u \geq 0
\end{aligned}$$

This is exactly Example 1.8.22

$$= \frac{1}{2}\sec u \tan u + \frac{1}{2}\log|\sec u + \tan u| + C$$

- Since we started with a function of x we need to finish with one. We know that $\sec u = x$ and then we can use trig identities

$$\tan^2 u = \sec^2 u - 1 = x^2 - 1 \qquad \text{so } \tan u = \pm\sqrt{x^2-1}$$

but we know

$$\tan u \geq 0 \qquad \text{so } \tan u = \sqrt{x^2-1}$$

Thus

$$\int \frac{x^2}{\sqrt{x^2-1}}\mathrm{d}x = \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\log|x+\sqrt{x^2-1}| + C$$

- The above holds when $x > 1$. We can confirm that it is also true when $x < -1$ by showing the right-hand side is a valid antiderivative of the integrand. To do so we must differentiate our answer. Notice that we do not need to consider the sign of $x + \sqrt{x^2-1}$ when we differentiate since we have already seen that

$$\frac{\mathrm{d}}{\mathrm{d}x}\log|x| = \frac{1}{x}$$

when either $x < 0$ or $x > 0$. So the following computation applies to both $x > 1$ and $x < -1$. The expressions become quite long so we differentiate each term separately:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[x\sqrt{x^2-1}\right] &= \left[\sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}}\right] \\
&= \frac{1}{\sqrt{x^2-1}}\left[(x^2-1)+x^2\right] \\
\frac{\mathrm{d}}{\mathrm{d}x}\log\left|x+\sqrt{x^2-1}\right| &= \frac{1}{x+\sqrt{x^2-1}} \cdot \left[1 + \frac{x}{\sqrt{x^2-1}}\right] \\
&= \frac{1}{x+\sqrt{x^2-1}} \cdot \frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}
\end{aligned}$$

$$= \frac{1}{\sqrt{x^2-1}}$$

Putting things together then gives us

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}x\sqrt{x^2-1}+\frac{1}{2}\log|x+\sqrt{x^2-1}|+C\right] \\
&= \frac{1}{2\sqrt{x^2-1}}\left[(x^2-1)+x^2+1\right]+0 \\
&= \frac{x^2}{\sqrt{x^2-1}}
\end{aligned}$$

This tells us that our answer for $x > 1$ is also valid when $x < -1$ and so

$$\int \frac{x^2}{\sqrt{x^2-1}}\mathrm{d}x = \frac{1}{2}x\sqrt{x^2-1}+\frac{1}{2}\log|x+\sqrt{x^2-1}|+C$$

when $x < -1$ and when $x > 1$.

In this example, we were lucky. The answer that we derived for $x > 1$ happened to be also valid for $x < -1$. This does not always happen with the $x = a\sec u$ substitution. When it doesn't, we have to apply separate $x > a$ and $x < -a$ analyses that are very similar to our $x > 1$ analysis above. Of course that doubles the tedium. So in the CLP-2 problem book, we will not pose questions that require separate $x > a$ and $x < -a$ computations.

Example 1.9.6

The method, as we have demonstrated it above, works when our integrand contains the square root of very specific families of quadratic polynomials. In fact, the same method works for more general quadratic polynomials — all we need to do is complete the square [1].

Example 1.9.7 $\int_3^5 \frac{\sqrt{x^2-2x-3}}{x-1}\mathrm{d}x$.

This time we have an integral with a square root in the integrand, but the argument of the square root, while a quadratic function of x, is not in one of the standard forms $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$. The reason that it is not in one of those forms is that the argument, x^2-2x-3, contains a term , namely $-2x$ that is of degree one in x. So we try to manipulate it into one of the standard forms by completing the square.

Solution:

- We first rewrite the quadratic polynomial x^2-2x-3 in the form $(x-a)^2+b$ for

1 If you have not heard of "completing the square" don't worry. It is not a difficult method and it will only take you a few moments to learn. It refers to rewriting a quadratic polynomial $P(x) = ax^2+bx+c$ as $P(x) = a(x+d)^2+e$ for new constants d, e.

some constants a, b. The easiest way to do this is to expand both expressions and compare coefficients of x:

$$x^2 - 2x - 3 = (x-a)^2 + b = (x^2 - 2ax + a^2) + b$$

So — if we choose $-2a = -2$ (so the coefficients of x^1 match) and $a^2 + b = -3$ (so the coefficients of x^0 match), then both expressions are equal. Hence we set $a = 1$ and $b = -4$. That is

$$x^2 - 2x - 3 = (x-1)^2 - 4$$

Many of you may have seen this method when learning to sketch parabolas.

- Once this is done we can convert the square root of the integrand into a standard form by making the simple substitution $y = x - 1$. Here goes

$$\begin{aligned}
&\int_3^5 \frac{\sqrt{x^2-2x-3}}{x-1}\mathrm{d}x \\
&= \int_3^5 \frac{\sqrt{(x-1)^2-4}}{x-1}\mathrm{d}x \\
&= \int_2^4 \frac{\sqrt{y^2-4}}{y}\mathrm{d}y && \text{with } y = x-1, \mathrm{d}y = \mathrm{d}x \\
&= \int_0^{\pi/3} \frac{\sqrt{4\sec^2 u - 4}}{2\sec u}\, 2\sec u \tan u \mathrm{d}u && \text{with } y = 2\sec u \\
& && \text{and } \mathrm{d}y = 2\sec u \tan u \mathrm{d}u
\end{aligned}$$

Notice that we could also do this in fewer steps by setting $(x-1) = 2\sec u, \mathrm{d}x = 2\sec u \tan u \mathrm{d}u$.

- To get the limits of integration we used that
    - the value of u that corresponds to $y = 2$ obeys $2 = y = 2\sec u = \frac{2}{\cos u}$ or $\cos u = 1$, so that $u = 0$ works and
    - the value of u that corresponds to $y = 4$ obeys $4 = y = 2\sec u = \frac{2}{\cos u}$ or $\cos u = \frac{1}{2}$, so that $u = \frac{\pi}{3}$ works.

- Now returning to the evaluation of the integral, we simplify and continue.

$$\begin{aligned}
\int_3^5 \frac{\sqrt{x^2-2x-3}}{x-1}\mathrm{d}x &= \int_0^{\pi/3} 2\sqrt{\sec^2 u - 1}\ \tan u \mathrm{d}u \\
&= 2\int_0^{\pi/3} \tan^2 u \mathrm{d}u && \text{since } \sec^2 u = 1 + \tan^2 u
\end{aligned}$$

In taking the square root of $\sec^2 u - 1 = \tan^2 u$ we used that $\tan u \geq 0$ on the range $0 \leq u \leq \frac{\pi}{3}$.

$$= 2\int_0^{\pi/3} \left[\sec^2 u - 1\right]\mathrm{d}u \qquad \text{since } \sec^2 u = 1 + \tan^2 u\text{, again}$$

$$= 2\Big[\tan u - u\Big]_0^{\pi/3}$$
$$= 2\big[\sqrt{3} - \frac{\pi}{3}\big]$$

Example 1.9.7

1.9.2 ➤ Exercises

Recall that we are using $\log x$ to denote the logarithm of x with base e. In other courses it is often denoted $\ln x$.

Exercises — Stage 1

1. $*$. For each of the following integrals, choose the substitution that is most beneficial for evaluating the integral.

a $\displaystyle\int \frac{2x^2}{\sqrt{9x^2-16}}\,\mathrm{d}x$

b $\displaystyle\int \frac{x^4-3}{\sqrt{1-4x^2}}\,\mathrm{d}x$

c $\displaystyle\int (25+x^2)^{-5/2}\,\mathrm{d}x$

2. For each of the following integrals, choose a trigonometric substitution that will eliminate the roots.

a $\displaystyle\int \frac{1}{\sqrt{x^2-4x+1}}\mathrm{d}x$

b $\displaystyle\int \frac{(x-1)^6}{(-x^2+2x+4)^{3/2}}\mathrm{d}x$

c $\displaystyle\int \frac{1}{\sqrt{4x^2+6x+10}}\mathrm{d}x$

d $\displaystyle\int \sqrt{x^2-x}\,\mathrm{d}x$

3. In each part of this question, assume θ is an angle in the interval $[0, \pi/2]$.

a If $\sin\theta = \dfrac{1}{20}$, what is $\cos\theta$?

b If $\tan\theta = 7$, what is $\csc\theta$?

c If $\sec\theta = \dfrac{\sqrt{x-1}}{2}$, what is $\tan\theta$?

4. Simplify the following expressions.

a $\sin\left(\arccos\left(\frac{x}{2}\right)\right)$

b $\sin\left(\arctan\left(\frac{1}{\sqrt{3}}\right)\right)$

c $\sec\left(\arcsin\left(\sqrt{x}\right)\right)$

Exercises — Stage 2

5. $*$. Evaluate $\displaystyle\int \frac{1}{(x^2+4)^{3/2}}\,\mathrm{d}x$.

6. $*$. Evaluate $\displaystyle\int_0^4 \frac{1}{(4+x^2)^{3/2}}\,\mathrm{d}x$. Your answer may not contain inverse trigonometric functions.

7. $*$. Evaluate $\displaystyle\int_0^{5/2} \frac{\mathrm{d}x}{\sqrt{25-x^2}}$.

8. $*$. Evaluate $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2+25}}$. You may use that $\displaystyle\int \sec x\;\mathrm{d}x = \log\left|\sec x + \tan x\right| + C$.

9. Evaluate $\displaystyle\int \frac{x+1}{\sqrt{2x^2+4x}}\,\mathrm{d}x$.

10. $*$. Evaluate $\displaystyle\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+16}}$.

11. $*$. Evaluate $\displaystyle\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-9}}$ for $x \geq 3$. Do not include any inverse trigonometric functions in your answer.

12. $*$. (a) Show that $\displaystyle\int_0^{\pi/4} \cos^4\theta\,\mathrm{d}\theta = (8+3\pi)/32$.

(b) Evaluate $\displaystyle\int_{-1}^{1} \frac{\mathrm{d}x}{(x^2+1)^3}$.

13. Evaluate $\displaystyle\int_{-\pi/12}^{\pi/12} \frac{15x^3}{(x^2+1)(9-x^2)^{5/2}}\mathrm{d}x$.

14. $*$. Evaluate $\displaystyle\int \sqrt{4-x^2}\,\mathrm{d}x$.

15. $*$. Evaluate $\displaystyle\int \frac{\sqrt{25x^2-4}}{x}\,\mathrm{d}x$ for $x > \frac{2}{5}$.

16. Evaluate $\displaystyle\int_{\sqrt{10}}^{\sqrt{17}} \frac{x^3}{\sqrt{x^2-1}}\,\mathrm{d}x$.

17. $*$. Evaluate $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{3-2x-x^2}}$.

18. Evaluate $\displaystyle\int \frac{1}{(2x-3)^3\sqrt{4x^2-12x+8}}\mathrm{d}x$ for $x > 2$.

19. Evaluate $\displaystyle\int_0^1 \frac{x^2}{(x^2+1)^{3/2}}\mathrm{d}x$.
You may use that $\int \sec x\mathrm{d}x = \log|\sec x + \tan x| + C$.

20. Evaluate $\displaystyle\int \frac{1}{(x^2+1)^2}\mathrm{d}x$.

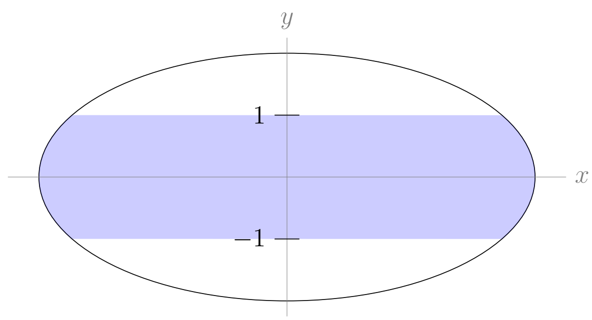
Exercises — Stage 3

21. Evaluate $\displaystyle\int \frac{x^2}{\sqrt{x^2-2x+2}}\mathrm{d}x$.
You may assume without proof that $\displaystyle\int \sec^3\theta\mathrm{d}\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\log|\sec\theta + \tan\theta| + C$.

22. Evaluate $\displaystyle\int \frac{1}{\sqrt{3x^2+5x}}\mathrm{d}x$.
You may use that $\int \sec x\mathrm{d}x = \log|\sec x + \tan x| + C$.

23. Evaluate $\displaystyle\int \frac{(1+x^2)^{3/2}}{x}\mathrm{d}x$. You may use the fact that $\displaystyle\int \csc\theta\mathrm{d}\theta = \log|\cot\theta - \csc\theta| + C$.

24. Below is the graph of the ellipse $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$. Find the area of the shaded region using the ideas from this section.



25. Let $f(x) = \dfrac{|x|}{\sqrt[4]{1-x^2}}$, and let R be the region between $f(x)$ and the x-axis over the interval $[-\frac{1}{2}, \frac{1}{2}]$.

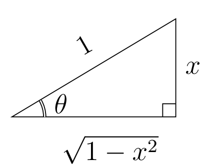
a Find the area of R.

b Find the volume of the solid formed by rotating R about the x-axis.

26. Evaluate $\displaystyle\int \sqrt{1+e^x}\,\mathrm{d}x$. You may use the antiderivative $\displaystyle\int \csc\theta\,\mathrm{d}\theta = \log|\cot\theta - \csc\theta| + C$.

27. Consider the following work.

$$\begin{aligned}
\int \frac{1}{1-x^2}\mathrm{d}x &= \int \frac{1}{1-\sin^2\theta}\cos\theta\mathrm{d}\theta && \text{using } x=\sin\theta, \mathrm{d}x = \cos\theta\mathrm{d}\theta\\
&= \int \frac{\cos\theta}{\cos^2\theta}\mathrm{d}\theta\\
&= \int \sec\theta\mathrm{d}\theta\\
&= \log|\sec\theta+\tan\theta| + C && \text{Example 1.8.19}\\
&= \log\left|\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right| + C && \text{using the triangle below}\\
&= \log\left|\frac{1+x}{\sqrt{1-x^2}}\right| + C
\end{aligned}$$



a Differentiate $\log\left|\dfrac{1+x}{\sqrt{1-x^2}}\right|$.

b True or false: $\displaystyle\int_2^3 \frac{1}{1-x^2}\mathrm{d}x = \left[\log\left|\frac{1+x}{\sqrt{1-x^2}}\right|\right]_{x=2}^{x=3}$

c Was the work in the question correct? Explain.

28.

a Suppose we are evaluating an integral that contains the term $\sqrt{a^2-x^2}$, where a is a positive constant, and we use the substitution $x = a\sin u$ (with inverse $u = \arcsin(x/a)$), so that

$$\sqrt{a^2-x^2} = \sqrt{a^2\cos^2 u} = |a\cos u|$$

Under what circumstances is $|a\cos u| \neq a\cos u$?

b Suppose we are evaluating an integral that contains the term $\sqrt{a^2+x^2}$, where a is a positive constant, and we use the substitution $x = a\tan u$ (with inverse $u = \arctan(x/a)$), so that

$$\sqrt{a^2+x^2} = \sqrt{a^2\sec^2 u} = |a\sec u|$$

Under what circumstances is $|a\sec u| \neq a\sec u$?

c Suppose we are evaluating an integral that contains the term $\sqrt{x^2-a^2}$, where a is a positive constant, and we use the substitution $x = a\sec u$ (with inverse $u = \operatorname{arcsec}(x/a) = \arccos(a/x)$), so that

$$\sqrt{x^2-a^2} = \sqrt{a^2\tan^2 u} = |a\tan u|$$

Under what circumstances is $|a\tan u| \neq a\tan u$?

1.10▲ Partial Fractions

Partial fractions is the name given to a technique of integration that may be used to integrate any rational function [1]. We already know how to integrate some simple rational functions

$$\int \frac{1}{x}\mathrm{d}x = \log|x| + C \qquad \int \frac{1}{1+x^2}\mathrm{d}x = \arctan(x) + C$$

Combining these with the substitution rule, we can integrate similar but more complicated rational functions:

$$\int \frac{1}{2x+3}\mathrm{d}x = \frac{1}{2}\log|2x+3| + C \qquad \int \frac{1}{3+4x^2}\mathrm{d}x = \frac{1}{2\sqrt{3}}\arctan\left(\frac{2x}{\sqrt{3}}\right) + C$$

1 Recall that a rational function is the ratio of two polynomials.

By summing such terms together we can integrate yet more complicated forms

$$\int \left[x + \frac{1}{x+1} + \frac{1}{x-1} \right] \mathrm{d}x = \frac{x^2}{2} + \log|x+1| + \log|x-1| + C$$

However we are not (typically) presented with a rational function nicely decomposed into neat little pieces. It is far more likely that the rational function will be written as the ratio of two polynomials. For example:

$$\int \frac{x^3+x}{x^2-1} \mathrm{d}x$$

In this specific example it is not hard to confirm that

$$x + \frac{1}{x+1} + \frac{1}{x-1} = \frac{x(x+1)(x-1) + (x-1) + (x+1)}{(x+1)(x-1)} = \frac{x^3+x}{x^2-1}$$

and hence

$$\begin{aligned} \int \frac{x^3+x}{x^2-1} \mathrm{d}x &= \int \left[x + \frac{1}{x+1} + \frac{1}{x-1} \right] \mathrm{d}x \\ &= \frac{x^2}{2} + \log|x+1| + \log|x-1| + C \end{aligned}$$

Of course going in this direction (from a sum of terms to a single rational function) is straightforward. To be useful we need to understand how to do this in reverse: decompose a given rational function into a sum of simpler pieces that we can integrate.

Suppose that $N(x)$ and $D(x)$ are polynomials. The basic strategy is to write $\frac{N(x)}{D(x)}$ as a sum of very simple, easy to integrate rational functions, namely

1 polynomials — we shall see below that these are needed when the degree [2] of $N(x)$ is equal to or strictly bigger than the degree of $D(x)$, and

2 rational functions of the particularly simple form $\frac{A}{(ax+b)^n}$ and

3 rational functions of the form $\frac{Ax+B}{(ax^2+bx+c)^m}$.

We already know how to integrate the first two forms, and we'll see how to integrate the third form in the near future.

To begin to explore this method of decomposition, let us go back to the example we just saw

$$x + \frac{1}{x+1} + \frac{1}{x-1} = \frac{x(x+1)(x-1) + (x-1) + (x+1)}{(x+1)(x-1)} = \frac{x^3+x}{x^2-1}$$

The technique that we will use is based on two observations:

1 The denominators on the left-hand side are the factors of the denominator $x^2-1 = (x-1)(x+1)$ on the right-hand side.

2 The degree of a polynomial is the largest power of x. For example, the degree of $2x^3+4x^2+6x+8$ is three.

2 Use $P(x)$ to denote the polynomial on the left hand side, and then use $N(x)$ and $D(x)$ to denote the numerator and denominator of the right hand side. That is

$$P(x) = x \qquad N(x) = x^3 + x \qquad D(x) = x^2 - 1.$$

Then the degree of $N(x)$ is the sum of the degrees of $P(x)$ and $D(x)$. This is because the highest degree term in $N(x)$ is x^3, which comes from multiplying $P(x)$ by $D(x)$, as we see in

$$x + \frac{1}{x+1} + \frac{1}{x-1} = \frac{\overbrace{x}^{P(x)}\overbrace{(x+1)(x-1)}^{D(x)} + (x-1) + (x+1)}{(x+1)(x-1)} = \frac{x^3+x}{x^2-1}$$

More generally, the presence of a polynomial on the left hand side is signalled on the right hand side by the fact that the degree of the numerator is at least as large as the degree of the denominator.

1.10.1 ▸▸ Partial fraction decomposition examples

Rather than writing up the technique — known as the partial fraction decomposition — in full generality, we will instead illustrate it through a sequence of examples.

Example 1.10.1 $\int \frac{x-3}{x^2-3x+2}\mathrm{d}x$.

In this example, we integrate $\frac{N(x)}{D(x)} = \frac{x-3}{x^2-3x+2}$.
Solution:

- *Step 1.* We first check to see if a polynomial $P(x)$ is needed. To do so, we check to see if the degree of the numerator, $N(x)$, is strictly smaller than the degree of the denominator $D(x)$. In this example, the numerator, $x-3$, has degree one and that is indeed strictly smaller than the degree of the denominator, $x^2 - 3x + 2$, which is two. In this case [a] we do not need to extract a polynomial $P(x)$ and we move on to step 2.

- *Step 2.* The second step is to factor the denominator

$$x^2 - 3x + 2 = (x-1)(x-2)$$

  In this example it is quite easy, but in future examples (and quite possibly in your homework, quizzes and exam) you will have to work harder to factor the denominator. In Appendix A.16 we have written up some simple tricks for factoring polynomials. We will illustrate them in Example 1.10.3 below.

- *Step 3.* The third step is to write $\frac{x-3}{x^2-3x+2}$ in the form

$$\frac{x-3}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2}$$

for some constants A and B. More generally, if the denominator consists of n different linear factors, then we decompose the ratio as

$$\text{rational function} = \frac{A_1}{\text{linear factor 1}} + \frac{A_2}{\text{linear factor 2}} + \cdots + \frac{A_n}{\text{linear factor n}}$$

To proceed we need to determine the values of the constants A, B and there are several different methods to do so. Here are two methods

- *Step 3 — Algebra Method.* This approach has the benefit of being conceptually clearer and easier, but the downside is that it is more tedious.

  To determine the values of the constants A, B, we put [b] the right-hand side back over the common denominator $(x-1)(x-2)$.

  $$\frac{x-3}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

  The fraction on the far left is the same as the fraction on the far right if and only if their numerators are the same.

  $$x-3 = A(x-2)+B(x-1)$$

  Write the right hand side as a polynomial in standard form (i.e. collect up all x terms and all constant terms)

  $$x-3 = (A+B)x + (-2A-B)$$

  For these two polynomials to be the same, the coefficient of x on the left hand side and the coefficient of x on the right hand side must be the same. Similarly the coefficients of x^0 (i.e. the constant terms) must match. This gives us a system of two equations.

  $$A+B=1 \qquad\qquad -2A-B=-3$$

  in the two unknowns A, B. We can solve this system by

  - using the first equation, namely $A+B=1$, to determine A in terms of B:

    $$A = 1-B$$

  - Substituting this into the remaining equation eliminates the A from second equation, leaving one equation in the one unknown B, which can then be solved for B:

    $$\begin{aligned} -2A-B &= -3 && \text{substitute } A=1-B \\ -2(1-B)-B &= -3 && \text{clean up} \\ -2+B &= -3 && \text{so } B=-1 \end{aligned}$$

- Once we know B, we can substitute it back into $A = 1 - B$ to get A.

$$A = 1 - B = 1 - (-1) = 2$$

Hence

$$\frac{x-3}{x^2-3x+2} = \frac{2}{x-1} - \frac{1}{x-2}$$

- *Step 3 — Sneaky Method.* This takes a little more work to understand, but it is more efficient than the algebra method.

  We wish to find A and B for which

$$\frac{x-3}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

  Note that the denominator on the left hand side has been written in factored form.

  - To determine A, we multiply both sides of the equation by A's denominator, which is $x-1$,

$$\frac{x-3}{x-2} = A + \frac{(x-1)B}{x-2}$$

    and then we completely eliminate B from the equation by evaluating at $x = 1$. This value of x is chosen to make $x - 1 = 0$.

$$\left.\frac{x-3}{x-2}\right|_{x=1} = A + \left.\frac{(x-1)B}{x-2}\right|_{x=1} \implies A = \frac{1-3}{1-2} = 2$$

  - To determine B, we multiply both sides of the equation by B's denominator, which is $x-2$,

$$\frac{x-3}{x-1} = \frac{(x-2)A}{x-1} + B$$

    and then we completely eliminate A from the equation by evaluating at $x = 2$. This value of x is chosen to make $x - 2 = 0$.

$$\left.\frac{x-3}{x-1}\right|_{x=2} = \left.\frac{(x-2)A}{x-1}\right|_{x=2} + B \implies B = \frac{2-3}{2-1} = -1$$

  Hence we have (the thankfully consistent answer)

$$\frac{x-3}{x^2-3x+2} = \frac{2}{x-1} - \frac{1}{x-2}$$

  Notice that no matter which method we use to find the constants we can easily check our answer by summing the terms back together:

$$\begin{aligned}\frac{2}{x-1} - \frac{1}{x-2} &= \frac{2(x-2)-(x-1)}{(x-2)(x-1)} \\ &= \frac{2x-4-x+1}{x^2-3x+2} = \frac{x-3}{x^2-3x+2} \checkmark\end{aligned}$$

- *Step 4.* The final step is to integrate.

$$\begin{aligned}\int \frac{x-3}{x^2-3x+2}\mathrm{d}x &= \int \frac{2}{x-1}\mathrm{d}x + \int \frac{-1}{x-2}\mathrm{d}x \\ &= 2\log|x-1| - \log|x-2| + C\end{aligned}$$

a We will soon get to an example (Example 1.10.2 in fact) in which the numerator degree is at least as large as the denominator degree — in that situation we have to extract a polynomial $P(x)$ before we can move on to step 2.

b That is, we take the decomposed form and sum it back together.

Example 1.10.1

Perhaps the first thing that you notice is that this process takes quite a few steps [3]. However no single step is all that complicated; it only takes practice. With that said, let's do another, slightly more complicated, one.

Example 1.10.2 $\int \frac{3x^3-8x^2+4x-1}{x^2-3x+2}\mathrm{d}x.$

In this example, we integrate $\frac{N(x)}{D(x)} = \frac{3x^3-8x^2+4x-1}{x^2-3x+2}$.

Solution:

- *Step 1.* We first check to see if the degree of the numerator $N(x)$ is strictly smaller than the degree of the denominator $D(x)$. In this example, the numerator, $3x^3-8x^2+4x-1$, has degree three and the denominator, x^2-3x+2, has degree two. As $3 \geq 2$, we have to implement the first step.

  The goal of the first step is to write $\frac{N(x)}{D(x)}$ in the form

$$\frac{N(x)}{D(x)} = P(x) + \frac{R(x)}{D(x)}$$

  with $P(x)$ being a polynomial and $R(x)$ being a polynomial of degree strictly smaller than the degree of $D(x)$. The right hand side is $\frac{P(x)D(x)+R(x)}{D(x)}$, so we have to express the numerator in the form $N(x) = P(x)D(x) + R(x)$, with $P(x)$ and $R(x)$ being polynomials and with the degree of R being strictly smaller than the degree of D. $P(x)D(x)$ is a sum of expressions of the form $ax^nD(x)$. We want to pull as many expressions of this form as possible out of the numerator $N(x)$, leaving only a low degree remainder $R(x)$.

  We do this using long division — the same long division you learned in school, but with the base 10 replaced by x.

3 Though, in fairness, we did step 3 twice — and that is the most tedious bit… Actually — sometimes factoring the denominator can be quite challenging. We'll consider this issue in more detail shortly.

- We start by observing that to get from the highest degree term in the denominator (x^2) to the highest degree term in the numerator ($3x^3$), we have to multiply it by $3x$. So we write,

$$\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & 3x & & & \\ \cline{2-5}
x^2-3x+2 & 3x^3 & -8x^2 & +4x & -1
\end{array}$$

  In the above expression, the denominator is on the left, the numerator is on the right and $3x$ is written above the highest order term of the numerator. Always put lower powers of x to the right of higher powers of x — this mirrors how you do long division with numbers; lower powers of ten sit to the right of higher powers of ten.

- Now we subtract $3x$ times the denominator, x^2-3x+2, which is $3x^3-9x^2+6x$, from the numerator.

$$\begin{array}{r|rrrrl}
\multicolumn{1}{r}{} & 3x & & & & \\ \cline{2-5}
x^2-3x+2 & 3x^3 & -8x^2 & +4x & -1 & \\
\multicolumn{1}{r}{} & 3x^3 & -9x^2 & +6x & & \longleftarrow 3x(x^2-3x+2) \\ \cline{2-5}
\multicolumn{1}{r}{} & & x^2 & -2x & -1 &
\end{array}$$

- This has left a remainder of x^2-2x-1. To get from the highest degree term in the denominator (x^2) to the highest degree term in the remainder (x^2), we have to multiply by 1. So we write,

$$\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & 3x & +\ 1 & & \\ \cline{2-5}
x^2-3x+2 & 3x^3 & -8x^2 & +4x & -1 \\
\multicolumn{1}{r}{} & 3x^3 & -9x^2 & +6x & \\ \cline{2-5}
\multicolumn{1}{r}{} & & x^2 & -2x & -1
\end{array}$$

- Now we subtract 1 times the denominator, x^2-3x+2, which is x^2-3x+2, from the remainder.

$$\begin{array}{r|rrrrl}
\multicolumn{1}{r}{} & 3x & +\ 1 & & & \\ \cline{2-5}
x^2-3x+2 & 3x^3 & -8x^2 & +4x & -1 & \\
\multicolumn{1}{r}{} & 3x^3 & -9x^2 & +6x & & \longleftarrow 3x(x^2-3x+2) \\ \cline{2-5}
\multicolumn{1}{r}{} & & x^2 & -2x & -1 & \\
\multicolumn{1}{r}{} & & x^2 & -3x & +2 & \longleftarrow 1\,(x^2-3x+2) \\ \cline{3-5}
\multicolumn{1}{r}{} & & & x & -3 &
\end{array}$$

- This leaves a remainder of $x-3$. Because the remainder has degree 1, which is smaller than the degree of the denominator (being degree 2), we stop.
- In this example, when we subtracted $3x(x^2-3x+2)$ and $1(x^2-3x+2)$ from $3x^3-8x^2+4x-1$ we ended up with $x-3$. That is,

$$3x^3-8x^2+4x-1\ -\ 3x(x^2-3x+2)\ -\ 1(x^2-3x+2)$$

$$= x - 3$$

or, collecting the two terms proportional to $(x^2 - 3x + 2)$

$$3x^3 - 8x^2 + 4x - 1 \;-\; (3x+1)(x^2 - 3x + 2) \;=\; x - 3$$

Moving the $(3x + 1)(x^2 - 3x + 2)$ to the right hand side and dividing the whole equation by $x^2 - 3x + 2$ gives

$$\frac{3x^3 - 8x^2 + 4x - 1}{x^2 - 3x + 2} \;=\; 3x + 1 \;+\; \frac{x - 3}{x^2 - 3x + 2}$$

And we can easily check this expression just by summing the two terms on the right-hand side.

We have written the integrand in the form $\frac{N(x)}{D(x)} = P(x) + \frac{R(x)}{D(x)}$, with the degree of $R(x)$ strictly smaller than the degree of $D(x)$, which is what we wanted. Observe that $R(x)$ is the final remainder of the long division procedure and $P(x)$ is at the top of the long division computation. This is the end of Step 1. Oof! You should definitely practice this step.

•

$$\begin{array}{rl|ll}
 & & 3x + 1 & \longleftarrow P(x) \\
\hline
D(x) \rightarrow x^2 - 3x + 2 & \big) & 3x^3 - 8x^2 + 4x - 1 & \longleftarrow N(x) \\
 & & 3x^3 - 9x^2 + 6x & \longleftarrow 3x \cdot D(x) \\
 & & x^2 - 2x - 1 & \longleftarrow N(x) - 3x \cdot D(x) \\
 & & x^2 - 3x + 2 & \longleftarrow 1 \cdot D(x) \\
 & & x - 3 & \longleftarrow R(x) = N(x) - (3x+1)D(x)
\end{array}$$

- *Step 2.* The second step is to factor the denominator

$$x^2 - 3x + 2 = (x - 1)(x - 2)$$

We already did this in Example 1.10.1.

- *Step 3.* The third step is to write $\frac{x-3}{x^2-3x+2}$ in the form

$$\frac{x - 3}{x^2 - 3x + 2} = \frac{A}{x - 1} + \frac{B}{x - 2}$$

for some constants A and B. We already did this in Example 1.10.1. We found $A = 2$ and $B = -1$.

- *Step 4.* The final step is to integrate.

$$\int \frac{3x^3 - 8x^2 + 4x - 1}{x^2 - 3x + 2}\mathrm{d}x$$

$$= \int \big[3x+1\big]\,\mathrm{d}x + \int \frac{2}{x-1}\,\mathrm{d}x + \int \frac{-1}{x-2}\,\mathrm{d}x$$
$$= \frac{3}{2}x^2 + x + 2\log|x-1| - \log|x-2| + C$$

You can see that the integration step is quite quick — almost all the work is in preparing the integrand.

Example 1.10.2

Here is a very solid example. It is quite long and the steps are involved. However please persist. No single step is too difficult.

Example 1.10.3 $\int \frac{x^4+5x^3+16x^2+26x+22}{x^3+3x^2+7x+5}\mathrm{d}x$.

In this example, we integrate $\frac{N(x)}{D(x)} = \frac{x^4+5x^3+16x^2+26x+22}{x^3+3x^2+7x+5}$.

Solution:

- *Step 1.* Again, we start by comparing the degrees of the numerator and denominator. In this example, the numerator, $x^4+5x^3+16x^2+26x+22$, has degree four and the denominator, x^3+3x^2+7x+5, has degree three. As $4 \geq 3$, we must execute the first step, which is to write $\frac{N(x)}{D(x)}$ in the form
$$\frac{N(x)}{D(x)} = P(x) + \frac{R(x)}{D(x)}$$
with $P(x)$ being a polynomial and $R(x)$ being a polynomial of degree strictly smaller than the degree of $D(x)$. This step is accomplished by long division, just as we did in Example 1.10.2. We'll go through the whole process in detail again.

  Actually — before you read on ahead, please have a go at the long division. It is good practice.

  - We start by observing that to get from the highest degree term in the denominator (x^3) to the highest degree term in the numerator (x^4), we have to multiply by x. So we write,

$$\begin{array}{r|l} & \quad x \\ x^3+3x^2+7x+5 & x^4+5x^3+16x^2+26x+22 \end{array}$$

  - Now we subtract x times the denominator x^3+3x^2+7x+5, which is $x^4+3x^3+7x^2+5x$, from the numerator.

$$\begin{array}{r|l l} & x & \\ x^3+3x^2+7x+5 & x^4+5x^3+16x^2+26x+22 & \\ & x^4+3x^3+\ \ 7x^2+\ \ 5x & \longleftarrow x(x^3+3x^2+7x+5) \\ \hline & \qquad 2x^3+\ \ 9x^2+21x+22 & \end{array}$$

- The remainder was $2x^3 + 9x^2 + 21x + 22$. To get from the highest degree term in the denominator (x^3) to the highest degree term in the remainder ($2x^3$), we have to multiply by 2. So we write,

$$\begin{array}{r|rrrrr} & & & & x+ & 2 \\ \cline{2-6} x^3+3x^2+7x+5 & x^4+ & 5x^3+ & 16x^2+ & 26x+ & 22 \\ & x^4+ & 3x^3+ & 7x^2+ & 5x & \\ \cline{2-6} & & 2x^3+ & 9x^2+ & 21x+ & 22 \end{array}$$

- Now we subtract 2 times the denominator $x^3 + 3x^2 + 7x + 5$, which is $2x^3 + 6x^2 + 14x + 10$, from the remainder.

$$\begin{array}{r|rrrrrl} & & & & x+ & 2 & \\ \cline{2-6} x^3+3x^2+7x+5 & x^4+ & 5x^3+ & 16x^2+ & 26x+ & 22 & \\ & x^4+ & 3x^3+ & 7x^2+ & 5x & & \longleftarrow x(x^3+3x^2+7x+5) \\ \cline{2-6} & & 2x^3+ & 9x^2+ & 21x+ & 22 & \\ & & 2x^3+ & 6x^2+ & 14x+ & 10 & \longleftarrow 2(x^3+3x^2+7x+5) \\ \cline{3-6} & & & 3x^2+ & 7x+ & 12 & \end{array}$$

- This leaves a remainder of $3x^2 + 7x + 12$. Because the remainder has degree 2, which is smaller than the degree of the denominator, which is 3, we stop.
- In this example, when we subtracted $x(x^3 + 3x^2 + 7x + 5)$ and $2(x^3 + 3x^2 + 7x + 5)$ from $x^4 + 5x^3 + 16x^2 + 26x + 22$ we ended up with $3x^2 + 7x + 12$. That is,

$$\begin{aligned} x^4 + 5x^3 + 16x^2 + 26x + 22 \ &- \ x(x^3 + 3x^2 + 7x + 5) \\ &- \ 2(x^3 + 3x^2 + 7x + 5) \\ = 3x^2 + 7x + 12 & \end{aligned}$$

or, collecting the two terms proportional to $(x^3 + 3x^2 + 7x + 5)$ we get

$$\begin{aligned} x^4 + 5x^3 + 16x^2 + 26x + 22 \ &- \ (x+2)(x^3 + 3x^2 + 7x + 5) \\ &= \ 3x^2 + 7x + 12 \end{aligned}$$

Moving the $(x + 2)(x^3 + 3x^2 + 7x + 5)$ to the right hand side and dividing the whole equation by $x^3 + 3x^2 + 7x + 5$ gives

$$\frac{x^4 + 5x^3 + 16x^2 + 26x + 22}{x^3 + 3x^2 + 7x + 5} = x + 2 + \frac{3x^2 + 7x + 12}{x^3 + 3x^2 + 7x + 5}$$

This is of the form $\frac{N(x)}{D(x)} = P(x) + \frac{R(x)}{D(x)}$, with the degree of $R(x)$ strictly smaller than the degree of $D(x)$, which is what we wanted. Observe, once again, that $R(x)$ is the final remainder of the long division procedure and $P(x)$ is at the top of the long division computation.

$$
\begin{array}{r|l}
 & x+2 \longleftarrow P(x) \\ \cline{2-2}
x^3+3x^2+7x+5 & x^4+5x^3+16x^2+26x+22 \\
 & x^4+3x^3+\ 7x^2+\ 5x \\ \cline{2-2}
 & \quad 2x^3+\ 9x^2+21x+22 \\
 & \quad 2x^3+\ 6x^2+14x+10 \\ \cline{2-2}
 & \qquad\quad 3x^2+\ 7x+12 \longleftarrow R(x)
\end{array}
$$

- *Step 2.* The second step is to factor the denominator $D(x) = x^3 + 3x^2 + 7x + 5$. In the "real world" factorisation of polynomials is often very hard. Fortunately [a], this is not the "real world" and there is a trick available to help us find this factorisation. The reader should take some time to look at Appendix A.16 before proceeding.
  - The trick exploits the fact that most polynomials that appear in homework assignments and on tests have integer coefficients and some integer roots. Any integer root of a polynomial that has integer coefficients, like $D(x) = x^3 + 3x^2 + 7x + 5$, must divide the constant term of the polynomial exactly. Why this is true is explained [b] in Appendix A.16.
  - So any integer root of $x^3 + 3x^2 + 7x + 5$ must divide 5 exactly. Thus the only integers which can be roots of $D(x)$ are ± 1 and ± 5. Of course, not all of these give roots of the polynomial — in fact there is no guarantee that any of them will be. We have to test each one.
  - To test if $+1$ is a root, we sub $x = 1$ into $D(x)$:

    $$D(1) = 1^3 + 3(1)^2 + 7(1) + 5 = 16$$

    As $D(1) \neq 0$, 1 is not a root of $D(x)$.
  - To test if -1 is a root, we sub it into $D(x)$:

    $$D(-1) = (-1)^3 + 3(-1)^2 + 7(-1) + 5 = -1 + 3 - 7 + 5 = 0$$

    As $D(-1) = 0$, -1 is a root of $D(x)$. As -1 is a root of $D(x)$, $\big(x - (-1)\big) = (x + 1)$ must factor $D(x)$ exactly. We can factor the $(x + 1)$ out of $D(x) = x^3 + 3x^2 + 7x + 5$ by long division once again.
  - Dividing $D(x)$ by $(x + 1)$ gives:

$$
\begin{array}{r|l}
 & x^2+2x+5 \\ \cline{2-2}
x+1 & x^3+3x^2+7x+5 \\
 & x^3+\ x^2 \qquad\qquad \longleftarrow x^2(x+1) \\ \cline{2-2}
 & \quad 2x^2+7x+5 \\
 & \quad 2x^2+2x \qquad \longleftarrow 2x(x+1) \\ \cline{2-2}
 & \qquad\quad 5x+5 \\
 & \qquad\quad 5x+5 \quad \longleftarrow 5(x+1) \\ \cline{2-2}
 & \qquad\qquad\quad 0
\end{array}
$$

This time, when we subtracted $x^2(x+1)$ and $2x(x+1)$ and $5(x+1)$ from x^3+3x^2+7x+5 we ended up with 0 — as we knew would happen, because we knew that $x+1$ divides x^3+3x^2+7x+5 exactly. Hence

$$x^3+3x^2+7x+5 \;-\; x^2(x+1) \;-\; 2x(x+1) \;-\; 5(x+1) \;=\; 0$$

or

$$x^3+3x^2+7x+5 \;=\; x^2(x+1) \;+\; 2x(x+1) \;+\; 5(x+1)$$

or

$$x^3+3x^2+7x+5 = (x^2+2x+5)(x+1)$$

- It isn't quite time to stop yet; we should attempt to factor the quadratic factor, x^2+2x+5. We can use the quadratic formula [c] to find the roots of x^2+2x+5:

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-2\pm\sqrt{4-20}}{2} = \frac{-2\pm\sqrt{-16}}{2}$$

  Since this expression contains the square root of a negative number the equation $x^2+2x+5=0$ has no real solutions; without the use of complex numbers, x^2+2x+5 cannot be factored.

We have reached the end of step 2. At this point we have

$$\frac{x^4+5x^3+16x^2+26x+22}{x^3+3x^2+7x+5} = x+2+\frac{3x^2+7x+12}{(x+1)(x^2+2x+5)}$$

- *Step 3.* The third step is to write $\frac{3x^2+7x+12}{(x+1)(x^2+2x+5)}$ in the form

$$\frac{3x^2+7x+12}{(x+1)(x^2+2x+5)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+2x+5}$$

for some constants A, B and C.

Note that the numerator, $Bx+C$ of the second term on the right hand side is not just a constant. It is of degree one, which is exactly one smaller than the degree of the denominator, x^2+2x+5. More generally, if the denominator consists of n different linear factors and m different quadratic factors, then we decompose the ratio as

$$\begin{aligned}\text{rational function} = &\frac{A_1}{\text{linear factor 1}}+\frac{A_2}{\text{linear factor 2}}+\cdots+\frac{A_n}{\text{linear factor n}}\\ &+\frac{B_1x+C_1}{\text{quadratic factor 1}}+\frac{B_2x+C_2}{\text{quadratic factor 2}}+\cdots\end{aligned}$$

$$+ \frac{B_m x + C_m}{\text{quadratic factor m}}$$

To determine the values of the constants A, B, C, we put the right hand side back over the common denominator $(x+1)(x^2+2x+5)$.

$$\begin{aligned}\frac{3x^2+7x+12}{(x+1)(x^2+2x+5)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+5} \\ &= \frac{A(x^2+2x+5)+(Bx+C)(x+1)}{(x+1)(x^2+2x+5)}\end{aligned}$$

The fraction on the far left is the same as the fraction on the far right if and only if their numerators are the same.

$$3x^2+7x+12 = A(x^2+2x+5)+(Bx+C)(x+1)$$

Again, as in Example 1.10.1, there are a couple of different ways to determine the values of A, B and C from this equation.

- *Step 3 — Algebra Method.* The conceptually clearest procedure is to write the right hand side as a polynomial in standard form (i.e. collect up all x^2 terms, all x terms and all constant terms)

  $$3x^2+7x+12 = (A+B)x^2 + (2A+B+C)x + (5A+C)$$

  For these two polynomials to be the same, the coefficient of x^2 on the left hand side and the coefficient of x^2 on the right hand side must be the same. Similarly the coefficients of x^1 must match and the coefficients of x^0 must match.

  This gives us a system of three equations

  $$A+B=3 \qquad 2A+B+C=7 \qquad 5A+C=12$$

  in the three unknowns A, B, C. We can solve this system by

  - using the first equation, namely $A+B=3$, to determine A in terms of B: $A=3-B$.
  - Substituting $A=3-B$ into the remaining two equations eliminates the A's from these two equations, leaving two equations in the two unknowns B and C.

    $$\begin{array}{llrr} & A=3-B & 2A+B+C=7 & 5A+C=12 \\ \Rightarrow & & 2(3-B)+B+C=7 & 5(3-B)+C=12 \\ \Rightarrow & & -B+C=1 & -5B+C=-3 \end{array}$$

  - Now we can use the equation $-B+C=1$, to determine B in terms of C: $B=C-1$.

- Substituting this into the remaining equation eliminates the B's leaving an equation in the one unknown C, which is easy to solve.

$$\begin{aligned} B = C - 1 \qquad & \qquad -5B + C = -3 \\ \Rightarrow \qquad & \qquad -5(C-1) + C = -3 \\ \Rightarrow \qquad & \qquad -4C = -8 \end{aligned}$$

- So $C = 2$, and then $B = C - 1 = 1$, and then $A = 3 - B = 2$. Hence

$$\frac{3x^2 + 7x + 12}{(x+1)(x^2+2x+5)} = \frac{2}{x+1} + \frac{x+2}{x^2+2x+5}$$

- *Step 3 — Sneaky Method.* While the above method is transparent, it is rather tedious. It is arguably better to use the second, sneakier and more efficient, procedure. In order for

$$3x^2 + 7x + 12 = A(x^2 + 2x + 5) + (Bx + C)(x + 1)$$

the equation must hold for all values of x.

- In particular, it must be true for $x = -1$. When $x = -1$, the factor $(x+1)$ multiplying $Bx+C$ is exactly zero. So B and C disappear from the equation, leaving us with an easy equation to solve for A:

$$\begin{aligned} 3x^2 + 7x + 12\Big|_{x=-1} &= \Big[A(x^2 + 2x + 5) + (Bx + C)(x + 1)\Big]_{x=-1} \\ &\Longrightarrow 8 = 4A \Longrightarrow A = 2 \end{aligned}$$

- Sub this value of A back in and simplify.

$$\begin{aligned} 3x^2 + 7x + 12 &= 2(x^2 + 2x + 5) + (Bx + C)(x + 1) \\ x^2 + 3x + 2 &= (Bx + C)(x + 1) \end{aligned}$$

Since $(x+1)$ is a factor on the right hand side, it must also be a factor on the left hand side.

$$\begin{aligned} (x+2)(x+1) &= (Bx + C)(x + 1) \\ \Rightarrow \quad (x+2) &= (Bx + C) \quad \Rightarrow \quad B = 1,\ C = 2 \end{aligned}$$

So again we find that

$$\frac{3x^2 + 7x + 12}{(x+1)(x^2+2x+5)} = \frac{2}{x+1} + \frac{x+2}{x^2+2x+5} \checkmark$$

Thus our integrand can be written as

$$\frac{x^4 + 5x^3 + 16x^2 + 26x + 22}{x^3 + 3x^2 + 7x + 5} = x + 2 + \frac{2}{x+1} + \frac{x+2}{x^2+2x+5}.$$

- *Step 4.* Now we can finally integrate! The first two pieces are easy.

$$\int (x+2)\mathrm{d}x = \frac{1}{2}x^2 + 2x \qquad \int \frac{2}{x+1}\mathrm{d}x = 2\log|x+1|$$

(We're leaving the arbitrary constant to the end of the computation.)

The final piece is a little harder. The idea is to complete the square [d] in the denominator

$$\frac{x+2}{x^2+2x+5} = \frac{x+2}{(x+1)^2+4}$$

and then make a change of variables to make the fraction look like $\frac{ay+b}{y^2+1}$. In this case

$$\frac{x+2}{(x+1)^2+4} = \frac{1}{4}\frac{x+2}{(\frac{x+1}{2})^2+1}$$

so we make the change of variables $y = \frac{x+1}{2}, \mathrm{d}y = \frac{\mathrm{d}x}{2},\ x = 2y-1, \mathrm{d}x = 2\,\mathrm{d}y$

$$\begin{aligned}\int \frac{x+2}{(x+1)^2+4}\mathrm{d}x &= \frac{1}{4}\int \frac{x+2}{\left(\frac{x+1}{2}\right)^2+1}\mathrm{d}x \\ &= \frac{1}{4}\int \frac{(2y-1)+2}{y^2+1}\,2\,\mathrm{d}y \ = \ \frac{1}{2}\int \frac{2y+1}{y^2+1}\,\mathrm{d}y \\ &= \int \frac{y}{y^2+1}\,\mathrm{d}y + \frac{1}{2}\int \frac{1}{y^2+1}\,\mathrm{d}y\end{aligned}$$

Both integrals are easily evaluated, using the substitution $u = y^2+1$, $\mathrm{d}u = 2y\,\mathrm{d}y$ for the first.

$$\begin{aligned}\int \frac{y}{y^2+1}\,\mathrm{d}y \ &= \ \int \frac{1}{u}\frac{\mathrm{d}u}{2} \ = \ \frac{1}{2}\log|u| = \frac{1}{2}\log(y^2+1) \\ &= \ \frac{1}{2}\log\left[\left(\frac{x+1}{2}\right)^2+1\right] \\ \frac{1}{2}\int \frac{1}{y^2+1}\,\mathrm{d}y \ &= \ \frac{1}{2}\arctan y \ = \ \frac{1}{2}\arctan\left(\frac{x+1}{2}\right)\end{aligned}$$

That's finally it. Putting all of the pieces together

$$\begin{aligned}\int \frac{x^4+5x^3+16x^2+26x+22}{x^3+3x^2+7x+5}\mathrm{d}x &= \frac{1}{2}x^2 + 2x + 2\log|x+1| \\ &+\frac{1}{2}\log\left[\left(\frac{x+1}{2}\right)^2+1\right] + \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C\end{aligned}$$

---

a One does not typically think of mathematics assignments or exams as nice kind places... The polynomials that appear in the "real world" are not so forgiving. Nature, red in tooth and claw

— to quote Tennyson inappropriately (especially when this author doesn't know any other words from the poem).

b Appendix A.16 contains several simple tricks for factoring polynomials. We recommend that you have a look at them.

c To be precise, the quadratic equation $ax^2 + bx + c = 0$ has solutions $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. The term $b^2 - 4ac$ is called the discriminant and it tells us about the number of solutions. If the discriminant is positive then there are two real solutions. When it is zero, there is a single solution. And if it is negative, there is no real solutions (you need complex numbers to say more than this).

d This same idea arose in Section 1.9. Given a quadratic written as $Q(x) = ax^2 + bx + c$ rewrite it as $Q(x) = a(x + d)^2 + e$. We can determine d and e by expanding and comparing coefficients of x: $ax^2 + bx + c = a(x^2 + 2dx + d^2) + e = ax^2 + 2dax + (e + ad^2)$. Hence $d = b/2a$ and $e = c - ad^2$.

Example 1.10.3

The best thing after working through a few a nice long examples is to do another nice long example — it is excellent practice [4]. We recommend that the reader attempt the problem before reading through our solution.

Example 1.10.4 $\int \frac{4x^3+23x^2+45x+27}{x^3+5x^2+8x+4}\mathrm{d}x$.

In this example, we integrate $\frac{N(x)}{D(x)} = \frac{4x^3+23x^2+45x+27}{x^3+5x^2+8x+4}$.

- *Step 1.* The degree of the numerator $N(x)$ is equal to the degree of the denominator $D(x)$, so the first step to write $\frac{N(x)}{D(x)}$ in the form

$$\frac{N(x)}{D(x)} = P(x) + \frac{R(x)}{D(x)}$$

with $P(x)$ being a polynomial (which should be of degree 0, i.e. just a constant) and $R(x)$ being a polynomial of degree strictly smaller than the degree of $D(x)$. By long division

$$\begin{array}{r|l} & 4 \\ \hline x^3+5x^2+8x+4 & 4x^3 + 23x^2 + 45x + 27 \\ & 4x^3 + 20x^2 + 32x + 16 \\ \hline & 3x^2 + 13x + 11 \end{array}$$

so

$$\frac{4x^3 + 23x^2 + 45x + 27}{x^3 + 5x^2 + 8x + 4} = 4 + \frac{3x^2 + 13x + 11}{x^3 + 5x^2 + 8x + 4}$$

4 At the risk of quoting Nietzsche, "That which does not kill us makes us stronger." Though this author always preferred the logically equivalent contrapositive — "That which does not make us stronger will kill us." However no one is likely to be injured by practicing partial fractions or looking up quotes on Wikipedia. Its also a good excuse to remind yourself of what a contrapositive is — though we will likely look at them again when we get to sequences and series.

- *Step 2.* The second step is to factorise $D(x) = x^3 + 5x^2 + 8x + 4$.
  - To start, we'll try and guess an integer root. Any integer root of $D(x)$ must divide the constant term, 4, exactly. Only ± 1, ± 2, ± 4 can be integer roots of $x^3 + 5x^2 + 8x + 4$.
  - We test to see if ± 1 are roots.

$$\begin{aligned} D(1) &= (1)^3 + 5(1)^2 + 8(1) + 4 \neq 0 && \Rightarrow x = 1 \text{ is not a root} \\ D(-1) &= (-1)^3 + 5(-1)^2 + 8(-1) + 4 = 0 && \Rightarrow x = -1 \text{ is a root} \end{aligned}$$

  So $(x + 1)$ must divide $x^3 + 5x^2 + 8x + 4$ exactly.
  - By long division

$$\begin{array}{r|l} & x^2 + 4x + \ 4 \\ \hline x+1 & x^3 + 5x^2 + 8x + 4 \\ & x^3 + \ x^2 \\ \hline & \quad 4x^2 + 8x + 4 \\ & \quad 4x^2 + 4x \\ \hline & \qquad\quad 4x + 4 \\ & \qquad\quad 4x + 4 \\ \hline & \qquad\qquad\ 0 \end{array}$$

  so

$$\begin{aligned} x^3 + 5x^2 + 8x + 4 &= (x+1)(x^2 + 4x + 4) \\ &= (x+1)(x+2)(x+2) \end{aligned}$$

  - Notice that we could have instead checked whether or not ± 2 are roots

$$\begin{aligned} D(2) &= (2)^3 + 5(2)^2 + 8(2) + 4 \neq 0 && \Rightarrow x = 2 \text{ is not a root} \\ D(-2) &= (-2)^3 + 5(-2)^2 + 8(-2) + 4 = 0 && \Rightarrow x = -2 \text{ is a root} \end{aligned}$$

  We now know that both -1 and -2 are roots of $x^3 + 5x^2 + 8x + 4$ and hence both $(x + 1)$ and $(x + 2)$ are factors of $x^3 + 5x^2 + 8x + 4$. Because $x^3 + 5x^2 + 8x + 4$ is of degree three and the coefficient of x^3 is 1, we must have $x^3 + 5x^2 + 8x + 4 = (x + 1)(x + 2)(x + a)$ for some constant a. Multiplying out the right hand side shows that the constant term is $2a$. So $2a = 4$ and $a = 2$.

This is the end of step 2. We now know that

$$\frac{4x^3 + 23x^2 + 45x + 27}{x^3 + 5x^2 + 8x + 4} = 4 + \frac{3x^2 + 13x + 11}{(x+1)(x+2)^2}$$

- *Step 3.* The third step is to write $\frac{3x^2+13x+11}{(x+1)(x+2)^2}$ in the form

$$\frac{3x^2+13x+11}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

for some constants A, B and C.

Note that there are two terms on the right hand arising from the factor $(x+2)^2$. One has denominator $(x+2)$ and one has denominator $(x+2)^2$. More generally, for each factor $(x+a)^n$ in the denominator of the rational function on the left hand side, we include

$$\frac{A_1}{x+a} + \frac{A_2}{(x+a)^2} + \cdots + \frac{A_n}{(x+a)^n}$$

in the partial fraction decomposition on the right hand side [a].

To determine the values of the constants A, B, C, we put the right hand side back over the common denominator $(x+1)(x+2)^2$.

$$\begin{aligned}\frac{3x^2+13x+11}{(x+1)(x+2)^2} &= \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}\\ &= \frac{A(x+2)^2 + B(x+1)(x+2) + C(x+1)}{(x+1)(x+2)^2}\end{aligned}$$

The fraction on the far left is the same as the fraction on the far right if and only if their numerators are the same.

$$3x^2+13x+11 = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$$

As in the previous examples, there are a couple of different ways to determine the values of A, B and C from this equation.

- *Step 3 — Algebra Method.* The conceptually clearest procedure is to write the right hand side as a polynomial in standard form (i.e. collect up all x^2 terms, all x terms and all constant terms)

$$3x^2+13x+11 = (A+B)x^2 + (4A+3B+C)x + (4A+2B+C)$$

For these two polynomials to be the same, the coefficient of x^2 on the left hand side and the coefficient of x^2 on the right hand side must be the same. Similarly the coefficients of x^1 and the coefficients of x^0 (i.e. the constant terms) must match. This gives us a system of three equations,

$$A+B=3 \qquad 4A+3B+C=13 \qquad 4A+2B+C=11$$

in the three unknowns A, B, C. We can solve this system by

- using the first equation, namely $A + B = 3$, to determine A in terms of B: $A = 3 - B$.
- Substituting this into the remaining equations eliminates the A, leaving two equations in the two unknown B, C.

$$4(3 - B) + 3B + C = 13 \qquad 4(3 - B) + 2B + C = 11$$

or

$$-B + C = 1 \qquad -2B + C = -1$$

- We can now solve the first of these equations, namely $-B + C = 1$, for B in terms of C, giving $B = C - 1$.
- Substituting this into the last equation, namely $-2B+C = -1$, gives $-2(C-1) + C = -1$ which is easily solved to give
- $C = 3$, and then $B = C - 1 = 2$ and then $A = 3 - B = 1$.

Hence

$$\begin{aligned}\frac{4x^3 + 23x^2 + 45x + 27}{x^3 + 5x^2 + 8x + 4} &= 4 + \frac{3x^2 + 13x + 11}{(x+1)(x+2)^2} \\ &= 4 + \frac{1}{x+1} + \frac{2}{x+2} + \frac{3}{(x+2)^2}\end{aligned}$$

- *Step 3 — Sneaky Method.* The second, sneakier, method for finding A, B and C exploits the fact that $3x^2 + 13x + 11 = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$ must be true for all values of x. In particular, it must be true for $x = -1$. When $x = -1$, the factor $(x+1)$ multiplying B and C is exactly zero. So B and C disappear from the equation, leaving us with an easy equation to solve for A:

$$\begin{aligned}3x^2 + 13x + 11\Big|_{x=-1} &= \Big[A(x+2)^2 + B(x+1)(x+2) + C(x+1)\Big]_{x=-1} \\ &\implies 1 = A\end{aligned}$$

Sub this value of A back in and simplify.

$$\begin{aligned}3x^2 + 13x + 11 &= (1)(x+2)^2 + B(x+1)(x+2) + C(x+1) \\ 2x^2 + 9x + 7 &= B(x+1)(x+2) + C(x+1) \\ &= (xB + 2B + C)(x+1)\end{aligned}$$

Since $(x+1)$ is a factor on the right hand side, it must also be a factor on the left hand side.

$$\begin{aligned}(2x+7)(x+1) &= (xB + 2B + C)(x+1) \\ &\Rightarrow \quad (2x+7) = (xB + 2B + C)\end{aligned}$$

For the coefficients of x to match, B must be 2. For the constant terms to match, $2B + C$ must be 7, so C must be 3. Hence we again have

$$\begin{aligned}\frac{4x^3 + 23x^2 + 45x + 27}{x^3 + 5x^2 + 8x + 4} &= 4 + \frac{3x^2 + 13x + 11}{(x+1)(x+2)^2} \\ &= 4 + \frac{1}{x+1} + \frac{2}{x+2} + \frac{3}{(x+2)^2}\end{aligned}$$

- *Step 4.* The final step is to integrate

$$\begin{aligned}&\int \frac{4x^3 + 23x^2 + 45x + 27}{x^3 + 5x^2 + 8x + 4} \mathrm{d}x \\ &= \int 4\mathrm{d}x + \int \frac{1}{x+1}\mathrm{d}x + \int \frac{2}{x+2}\mathrm{d}x + \int \frac{3}{(x+2)^2}\mathrm{d}x \\ &= 4x + \log|x+1| + 2\log|x+2| - \frac{3}{x+2} + C\end{aligned}$$

a This is justified in the (optional) subsection "Justification of the Partial Fraction Decompositions" below.

Example 1.10.4

The method of partial fractions is not just confined to the problem of integrating rational functions. There are other integrals — such as $\int \sec x \mathrm{d}x$ and $\int \sec^3 x \mathrm{d}x$ — that can be transformed (via substitutions) into integrals of rational functions. We encountered both of these integrals in Sections 1.8 and 1.9 on trigonometric integrals and substitutions.

Example 1.10.5 $\int \sec x \mathrm{d}x$.

Solution: In this example, we integrate $\sec x$. It is not yet clear what this integral has to do with partial fractions. To get to a partial fractions computation, we first make one of our old substitutions.

$$\begin{aligned}\int \sec x \mathrm{d}x &= \int \frac{1}{\cos x}\mathrm{d}x && \text{massage the expression a little} \\ &= \int \frac{\cos x}{\cos^2 x}\mathrm{d}x && \text{substitute } u = \sin x,\ \mathrm{d}u = \cos x\mathrm{d}x \\ &= -\int \frac{\mathrm{d}u}{u^2 - 1} && \text{and use } \cos^2 x = 1 - \sin^2 x = 1 - u^2\end{aligned}$$

So we now have to integrate $\frac{1}{u^2-1}$, which is a rational function of u, and so is perfect for partial fractions.

- *Step 1.* The degree of the numerator, 1, is zero, which is strictly smaller than the degree of the denominator, $u^2 - 1$, which is two. So the first step is skipped.

- *Step 2.* The second step is to factor the denominator:
$$u^2 - 1 = (u-1)(u+1)$$
- *Step 3.* The third step is to write $\frac{1}{u^2-1}$ in the form
$$\frac{1}{u^2-1} = \frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$$
for some constants A and B.
- *Step 3 — Sneaky Method.*
  - Multiply through by the denominator to get
$$1 = A(u+1) + B(u-1)$$
This equation must be true for all u.
  - If we now set $u = 1$ then we eliminate B from the equation leaving us with
$$1 = 2A \qquad \text{so } A = \frac{1}{2}.$$
  - Similarly, if we set $u = -1$ then we eliminate A, leaving
$$1 = -2B \qquad \text{which implies } B = -\frac{1}{2}.$$

We have now found that $A = \frac{1}{2}, B = -\frac{1}{2}$, so
$$\frac{1}{u^2-1} = \frac{1}{2}\left[\frac{1}{u-1} - \frac{1}{u+1}\right].$$
- It is always a good idea to check our work.
$$\frac{\frac{1}{2}}{u-1} + \frac{-\frac{1}{2}}{u+1} = \frac{\frac{1}{2}(u+1) - \frac{1}{2}(u-1)}{(u-1)(u+1)} = \frac{1}{(u-1)(u+1)} \checkmark$$
- *Step 4.* The final step is to integrate.
$$\begin{aligned}
\int \sec x \mathrm{d}x &= -\int \frac{\mathrm{d}u}{u^2-1} && \text{after substitution} \\
&= -\frac{1}{2}\int \frac{\mathrm{d}u}{u-1} + \frac{1}{2}\int \frac{\mathrm{d}u}{u+1} && \text{partial fractions} \\
&= -\frac{1}{2}\log|u-1| + \frac{1}{2}\log|u+1| + C \\
&= -\frac{1}{2}\log|\sin(x)-1| + \frac{1}{2}\log|\sin(x)+1| + C && \text{rearrange a little} \\
&= \frac{1}{2}\log\left|\frac{1+\sin x}{1-\sin x}\right| + C
\end{aligned}$$

Notice that since $-1 \le \sin x \le 1$, we are free to drop the absolute values in the last line if we wish.

Example 1.10.5

Another example in the same spirit, though a touch harder. Again, we saw this problem in Section 1.8 and 1.9.

Example 1.10.6 $\int \sec^3 x\mathrm{d}x$.

Solution:

- We'll start by converting it into the integral of a rational function using the substitution $u = \sin x$, $\mathrm{d}u = \cos x\mathrm{d}x$.

$$\begin{aligned}
\int \sec^3 x\mathrm{d}x &= \int \frac{1}{\cos^3 x}\mathrm{d}x && \text{massage this a little} \\
&= \int \frac{\cos x}{\cos^4 x}\mathrm{d}x && \text{replace } \cos^2 x = 1 - \sin^2 x = 1 - u^2 \\
&= \int \frac{\cos x\mathrm{d}x}{\left[1 - \sin^2 x\right]^2} \\
&= \int \frac{\mathrm{d}u}{[1 - u^2]^2}
\end{aligned}$$

- We could now find the partial fraction decomposition of the integrand $\frac{1}{[1-u^2]^2}$ by executing the usual four steps. But it is easier to use

$$\frac{1}{u^2 - 1} = \frac{1}{2}\Big[\frac{1}{u-1} - \frac{1}{u+1}\Big]$$

which we worked out in Example 1.10.5 above.

- Squaring this gives

$$\begin{aligned}
\frac{1}{\left[1 - u^2\right]^2} &= \frac{1}{4}\Big[\frac{1}{u-1} - \frac{1}{u+1}\Big]^2 \\
&= \frac{1}{4}\Big[\frac{1}{(u-1)^2} - \frac{2}{(u-1)(u+1)} + \frac{1}{(u+1)^2}\Big] \\
&= \frac{1}{4}\Big[\frac{1}{(u-1)^2} - \frac{1}{u-1} + \frac{1}{u+1} + \frac{1}{(u+1)^2}\Big]
\end{aligned}$$

where we have again used $\frac{1}{u^2-1} = \frac{1}{2}\left[\frac{1}{u-1} - \frac{1}{u+1}\right]$ in the last step.

- It only remains to do the integrals and simplify.

$$\begin{aligned}
\int \sec^3 x\mathrm{d}x &= \frac{1}{4}\int \Big[\frac{1}{(u-1)^2} - \frac{1}{u-1} + \frac{1}{u+1} + \frac{1}{(u+1)^2}\Big]\mathrm{d}u \\
&= \frac{1}{4}\Big[-\frac{1}{u-1} - \log|u-1| + \log|u+1| - \frac{1}{u+1}\Big] + C \\
& \qquad \text{group carefully} \\
&= \frac{-1}{4}\Big[\frac{1}{u-1} + \frac{1}{u+1}\Big] + \frac{1}{4}\Big[\log|u+1| - \log|u-1|\Big] + C
\end{aligned}$$

$$\begin{aligned}
&\quad \text{sum carefully}\\
&= -\frac{1}{4}\frac{2u}{u^2-1} + \frac{1}{4}\log\left|\frac{u+1}{u-1}\right| + C\\
&\quad \text{clean up}\\
&= \frac{1}{2}\frac{u}{1-u^2} + \frac{1}{4}\log\left|\frac{u+1}{u-1}\right| + C\\
&\quad \text{put } u = \sin x\\
&= \frac{1}{2}\frac{\sin x}{\cos^2 x} + \frac{1}{4}\log\left|\frac{\sin x+1}{\sin x-1}\right| + C
\end{aligned}$$

Example 1.10.6

1.10.2 ➤ The form of partial fraction decompositions

In the examples above we used the partial fractions method to decompose rational functions into easily integrated pieces. Each of those examples was quite involved and we had to spend quite a bit of time factoring and doing long division. The key step in each of the computations was Step 3 — in that step we decomposed the rational function $\frac{N(x)}{D(x)}$ (or $\frac{R(x)}{D(x)}$), for which the degree of the numerator is strictly smaller than the degree of the denominator, into a sum of particularly simple rational functions, like $\frac{A}{x-a}$. We did not, however, give a systematic description of those decompositions.

In this subsection we fill that gap by describing the general [5] form of partial fraction decompositions. The justification of these forms is not part of the course, but the interested reader is invited to read the next (optional) subsection where such justification is given. In the following it is assumed that

- $N(x)$ and $D(x)$ are polynomials with the degree of $N(x)$ strictly smaller than the degree of $D(x)$.
- K is a constant.
- $a_1, a_2, \cdots, a_j$ are all different numbers.
- $m_1, m_2, \cdots, m_j$, and $n_1, n_2, \cdots, n_k$ are all strictly positive integers.
- $x^2+b_1x+c_1,\ x^2+b_2x+c_2,\ \cdots,\ x^2+b_kx+c_k$ are all different.

5 Well — not the completely general form, in the sense that we are not allowing the use of complex numbers. As a result we have to use both linear and quadratic factors in the denominator. If we could use complex numbers we would be able to restrict ourselves to linear factors.

1.10.2.1 ▸▸▸ Simple linear factor case

If the denominator $D(x) = K(x-a_1)(x-a_2)\cdots(x-a_j)$ is a product of j different linear factors, then

Equation 1.10.7

$$\frac{N(x)}{D(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_j}{x-a_j}$$

We can then integrate each term

$$\int \frac{A}{x-a}\mathrm{d}x = A\log|x-a| + C.$$

1.10.2.2 ▸▸▸ General linear factor case

If the denominator $D(x) = K(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_j)^{m_j}$ then

Claim 1.10.8

$$\begin{aligned}\frac{N(x)}{D(x)} &= \frac{A_{1,1}}{x-a_1} + \frac{A_{1,2}}{(x-a_1)^2} + \cdots + \frac{A_{1,m_1}}{(x-a_1)^{m_1}}\\ &\quad + \frac{A_{2,1}}{x-a_2} + \frac{A_{2,2}}{(x-a_2)^2} + \cdots + \frac{A_{2,m_2}}{(x-a_2)^{m_2}} + \cdots\\ &\quad + \frac{A_{j,1}}{x-a_j} + \frac{A_{j,2}}{(x-a_j)^2} + \cdots + \frac{A_{j,m_j}}{(x-a_j)^{m_j}}\end{aligned}$$

Notice that we could rewrite each line as

$$\begin{aligned}\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_m}{(x-a)^m} &= \frac{A_1(x-a)^{m-1} + A_2(x-a)^{m-2} + \cdots + A_m}{(x-a)^m}\\ &= \frac{B_1x^{m-1} + B_2x^{m-2} + \cdots + B_m}{(x-a)^m}\end{aligned}$$

which is a polynomial whose degree, $m-1$, is strictly smaller than that of the denominator $(x-a)^m$. But the form of Equation 1.10.8 is preferable because it is easier to integrate.

$$\int \frac{A}{x-a}\mathrm{d}x = A\log|x-a| + C$$

$$\int \frac{A}{(x-a)^k}\mathrm{d}x = -\frac{1}{k-1}\cdot\frac{A}{(x-a)^{k-1}} \qquad \text{provided } k>1.$$

1.10.2.3 ▸▸▸ Simple linear and quadratic factor case

If $D(x) = K(x-a_1)\cdots(x-a_j)(x^2+b_1x+c_1)\cdots(x^2+b_kx+c_k)$ then

Claim 1.10.9

$$\frac{N(x)}{D(x)} = \frac{A_1}{x-a_1}+\cdots+\frac{A_j}{x-a_j}+\frac{B_1x+C_1}{x^2+b_1x+c_1}+\cdots+\frac{B_kx+C_k}{x^2+b_kx+c_k}$$

Note that the numerator of each term on the right hand side has degree one smaller than the degree of the denominator.

The quadratic terms $\frac{Bx+C}{x^2+bx+c}$ are integrated in a two-step process that is best illustrated with a simple example (see also Example 1.10.3 above).

Example 1.10.10 $\int \frac{2x+7}{x^2+4x+13}\mathrm{d}x.$

Solution:

- Start by completing the square in the denominator:

$$\begin{aligned} x^2+4x+13 &= (x+2)^2+9 && \text{and thus}\\ \frac{2x+7}{x^2+4x+13} &= \frac{2x+7}{(x+2)^2+3^2} \end{aligned}$$

- Now set $y=(x+2)/3, \mathrm{d}y=\frac{1}{3}\mathrm{d}x$, or equivalently $x=3y-2, \mathrm{d}x=3\mathrm{d}y$:

$$\begin{aligned} \int \frac{2x+7}{x^2+4x+13}\mathrm{d}x &= \int \frac{2x+7}{(x+2)^2+3^2}\mathrm{d}x\\ &= \int \frac{6y-4+7}{3^2y^2+3^2}\cdot 3\mathrm{d}y\\ &= \int \frac{6y+3}{3(y^2+1)}\mathrm{d}y\\ &= \int \frac{2y+1}{y^2+1}\mathrm{d}y \end{aligned}$$

  Notice that we chose 3 in $y=(x+2)/3$ precisely to transform the denominator into the form y^2+1.

- Now almost always the numerator will be a linear polynomial of y and we decompose as follows

$$\begin{aligned}\int \frac{2x+7}{x^2+4x+13}\mathrm{d}x &= \int \frac{2y+1}{y^2+1}\mathrm{d}y \\ &= \int \frac{2y}{y^2+1}\mathrm{d}y + \int \frac{1}{y^2+1}\mathrm{d}y \\ &= \log|y^2+1| + \arctan y + C \\ &= \log\left|\left(\frac{x+2}{3}\right)^2+1\right| + \arctan\left(\frac{x+2}{3}\right) + C\end{aligned}$$

Example 1.10.10

1.10.2.4 ▸▸▸ Optional — General linear and quadratic factor case

If $D(x) = K(x-a_1)^{m_1}\cdots(x-a_j)^{m_j}(x^2+b_1x+c_1)^{n_1}\cdots(x^2+b_kx+c_k)^{n_k}$

Claim 1.10.11

$$\begin{aligned}\frac{N(x)}{D(x)} &= \frac{A_{1,1}}{x-a_1} + \frac{A_{1,2}}{(x-a_1)^2} + \cdots + \frac{A_{1,m_1}}{(x-a_1)^{m_1}} + \cdots \\ &+ \frac{A_{j,1}}{x-a_j} + \frac{A_{j,2}}{(x-a_j)^2} + \cdots + \frac{A_{j,m_j}}{(x-a_j)^{m_j}} \\ &+ \frac{B_{1,1}x+C_{1,1}}{x^2+b_1x+c_1} + \frac{B_{1,2}x+C_{1,2}}{(x^2+b_1x+c_1)^2} + \cdots + \frac{B_{1,n_1}x+C_{1,n_1}}{(x^2+b_1x+c_1)^{n_1}} + \cdots \\ &+ \frac{B_{k,1}x+C_{k,1}}{x^2+b_kx+c_k} + \frac{B_{k,2}x+C_{k,2}}{(x^2+b_kx+c_k)^2} + \cdots + \frac{B_{k,n_k}x+C_{1,n_k}}{(x^2+b_kx+c_k)^{n_k}}\end{aligned}$$

We have already seen how to integrate the simple and general linear terms, and the simple quadratic terms. Integrating general quadratic terms is not so straightforward.

Example 1.10.12 $\int \frac{\mathrm{d}x}{(x^2+1)^n}$.

This example is not so easy, so it should definitely be considered optional.
Solution: In what follows write

$$I_n = \int \frac{\mathrm{d}x}{(x^2+1)^n}.$$

- When $n=1$ we know that

$$\int \frac{\mathrm{d}x}{x^2+1} = \arctan x + C$$

- Now assume that $n > 1$, then

$$\begin{aligned}\int \frac{1}{(x^2+1)^n}\mathrm{d}x &= \int \frac{(x^2+1-x^2)}{(x^2+1)^n}\mathrm{d}x && \text{sneaky}\\ &= \int \frac{1}{(x^2+1)^{n-1}}\mathrm{d}x - \int \frac{x^2}{(x^2+1)^n}\mathrm{d}x\\ &= I_{n-1} - \int \frac{x^2}{(x^2+1)^n}\mathrm{d}x\end{aligned}$$

So we can write I_n in terms of I_{n-1} and this second integral.

- We can use integration by parts to compute the second integral:

$$\int \frac{x^2}{(x^2+1)^n}\mathrm{d}x = \int \frac{x}{2} \cdot \frac{2x}{(x^2+1)^n}\mathrm{d}x \qquad \text{sneaky}$$

We set $u = x/2$ and $\mathrm{d}v = \frac{2x}{(x^2+1)^n}\mathrm{d}x$, which gives $\mathrm{d}u = \frac{1}{2}\mathrm{d}x$ and $v = -\frac{1}{n-1} \cdot \frac{1}{(x^2+1)^{n-1}}$. You can check v by differentiating. Integration by parts gives

$$\begin{aligned}&\int \frac{x}{2} \cdot \frac{2x}{(x^2+1)^n}\mathrm{d}x\\ &\qquad = -\frac{x}{2(n-1)(x^2+1)^{n-1}} + \int \frac{\mathrm{d}x}{2(n-1)(x^2+1)^{n-1}}\\ &\qquad = -\frac{x}{2(n-1)(x^2+1)^{n-1}} + \frac{1}{2(n-1)} \cdot I_{n-1}\end{aligned}$$

- Now put everything together:

$$\begin{aligned}I_n &= \int \frac{1}{(x^2+1)^n}\mathrm{d}x\\ &= I_{n-1} + \frac{x}{2(n-1)(x^2+1)^{n-1}} - \frac{1}{2(n-1)} \cdot I_{n-1}\\ &= \frac{2n-3}{2(n-1)}I_{n-1} + \frac{x}{2(n-1)(x^2+1)^{n-1}}\end{aligned}$$

- We can then use this recurrence to write down I_n for the first few n:

$$\begin{aligned}I_2 &= \frac{1}{2}I_1 + \frac{x}{2(x^2+1)} + C\\ &= \frac{1}{2}\arctan x + \frac{x}{2(x^2+1)}\\ I_3 &= \frac{3}{4}I_2 + \frac{x}{4(x^2+1)^2}\\ &= \frac{3}{8}\arctan x + \frac{3x}{8(x^2+1)} + \frac{x}{4(x^2+1)^2} + C\end{aligned}$$

$$\begin{aligned}I_4 &= \frac{5}{6}I_3 + \frac{x}{6(x^2+1)^3}\\ &= \frac{5}{16}\arctan x + \frac{5x}{16(x^2+1)} + \frac{5x}{24(x^2+1)^2} + \frac{x}{6(x^2+1)^3} + C\end{aligned}$$

and so forth. You can see why partial fraction questions involving denominators with repeated quadratic factors do not often appear on exams.

Example 1.10.12

1.10.3 ▸▸ Optional — Justification of the partial fraction decompositions

We will now see the justification for the form of the partial fraction decompositions. We will only consider the case in which the denominator has only linear factors. The arguments when there are quadratic factors too are similar [6].

1.10.3.1 ▸▸▸ Simple linear factor case

In the most common partial fraction decomposition, we split up

$$\frac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)}$$

into a sum of the form

$$\frac{A_1}{x-a_1} + \cdots + \frac{A_d}{x-a_d}$$

We now show that this decomposition can always be achieved, under the assumptions that the a_i's are all different and $N(x)$ is a polynomial of degree at most $d-1$. To do so, we shall repeatedly apply the following Lemma.

Lemma 1.10.13

Let $N(x)$ and $D(x)$ be polynomials of degree n and d respectively, with $n \le d$. Suppose that a is NOT a zero of $D(x)$. Then there is a polynomial $P(x)$ of degree $p < d$ and a number A such that

$$\frac{N(x)}{D(x)\,(x-a)} = \frac{P(x)}{D(x)} + \frac{A}{x-a}$$

6 In fact, quadratic factors are completely avoidable because, if we use complex numbers, then every polynomial can be written as a product of linear factors. This is the fundamental theorem of algebra.

Proof.

- To save writing, let $z = x - a$. We then write $\tilde{N}(z) = N(z+a)$ and $\tilde{D}(z) = D(z+a)$, which are again polynomials of degree n and d respectively. We also know that $\tilde{D}(0) = D(a) \neq 0$.
- In order to complete the proof we need to find a polynomial $\tilde{P}(z)$ of degree $p < d$ and a number A such that

$$\frac{\tilde{N}(z)}{\tilde{D}(z)\, z} = \frac{\tilde{P}(z)}{\tilde{D}(z)} + \frac{A}{z} = \frac{\tilde{P}(z)z + A\tilde{D}(z)}{\tilde{D}(z)\, z}$$

  or equivalently, such that

$$\tilde{P}(z)z + A\tilde{D}(z) = \tilde{N}(z).$$

- Now look at the polynomial on the left hand side. Every term in $\tilde{P}(z)z$, has at least one power of z. So the constant term on the left hand side is exactly the constant term in $A\tilde{D}(z)$, which is equal to $A\tilde{D}(0)$. The constant term on the right hand side is equal to $\tilde{N}(0)$. So the constant terms on the left and right hand sides are the same if we choose $A = \frac{\tilde{N}(0)}{\tilde{D}(0)}$. Recall that $\tilde{D}(0)$ cannot be zero, so A is well defined.
- Now move $A\tilde{D}(z)$ to the right hand side.

$$\tilde{P}(z)z = \tilde{N}(z) - A\tilde{D}(z)$$

  The constant terms in $\tilde{N}(z)$ and $A\tilde{D}(z)$ are the same, so the right hand side contains no constant term and the right hand side is of the form $\tilde{N}_1(z)z$ for some polynomial $\tilde{N}_1(z)$.
- Since $\tilde{N}(z)$ is of degree at most d and $A\tilde{D}(z)$ is of degree exactly d, $\tilde{N}_1$ is a polynomial of degree $d-1$. It now suffices to choose $\tilde{P}(z) = \tilde{N}_1(z)$.

$\square$

Now back to

$$\frac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)}$$

Apply Lemma 1.10.13, with $D(x) = (x-a_2)\times\cdots\times(x-a_d)$ and $a = a_1$. It says

$$\frac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)} = \frac{A_1}{x-a_1} + \frac{P(x)}{(x-a_2)\times\cdots\times(x-a_d)}$$

for some polynomial P of degree at most $d-2$ and some number A_1.

Apply Lemma 1.10.13 a second time, with $D(x) = (x-a_3)\times\cdots\times(x-a_d)$, $N(x) =$

$P(x)$ and $a = a_2$. It says

$$\frac{P(x)}{(x-a_2)\times\cdots\times(x-a_d)} = \frac{A_2}{x-a_2} + \frac{Q(x)}{(x-a_3)\times\cdots\times(x-a_d)}$$

for some polynomial Q of degree at most $d-3$ and some number A_2.

At this stage, we know that

$$\frac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \frac{Q(x)}{(x-a_3)\times\cdots\times(x-a_d)}$$

If we just keep going, repeatedly applying Lemma 1, we eventually end up with

$$\frac{N(x)}{(x-a_1)\times\cdots\times(x-a_d)} = \frac{A_1}{x-a_1} + \cdots + \frac{A_d}{x-a_d}$$

as required.

1.10.3.2 ▸▸▸ The general case with linear factors

Now consider splitting

$$\frac{N(x)}{(x-a_1)^{n_1}\times\cdots\times(x-a_d)^{n_d}}$$

into a sum of the form [7]

$$\left[\frac{A_{1,1}}{x-a_1} + \cdots + \frac{A_{1,n_1}}{(x-a_1)^{n_1}}\right] + \cdots + \left[\frac{A_{d,1}}{x-a_d} + \cdots + \frac{A_{d,n_d}}{(x-a_d)^{n_d}}\right]$$

We now show that this decomposition can always be achieved, under the assumptions that the a_i's are all different and $N(x)$ is a polynomial of degree at most $n_1+\cdots+n_d-1$. To do so, we shall repeatedly apply the following Lemma.

Lemma 1.10.14

Let $N(x)$ and $D(x)$ be polynomials of degree n and d respectively, with $n < d+m$. Suppose that a is NOT a zero of $D(x)$. Then there is a polynomial $P(x)$ of degree $p < d$ and numbers $A_1, \cdots, A_m$ such that

$$\frac{N(x)}{D(x)\,(x-a)^m} = \frac{P(x)}{D(x)} + \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_m}{(x-a)^m}$$

7 If we allow ourselves to use complex numbers as roots, this is the general case. We don't need to consider quadratic (or higher) factors since all polynomials can be written as products of linear factors with complex coefficients.

Proof.

- As we did in the proof of the previous lemma, we write $z = x - a$. Then $\tilde{N}(z) = N(z + a)$ and $\tilde{D}(z) = D(z + a)$ are polynomials of degree n and d respectively, $\tilde{D}(0) = D(a) \neq 0$.

- In order to complete the proof we have to find a polynomial $\tilde{P}(z)$ of degree $p < d$ and numbers $A_1, \cdots, A_m$ such that
$$\begin{aligned}\frac{\tilde{N}(z)}{\tilde{D}(z)\, z^m} &= \frac{\tilde{P}(z)}{\tilde{D}(z)} + \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_m}{z^m} \\ &= \frac{\tilde{P}(z)z^m + A_1 z^{m-1}\tilde{D}(z) + A_2 z^{m-2}\tilde{D}(z) + \cdots + A_m\tilde{D}(z)}{\tilde{D}(z)\, z^m}\end{aligned}$$
or equivalently, such that
$$\begin{aligned}\tilde{P}(z)z^m + A_1 z^{m-1}\tilde{D}(z) + A_2 z^{m-2}\tilde{D}(z) + \cdots + A_{m-1}z\tilde{D}(z) + A_m\tilde{D}(z) \\ = \tilde{N}(z)\end{aligned}$$

- Now look at the polynomial on the left hand side. Every single term on the left hand side, except for the very last one, $A_m\tilde{D}(z)$, has at least one power of z. So the constant term on the left hand side is exactly the constant term in $A_m\tilde{D}(z)$, which is equal to $A_m\tilde{D}(0)$. The constant term on the right hand side is equal to $\tilde{N}(0)$. So the constant terms on the left and right hand sides are the same if we choose $A_m = \frac{\tilde{N}(0)}{\tilde{D}(0)}$. Recall that $\tilde{D}(0) \neq 0$ so A_m is well defined.

- Now move $A_m\tilde{D}(z)$ to the right hand side.
$$\begin{aligned}\tilde{P}(z)z^m + A_1 z^{m-1}\tilde{D}(z) + A_2 z^{m-2}\tilde{D}(z) + \cdots + A_{m-1}z\tilde{D}(z) \\ = \tilde{N}(z) - A_m\tilde{D}(z)\end{aligned}$$
The constant terms in $\tilde{N}(z)$ and $A_m\tilde{D}(z)$ are the same, so the right hand side contains no constant term and the right hand side is of the form $\tilde{N}_1(z)z$ with $\tilde{N}_1$ a polynomial of degree at most $d + m - 2$. (Recall that $\tilde{N}$ is of degree at most $d + m - 1$ and $\tilde{D}$ is of degree at most d.) Divide the whole equation by z to get
$$\tilde{P}(z)z^{m-1} + A_1 z^{m-2}\tilde{D}(z) + A_2 z^{m-3}\tilde{D}(z) + \cdots + A_{m-1}\tilde{D}(z) = \tilde{N}_1(z).$$

- Now, we can repeat the previous argument. The constant term on the left hand side, which is exactly equal to $A_{m-1}\tilde{D}(0)$ matches the constant term on the right hand side, which is equal to $\tilde{N}_1(0)$ if we choose $A_{m-1} = \frac{\tilde{N}_1(0)}{\tilde{D}(0)}$. With this choice of A_{m-1}
$$\begin{aligned}\tilde{P}(z)z^{m-1} + A_1 z^{m-2}\tilde{D}(z) + A_2 z^{m-3}\tilde{D}(z) + \cdots + A_{m-2}z\tilde{D}(z) \\ = \tilde{N}_1(z) - A_{m-1}\tilde{D}(z) = \tilde{N}_2(z)z\end{aligned}$$
with $\tilde{N}_2$ a polynomial of degree at most $d+m-3$. Divide by z and continue.

- After m steps like this, we end up with

$$\tilde{P}(z)z = \tilde{N}_{m-1}(z) - A_1\tilde{D}(z)$$

after having chosen $A_1 = \frac{\tilde{N}_{m-1}(0)}{\tilde{D}(0)}$.

- There is no constant term on the right side so that $\tilde{N}_{m-1}(z) - A_1\tilde{D}(z)$ is of the form $\tilde{N}_m(z)z$ with $\tilde{N}_m$ a polynomial of degree $d-1$. Choosing $\tilde{P}(z) = \tilde{N}_m(z)$ completes the proof.

□

Now back to

$$\frac{N(x)}{(x-a_1)^{n_1} \times \cdots \times (x-a_d)^{n_d}}$$

Apply Lemma 1.10.14, with $D(x) = (x-a_2)^{n_2} \times \cdots \times (x-a_d)^{n_d}$, $m = n_1$ and $a = a_1$. It says

$$\frac{N(x)}{(x-a_1)^{n_1} \times \cdots \times (x-a_d)^{n_d}}$$
$$= \frac{A_{1,1}}{x-a_1} + \frac{A_{1,2}}{(x-a_1)^2} + \cdots + \frac{A_{1,n_1}}{(x-a)^{n_1}} + \frac{P(x)}{(x-a_2)^{n_2} \times \cdots \times (x-a_d)^{n_d}}$$

Apply Lemma 1.10.14 a second time, with $D(x) = (x-a_3)^{n_3} \times \cdots \times (x-a_d)^{n_d}$, $N(x) = P(x)$, $m = n_2$ and $a = a_2$. And so on. Eventually, we end up with

$$\left[\frac{A_{1,1}}{x-a_1} + \cdots + \frac{A_{1,n_1}}{(x-a_1)^{n_1}}\right] + \cdots + \left[\frac{A_{d,1}}{x-a_d} + \cdots + \frac{A_{d,n_d}}{(x-a_d)^{n_d}}\right]$$

which is exactly what we were trying to show.

1.10.3.3 ▸▸▸ Really Optional — The Fully General Case

We are now going to see that, in general, if $N(x)$ and $D(x)$ are polynomials with the degree of N being strictly smaller than the degree of D (which we'll denote $\deg(N) < \deg(D)$) and if

$$D(x) = K(x-a_1)^{m_1} \cdots (x-a_j)^{m_j}(x^2+b_1x+c_1)^{n_1} \cdots (x^2+b_kx+c_k)^{n_k} \qquad (\star)$$

(with $b_\ell^2 - 4c_\ell < 0$ for all $1 \le \ell \le k$ so that no quadratic factor can be written as a product of linear factors with real coefficients) then there are real numbers $A_{i,j}$, $B_{i,j}$, $C_{i,j}$ such that

$$\frac{N(x)}{D(x)} = \frac{A_{1,1}}{x-a_1} + \frac{A_{1,2}}{(x-a_1)^2} + \cdots + \frac{A_{1,m_1}}{(x-a_1)^{m_1}} + \cdots$$

$$
\begin{aligned}
&+\frac{A_{j,1}}{x-a_j}+\frac{A_{j,2}}{(x-a_j)^2}+\cdots+\frac{A_{j,m_j}}{(x-a_j)^{m_j}}\\
&+\frac{B_{1,1}x+C_{1,1}}{x^2+b_1x+c_1}+\frac{B_{1,2}x+C_{1,2}}{(x^2+b_1x+c_1)^2}+\cdots+\frac{B_{1,n_1}x+C_{1,n_1}}{(x^2+b_1x+c_1)^{n_1}}+\cdots\\
&+\frac{B_{k,1}x+C_{k,1}}{x^2+b_kx+c_k}+\frac{B_{k,2}x+C_{k,2}}{(x^2+b_kx+c_k)^2}+\cdots+\frac{B_{k,n_k}x+C_{1,n_k}}{(x^2+b_kx+c_k)^{n_k}}
\end{aligned}
$$

This was Equation 1.10.11. We start with two simpler results, that we'll use repeatedly to get Equation 1.10.11. In the first simpler result, we consider the fraction $\frac{P(x)}{Q_1(x)\,Q_2(x)}$ with $P(x)$, $Q_1(x)$ and $Q_2(x)$ being polynomials with real coefficients and we are going to assume that when $P(x)$, $Q_1(x)$ and $Q_2(x)$ are factored as in ($\star$), no two of them have a common linear or quadratic factor. As an example, no two of

$$
\begin{aligned}
P(x)&=2(x-3)(x-4)(x^2+3x+3)\\
Q_1(x)&=2(x-1)(x^2+2x+2)\\
Q_2(x)&=2(x-2)(x^2+2x+3)
\end{aligned}
$$

have such a common factor. But, for

$$
\begin{aligned}
P(x)&=2(x-3)(x-4)(x^2+x+1)\\
Q_1(x)&=2(x-1)(x^2+2x+2)\\
Q_2(x)&=2(x-2)(x^2+x+1)
\end{aligned}
$$

$P(x)$ and $Q_2(x)$ have the common factor x^2+x+1.

Lemma 1.10.15

Let $P(x)$, $Q_1(x)$ and $Q_2(x)$ be polynomials with real coefficients and with $\deg(P)<\deg(Q_1Q_2)$. Assume that no two of $P(x)$, $Q_1(x)$ and $Q_2(x)$ have a common linear or quadratic factor. Then there are polynomials P_1, P_2 with $\deg(P_1)<\deg(Q_1)$, $\deg(P_2)<\deg(Q_2)$, and

$$\frac{P(x)}{Q_1(x)\,Q_2(x)}=\frac{P_1(x)}{Q_1(x)}+\frac{P_2(x)}{Q_2(x)}$$

Proof. We are to find polynomials P_1 and P_2 that obey

$$P(x)=P_1(x)\,Q_2(x)+P_2(x)\,Q_1(x)$$

Actually, we are going to find polynomials p_1 and p_2 that obey

$$p_1(x)\,Q_1(x)+p_2(x)\,Q_2(x)=C \tag{$\star\star$}$$

for some nonzero constant C, and then just multiply ($\star\star$) by $\frac{P(x)}{C}$. To find p_1, p_2 and C we are going to use something called the Euclidean algorithm. It is an algorithm[a] that is used to efficiently find the greatest common divisors of two numbers. Because $Q_1(x)$ and $Q_2(x)$ have no common factors of degree 1 or 2, their ``greatest common divisor" has degree 0, i.e. is a constant.

- The first step is to apply long division to $\frac{Q_1(x)}{Q_2(x)}$ to find polynomials $n_0(x)$ and $r_0(x)$ such that

$$\frac{Q_1(x)}{Q_2(x)} = n_0(x) + \frac{r_0(x)}{Q_2(x)} \qquad \text{with } \deg(r_0) < \deg(Q_2)$$

or, equivalently,

$$Q_1(x) = n_0(x)\,Q_2(x) + r_0(x) \qquad \text{with } \deg(r_0) < \deg(Q_2)$$

- The second step is to apply long division to $\frac{Q_2(x)}{r_0(x)}$ to find polynomials $n_1(x)$ and $r_1(x)$ such that

$$Q_2(x) = n_1(x)\,r_0(x) + r_1(x) \qquad \text{with } \deg(r_1) < \deg(r_0) \text{ or } r_1(x) = 0$$

- The third step (assuming that $r_1(x)$ was not zero) is to apply long division to $\frac{r_0(x)}{r_1(x)}$ to find polynomials $n_2(x)$ and $r_2(x)$ such that

$$r_0(x) = n_2(x)\,r_1(x) + r_2(x) \qquad \text{with } \deg(r_2) < \deg(r_1) \text{ or } r_2(x) = 0$$

- And so on.

As the degree of the remainder $r_i(x)$ decreases by at least one each time i is increased by one, the above iteration has to terminate with some $r_{\ell+1}(x) = 0$. That is, we choose ℓ to be index of the last nonzero remainder. Here is a summary of all of the long division steps.

$$\begin{aligned}
Q_1(x) &= n_0(x)\,Q_2(x) + r_0(x) && \text{with } \deg(r_0) < \deg(Q_2)\\
Q_2(x) &= n_1(x)\,r_0(x) + r_1(x) && \text{with } \deg(r_1) < \deg(r_0)\\
r_0(x) &= n_2(x)\,r_1(x) + r_2(x) && \text{with } \deg(r_2) < \deg(r_1)\\
r_1(x) &= n_3(x)\,r_2(x) + r_3(x) && \text{with } \deg(r_3) < \deg(r_2)\\
&\ \ \vdots\\
r_{\ell-2}(x) &= n_\ell(x)\,r_{\ell-1}(x) + r_\ell(x) && \text{with } \deg(r_\ell) < \deg(r_{\ell-1})\\
r_{\ell-1}(x) &= n_{\ell+1}(x)\,r_\ell(x) + r_{\ell+1}(x) && \text{with } r_{\ell+1} = 0
\end{aligned}$$

Now we are going to take a closer look at all of the different remainders that we have generated.

- From first long division step, namely $Q_1(x) = n_0(x)\,Q_2(x) + r_0(x)$ we have that the remainder

$$r_0(x) = Q_1(x) - n_0(x)\,Q_2(x)$$

- From the second long division step, namely $Q_2(x) = n_1(x)\,r_0(x) + r_1(x)$ we have that the remainder

$$\begin{aligned} r_1(x) &= Q_2(x) - n_1(x)\,r_0(x) = Q_2(x) - n_1(x)\big[Q_1(x) - n_0(x)\,Q_2(x)\big] \\ &= A_1(x)\,Q_1(x) + B_1(x)\,Q_2(x) \end{aligned}$$

  with $A_1(x) = -n_1(x)$ and $B_1(x) = 1 + n_0(x)\,n_1(x)$.

- From the third long division step (assuming that $r_1(x)$ was not zero), namely $r_0(x) = n_2(x)\,r_1(x) + r_2(x)$, we have that the remainder

$$\begin{aligned} r_2(x) &= r_0(x) - n_2(x)\,r_1(x) \\ &= \big[Q_1(x) - n_0(x)\,Q_2(x)\big] - n_2(x)\big[A_1(x)\,Q_1(x) + B_1(x)\,Q_2(x)\big] \\ &= A_2(x)\,Q_1(x) + B_2(x)\,Q_2(x) \end{aligned}$$

  with $A_2(x) = 1 - n_2(x)\,A_1(x)$ and $B_2(x) = -n_0(x) - n_2(x)\,B_1(x)$.

- And so on. Continuing in this way, we conclude that the final nonzero remainder $r_\ell(x) = A_\ell(x)\,Q_1(x) + B_\ell(x)\,Q_2(x)$ for some polynomials A_ℓ and B_ℓ.

Now the last nonzero remainder $r_\ell(x)$ has to be a nonzero constant C because

- it is nonzero by the definition of $r_\ell(x)$ and
- if $r_\ell(x)$ were a polynomial of degree at least one, then
  - $r_\ell(x)$ would be a factor of $r_{\ell-1}(x)$ because $r_{\ell-1}(x) = n_{\ell+1}(x)\,r_\ell(x)$ and
  - $r_\ell(x)$ would be a factor of $r_{\ell-2}(x)$ because $r_{\ell-2}(x) = n_\ell(x)\,r_{\ell-1}(x) + r_\ell(x)$ and
  - $r_\ell(x)$ would be a factor of $r_{\ell-3}(x)$ because $r_{\ell-3}(x) = n_{\ell-1}(x)\,r_{\ell-2}(x) + r_{\ell-1}(x)$ and
  - $\cdots$ and $\cdots$
  - $r_\ell(x)$ would be a factor of $r_1(x)$ because $r_1(x) = n_3(x)\,r_2(x) + r_3(x)$ and
  - $r_\ell(x)$ would be a factor of $r_0(x)$ because $r_0(x) = n_2(x)\,r_1(x) + r_2(x)$ and
  - $r_\ell(x)$ would be a factor of $Q_2(x)$ because $Q_2(x) = n_1(x)\,r_0(x) + r_1(x)$ and
  - $r_\ell(x)$ would be a factor of $Q_1(x)$ because $Q_1(x) = n_0(x)\,Q_2(x) + r_0(x)$
- so that $r_\ell(x)$ would be a common factor for $Q_1(x)$ and $Q_2(x)$, in contradiction to the hypothesis that no two of $P(x)$, $Q_1(x)$ and $Q_2(x)$ have a common linear or quadratic factor.

We now have that $A_\ell(x)\,Q_1(x) + B_\ell(x)\,Q_2(x) = r_\ell(x) = C$. Multiplying by $\frac{P(x)}{C}$ gives

$$\tilde{P}_2(x)\,Q_1(x) + \tilde{P}_1(x)\,Q_2(x) = P(x) \quad \text{or} \quad \frac{\tilde{P}_1(x)}{Q_1(x)} + \frac{\tilde{P}_2(x)}{Q_2(x)} = \frac{P(x)}{Q_1(x)\,Q_2(x)}$$

with $\tilde{P}_2(x) = \frac{P(x)\,A_\ell(x)}{C}$ and $\tilde{P}_1(x) = \frac{P(x)\,B_\ell(x)}{C}$. We're not quite done, because there is still the danger that $\deg(\tilde{P}_1) \geq \deg(Q_1)$ or $\deg(\tilde{P}_2) \geq \deg(Q_2)$. To deal with that possibility, we long divide $\frac{\tilde{P}_1(x)}{Q_1(x)}$ and call the remainder $P_1(x)$.

$$\frac{\tilde{P}_1(x)}{Q_1(x)} = N(x) + \frac{P_1(x)}{Q_1(x)} \qquad \text{with } \deg(P_1) < \deg(Q_1)$$

Therefore we have that

$$\begin{aligned} \frac{P(x)}{Q_1(x)\,Q_2(x)} &= \frac{P_1(x)}{Q_1(x)} + N(x) + \frac{\tilde{P}_2(x)}{Q_2(x)} \\ &= \frac{P_1(x)}{Q_1(x)} + \frac{\tilde{P}_2(x) + N(x)Q_2(x)}{Q_2(x)} \end{aligned}$$

Denoting $P_2(x) = \tilde{P}_2(x) + N(x)Q_2(x)$ gives $\frac{P}{Q_1\,Q_2} = \frac{P_1}{Q_1} + \frac{P_2}{Q_2}$ and since $\deg(P_1) <$ $\deg(Q_1)$, the only thing left to prove is that $\deg(P_2) < \deg(Q_2)$. We assume that $\deg(P_2) \geq \deg(Q_2)$ and look for a contradiction. We have

$$\begin{aligned} &\deg(P_2 Q_1) \geq \deg(Q_1 Q_2) > \deg(P_1 Q_2) \\ &\implies \deg(P) = \deg(P_1 Q_2 + P_2 Q_1) = \deg(P_2 Q_1) \geq \deg(Q_1 Q_2) \end{aligned}$$

which contradicts the hypothesis that $\deg(P) < \deg(Q_1 Q_2)$ and the proof is complete.

□

a It appears in Euclid's Elements, which was written about 300 BC, and it was probably known even before that.

For the second of the two simpler results, that we'll shortly use repeatedly to get Equation 1.10.11, we consider $\frac{P(x)}{(x-a)^m}$ and $\frac{P(x)}{(x^2+bx+c)^m}$.

Lemma 1.10.16

Let $m \geq 2$ be an integer, and let $Q(x)$ be either $x - a$ or $x^2 + bx + c$, with a, b and c being real numbers. Let $P(x)$ be a polynomial with real coefficients, which does not contain $Q(x)$ as a factor, and with $\deg(P) < \deg(Q^m) = m\deg(Q)$. Then, for each $1 \leq i \leq m$, there is a polynomial P_i with $\deg(P_i) < \deg(Q)$ or $P_i = 0$, such that

$$\frac{P(x)}{Q(x)^m} = \frac{P_1(x)}{Q(x)} + \frac{P_2(x)}{Q(x)^2} + \frac{P_3(x)}{Q(x)^3} + \cdots + \frac{P_{m-1}(x)}{Q(x)^{m-1}} + \frac{P_m(x)}{Q(x)^m}.$$

In particular, if $Q(x) = x - a$, then each $P_i(x)$ is just a constant A_i, and if $Q(x) = x^2 + bx + c$, then each $P_i(x)$ is a polynomial $B_i x + C_i$ of degree at most one.

Proof. We simply repeatedly use long divison to get

$$\begin{aligned}
\frac{P(x)}{Q(x)^m} &= \frac{P(x)}{Q(x)} \frac{1}{Q(x)^{m-1}} = \left\{ n_1(x) + \frac{r_1(x)}{Q(x)} \right\} \frac{1}{Q(x)^{m-1}} \\
&= \frac{r_1(x)}{Q(x)^m} + \frac{n_1(x)}{Q(x)} \frac{1}{Q(x)^{m-2}} \\
&= \frac{r_1(x)}{Q(x)^m} + \left\{ n_2(x) + \frac{r_2(x)}{Q(x)} \right\} \frac{1}{Q(x)^{m-2}} \\
&= \frac{r_1(x)}{Q(x)^m} + \frac{r_2(x)}{Q(x)^{m-1}} + \frac{n_2(x)}{Q(x)} \frac{1}{Q(x)^{m-3}} \\
&\vdots \\
&= \frac{r_1(x)}{Q(x)^m} + \frac{r_2(x)}{Q(x)^{m-1}} + \cdots + \frac{r_{m-2}(x)}{Q(x)^3} + \frac{n_{m-2}(x)}{Q(x)} \frac{1}{Q(x)} \\
&= \frac{r_1(x)}{Q(x)^m} + \frac{r_2(x)}{Q(x)^{m-1}} + \cdots + \frac{r_{m-2}(x)}{Q(x)^3} + \\
&\qquad\qquad \left\{ n_{m-1}(x) + \frac{r_{m-1}(x)}{Q(x)} \right\} \frac{1}{Q(x)} \\
&= \frac{r_1(x)}{Q(x)^m} + \frac{r_2(x)}{Q(x)^{m-1}} + \cdots + \frac{r_{m-2}(x)}{Q(x)^3} + \frac{r_{m-1}(x)}{Q(x)^2} + \frac{n_{m-1}(x)}{Q(x)}
\end{aligned}$$

By the rules of long division every $\deg(r_i) < \deg(Q)$. It is also true that the final numerator, n_{m-1}, has $\deg(n_{m-1}) < \deg(Q)$ — that is, we kept dividing by Q until the degree of the quotient was less than the degree of Q. To see this, note that $\deg(P) < m\deg(Q)$ and

$$\begin{aligned}
\deg(n_1) &= \deg(P) - \deg(Q) \\
\deg(n_2) &= \deg(n_1) - \deg(Q) = \deg(P) - 2\deg(Q) \\
&\vdots \\
\deg(n_{m-1}) &= \deg(n_{m-2}) - \deg(Q) = \deg(P) - (m-1)\deg(Q) \\
&< m\deg(Q) - (m-1)\deg(Q) \\
&= \deg(Q)
\end{aligned}$$

So, if $\deg(Q) = 1$, then $r_1, r_2, \ldots, r_{m-1}, n_{m-1}$ are all real numbers, and if $\deg(Q) = 2$, then $r_1, r_2, \ldots, r_{m-1}, n_{m-1}$ all have degree at most one.

$\square$

We are now in a position to get Equation 1.10.11. We use $(\star)$ to factor[8] $D(x) = (x - a_1)^{m_1} Q_2(x)$ and use Lemma 1.10.15 to get

$$\frac{N(x)}{D(x)} = \frac{N(x)}{(x-a_1)^{m_1} Q_2(x)} = \frac{P_1(x)}{(x-a_1)^{m_1}} + \frac{P_2(x)}{Q_2(x)}$$

where $\deg(P_1) < m_1$, and $\deg(P_2) < \deg(Q_2)$. Then we use Lemma Lemma 1.10.16 to get

$$\frac{N(x)}{D(x)} = \frac{P_1(x)}{(x-a_1)^{m_1}} + \frac{P_2(x)}{Q_2(x)} = \frac{A_{1,1}}{x-a_1} + \frac{A_{1,2}}{(x-a_1)^2} + \cdots + \frac{A_{1,m_1}}{(x-a_1)^{m_1}} + \frac{P_2(x)}{Q_2(x)}$$

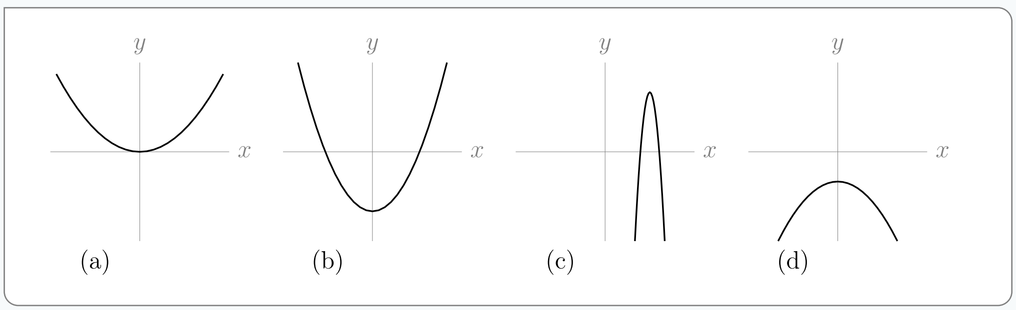
We continue working on $\frac{P_2(x)}{Q_2(x)}$ in this way, pulling off of the denominator one $(x - a_i)^{m_i}$ or one $(x^2 + b_i x + c_i)^{n_i}$ at a time, until we exhaust all of the factors in the denominator $D(x)$.

1.10.4 ➤➤ Exercises

Recall that we are using $\log x$ to denote the logarithm of x with base e. In other courses it is often denoted $\ln x$.

Exercises — Stage 1

1. Below are the graphs of four different quadratic functions. For each quadratic function, decide whether it is: (i) irreducible, (ii) the product of two distinct linear factors, or (iii) the product of a repeated linear factor (and possibly a constant).



8 This is assuming that there is at least one linear factor. If not, we factor $D(x) = (x^2 + b_1 x + c_1)^{n_1} Q_2(x)$ instead.

2. $*$. Write out the ***general form*** of the partial-fractions decomposition of $\dfrac{x^3+3}{(x^2-1)^2(x^2+1)}$. You need not determine the values of any of the coefficients.

3. $*$. Find the coefficient of $\dfrac{1}{x-1}$ in the partial fraction decomposition of $\dfrac{3x^3-2x^2+11}{x^2(x-1)(x^2+3)}$.

4. Re-write the following rational functions as the sum of a polynomial and a rational function whose numerator has a strictly smaller degree than its denominator. (Remember our method of partial fraction decomposition of a rational function only works when the degree of the numerator is strictly smaller than the degree of the denominator.)

a $\dfrac{x^3+2x+2}{x^2+1}$

b $\dfrac{15x^4+6x^3+34x^2+4x+20}{5x^2+2x+8}$

c $\dfrac{2x^5+9x^3+12x^2+10x+30}{2x^2+5}$

5. Factor the following polynomials into linear and irreducible factors.

a $5x^3-3x^2-10x+6$

b x^4-3x^2-5

c $x^4-4x^3-10x^2-11x-6$

d $2x^4+12x^3-x^2-52x+15$

6. Here is a fact:

Suppose we have a rational function with a repeated linear factor $(ax+b)^n$ in the denominator, and the degree of the numerator is strictly less than the degree of the denominator. In the partial fraction decomposition, we can replace the terms

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+\cdots+\frac{A_n}{(ax+b)^n} \tag{1}$$

with the single term

$$\frac{B_1+B_2x+B_3x^2+\cdots+B_nx^{n-1}}{(ax+b)^n} \tag{2}$$

and still be guaranteed to find a solution.

Why do we use the sum in (1), rather than the single term in (2), in partial fraction decomposition?

Exercises — Stage 2

7. *. Evaluate $\displaystyle\int_1^2 \frac{\mathrm{d}x}{x+x^2}$.

8. *. Calculate $\displaystyle\int \frac{1}{x^4+x^2}\,\mathrm{d}x$.

9. *. Calculate $\displaystyle\int \frac{12x+4}{(x-3)(x^2+1)}\,\mathrm{d}x$.

10. *. Evaluate the following indefinite integral using partial fraction:

$$F(x) = \int \frac{3x^2-4}{(x-2)(x^2+4)}\,\mathrm{d}x.$$

11. *. Evaluate $\displaystyle\int \frac{x-13}{x^2-x-6}\mathrm{d}x$.

12. *. Evaluate $\displaystyle\int \frac{5x+1}{x^2+5x+6}\mathrm{d}x$.

13. Evaluate $\displaystyle\int \frac{5x^2-3x-1}{x^2-1}\mathrm{d}x$.

14. Evaluate $\displaystyle\int \frac{4x^4+14x^2+2}{4x^4+x^2}\mathrm{d}x$.

15. Evaluate $\displaystyle\int \frac{x^2+2x-1}{x^4-2x^3+x^2}\mathrm{d}x$.

16. Evaluate $\displaystyle\int \frac{3x^2-4x-10}{2x^3-x^2-8x+4}\mathrm{d}x$.

17. Evaluate $\displaystyle\int_0^1 \frac{10x^2+24x+8}{2x^3+11x^2+6x+5}\mathrm{d}x$.

Exercises — Stage 3 In Questions 18 and 19, we use partial fraction to find the antiderivatives of two important functions: cosecant, and cosecant cubed.The purpose of performing a partial fraction decomposition is to manipulate an integrand into a form that is easily integrable. These "easily integrable" forms are rational functions whose denominator is a power of a linear function, or of an irreducible quadratic function. In Questions 20 through 23, we explore the integration of rational functions whose denominators involve irreducible quadratics.In Questions 24 through 26, we use substitution to turn a non-rational integrand into a rational integrand, then evaluate the resulting integral using partial fraction. Till now, the partial fraction problems you've seen have all looked largely the same, but keep in mind that a partial fraction decomposition can be a small step in a larger problem.

18. Using the method of Example 1.10.5, integrate $\displaystyle\int \csc x \, dx$.

19. Using the method of Example 1.10.6, integrate $\displaystyle\int \csc^3 x \, dx$.

20. Evaluate $\displaystyle\int_1^2 \frac{3x^3 + 15x^2 + 35x + 10}{x^4 + 5x^3 + 10x^2} dx$.

21. Evaluate $\displaystyle\int \left(\frac{3}{x^2+2} + \frac{x-3}{(x^2+2)^2} \right) dx$.

22. Evaluate $\displaystyle\int \frac{1}{(1+x^2)^3} dx$.

23. Evaluate $\displaystyle\int \left(3x + \frac{3x+1}{x^2+5} + \frac{3x}{(x^2+5)^2} \right) dx$.

24. Evaluate $\displaystyle\int \frac{\cos\theta}{3\sin\theta + \cos^2\theta - 3} d\theta$.

25. Evaluate $\displaystyle\int \frac{1}{e^{2t} + e^t + 1} dt$.

26. Evaluate $\displaystyle\int \sqrt{1+e^x} dx$ using partial fraction.

27. $*$. The region R is the portion of the first quadrant where $3 \le x \le 4$ and $0 \le y \le \dfrac{10}{\sqrt{25-x^2}}$.

a Sketch the region R.

b Determine the volume of the solid obtained by revolving R around the x-axis.

c Determine the volume of the solid obtained by revolving R around the y-axis.

28. Find the area of the finite region bounded by the curves $y = \dfrac{4}{3+x^2}$, $y = \dfrac{2}{x(x+1)}$, $x = \dfrac{1}{4}$, and $x = 3$.

29. Let $F(x) = \displaystyle\int_1^x \frac{1}{t^2-9} dt$.

a Give a formula for $F(x)$ that does not involve an integral.

b Find $F'(x)$.

1.11 ▲ Numerical Integration

By now the reader will have come to appreciate that integration is generally quite a bit more difficult than differentiation. There are a great many simple-looking integrals, such as $\int e^{-x^2} \mathrm{d}x$, that are either very difficult or even impossible to express in terms of standard functions [1]. Such integrals are not merely mathematical curiosities, but arise very naturally in many contexts. For example, the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t$$

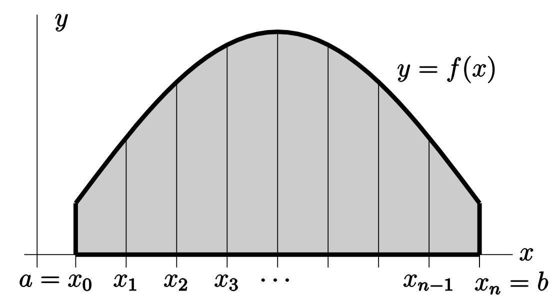
is extremely important in many areas of mathematics, and also in many practical applications of statistics.

In such applications we need to be able to evaluate this integral (and many others) at a given numerical value of x. In this section we turn to the problem of how to find (approximate) numerical values for integrals, without having to evaluate them algebraically. To develop these methods we return to Riemann sums and our geometric interpretation of the definite integral as the signed area.

We start by describing (and applying) three simple algorithms for generating, numerically, approximate values for the definite integral $\int_a^b f(x)\,\mathrm{d}x$. In each algorithm, we begin in much the same way as we approached Riemann sums.

- We first select an integer $n > 0$, called the "number of steps".
- We then divide the interval of integration, $a \le x \le b$, into n equal subintervals, each of length $\Delta x = \frac{b-a}{n}$. The first subinterval runs from $x_0 = a$ to $x_1 = a + \Delta x$. The second runs from x_1 to $x_2 = a + 2\Delta x$, and so on. The last runs from $x_{n-1} = b - \Delta x$ to $x_n = b$.

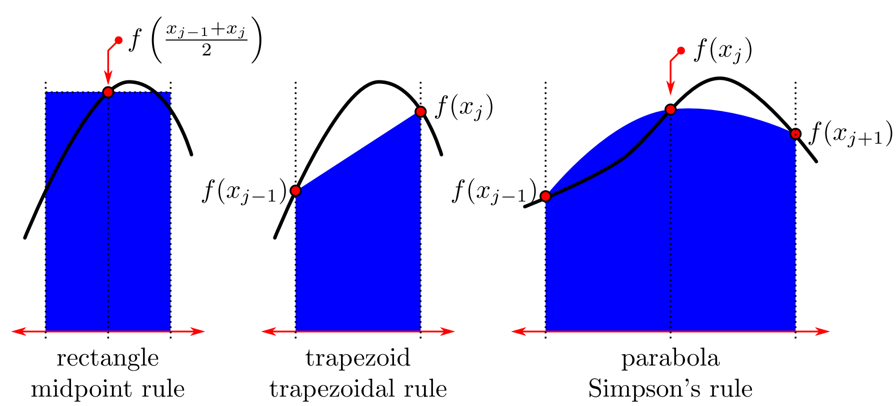
1 We apologise for being a little sloppy here — but we just want to say that it can be very hard or even impossible to write some integrals as some finite sized expression involving polynomials, exponentials, logarithms and trigonometric functions. We don't want to get into a discussion of computability, though that is a very interesting topic.



This splits the original integral into n pieces:

$$\int_a^b f(x)\,\mathrm{d}x = \int_{x_0}^{x_1} f(x)\,\mathrm{d}x + \int_{x_1}^{x_2} f(x)\,\mathrm{d}x + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,\mathrm{d}x$$

Each subintegral $\int_{x_{j-1}}^{x_j} f(x)\,\mathrm{d}x$ is approximated by the area of a simple geometric figure. The three algorithms we consider approximate the area by rectangles, trapezoids and parabolas (respectively).



We will explain these rules in detail below, but we give a brief overview here:

1 The midpoint rule approximates each subintegral by the area of a rectangle of height given by the value of the function at the midpoint of the subinterval

$$\int_{x_{j-1}}^{x_j} f(x)\mathrm{d}x \approx f\left(\frac{x_{j-1}+x_j}{2}\right)\Delta x$$

This is illustrated in the leftmost figure above.

2 The trapezoidal rule approximates each subintegral by the area of a trapezoid with vertices at $(x_{j-1}, 0), (x_{j-1}, f(x_{j-1})), (x_j, f(x_j)), (x_j, 0)$:

$$\int_{x_{j-1}}^{x_j} f(x)\mathrm{d}x \approx \frac{1}{2}\left[f(x_{j-1}) + f(x_j)\right]\Delta x$$

The trapezoid is illustrated in the middle figure above. We shall derive the formula for the area shortly.

3 Simpson's rule approximates two adjacent subintegrals by the area under a parabola that passes through the points $(x_{j-1}, f(x_{j-1}))$, $(x_j, f(x_j))$ and $(x_{j+1}, f(x_{j+1}))$:

$$\int_{x_{j-1}}^{x_{j+1}} f(x)\mathrm{d}x \approx \frac{1}{3}\left[f(x_{j-1}) + 4f(x_j) + f(x_{j+1})\right]\Delta x$$

The parabola is illustrated in the right hand figure above. We shall derive the formula for the area shortly.

Definition 1.11.1 Midpoints.

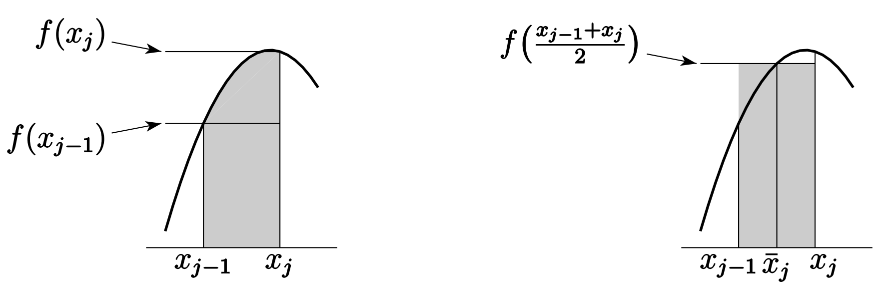
In what follows we need to refer to the midpoint between x_{j-1} and x_j very frequently. To save on writing (and typing) we introduce the notation

$$\bar{x}_j = \frac{1}{2}\left(x_{j-1} + x_j\right).$$

1.11.1 ▸▸ The midpoint rule

The integral $\int_{x_{j-1}}^{x_j} f(x)\,\mathrm{d}x$ represents the area between the curve $y = f(x)$ and the x-axis with x running from x_{j-1} to x_j. The width of this region is $x_j - x_{j-1} = \Delta x$. The height varies over the different values that $f(x)$ takes as x runs from x_{j-1} to x_j.

The midpoint rule approximates this area by the area of a rectangle of width $x_j - x_{j-1} = \Delta x$ and height $f(\bar{x}_j)$ which is the exact height at the midpoint of the range covered by x.



The area of the approximating rectangle is $f(\bar{x}_j)\Delta x$, and the midpoint rule approximates each subintegral by

$$\int_{x_{j-1}}^{x_j} f(x)\,\mathrm{d}x \approx f(\bar{x}_j)\Delta x.$$

Applying this approximation to each subinterval and summing gives us the following approximation of the full integral:

$$\int_a^b f(x)\,\mathrm{d}x = \int_{x_0}^{x_1} f(x)\,\mathrm{d}x + \int_{x_1}^{x_2} f(x)\,\mathrm{d}x + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,\mathrm{d}x$$

$$\approx f(\bar{x}_1)\Delta x + f(\bar{x}_2)\Delta x + \cdots + f(\bar{x}_n)\Delta x$$

So notice that the approximation is the sum of the function evaluated at the midpoint of each interval and then multiplied by Δx. Our other approximations will have similar forms.

In summary:

Equation 1.11.2 The midpoint rule.

The midpoint rule approximation is

$$\int_a^b f(x)\,\mathrm{d}x \approx \Big[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)\Big]\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and

$$\begin{array}{llllll} x_0 = a & x_1 = a + \Delta x & x_2 = a + 2\Delta x & \cdots & x_{n-1} = b - \Delta x & x_n = b \\ & \bar{x}_1 = \frac{x_0+x_1}{2} & \bar{x}_2 = \frac{x_1+x_2}{2} & \cdots & \bar{x}_{n-1} = \frac{x_{n-2}+x_{n-1}}{2} & \bar{x}_n = \frac{x_{n-1}+x_n}{2} \end{array}$$

Example 1.11.3 $\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x$.

We approximate the above integral using the midpoint rule with $n = 8$ step.

Solution:

- First we set up all the x-values that we will need. Note that $a = 0$, $b = 1$, $\Delta x = \frac{1}{8}$ and

$$x_0 = 0 \qquad x_1 = \tfrac{1}{8} \qquad x_2 = \tfrac{2}{8} \qquad \cdots \qquad x_7 = \tfrac{7}{8} \qquad x_8 = \tfrac{8}{8} = 1$$

  Consequently

$$\bar{x}_1 = \tfrac{1}{16} \qquad \bar{x}_2 = \tfrac{3}{16} \qquad \bar{x}_3 = \tfrac{5}{16} \qquad \cdots \qquad \bar{x}_8 = \tfrac{15}{16}$$

- We now apply Equation 1.11.2 to the integrand $f(x) = \frac{4}{1+x^2}$:

$$\begin{aligned}\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x &\approx \Bigg[\overbrace{\frac{4}{1+\bar{x}_1^2}}^{f(\bar{x}_1)} + \overbrace{\frac{4}{1+\bar{x}_2^2}}^{f(\bar{x}_2)} + \cdots + \overbrace{\frac{4}{1+\bar{x}_7^2}}^{f(\bar{x}_{n-1})} + \overbrace{\frac{4}{1+\bar{x}_8^2}}^{f(\bar{x}_n)}\Bigg]\Delta x \\ &= \Bigg[\frac{4}{1+\frac{1}{16^2}} + \frac{4}{1+\frac{3^2}{16^2}} + \frac{4}{1+\frac{5^2}{16^2}} + \frac{4}{1+\frac{7^2}{16^2}} + \frac{4}{1+\frac{9^2}{16^2}} \\ &\qquad + \frac{4}{1+\frac{11^2}{16^2}} + \frac{4}{1+\frac{13^2}{16^2}} + \frac{4}{1+\frac{15^2}{16^2}}\Bigg]\frac{1}{8} \\ &= \big[3.98444 + 3.86415 + 3.64413 + 3.35738 + 3.03858 +\end{aligned}$$

$$2.71618 + 2.40941 + 2.12890]\frac{1}{8}$$

$$= 3.1429$$

where we have rounded to four decimal places.

- In this case we can compute the integral exactly (which is one of the reasons it was chosen as a first example):

$$\int_0^1 \frac{4}{1+x^2}\mathrm{d}x = 4\arctan x\Big|_0^1 = \pi$$

- So the error in the approximation generated by eight steps of the midpoint rule is

$$|3.1429 - \pi| = 0.0013$$

- The relative error is then

$$\frac{|\text{approximate} - \text{exact}|}{\text{exact}} = \frac{|3.1429 - \pi|}{\pi} = 0.0004$$

That is the error is 0.0004 times the actual value of the integral.

- We can write this as a percentage error by multiplying it by 100

$$\text{percentage error} = 100 \times \frac{|\text{approximate} - \text{exact}|}{\text{exact}} = 0.04\%$$

That is, the error is about 0.04% of the exact value.

Example 1.11.3

The midpoint rule gives us quite good estimates of the integral without too much work — though it is perhaps a little tedious to do by hand [2]. Of course, it would be very helpful to quantify what we mean by "good" in this context and that requires us to discuss errors.

Definition 1.11.4

Suppose that α is an approximation to A. This approximation has

- absolute error $|A-\alpha|$ and
- relative error $\frac{|A-\alpha|}{|A|}$ and
- percentage error $100\frac{|A-\alpha|}{|A|}$

2 Thankfully it is very easy to write a program to apply the midpoint rule.

We will discuss errors further in Section 1.11.4 below.

Example 1.11.5 $\int_0^\pi \sin x \, dx$.

As a second example, we apply the midpoint rule with $n = 8$ steps to the above integral.

- We again start by setting up all the x-values that we will need. So $a = 0$, $b = \pi$, $\Delta x = \frac{\pi}{8}$ and

$$x_0 = 0 \qquad x_1 = \tfrac{\pi}{8} \qquad x_2 = \tfrac{2\pi}{8} \qquad \cdots \qquad x_7 = \tfrac{7\pi}{8} \qquad x_8 = \tfrac{8\pi}{8} = \pi$$

  Consequently,

$$\bar{x}_1 = \tfrac{\pi}{16} \qquad \bar{x}_2 = \tfrac{3\pi}{16} \qquad \cdots \qquad \bar{x}_7 = \tfrac{13\pi}{16} \qquad \bar{x}_8 = \tfrac{15\pi}{16}$$

- Now apply Equation 1.11.2 to the integrand $f(x) = \sin x$:

$$\begin{aligned}
&\int_0^\pi \sin x \, dx \approx \Big[\sin(\bar{x}_1) + \sin(\bar{x}_2) + \cdots + \sin(\bar{x}_8)\Big]\Delta x \\
&= \Big[\sin(\tfrac{\pi}{16}) + \sin(\tfrac{3\pi}{16}) + \sin(\tfrac{5\pi}{16}) + \sin(\tfrac{7\pi}{16}) + \sin(\tfrac{9\pi}{16}) + \\
&\qquad\qquad \sin(\tfrac{11\pi}{16}) + \sin(\tfrac{13\pi}{16}) + \sin(\tfrac{15\pi}{16})\Big]\tfrac{\pi}{8} \\
&= \Big[0.1951 + 0.5556 + 0.8315 + 0.9808 + 0.9808 + \\
&\qquad\qquad 0.8315 + 0.5556 + 0.1951\Big] \times 0.3927 \\
&= 5.1260 \times 0.3927 = 2.013
\end{aligned}$$

- Again, we have chosen this example so that we can compare it against the exact value:

$$\int_0^\pi \sin x dx = \big[-\cos x\big]_0^\pi = -\cos\pi + \cos 0 = 2.$$

- So with eight steps of the midpoint rule we achieved

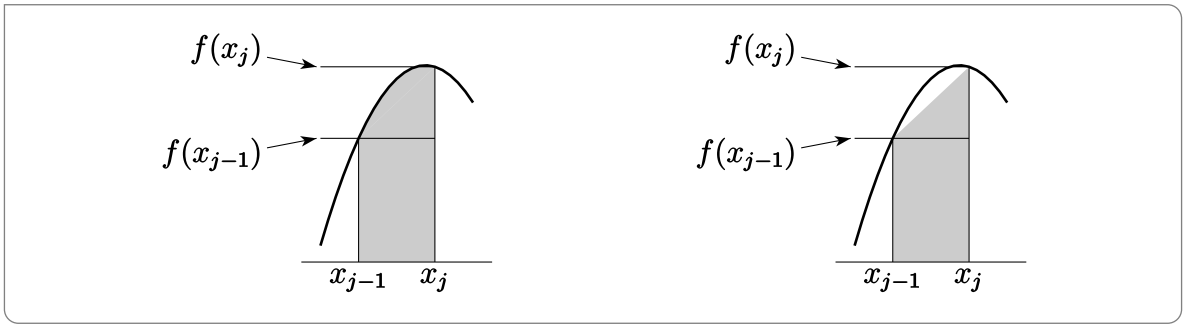
$$\begin{aligned}
\text{absolute error} &= |2.013 - 2| = 0.013 \\
\text{relative error} &= \frac{|2.013 - 2|}{2} = 0.0065 \\
\text{percentage error} &= 100 \times \frac{|2.013 - 2|}{2} = 0.65\%
\end{aligned}$$

  With little work we have managed to estimate the integral to within 1% of its true value.

1.11.2 ➤ The trapezoidal rule

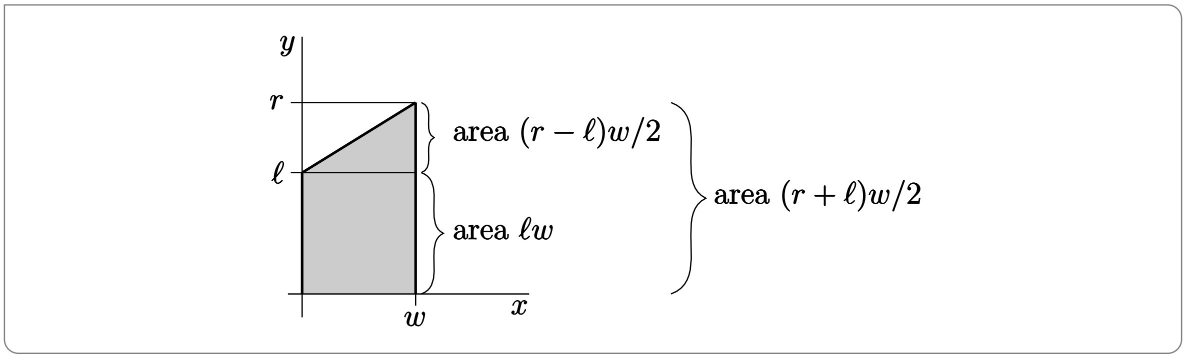
Consider again the area represented by the integral $\int_{x_{j-1}}^{x_j} f(x)\,\mathrm{d}x$. The trapezoidal rule [3] (unsurprisingly) approximates this area by a trapezoid [4] whose vertices lie at

$$(x_{j-1},0),(x_{j-1},f(x_{j-1})),(x_j,f(x_j)) \text{ and } (x_j,0).$$



The trapezoidal approximation of the integral $\int_{x_{j-1}}^{x_j} f(x)\,\mathrm{d}x$ is the shaded region in the figure on the right above. It has width $x_j - x_{j-1} = \Delta x$. Its left hand side has height $f(x_{j-1})$ and its right hand side has height $f(x_j)$.

As the figure below shows, the area of a trapezoid is its width times its average height.



So the trapezoidal rule approximates each subintegral by

$$\int_{x_{j-1}}^{x_j} f(x)\,\mathrm{d}x \approx \tfrac{f(x_{j-1})+f(x_j)}{2}\Delta x$$

Applying this approximation to each subinterval and then summing the result gives us the following approximation of the full integral

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= \int_{x_0}^{x_1} f(x)\,\mathrm{d}x + \int_{x_1}^{x_2} f(x)\,\mathrm{d}x + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,\mathrm{d}x \\ &\approx \tfrac{f(x_0)+f(x_1)}{2}\Delta x + \tfrac{f(x_1)+f(x_2)}{2}\Delta x + \cdots + \tfrac{f(x_{n-1})+f(x_n)}{2}\Delta x\end{aligned}$$

3 This method is also called the “trapezoid rule” and “trapezium rule”.
4 A trapezoid is a four sided polygon, like a rectangle. But, unlike a rectangle, the top and bottom of a trapezoid need not be parallel.

$$= \left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right]\Delta x$$

So notice that the approximation has a very similar form to the midpoint rule, excepting that

- we evaluate the function at the x_j's rather than at the midpoints, and
- we multiply the value of the function at the endpoints x_0, x_n by $\frac{1}{2}$.

In summary:

Equation 1.11.6 The trapezoidal rule.

The trapezoidal rule approximation is

$$\int_a^b f(x)\,\mathrm{d}x \approx \left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right]\Delta x$$

where

$$\Delta x = \tfrac{b-a}{n}, \quad x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \cdots, \quad x_{n-1} = b - \Delta x, \quad x_n = b$$

To compare and contrast we apply the trapezoidal rule to the examples we did above with the midpoint rule.

Example 1.11.7 $\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x$ — using the trapezoidal rule.

Solution: We proceed very similarly to Example 1.11.3 and again use $n = 8$ steps.

- We again have $f(x) = \frac{4}{1+x^2}$, $a = 0$, $b = 1$, $\Delta x = \frac{1}{8}$ and

$$x_0 = 0 \qquad x_1 = \tfrac{1}{8} \qquad x_2 = \tfrac{2}{8} \qquad \cdots \qquad x_7 = \tfrac{7}{8} \qquad x_8 = \tfrac{8}{8} = 1$$

- Applying the trapezoidal rule, Equation 1.11.6, gives

$$\begin{aligned}
\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x &\approx \left[\frac{1}{2}\overbrace{\frac{4}{1+x_0^2}}^{f(x_0)} + \overbrace{\frac{4}{1+x_1^2}}^{f(x_1)} + \cdots + \overbrace{\frac{4}{1+x_7^2}}^{f(x_{n-1})} + \frac{1}{2}\overbrace{\frac{4}{1+x_8^2}}^{f(x_n)}\right]\Delta x \\
&= \left[\frac{1}{2}\frac{4}{1+0^2} + \frac{4}{1+\frac{1}{8^2}} + \frac{4}{1+\frac{2^2}{8^2}} + \frac{4}{1+\frac{3^2}{8^2}}\right. \\
&\qquad \left. + \frac{4}{1+\frac{4^2}{8^2}} + \frac{4}{1+\frac{5^2}{8^2}} + \frac{4}{1+\frac{6^2}{8^2}} + \frac{4}{1+\frac{7^2}{8^2}} + \frac{1}{2}\frac{4}{1+\frac{8^2}{8^2}}\right]\frac{1}{8} \\
&= \left[\frac{1}{2} \times 4 + 3.939 + 3.765 + 3.507\right. \\
&\qquad \left. + 3.2 + 2.876 + 2.56 + 2.266 + \frac{1}{2} \times 2\right]\frac{1}{8}
\end{aligned}$$

$$= 3.139$$

to three decimal places.

- The exact value of the integral is still π. So the error in the approximation generated by eight steps of the trapezoidal rule is $|3.139 - \pi| = 0.0026$, which is $100\frac{|3.139-\pi|}{\pi}\% = 0.08\%$ of the exact answer. Notice that this is roughly twice the error that we achieved using the midpoint rule in Example 1.11.3.

Example 1.11.7

Let us also redo Example 1.11.5 using the trapezoidal rule.

Example 1.11.8 $\int_0^\pi \sin x \, \mathrm{d}x$ — using the trapezoidal rule.

Solution: We proceed very similarly to Example 1.11.5 and again use $n = 8$ steps.

- We again have $a = 0$, $b = \pi$, $\Delta x = \frac{\pi}{8}$ and

$$x_0 = 0 \qquad x_1 = \tfrac{\pi}{8} \qquad x_2 = \tfrac{2\pi}{8} \qquad \cdots \qquad x_7 = \tfrac{7\pi}{8} \qquad x_8 = \tfrac{8\pi}{8} = \pi$$

- Applying the trapezoidal rule, Equation 1.11.6, gives

$$\begin{aligned}
\int_0^\pi \sin x \, \mathrm{d}x &\approx \Big[\frac{1}{2}\sin(x_0) + \sin(x_1) + \cdots + \sin(x_7) + \frac{1}{2}\sin(x_8)\Big]\Delta x \\
&= \Big[\frac{1}{2}\sin 0 + \sin\tfrac{\pi}{8} + \sin\tfrac{2\pi}{8} + \sin\tfrac{3\pi}{8} + \sin\tfrac{4\pi}{8} + \sin\tfrac{5\pi}{8} \\
&\qquad + \sin\tfrac{6\pi}{8} + \sin\tfrac{7\pi}{8} + \frac{1}{2}\sin\tfrac{8\pi}{8}\Big]\tfrac{\pi}{8} \\
&= \Big[\frac{1}{2}\times 0 + 0.3827 + 0.7071 + 0.9239 + 1.0000 + 0.9239 + \\
&\qquad 0.7071 + 0.3827 + \frac{1}{2}\times 0\Big] \times 0.3927 \\
&= 5.0274 \times 0.3927 = 1.974
\end{aligned}$$

- The exact answer is $\int_0^\pi \sin x \, \mathrm{d}x = -\cos x\Big|_0^\pi = 2$. So with eight steps of the trapezoidal rule we achieved $100\frac{|1.974-2|}{2} = 1.3\%$ accuracy. Again this is approximately twice the error we achieved in Example 1.11.5 using the midpoint rule.

These two examples suggest that the midpoint rule is more accurate than the trapezoidal rule. Indeed, this observation is born out by a rigorous analysis of the error — see Section 1.11.4.

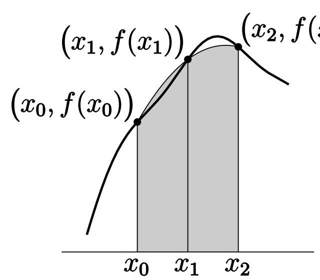
1.11.3 ▸▸ Simpson's Rule

When we use the trapezoidal rule we approximate the area $\int_{x_{j-1}}^{x_j} f(x)\mathrm{d}x$ by the area between the x-axis and a straight line that runs from $(x_{j-1}, f(x_{j-1}))$ to $(x_j, f(x_j))$ — that is, we approximate the function $f(x)$ on this interval by a linear function that agrees with the function at each endpoint. An obvious way to extend this — just as we did when extending linear approximations to quadratic approximations in our differential calculus course — is to approximate the function with a quadratic. This is precisely what Simpson's [5] rule does.

Simpson's rule approximates the integral over two neighbouring subintervals by the area between a parabola and the x-axis. In order to describe this parabola we need 3 distinct points (which is why we approximate two subintegrals at a time). That is, we approximate

$$\int_{x_0}^{x_1} f(x)\,\mathrm{d}x + \int_{x_1}^{x_2} f(x)\,\mathrm{d}x = \int_{x_0}^{x_2} f(x)\,\mathrm{d}x$$

by the area bounded by the parabola that passes through the three points $\big(x_0, f(x_0)\big)$, $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$, the x-axis and the vertical lines $x = x_0$ and $x = x_2$.



We repeat this on the next pair of subintervals and approximate $\int_{x_2}^{x_4} f(x)\,\mathrm{d}x$ by the area between the x-axis and the part of a parabola with $x_2 \le x \le x_4$. This parabola passes through the three points $\big(x_2, f(x_2)\big)$, $\big(x_3, f(x_3)\big)$ and $\big(x_4, f(x_4)\big)$. And so on. Because Simpson's rule does the approximation two slices at a time, *n must be even.*

To derive Simpson's rule formula, we first find the equation of the parabola that passes through the three points $\big(x_0, f(x_0)\big)$, $\big(x_1, f(x_1)\big)$ and $\big(x_2, f(x_2)\big)$. Then we find the area between the x-axis and the part of that parabola with $x_0 \le x \le x_2$. To simplify this computation consider a parabola passing through the points $(-h, y_{-1}), (0, y_0)$ and (h, y_1).

Write the equation of the parabola as

$$y = Ax^2 + Bx + C$$

Then the area between it and the x-axis with x running from $-h$ to h is

$$\int_{-h}^{h} \big[Ax^2 + Bx + C\big]\mathrm{d}x = \left[\frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\right]_{-h}^{h}$$

5 Simpson's rule is named after the 18th century English mathematician Thomas Simpson, despite its use a century earlier by the German mathematician and astronomer Johannes Kepler. In many German texts the rule is often called Kepler's rule.

$$= \frac{2A}{3}h^3 + 2Ch \qquad \text{it is helpful to write it as}$$
$$= \frac{h}{3}\left(2Ah^2 + 6C\right)$$

Now, the the three points $(-h, y_{-1}), (0, y_0)$ and (h, y_1) lie on this parabola if and only if

$$\begin{aligned} Ah^2 - Bh + C &= y_{-1} && \text{at } (-h, y_{-1}) \\ C &= y_0 && \text{at } (0, y_0) \\ Ah^2 + Bh + C &= y_1 && \text{at } (h, y_1) \end{aligned}$$

Adding the first and third equations together gives us

$$2Ah^2 + (B - B)h + 2C = y_{-1} + y_1$$

To this we add four times the middle equation

$$2Ah^2 + 6C = y_{-1} + 4y_0 + y_1.$$

This means that

$$\begin{aligned} \text{area} &= \int_{-h}^{h} \left[Ax^2 + Bx + C\right] \mathrm{d}x = \frac{h}{3}\left(2Ah^2 + 6C\right) \\ &= \frac{h}{3}\left(y_{-1} + 4y_0 + y_1\right) \end{aligned}$$

Note that here

- h is one half of the length of the x-interval under consideration
- y_{-1} is the height of the parabola at the left hand end of the interval under consideration
- y_0 is the height of the parabola at the middle point of the interval under consideration
- y_1 is the height of the parabola at the right hand end of the interval under consideration

So Simpson's rule approximates

$$\int_{x_0}^{x_2} f(x)\,\mathrm{d}x \approx \tfrac{1}{3}\Delta x\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

and

$$\int_{x_2}^{x_4} f(x)\,\mathrm{d}x \approx \tfrac{1}{3}\Delta x\left[f(x_2) + 4f(x_3) + f(x_4)\right]$$

and so on. Summing these all together gives:

$$\int_a^b f(x)\,\mathrm{d}x = \int_{x_0}^{x_2} f(x)\,\mathrm{d}x + \int_{x_2}^{x_4} f(x)\,\mathrm{d}x + \int_{x_4}^{x_6} f(x)\,\mathrm{d}x + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,\mathrm{d}x$$

$$\approx \tfrac{\Delta x}{3}\big[f(x_0)+4f(x_1)+f(x_2)\big] + \tfrac{\Delta x}{3}\big[f(x_2)+4f(x_3)+f(x_4)\big]$$
$$+ \tfrac{\Delta x}{3}\big[f(x_4)+4f(x_5)+f(x_6)\big] + \cdots + \tfrac{\Delta x}{3}\big[f(x_{n-2})+4f(x_{n-1})+f(x_n)\big]$$
$$= \Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big]\tfrac{\Delta x}{3}$$

In summary

Equation 1.11.9 Simpson's rule.

The Simpson's rule approximation is

$$\int_a^b f(x)\,\mathrm{d}x \approx \Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+\cdots$$
$$\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big]\tfrac{\Delta x}{3}$$

where n is even and

$$\Delta x = \tfrac{b-a}{n},\quad x_0=a,\quad x_1=a+\Delta x,\quad x_2=a+2\Delta x,\quad \cdots,\quad x_{n-1}=b-\Delta x,\quad x_n=b$$

Notice that Simpson's rule requires essentially no more work than the trapezoidal rule. In both rules we must evaluate $f(x)$ at $x=x_0,x_1,\cdots,x_n$, but we add those terms multiplied by different constants [6].

Let's put it to work on our two running examples.

Example 1.11.10 $\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x$ **— using Simpson's rule.**

Solution: We proceed almost identically to Example 1.11.7 and again use $n=8$ steps.

- We have the same $\Delta, a, b, x_0, \cdots, x_n$ as Example 1.11.7.
- Applying Equation 1.11.9 gives

$$\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x$$
$$\approx \Bigg[\frac{4}{1+0^2}+4\frac{4}{1+\frac{1}{8^2}}+2\frac{4}{1+\frac{2^2}{8^2}}+4\frac{4}{1+\frac{3^2}{8^2}}+2\frac{4}{1+\frac{4^2}{8^2}}$$
$$+4\frac{4}{1+\frac{5^2}{8^2}}+2\frac{4}{1+\frac{6^2}{8^2}}+4\frac{4}{1+\frac{7^2}{8^2}}+\frac{4}{1+\frac{8^2}{8^2}}\Bigg]\frac{1}{8\times 3}$$
$$= \Big[4+4\times 3.938461538+2\times 3.764705882+4\times 3.506849315+2\times 3.2$$

6 There is an easy generalisation of Simpson's rule that uses cubics instead of parabolas. It is known as Simpson's second rule and Simpson's $\frac{3}{8}$ rule. While one can push this approach further (using quartics, quintics etc), it can sometimes lead to larger errors — the interested reader should look up Runge's phenomenon.

$$+ 4 \times 2.876404494 + 2 \times 2.56 + 4 \times 2.265486726 + 2\Big] \frac{1}{8 \times 3}$$
$$= 3.14159250$$

to eight decimal places.

- This agrees with π (the exact value of the integral) to six decimal places. So the error in the approximation generated by eight steps of Simpson's rule is $|3.14159250 - \pi| = 1.5 \times 10^{-7}$, which is $100\frac{|3.14159250-\pi|}{\pi}\% = 5 \times 10^{-6}\%$ of the exact answer.

Example 1.11.10

It is striking that the absolute error approximating with Simpson's rule is so much smaller than the error from the midpoint and trapezoidal rules.

midpoint error	$= 0.0013$
trapezoid error	$= 0.0026$
Simpson error	$= 0.00000015$

Buoyed by this success, we will also redo Example 1.11.8 using Simpson's rule.

Example 1.11.11 $\int_0^\pi \sin x \,\mathrm{d}x$ — Simpson's rule.

Solution: We proceed almost identically to Example 1.11.8 and again use $n = 8$ steps.

- We have the same $\Delta, a, b, x_0, \cdots, x_n$ as Example 1.11.7.
- Applying Equation 1.11.9 gives

$$\begin{aligned}
\int_0^\pi \sin x \,\mathrm{d}x & \\
&\approx \Big[\sin(x_0) + 4\sin(x_1) + 2\sin(x_2) + \cdots + 4\sin(x_7) + \sin(x_8)\Big]\tfrac{\Delta x}{3}\\
&= \Big[\sin(0) + 4\sin(\tfrac{\pi}{8}) + 2\sin(\tfrac{2\pi}{8}) + 4\sin(\tfrac{3\pi}{8}) + 2\sin(\tfrac{4\pi}{8})\\
&\qquad + 4\sin(\tfrac{5\pi}{8}) + 2\sin(\tfrac{6\pi}{8}) + 4\sin(\tfrac{7\pi}{8}) + \sin(\tfrac{8\pi}{8})\Big]\tfrac{\pi}{8\times 3}\\
&= \Big[0 + 4 \times 0.382683 + 2 \times 0.707107 + 4 \times 0.923880 + 2 \times 1.0\\
&\qquad + 4 \times 0.923880 + 2 \times 0.707107 + 4 \times 0.382683 + 0\Big]\tfrac{\pi}{8\times 3}\\
&= 15.280932 \times 0.130900\\
&= 2.00027
\end{aligned}$$

- With only eight steps of Simpson's rule we achieved $100\frac{2.00027-2}{2} = 0.014\%$ accuracy.

Again we contrast the error we achieved with the other two rules:

$$\begin{aligned} \text{midpoint error} &= 0.013 \\ \text{trapezoid error} &= 0.026 \\ \text{Simpson error} &= 0.00027 \end{aligned}$$

This completes our derivation of the midpoint, trapezoidal and Simpson's rules for approximating the values of definite integrals. So far we have not attempted to see how efficient and how accurate the algorithms are in general. That's our next task.

1.11.4 ▸▸ Three Simple Numerical Integrators — Error Behaviour

Now we are armed with our three (relatively simple) methods for numerical integration we should give thought to how practical they might be in the real world [7]. Two obvious considerations when deciding whether or not a given algorithm is of any practical value are

a the amount of computational effort required to execute the algorithm and

b the accuracy that this computational effort yields.

For algorithms like our simple integrators, the bulk of the computational effort usually goes into evaluating the function $f(x)$. The number of evaluations of $f(x)$ required for n steps of the midpoint rule is n, while the number required for n steps of the trapezoidal and Simpson's rules is $n+1$. So all three of our rules require essentially the same amount of effort — one evaluation of $f(x)$ per step.

To get a first impression of the error behaviour of these methods, we apply them to a problem whose answer we know exactly:

$$\int_0^\pi \sin x \,\mathrm{d}x = -\cos x\Big|_0^\pi = 2.$$

To be a little more precise, we would like to understand how the errors of the three methods change as we increase the effort we put in (as measured by the number of steps n). The following table lists the error in the approximate value for this number generated by our three rules applied with three different choices of n. It also lists the number of evaluations of f required to compute the approximation.

	Midpoint		***Trapezoidal***		***Simpson's***	
n	error	# evals	error	# evals	error	# evals
10	8.2×10^{-3}	10	1.6×10^{-2}	11	1.1×10^{-4}	11
100	8.2×10^{-5}	100	1.6×10^{-4}	101	1.1×10^{-8}	101
1000	8.2×10^{-7}	1000	1.6×10^{-6}	1001	1.1×10^{-12}	1001

Observe that

7 Indeed, even beyond the "real world" of many applications in first year calculus texts, some of the methods we have described are used by actual people (such as ship builders, engineers and surveyors) to estimate areas and volumes of actual objects!

- Using 101 evaluations of f worth of Simpson's rule gives an error 75 times smaller than 1000 evaluations of f worth of the midpoint rule.
- The trapezoidal rule error with n steps is about twice the midpoint rule error with n steps.
- With the midpoint rule, increasing the number of steps by a factor of 10 appears to reduce the error by about a factor of $100 = 10^2 = n^2$.
- With the trapezoidal rule, increasing the number of steps by a factor of 10 appears to reduce the error by about a factor of $10^2 = n^2$.
- With Simpson's rule, increasing the number of steps by a factor of 10 appears to reduce the error by about a factor of $10^4 = n^4$.

So it looks like

$$\begin{aligned}
\text{approx value of } \int_a^b f(x)\,\mathrm{d}x \text{ given by } n \text{ midpoint steps } &\approx \int_a^b f(x)\,\mathrm{d}x + K_M \cdot \frac{1}{n^2}\\
\text{approx value of } \int_a^b f(x)\,\mathrm{d}x \text{ given by } n \text{ trapezoidal steps} &\approx \int_a^b f(x)\,\mathrm{d}x + K_T \cdot \frac{1}{n^2}\\
\text{approx value of } \int_a^b f(x)\,\mathrm{d}x \text{ given by } n \text{ Simpson's steps } &\approx \int_a^b f(x)\,\mathrm{d}x + K_M \cdot \frac{1}{n^4}
\end{aligned}$$

with some constants K_M, K_T and K_S. It also seems that $K_T \approx 2K_M$.

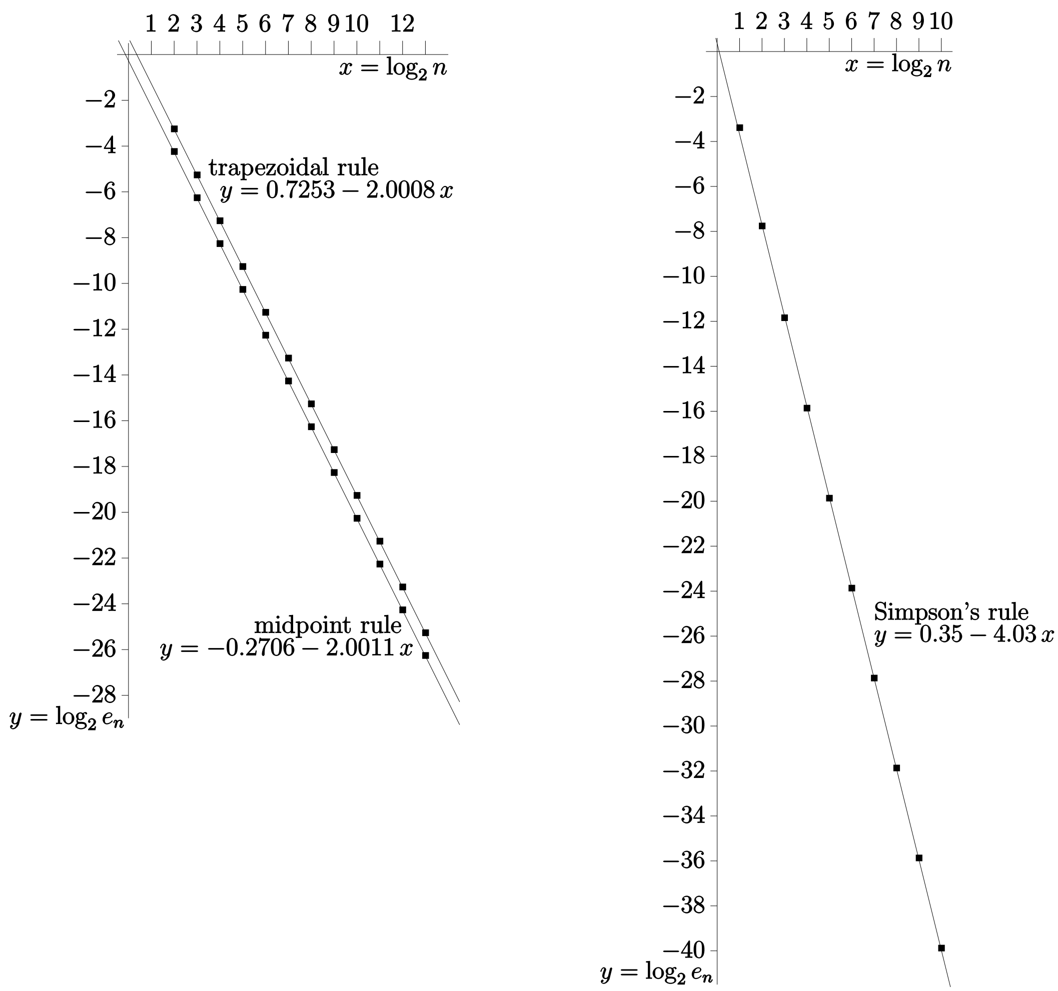


Figure 1.11.12: A log-log plot of the error in the n step approximation to $\displaystyle\int_0^\pi \sin x \,\mathrm{d}x$.

To test these conjectures for the behaviour of the errors we apply our three rules with about ten different choices of n of the form $n = 2^m$ with m integer. Figure 1.11.12 contains two graphs of the results. The left-hand plot shows the results for the midpoint and trapezoidal rules and the right-hand plot shows the results for Simpson's rule.

For each rule we are expecting (based on our conjectures above) that the error

$$e_n = |\text{exact value} \ - \ \text{approximate value}|$$

with n steps is (roughly) of the form

$$e_n = K\frac{1}{n^k}$$

for some constants K and k. We would like to test if this is really the case, by graphing $Y = e_n$ against $X = n$ and seeing if the graph "looks right". But it is not easy to tell whether or not a given curve really is $Y = \frac{K}{X^k}$, for some specific k, by just looking at it. However, your eye is pretty good at determining whether or not a graph is a straight line. Fortunately, there is a little trick that turns the curve $Y = \frac{K}{X^k}$ into a straight line — no matter what k is.

Instead of plotting Y against X, we plot $\log Y$ against $\log X$. This transformation [8] works because when $Y = \frac{K}{X^k}$

$$\log Y = \log K - k \log X$$

So plotting $y = \log Y$ against $x = \log X$ gives the straight line $y = \log K - kx$, which has slope $-k$ and y-intercept $\log K$.

The three graphs in Figure 1.11.12 plot $y = \log_2 e_n$ against $x = \log_2 n$ for our three rules. Note that we have chosen to use logarithms [9] with this "unusual base" because it makes it very clear how much the error is improved if we *double* the number of steps used. To be more precise — one unit step along the x-axis represents changing $n \mapsto 2n$. For example, applying Simpson's rule with $n = 2^4$ steps results in an error of 0000166, so the point $(x = \log_2 2^4 = 4, y = \log_2 0000166 = \frac{\log 0000166}{\log 2} = -15.8)$ has been included on the graph. Doubling the effort used — that is, doubling the number of steps to $n = 2^5$— results in an error of 0.00000103. So, the data point $(x = \log_2 2^5 = 5\ ,\ y = \log_2 0.00000103 = \frac{\ln 0.00000103}{\ln 2} = -19.9)$ lies on the graph. Note that the x-coordinates of these points differ by 1 unit.

For each of the three sets of data points, a straight line has also been plotted "through" the data points. A procedure called linear regression [10] has been used to decide precisely which straight line to plot. It provides a formula for the slope and y-intercept of the straight line which "best fits" any given set of data points. From the three lines, it sure looks like $k = 2$ for the midpoint and trapezoidal rules and $k = 4$ for Simpson's rule. It also looks like the ratio between the value of K for the trapezoidal rule, namely $K = 2^{0.7253}$, and the value of K for the midpoint rule, namely $K = 2^{-0.2706}$, is pretty close to 2: $2^{0.7253}/2^{-0.2706} = 2^{0.9959}$.

The intuition, about the error behaviour, that we have just developed is in fact correct — provided the integrand $f(x)$ is reasonably smooth. To be more precise

Theorem 1.11.13 Numerical integration errors.

Assume that $|f''(x)| \leq M$ for all $a \leq x \leq b$. Then

the total error introduced by the midpoint rule is bounded by $\frac{M}{24}\frac{(b-a)^3}{n^2}$

8 There is a variant of this trick that works even when you don't know the answer to the integral ahead of time. Suppose that you suspect that the approximation satisfies $M_n = A + K\frac{1}{n^k}$ where A is the exact value of the integral and suppose that you don't know the values of A, K and k. Then $M_n - M_{2n} = K\frac{1}{n^k} - K\frac{1}{(2n)^k} = K\big(1 - \frac{1}{2^k}\big)\frac{1}{n^k}$ so plotting $y = \log(M_n - M_{2n})$ against $x = \log n$ gives the straight line $y = \log\big[K\big(1 - \frac{1}{2^k}\big)\big] - kx$.

9 Now is a good time for a quick revision of logarithms — see "Whirlwind review of logarithms" in Section 2.7 of the CLP-1 text.

10 Linear regression is not part of this course as its derivation requires some multivariable calculus. It is a very standard technique in statistics.

and

the total error introduced by the trapezoidal rule is bounded by $\dfrac{M}{12}\dfrac{(b-a)^3}{n^2}$

when approximating $\displaystyle\int_a^b f(x)\mathrm{d}x$. Further, if $|f^{(4)}(x)| \le L$ for all $a \le x \le b$, then

the total error introduced by Simpson's rule is bounded by $\dfrac{L}{180}\dfrac{(b-a)^5}{n^4}$.

The first of these error bounds in proven in the following (optional) section. Here are some examples which illustrate how they are used. First let us check that the above result is consistent with our data in Figure 1.11.12

Example 1.11.14 Midpoint rule error approximating $\int_0^\pi \sin x\,\mathrm{d}x$.

- The integral $\int_0^\pi \sin x\,\mathrm{d}x$ has $b-a=\pi$.
- The second derivative of the integrand satisfies

$$\left|\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin x\right| = |-\sin x| \le 1$$

  So we take $M=1$.
- So the error, e_n, introduced when n steps are used is bounded by

$$\begin{aligned}|e_n| &\le \frac{M}{24}\frac{(b-a)^3}{n^2}\\ &= \frac{\pi^3}{24}\frac{1}{n^2}\\ &\approx 1.29\frac{1}{n^2}\end{aligned}$$

- The data in the graph in Figure 1.11.12 gives

$$|e_n| \approx 2^{-.2706}\frac{1}{n^2} = 0.83\frac{1}{n^2}$$

  which is consistent with the bound $|e_n| \le \frac{\pi^3}{24}\frac{1}{n^2}$.

In a typical application we would be asked to evaluate a given integral to some specified accuracy. For example, if you are manufacturer and your machinery can only cut materials to an accuracy of $\frac{1}{10}^{\text{th}}$ of a millimeter, there is no point in making design specifications more accurate than $\frac{1}{10}^{\text{th}}$ of a millimeter.

Example 1.11.15 How many steps for a given accuracy?

Suppose, for example, that we wish to use the midpoint rule to evaluate [a]

$$\int_0^1 e^{-x^2}\mathrm{d}x$$

to within an accuracy of 10^{-6}.

Solution:

- The integral has $a = 0$ and $b = 1$.
- The first two derivatives of the integrand are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}e^{-x^2} &= -2xe^{-x^2} && \text{and} \\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}e^{-x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\big(-2xe^{-x^2}\big) = -2e^{-x^2} + 4x^2e^{-x^2} = 2(2x^2-1)e^{-x^2} \end{aligned}$$

- As x runs from 0 to 1, $2x^2 - 1$ increases from -1 to 1, so that

$$0 \le x \le 1 \implies |2x^2-1| \le 1,\ e^{-x^2} \le 1 \implies \left|2(2x^2-1)e^{-x^2}\right| \le 2$$

  So we take $M = 2$.

- The error introduced by the n step midpoint rule is at most

$$\begin{aligned} e_n &\le \frac{M}{24}\frac{(b-a)^3}{n^2} \\ &\le \frac{2}{24}\frac{(1-0)^3}{n^2} = \frac{1}{12n^2} \end{aligned}$$

- We need this error to be smaller than 10^{-6} so

$$\begin{aligned} e_n \le \frac{1}{12n^2} &\le 10^{-6} && \text{and so} \\ 12n^2 &\ge 10^6 && \text{clean up} \\ n^2 &\ge \frac{10^6}{12} = 83333.3 && \text{square root both sides} \\ n &\ge 288.7 \end{aligned}$$

  So 289 steps of the midpoint rule will do the job.

- In fact $n = 289$ results in an error of about 3.7×10^{-7}.

a This is our favourite running example of an integral that cannot be evaluated algebraically — we need to use numerical methods.

That seems like far too much work, and the trapezoidal rule will have twice the error. So we should look at Simpson's rule.

Example 1.11.16 How many steps using Simpson's rule?

Suppose now that we wish evaluate $\int_0^1 e^{-x^2}\,\mathrm{d}x$ to within an accuracy of 10^{-6} — but now using Simpson's rule. How many steps should we use?

Solution:

- Again we have $a = 0, b = 1$.
- We then need to bound $\frac{\mathrm{d}^4}{\mathrm{d}x^4}e^{-x^2}$ on the domain of integration, $0 \le x \le 1$.

$$\begin{aligned}\frac{\mathrm{d}^3}{\mathrm{d}x^3}e^{-x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{2(2x^2-1)e^{-x^2}\right\} = 8xe^{-x^2} - 4x(2x^2-1)e^{-x^2}\\ &= 4(-2x^3+3x)e^{-x^2}\\ \frac{\mathrm{d}^4}{\mathrm{d}x^4}e^{-x^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{4(-2x^3+3x)e^{-x^2}\right\}\\ &= 4(-6x^2+3)e^{-x^2} - 8x(-2x^3+3x)e^{-x^2}\\ &= 4(4x^4-12x^2+3)e^{-x^2}\end{aligned}$$

- Now, for any x, $e^{-x^2} \le 1$. Also, for $0 \le x \le 1$,

$$\begin{aligned} 0 &\le x^2, x^4 \le 1 && \text{so}\\ 3 &\le 4x^4+3 \le 7 && \text{and}\\ -12 &\le -12x^2 \le 0 && \text{adding these together gives}\\ -9 &\le 4x^4-12x^2+3 \le 7 \end{aligned}$$

  Consequently, $|4x^4 - 12x^2 + 3|$ is bounded by 9 and so

$$\left|\frac{\mathrm{d}^4}{\mathrm{d}x^4}e^{-x^2}\right| \le 4 \times 9 = 36$$

  So take $L = 36$.

- The error introduced by the n step Simpson's rule is at most

$$\begin{aligned} e_n &\le \frac{L}{180}\frac{(b-a)^5}{n^4}\\ &\le \frac{36}{180}\frac{(1-0)^5}{n^4} = \frac{1}{5n^4}\end{aligned}$$

- In order for this error to be no more than 10^{-6} we require n to satisfy

$$e_n \le \frac{1}{5n^4} \le 10^{-6} \qquad \text{and so}$$

$$
\begin{aligned}
5n^4 &\geq 10^6 \\
n^4 &\geq 200000 && \text{take fourth root} \\
n &\geq 21.15
\end{aligned}
$$

So 22 steps of Simpson's rule will do the job.

- $n = 22$ steps actually results in an error of 3.5×10^{-8}. The reason that we get an error so much smaller than we need is that we have overestimated the number of steps required. This, in turn, occurred because we made quite a rough bound of $\left|\frac{d^4}{dx^4} f(x)\right| \leq 36$. If we are more careful then we will get a slightly smaller n. It actually turns out [a] that you only need $n = 10$ to approximate within 10^{-6}.

a The authors tested this empirically.

Example 1.11.16

1.11.5 ➤➤ Optional — An error bound for the midpoint rule

We now try develop some understanding as to why we got the above experimental results. We start with the error generated by a single step of the midpoint rule. That is, the error introduced by the approximation

$$\int_{x_0}^{x_1} f(x)\, dx \approx f(\bar{x}_1)\Delta x \qquad \text{where } \Delta x = x_1 - x_0,\ \bar{x}_1 = \tfrac{x_0+x_1}{2}$$

To do this we are going to need to apply integration by parts in a sneaky way. Let us start by considering [11] a subinterval $\alpha \leq x \leq \beta$ and let's call the width of the subinterval $2q$ so that $\beta = \alpha + 2q$. If we were to now apply the midpoint rule to this subinterval, then we would write

$$\int_\alpha^\beta f(x)dx \approx 2q \cdot f(\alpha + q) = qf(\alpha + q) + qf(\beta - q)$$

since the interval has width $2q$ and the midpoint is $\alpha + q = \beta - q$.

The sneaky trick we will employ is to write

$$\int_\alpha^\beta f(x)dx = \int_\alpha^{\alpha+q} f(x)dx + \int_{\beta-q}^\beta f(x)dx$$

and then examine each of the integrals on the right-hand side (using integration by parts) and show that they are each of the form

$$\int_\alpha^{\alpha+q} f(x)dx \approx qf(\alpha + q) + \text{small error term}$$

11 We chose this interval so that we didn't have lots of subscripts floating around in the algebra.

$$\int_{\beta-q}^{\beta} f(x)\mathrm{d}x \approx qf(\beta-q) + \text{small error term}$$

Let us apply integration by parts to $\int_\alpha^{\alpha+q} f(x)\mathrm{d}x$ — with $u = f(x), \mathrm{d}v = \mathrm{d}x$ so $\mathrm{d}u = f'(x)\mathrm{d}x$ and we will make the slightly non-standard choice of $v = x - \alpha$:

$$\begin{aligned}\int_\alpha^{\alpha+q} f(x)\mathrm{d}x &= \big[(x-\alpha)f(x)\big]_\alpha^{\alpha+q} - \int_\alpha^{\alpha+q} (x-\alpha)f'(x)\mathrm{d}x \\ &= qf(\alpha+q) - \int_\alpha^{\alpha+q} (x-\alpha)f'(x)\mathrm{d}x\end{aligned}$$

Notice that the first term on the right-hand side is the term we need, and that our non-standard choice of v allowed us to avoid introducing an $f(\alpha)$ term.

Now integrate by parts again using $u = f'(x), \mathrm{d}v = (x-\alpha)\mathrm{d}x$, so $\mathrm{d}u = f''(x), v = \frac{(x-\alpha)^2}{2}$:

$$\begin{aligned}\int_\alpha^{\alpha+q} f(x)\mathrm{d}x &= qf(\alpha+q) - \int_\alpha^{\alpha+q} (x-\alpha)f'(x)\mathrm{d}x \\ &= qf(\alpha+q) - \left[\frac{(x-\alpha)^2}{2}f'(x)\right]_\alpha^{\alpha+q} + \int_\alpha^{\alpha+q} \frac{(x-\alpha)^2}{2}f''(x)\mathrm{d}x \\ &= qf(\alpha+q) - \frac{q^2}{2}f'(\alpha+q) + \int_\alpha^{\alpha+q} \frac{(x-\alpha)^2}{2}f''(x)\mathrm{d}x\end{aligned}$$

To obtain a similar expression for the other integral, we repeat the above steps and obtain:

$$\int_{\beta-q}^{\beta} f(x)\mathrm{d}x = qf(\beta-q) + \frac{q^2}{2}f'(\beta-q) + \int_{\beta-q}^{\beta} \frac{(x-\beta)^2}{2}f''(x)\mathrm{d}x$$

Now add together these two expressions

$$\begin{aligned}\int_\alpha^{\alpha+q} f(x)\mathrm{d}x + \int_{\beta-q}^{\beta} f(x)\mathrm{d}x &= qf(\alpha+q) + qf(\beta-q) + \frac{q^2}{2}\left(f'(\beta-q) - f'(\alpha+q)\right) \\ &\quad + \int_\alpha^{\alpha+q} \frac{(x-\alpha)^2}{2}f''(x)\mathrm{d}x + \int_{\beta-q}^{\beta} \frac{(x-\beta)^2}{2}f''(x)\mathrm{d}x\end{aligned}$$

Then since $\alpha + q = \beta - q$ we can combine the integrals on the left-hand side and eliminate some terms from the right-hand side:

$$\int_\alpha^{\beta} f(x)\mathrm{d}x = 2qf(\alpha+q) + \int_\alpha^{\alpha+q} \frac{(x-\alpha)^2}{2}f''(x)\mathrm{d}x + \int_{\beta-q}^{\beta} \frac{(x-\beta)^2}{2}f''(x)\mathrm{d}x$$

Rearrange this expression a little and take absolute values

$$\left|\int_\alpha^{\beta} f(x)\mathrm{d}x - 2qf(\alpha+q)\right| \leq \left|\int_\alpha^{\alpha+q} \frac{(x-\alpha)^2}{2}f''(x)\mathrm{d}x\right| + \left|\int_{\beta-q}^{\beta} \frac{(x-\beta)^2}{2}f''(x)\mathrm{d}x\right|$$

where we have also made use of the triangle inequality [12]. By assumption $|f''(x)| \leq M$ on the interval $\alpha \leq x \leq \beta$, so

$$\begin{aligned}\left|\int_\alpha^\beta f(x)\mathrm{d}x - 2qf(\alpha+q)\right| &\leq M\int_\alpha^{\alpha+q}\frac{(x-\alpha)^2}{2}\mathrm{d}x + M\int_{\beta-q}^{\beta}\frac{(x-\beta)^2}{2}\mathrm{d}x \\ &= \frac{Mq^3}{3} = \frac{M(\beta-\alpha)^3}{24}\end{aligned}$$

where we have used $q = \frac{\beta-\alpha}{2}$ in the last step.

Thus on any interval $x_i \leq x \leq x_{i+1} = x_i + \Delta x$

$$\left|\int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x - \Delta x f\left(\frac{x_i+x_{i+1}}{2}\right)\right| \leq \frac{M}{24}(\Delta x)^3$$

Putting everything together we see that the error using the midpoint rule is bounded by

$$\begin{aligned}&\left|\int_a^b f(x)\mathrm{d}x - \left[f(\bar{x}_1)+f(\bar{x}_2)+\cdots+f(\bar{x}_n)\right]\Delta x\right| \\ &\leq \left|\int_{x_0}^{x_1} f(x)\mathrm{d}x - \Delta x f(\bar{x}_1)\right| + \cdots + \left|\int_{x_{n-1}}^{x_n} f(x)\mathrm{d}x - \Delta x f(\bar{x}_n)\right| \\ &\leq n\times\frac{M}{24}(\Delta x)^3 = n\times\frac{M}{24}\left(\frac{b-a}{n}\right)^3 = \frac{M(b-a)^3}{24n^2}\end{aligned}$$

as required.

A very similar analysis shows that, as was stated in Theorem 1.11.13 above,

- the total error introduced by the trapezoidal rule is bounded by $\dfrac{M}{12}\dfrac{(b-a)^3}{n^2}$,
- the total error introduced by Simpson's rule is bounded by $\dfrac{M}{180}\dfrac{(b-a)^5}{n^4}$

1.11.6 ▸▸ Exercises

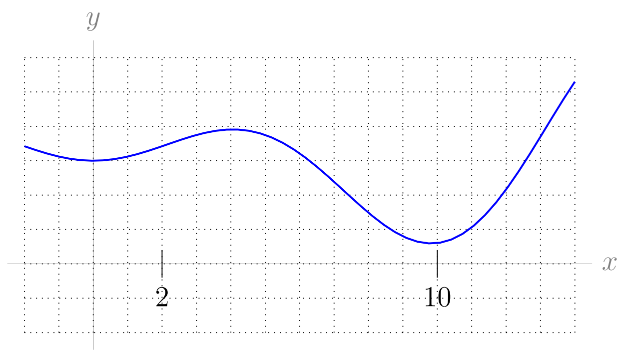
Recall that we are using $\log x$ to denote the logarithm of x with base e. In other courses it is often denoted $\ln x$.

Exercises — Stage 1

1. Suppose we approximate an object to have volume $1.5\mathrm{m}^3$, when its exact volume is $1.387\mathrm{m}^3$. Give the relative error, absolute error, and percent error of our approximation.

12 The triangle inequality says that for any real numbers x, y $|x+y| \leq |x| + |y|$.

2. Consider approximating $\displaystyle\int_2^{10} f(x)\mathrm{d}x$, where $f(x)$ is the function in the graph below.



a Draw the rectangles associated with the midpoint rule approximation and $n = 4$.

b Draw the trapezoids associated with the trapezoidal rule approximation and $n = 4$.

You don't have to give an approximation.

3. Let $f(x) = -\dfrac{1}{12}x^4 + \dfrac{7}{6}x^3 - 3x^2$.

a Find a reasonable value M such that $|f''(x)| \le M$ for all $1 \le x \le 6$.

b Find a reasonable value L such that $|f^{(4)}(x)| \le L$ for all $1 \le x \le 6$.

4. Let $f(x) = x\sin x + 2\cos x$. Find a reasonable value M such that $|f''(x)| \le M$ for all $-3 \le x \le 2$.

5. Consider the quantity $A = \displaystyle\int_{-\pi}^{\pi} \cos x\mathrm{d}x$.

a Find the upper bound on the error using Simpson's rule with $n = 4$ to approximate A using Theorem 1.11.13 in the text.

b Find the Simpson's rule approximation of A using $n = 4$.

c What is the (actual) absolute error in the Simpson's rule approximation of A with $n = 4$?

6. Give a function $f(x)$ such that:

- $f''(x) \le 3$ for every x in $[0, 1]$, and

- the error using the trapezoidal rule approximating $\int_0^1 f(x)\mathrm{d}x$ with $n = 2$ intervals is exactly $\frac{1}{16}$.

7. Suppose my mother is under 100 years old, and I am under 200 years old. [a] Who is older?

[a] We're going somewhere with this.

8.

a True or False: for fixed positive constants M, n, a, and b, with $b > a$,

$$\frac{M}{24}\frac{(b-a)^3}{n^2} \le \frac{M}{12}\frac{(b-a)^3}{n^2}$$

b True or False: for a function $f(x)$ and fixed constants n, a, and b, with $b > a$, the n-interval midpoint approximation of $\int_a^b f(x)\mathrm{d}x$ is more accurate than the n-interval trapezoidal approximation.

9. ***.** Decide whether the following statement is true or false. If false, provide a counterexample. If true, provide a brief justification.

When $f(x)$ is positive and concave up, any trapezoidal rule approximation for $\int_a^b f(x)\,\mathrm{d}x$ will be an upper estimate for $\int_a^b f(x)\,\mathrm{d}x$.

10. Give a polynomial $f(x)$ with the property that the Simpson's rule approximation of $\int_a^b f(x)\mathrm{d}x$ is exact for all a, b, and n.

Exercises — Stage 2 Questions 11 and 12 ask you to approximate a given integral using the formulas in Equations 1.11.2, 1.11.6, and 1.11.9 in the text.Questions 13 though 17 ask you to approximate a quantity based on observed data.In Questions 18 through 24, we practice finding error bounds for our approximations.

11. Write out all three approximations of $\int_0^{30} \frac{1}{x^3+1}\mathrm{d}x$ with $n = 6$. (That is: midpoint, trapezoidal, and Simpson's.) You do not need to simplify your answers.

12. $*$. Find the midpoint rule approximation to $\int_0^\pi \sin x \,dx$ with $n = 3$.

13. $*$. The solid V is 40 cm high and the horizontal cross sections are circular disks. The table below gives the diameters of the cross sections in centimeters at 10 cm intervals. Use the trapezoidal rule to estimate the volume of V.

height	0	10	20	30	40
diameter	24	16	10	6	4

14. $*$. A 6 metre long cedar log has cross sections that are approximately circular. The diameters of the log, measured at one metre intervals, are given below:

metres from left end of log	0	1	2	3	4	5	6
diameter in metres	1.2	1	0.8	0.8	1	1	1.2

Use Simpson's Rule to estimate the volume of the log.

15. $*$. The circumference of an 8 metre high tree at different heights above the ground is given in the table below. Assume that all horizontal cross–sections of the tree are circular disks.

height (metres)	0	2	4	6	8
circumference (metres)	1.2	1.1	1.3	0.9	0.2

Use Simpson's rule to approximate the volume of the tree.

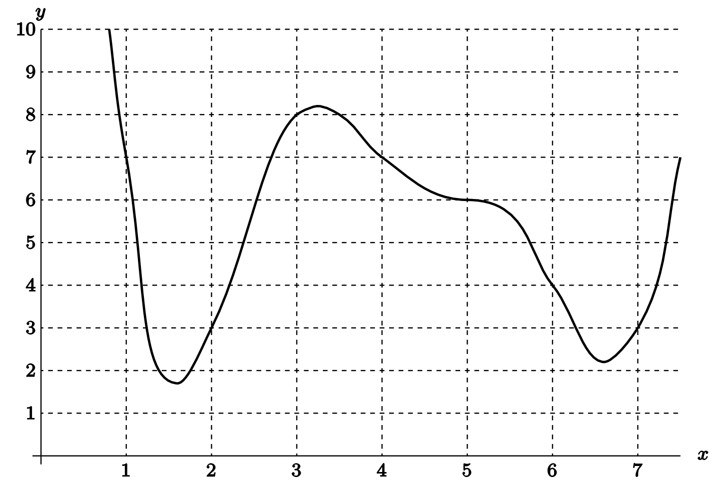
16. $*$. By measuring the areas enclosed by contours on a topographic map, a geologist determines the cross sectional areas A in m^2 of a 60 m high hill. The table below gives the cross sectional area $A(h)$ at various heights h. The volume of the hill is $V = \int_0^{60} A(h)\,dh$.

h	0	10	20	30	40	50	60
A	10,200	9,200	8,000	7,100	4,500	2,400	100

a If the geologist uses the Trapezoidal Rule to estimate the volume of the hill, what will be their estimate, to the nearest $1{,}000\text{m}^3$?

b What will be the geologist's estimate of the volume of the hill if they use Simpson's Rule instead of the Trapezoidal Rule?

17. $*$. The graph below applies to both parts (a) and (b).



a Use the Trapezoidal Rule, with $n = 4$, to estimate the area under the graph between $x = 2$ and $x = 6$. Simplify your answer completely.

b Use Simpson's Rule, with $n = 4$, to estimate the area under the graph between $x = 2$ and $x = 6$.

18. $*$. The integral $\displaystyle\int_{-1}^{1} \sin(x^2)\,\mathrm{d}x$ is estimated using the Midpoint Rule with 1000 intervals. Show that the absolute error in this approximation is at most $2\cdot10^{-6}$. You may use the fact that when approximating $\int_a^b f(x)\,\mathrm{d}x$ with the Midpoint Rule using n points, the absolute value of the error is at most $M(b-a)^3/24n^2$ when $|f''(x)| \le M$ for all $x \in [a, b]$.

19. $*$. The total error using the midpoint rule with n subintervals to approximate the integral of $f(x)$ over $[a, b]$ is bounded by $\dfrac{M(b-a)^3}{(24n^2)}$, if $|f''(x)| \le M$ for all $a \le x \le b$.

Using this bound, if the integral $\displaystyle\int_{-2}^{1} 2x^4\,\mathrm{d}x$ is approximated using the midpoint rule with 60 subintervals, what is the largest possible error between the approximation M_{60} and the true value of the integral?

20. $*$. Both parts of this question concern the integral $I = \displaystyle\int_0^2 (x-3)^5\,\mathrm{d}x$.

a Write down the Simpson's Rule approximation to I with $n = 6$. Leave your answer in calculator-ready form.

b Which method of approximating I results in a smaller error bound: the Midpoint Rule with $n = 100$ intervals, or Simpson's Rule with $n = 10$

intervals? You may use the formulas

$$|E_M| \le \frac{M(b-a)^3}{24n^2} \qquad \text{and} \qquad |E_S| \le \frac{L(b-a)^5}{180n^4},$$

where M is an upper bound for $|f''(x)|$ and L is an upper bound for $|f^{(4)}(x)|$, and E_M and E_S are the absolute errors arising from the midpoint rule and Simpson's rule, respectively.

21. $*$. Find a bound for the error in approximating $\displaystyle\int_1^5 \frac{1}{x}\,\mathrm{d}x$ using Simpson's rule with $n = 4$. Do not write down the Simpson's rule approximation S_4.
In general the error in approximating $\int_a^b f(x)\mathrm{d}x$ using Simpson's rule with n steps is bounded by $\dfrac{L(b-a)}{180}(\Delta x)^4$ where $\Delta x = \dfrac{b-a}{n}$ and $L \ge |f^{(4)}(x)|$ for all $a \le x \le b$.

22. $*$. Find a bound for the error in approximating

$$\int_0^1 \left(e^{-2x} + 3x^3\right) \mathrm{d}x$$

using Simpson's rule with $n = 6$. Do not write down the Simpson's rule approximation S_n.
In general, the error in approximating $\int_a^b f(x)\mathrm{d}x$ using Simpson's rule with n steps is bounded by $\dfrac{L(b-a)}{180}(\Delta x)^4$ where $\Delta x = \dfrac{b-a}{n}$ and $L \ge |f^{(4)}(x)|$ for all $a \le x \le b$.

23. $*$. Let $I = \displaystyle\int_1^2 (1/x)\,\mathrm{d}x$.

a Write down the trapezoidal approximation T_4 for I. *You do not need to simplify your answer.*

b Write down the Simpson's approximation S_4 for I. *You do not need to simplify your answer.*

c Without computing I, find an upper bound for $|I - S_4|$. You may use the fact that if $\left|f^{(4)}(x)\right| \le L$ on the interval $[a, b]$, then the error in using S_n to approximate $\int_a^b f(x)\,\mathrm{d}x$ has absolute value less than or equal to $L(b-a)^5/180n^4$.

24. $*$. A function $s(x)$ satisfies $s(0) = 1.00664$, $s(2) = 1.00543$, $s(4) = 1.00435$, $s(6) = 1.00331$, $s(8) = 1.00233$. Also, it is known to satisfy $\left|s^{(k)}(x)\right| \le \dfrac{k}{1000}$ for $0 \le x \le 8$ and all positive integers k.

a Find the best Trapezoidal Rule and Simpson's Rule approximations that you can for $I = \int_0^8 s(x)\mathrm{d}x$.

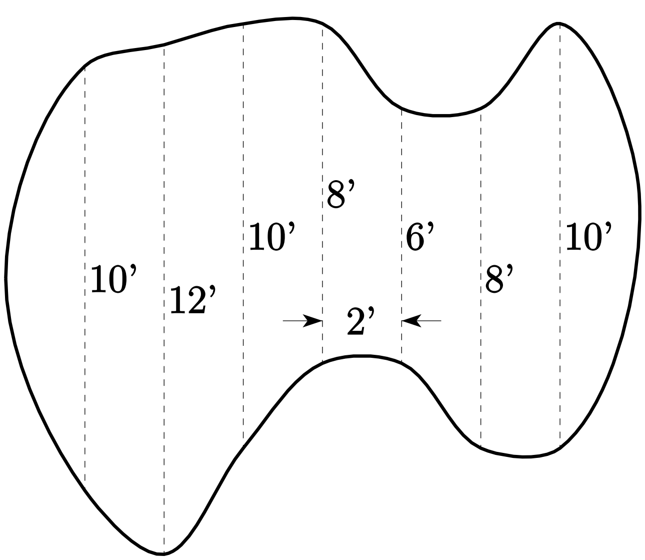
b Determine the maximum possible sizes of errors in the approximations you gave in part (a). Recall that if a function $f(x)$ satisfies $\left|f^{(k)}(x)\right| \le K_k$ on $[a, b]$, then

$$\left|\int_a^b f(x)\mathrm{d}x - T_n\right| \le \frac{K_2(b-a)^3}{12n^2} \quad \text{and} \quad \left|\int_a^b f(x)\mathrm{d}x - S_n\right| \le \frac{K_4(b-a)^5}{180n^4}$$

25. $*$. Consider the trapezoidal rule for making numerical approximations to $\int_a^b f(x)\mathrm{d}x$. The error for the trapezoidal rule satisfies $|E_T| \le \dfrac{M(b-a)^3}{12n^2}$, where $|f''(x)| \le M$ for $a \le x \le b$. If $-2 < f''(x) < 0$ for $1 \le x \le 4$, find a value of n to guarantee the trapezoidal rule will give an approximation for $\int_1^4 f(x)\mathrm{d}x$ with absolute error, $|E_T|$, less than 0.001.

Exercises — Stage 3

26. $*$. A swimming pool has the shape shown in the figure below. The vertical cross–sections of the pool are semi–circular disks. The distances in feet across the pool are given in the figure at 2–foot intervals along the sixteen–foot length of the pool. Use Simpson's Rule to estimate the volume of the pool.



27. $*$. A piece of wire 1m long with radius 1mm is made in such a way that the density varies in its cross–section, but is radially symmetric (that is, the local density $g(r)$ in kg/m^3 depends only on the distance r in mm from the centre of the wire). Take as given that the total mass W of the wire in kg is given by

$$W = 2\pi 10^{-6} \int_0^1 r g(r)\,\mathrm{d}r$$

Data from the manufacturer is given below:

r	0	1/4	1/2	3/4	1
$g(r)$	8051	8100	8144	8170	8190

a Find the best Trapezoidal Rule approximation that you can for W based on the data in the table.

b Suppose that it is known that $|g'(r)| < 200$ and $|g''(r)| < 150$ for all values of r. Determine the maximum possible size of the error in the approximation you gave in part (a). Recall that if a function $f(x)$ satisfies $|f''(x)| \le M$ on $[a, b]$, then

$$|I - T_n| \le \frac{M(b-a)^3}{12n^2}$$

where $I = \int_a^b f(x)\,\mathrm{d}x$ and T_n is the Trapezoidal Rule approximation to I using n subintervals.

28. $*$. Simpson's rule can be used to approximate $\log 2$, since $\log 2 = \displaystyle\int_1^2 \frac{1}{x}\,\mathrm{d}x$.

a Use Simpson's rule with 6 subintervals to approximate $\log 2$.

b How many subintervals are required in order to guarantee that the absolute error is less than 0.00001?

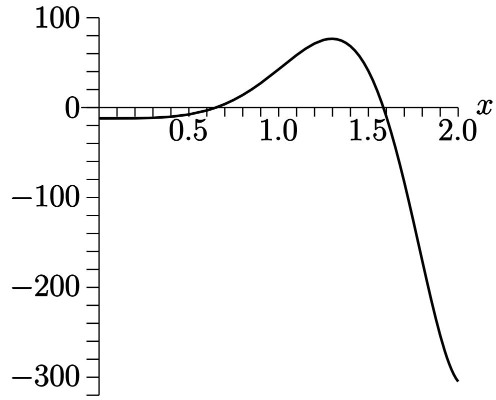
Note that if E_n is the error using n subintervals, then $|E_n| \le \dfrac{L(b-a)^5}{180n^4}$ where L is the maximum absolute value of the fourth derivative of the function being integrated and a and b are the end points of the interval.

29. $*$. Let $I = \displaystyle\int_0^2 \cos(x^2)\mathrm{d}x$ and let S_n be the Simpson's rule approximation to I using n subintervals.

a Estimate the maximum absolute error in using S_8 to approximate I.

b How large should n be in order to ensure that $|I - S_n| \le 0.0001$?

Note: The graph of $f''''(x)$, where $f(x) = \cos(x^2)$, is shown below. The absolute error in the Simpson's rule approximation is bounded by $\dfrac{L(b-a)^5}{180n^4}$ when $|f''''(x)| \le L$ on the interval $[a,b]$.



30. $*$. Define a function $f(x)$ and an integral I by

$$f(x) = \int_0^{x^2} \sin(\sqrt{t})\,\mathrm{d}t, \qquad I = \int_0^1 f(t)\,\mathrm{d}t$$

Estimate how many subdivisions are needed to calculate I to five decimal places of accuracy using the trapezoidal rule.

Note that if E_n is the error using n subintervals, then $|E_n| \le \dfrac{M(b-a)^3}{12n^2}$, where M is the maximum absolute value of the second derivative of the function being integrated and a and b are the limits of integration.

31. Let $f(x)$ be a function [a] with $f''(x) = \dfrac{x^2}{x+1}$.

a Show that $|f''(x)| \le 1$ whenever x is in the interval $[0,1]$.

b Find the maximum value of $|f''(x)|$ over the interval $[0,1]$.

c Assuming $M = 1$, how many intervals should you use to approximate $\int_0^1 f(x)\mathrm{d}x$ to within 10^{-5}?

d Using the value of M you found in (b), how many intervals should you use to approximate $\int_0^1 f(x)\mathrm{d}x$ to within 10^{-5}?

a For example, $f(x) = \frac{1}{6}x^3 - \frac{1}{2}x^2 + (1+x)\log|x+1|$ will do, but you don't need to know what $f(x)$ is for this problem.

32. Approximate the function $\log x$ with a rational function by approximating the integral $\displaystyle\int_1^x \frac{1}{t}\mathrm{d}t$ using Simpson's rule. Your rational function $f(x)$ should approximate $\log x$ with an error of not more than 0.1 for any x in the interval $[1, 3]$.

33. Using an approximation of the area under the curve $\dfrac{1}{x^2+1}$, show that the constant $\arctan 2$ is in the interval $\left[\dfrac{\pi}{4}+0.321, \dfrac{\pi}{4}+0.323\right]$.
You may assume use without proof that $\dfrac{\mathrm{d}^4}{\mathrm{d}x^4}\left\{\dfrac{1}{1+x^2}\right\} = \dfrac{24(5x^4-10x^2+1)}{(x^2+1)^5}$.
You may use a calculator, but only to add, subtract, multiply, and divide.

1.12▲ Improper Integrals

1.12.1 ▸▸ Definitions

To this point we have only considered nicely behaved integrals $\int_a^b f(x)\mathrm{d}x$. Though the algebra involved in some of our examples was quite difficult, all the integrals had

- finite limits of integration a and b, and
- a bounded integrand $f(x)$ (and in fact continuous except possibly for finitely many jump discontinuities).

Not all integrals we need to study are quite so nice.

Definition 1.12.1

An integral having either an infinite limit of integration or an unbounded integrand is called an improper integral.

Two examples are

$$\int_0^\infty \frac{\mathrm{d}x}{1+x^2} \qquad \text{and} \qquad \int_0^1 \frac{\mathrm{d}x}{x}$$

The first has an infinite domain of integration and the integrand of the second tends to ∞ as x approaches the left end of the domain of integration. We'll start with an example that illustrates the traps that you can fall into if you treat such integrals sloppily. Then we'll see how to treat them carefully.

Example 1.12.2 $\int_{-1}^{1} \frac{1}{x^2} \mathrm{d}x$.

Consider the integral

$$\int_{-1}^{1} \frac{1}{x^2} \mathrm{d}x$$

If we "do" this integral completely naively then we get

$$\begin{aligned} \int_{-1}^{1} \frac{1}{x^2}\, \mathrm{d}x &= \left.\frac{x^{-1}}{-1}\right|_{-1}^{1} \\ &= \frac{1}{-1} - \frac{-1}{-1} \\ &= -2 \end{aligned}$$

which is ***wrong*** [a]. In fact, the answer is ridiculous. The integrand $\frac{1}{x^2} > 0$, so the integral has to be positive.

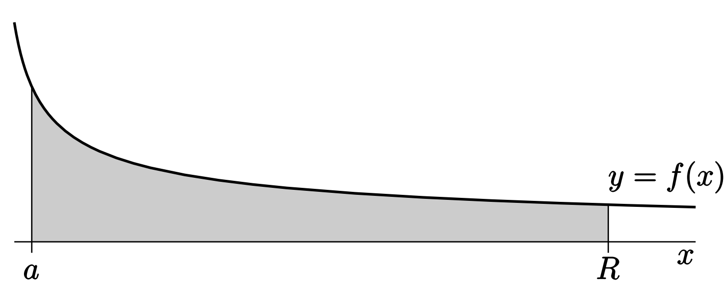
The flaw in the argument is that the fundamental theorem of calculus, which says that

if $F'(x) = f(x)$ then $\int_a^b f(x)\, \mathrm{d}x = F(b) - F(a)$

is applicable only when $F'(x)$ exists and equals $f(x)$ for *all* $a \le x \le b$. In this case $F'(x) = \frac{1}{x^2}$ does not exist for $x = 0$. The given integral is improper. We'll see later that the correct answer is $+\infty$.

a Very wrong. But it is not an example of "not even wrong" — which is a phrase attributed to the physicist Wolfgang Pauli who was known for his harsh critiques of sloppy arguments. The phrase is typically used to describe arguments that are so incoherent that not only can one not prove they are true, but they lack enough coherence to be able to show they are false. The interested reader should do a little searchengineing and look at the concept of falisfyability.

Let us put this example to one side for a moment and turn to the integral $\int_a^\infty \frac{\mathrm{d}x}{1+x^2}$. In this case, the integrand is bounded but the domain of integration extends to $+\infty$. We can evaluate this integral by sneaking up on it. We compute it on a bounded domain of integration, like $\int_a^R \frac{\mathrm{d}x}{1+x^2}$, and then take the limit $R \to \infty$.



Let us put this into practice:

Example 1.12.3 $\int_a^\infty \frac{\mathrm{d}x}{1+x^2}$.

Solution:

- Since the domain extends to $+\infty$ we first integrate on a finite domain

$$\int_a^R \frac{\mathrm{d}x}{1+x^2} = \arctan x \Big|_a^R$$
$$= \arctan R - \arctan a$$

- We then take the limit as $R \to +\infty$:

$$\int_a^\infty \frac{\mathrm{d}x}{1+x^2} = \lim_{R\to\infty} \int_a^R \frac{\mathrm{d}x}{1+x^2}$$
$$= \lim_{R\to\infty} \big[\arctan R - \arctan a\big]$$
$$= \frac{\pi}{2} - \arctan a.$$

To be more precise, we actually formally *define* an integral with an infinite domain as the limit of the integral with a finite domain as we take one or more of the limits of integration to infinity.

Definition 1.12.4 Improper integral with infinite domain of integration.

a If the integral $\int_a^R f(x)\mathrm{d}x$ exists for all $R > a$, then

$$\int_a^\infty f(x)\mathrm{d}x = \lim_{R\to\infty} \int_a^R f(x)\mathrm{d}x$$

when the limit exists (and is finite).

b If the integral $\int_r^b f(x)\mathrm{d}x$ exists for all $r < b$, then

$$\int_{-\infty}^b f(x)\mathrm{d}x = \lim_{r\to-\infty} \int_r^b f(x)\mathrm{d}x$$

when the limit exists (and is finite).

c If the integral $\int_r^R f(x)\mathrm{d}x$ exists for all $r < R$, then

$$\int_{-\infty}^\infty f(x)\mathrm{d}x = \lim_{r\to-\infty} \int_r^c f(x)\mathrm{d}x + \lim_{R\to\infty} \int_c^R f(x)\mathrm{d}x$$

when both limits exist (and are finite). Any c can be used.

When the limit(s) exist, the integral is said to be convergent. Otherwise it is said to be divergent.

We must also be able to treat an integral like $\int_0^1 \frac{\mathrm{d}x}{x}$ that has a finite domain of integration but whose integrand is unbounded near one limit of integration [1] Our approach is similar — we sneak up on the problem. We compute the integral on a smaller domain, such as $\int_t^1 \frac{\mathrm{d}x}{x}$, with $t > 0$, and then take the limit $t \to 0+$.

Example 1.12.5 $\int_0^1 \frac{1}{x}\mathrm{d}x$.

Solution:

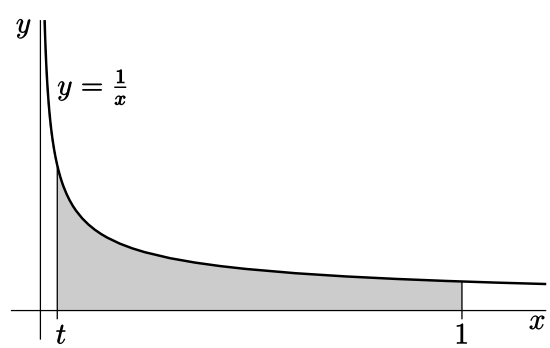
- Since the integrand is unbounded near $x = 0$, we integrate on the smaller domain $t \le x \le 1$ with $t > 0$:

$$\int_t^1 \frac{1}{x}\mathrm{d}x = \log|x|\Big|_t^1 = -\log|t|$$

- We then take the limit as $t \to 0^+$ to obtain

$$\int_0^1 \frac{1}{x}\mathrm{d}x = \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\mathrm{d}x = \lim_{t\to 0^+} -\log|t| = +\infty$$

  Thus this integral diverges to $+\infty$.



Indeed, we *define* integrals with unbounded integrands via this process:

Definition 1.12.6 Improper integral with unbounded integrand.

a If the integral $\int_t^b f(x)\mathrm{d}x$ exists for all $a < t < b$, then

$$\int_a^b f(x)\mathrm{d}x = \lim_{t\to a+} \int_t^b f(x)\mathrm{d}x$$

when the limit exists (and is finite).

1 This will, in turn, allow us to deal with integrals whose integrand is unbounded somewhere inside the domain of integration.

b If the integral $\int_a^T f(x)\mathrm{d}x$ exists for all $a < T < b$, then

$$\int_a^b f(x)\mathrm{d}x = \lim_{T\to b-} \int_a^T f(x)\mathrm{d}x$$

when the limit exists (and is finite).

c Let $a < c < b$. If the integrals $\int_a^T f(x)\mathrm{d}x$ and $\int_t^b f(x)\mathrm{d}x$ exist for all $a < T < c$ and $c < t < b$, then

$$\int_a^b f(x)\mathrm{d}x = \lim_{T\to c-} \int_a^T f(x)\mathrm{d}x + \lim_{t\to c+} \int_t^b f(x)\mathrm{d}x$$

when both limit exist (and are finite).

When the limit(s) exist, the integral is said to be convergent. Otherwise it is said to be divergent.

Notice that (c) is used when the integrand is unbounded at some point in the middle of the domain of integration, such as was the case in our original example

$$\int_{-1}^1 \frac{1}{x^2}\mathrm{d}x$$

A quick computation shows that this integral diverges to $+\infty$

$$\begin{aligned}\int_{-1}^1 \frac{1}{x^2}\mathrm{d}x &= \lim_{a\to 0^-} \int_{-1}^a \frac{1}{x^2}\mathrm{d}x + \lim_{b\to 0^+} \int_b^1 \frac{1}{x^2}\mathrm{d}x \\ &= \lim_{a\to 0^-} \left[1 - \frac{1}{a}\right] + \lim_{b\to 0^+} \left[\frac{1}{b} - 1\right] \\ &= +\infty\end{aligned}$$

More generally, if an integral has more than one “source of impropriety” (for example an infinite domain of integration and an integrand with an unbounded integrand or multiple infinite discontinuities) then you split it up into a sum of integrals with a single “source of impropriety” in each. For the integral, as a whole, to converge every term in that sum has to converge.

For example

Example 1.12.7 $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x-2)x^2}$.

Consider the integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x-2)x^2}$$

- The domain of integration that extends to both $+\infty$ and $-\infty$.

- The integrand is singular (i.e. becomes infinite) at $x = 2$ and at $x = 0$.
- So we would write the integral as

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x-2)x^2} = \int_{-\infty}^{a} \frac{\mathrm{d}x}{(x-2)x^2} + \int_{a}^{0} \frac{\mathrm{d}x}{(x-2)x^2} + \int_{0}^{b} \frac{\mathrm{d}x}{(x-2)x^2} + \int_{b}^{2} \frac{\mathrm{d}x}{(x-2)x^2} + \int_{2}^{c} \frac{\mathrm{d}x}{(x-2)x^2} + \int_{c}^{\infty} \frac{\mathrm{d}x}{(x-2)x^2}$$

  where

  - a is any number strictly less than 0,
  - b is any number strictly between 0 and 2, and
  - c is any number strictly bigger than 2.

  So, for example, take $a = -1, b = 1, c = 3$.

- When we examine the right-hand side we see that
  - the first integral has domain of integration extending to $-\infty$
  - the second integral has an integrand that becomes unbounded as $x \to 0-$,
  - the third integral has an integrand that becomes unbounded as $x \to 0+$,
  - the fourth integral has an integrand that becomes unbounded as $x \to 2-$,
  - the fifth integral has an integrand that becomes unbounded as $x \to 2+$, and
  - the last integral has domain of integration extending to $+\infty$.
- Each of these integrals can then be expressed as a limit of an integral on a small domain.

Example 1.12.7

1.12.2 ⏩ Examples

With the more formal definitions out of the way, we are now ready for some (important) examples.

Example 1.12.8 $\int_1^\infty \frac{\mathrm{d}x}{x^p}$ with $p > 0$.

Solution:

- Fix any $p > 0$.
- The domain of the integral $\int_1^\infty \frac{\mathrm{d}x}{x^p}$ extends to $+\infty$ and the integrand $\frac{1}{x^p}$ is contin-

uous and bounded on the whole domain.

- So we write this integral as the limit
$$\int_1^\infty \frac{\mathrm{d}x}{x^p} = \lim_{R\to\infty} \int_1^R \frac{\mathrm{d}x}{x^p}$$
- The antiderivative of $1/x^p$ changes when $p = 1$, so we will split the problem into three cases, $p > 1$, $p = 1$ and $p < 1$.
- When $p > 1$,
$$\begin{aligned}\int_1^R \frac{\mathrm{d}x}{x^p} &= \frac{1}{1-p} x^{1-p}\Big|_1^R \\ &= \frac{R^{1-p}-1}{1-p}\end{aligned}$$
Taking the limit as $R \to \infty$ gives
$$\begin{aligned}\int_1^\infty \frac{\mathrm{d}x}{x^p} &= \lim_{R\to\infty} \int_1^R \frac{\mathrm{d}x}{x^p} \\ &= \lim_{R\to\infty} \frac{R^{1-p}-1}{1-p} \\ &= \frac{-1}{1-p} = \frac{1}{p-1}\end{aligned}$$
since $1 - p < 0$.
- Similarly when $p < 1$ we have
$$\int_1^\infty \frac{\mathrm{d}x}{x^p} = \lim_{R\to\infty} \int_1^R \frac{\mathrm{d}x}{x^p} \qquad = \lim_{R\to\infty} \frac{R^{1-p}-1}{1-p}$$
$$= +\infty$$
because $1 - p > 0$ and the term R^{1-p} diverges to $+\infty$.
- Finally when $p = 1$
$$\int_1^R \frac{\mathrm{d}x}{x} = \log|R| - \log 1 = \log R$$
Then taking the limit as $R \to \infty$ gives us
$$\int_1^\infty \frac{\mathrm{d}x}{x^p} = \lim_{R\to\infty} \log|R| = +\infty.$$
- So summarising, we have
$$\int_1^\infty \frac{\mathrm{d}x}{x^p} = \begin{cases} \text{divergent} & \text{if } p \le 1 \\ \frac{1}{p-1} & \text{if } p > 1 \end{cases}$$

Example 1.12.8

Example 1.12.9 $\int_0^1 \frac{\mathrm{d}x}{x^p}$ with $p > 0$.

Solution:

- Again fix any $p > 0$.
- The domain of integration of the integral $\int_0^1 \frac{\mathrm{d}x}{x^p}$ is finite, but the integrand $\frac{1}{x^p}$ becomes unbounded as x approaches the left end, 0, of the domain of integration.
- So we write this integral as

$$\int_0^1 \frac{\mathrm{d}x}{x^p} = \lim_{t \to 0+} \int_t^1 \frac{\mathrm{d}x}{x^p}$$

- Again, the antiderivative changes at $p = 1$, so we split the problem into three cases.
- When $p > 1$ we have

$$\begin{aligned} \int_t^1 \frac{\mathrm{d}x}{x^p} &= \frac{1}{1-p} x^{1-p} \bigg|_t^1 \\ &= \frac{1 - t^{1-p}}{1-p} \end{aligned}$$

  Since $1 - p < 0$ when we take the limit as $t \to 0$ the term t^{1-p} diverges to $+\infty$ and we obtain

$$\int_0^1 \frac{\mathrm{d}x}{x^p} = \lim_{t \to 0^+} \frac{1 - t^{1-p}}{1-p} = +\infty$$

- When $p = 1$ we similarly obtain

$$\begin{aligned} \int_0^1 \frac{\mathrm{d}x}{x} &= \lim_{t \to 0+} \int_t^1 \frac{\mathrm{d}x}{x} \\ &= \lim_{t \to 0+} \big(-\log|t| \big) \\ &= +\infty \end{aligned}$$

- Finally, when $p < 1$ we have

$$\begin{aligned} \int_0^1 \frac{\mathrm{d}x}{x^p} &= \lim_{t \to 0^+} \int_t^1 \frac{\mathrm{d}x}{x^p} \\ &= \lim_{t \to 0^+} \frac{1 - t^{1-p}}{1-p} = \frac{1}{1-p} \end{aligned}$$

  since $1 - p > 0$.
- In summary

$$\int_0^1 \frac{\mathrm{d}x}{x^p} = \begin{cases} \frac{1}{1-p} & \text{if } p < 1 \\ \text{divergent} & \text{if } p \geq 1 \end{cases}$$

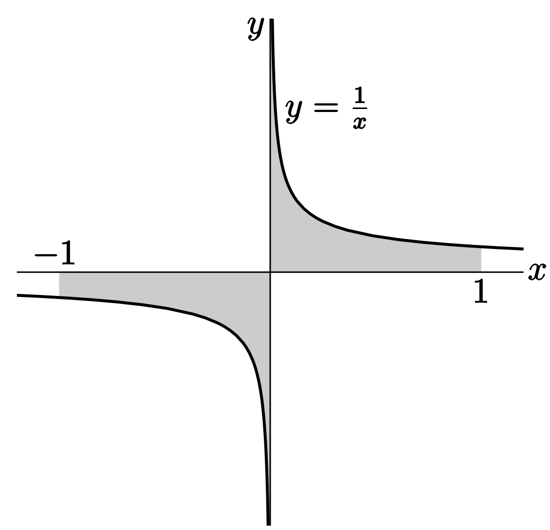
Example 1.12.10 $\int_0^\infty \frac{\mathrm{d}x}{x^p}$ with $p > 0$.

Solution:

- Yet again fix $p > 0$.
- This time the domain of integration of the integral $\int_0^\infty \frac{\mathrm{d}x}{x^p}$ extends to $+\infty$, and in addition the integrand $\frac{1}{x^p}$ becomes unbounded as x approaches the left end, 0, of the domain of integration.
- So we split the domain in two — given our last two examples, the obvious place to cut is at $x = 1$:
$$\int_0^\infty \frac{\mathrm{d}x}{x^p} = \int_0^1 \frac{\mathrm{d}x}{x^p} + \int_1^\infty \frac{\mathrm{d}x}{x^p}$$
- We saw, in Example 1.12.9, that the first integral diverged whenever $p \geq 1$, and we also saw, in Example 1.12.8, that the second integral diverged whenever $p \leq 1$.
- So the integral $\int_0^\infty \frac{\mathrm{d}x}{x^p}$ diverges for all values of p.

Example 1.12.11 $\int_{-1}^1 \frac{\mathrm{d}x}{x}$.

This is a pretty subtle example. Look at the sketch below:



This suggests that the signed area to the left of the y-axis should exactly cancel the area to the right of the y-axis making the value of the integral $\int_{-1}^1 \frac{\mathrm{d}x}{x}$ exactly zero. But both of the integrals

$$\int_0^1 \frac{\mathrm{d}x}{x} = \lim_{t\to 0+} \int_t^1 \frac{\mathrm{d}x}{x} = \lim_{t\to 0+} \Big[\log x\Big]_t^1 = \lim_{t\to 0+} \log\frac{1}{t} = +\infty$$

$$\int_{-1}^{0} \frac{\mathrm{d}x}{x} = \lim_{T\to 0-} \int_{-1}^{T} \frac{\mathrm{d}x}{x} = \lim_{T\to 0-} \Big[\log|x|\Big]_{-1}^{T} = \lim_{T\to 0-} \log|T| \ = -\infty$$

diverge so $\int_{-1}^{1} \frac{\mathrm{d}x}{x}$ diverges. Don't make the mistake of thinking that $\infty - \infty = 0$. ***It is undefined***. And it is undefined for good reason.
For example, we have just seen that the area to the right of the y-axis is

$$\lim_{t\to 0+} \int_{t}^{1} \frac{\mathrm{d}x}{x} = +\infty$$

and that the area to the left of the y-axis is (substitute $-7t$ for T above)

$$\lim_{t\to 0+} \int_{-1}^{-7t} \frac{\mathrm{d}x}{x} = -\infty$$

If $\infty - \infty = 0$, the following limit should be 0.

$$\begin{aligned}
\lim_{t\to 0+} \left[\int_{t}^{1} \frac{\mathrm{d}x}{x} + \int_{-1}^{-7t} \frac{\mathrm{d}x}{x}\right] &= \lim_{t\to 0+} \left[\log\frac{1}{t} + \log|-7t|\right] \\
&= \lim_{t\to 0+} \left[\log\frac{1}{t} + \log(7t)\right] \\
&= \lim_{t\to 0+} \Big[-\log t + \log 7 + \log t\Big] = \lim_{t\to 0+} \log 7 \\
&= \log 7
\end{aligned}$$

This appears to give $\infty - \infty = \log 7$. Of course the number 7 was picked at random. You can make $\infty - \infty$ be any number at all, by making a suitable replacement for 7.

Example 1.12.11

Example 1.12.12 Example 1.12.2 revisited.

The careful computation of the integral of Example 1.12.2 is

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{x^2}\mathrm{d}x &= \lim_{T\to 0-} \int_{-1}^{T} \frac{1}{x^2}\mathrm{d}x + \lim_{t\to 0+} \int_{t}^{1} \frac{1}{x^2}\mathrm{d}x \\
&= \lim_{T\to 0-} \left[-\frac{1}{x}\right]_{-1}^{T} + \lim_{t\to 0+} \left[-\frac{1}{x}\right]_{t}^{1} \\
&= \infty + \infty
\end{aligned}$$

Hence the integral diverges to $+\infty$.

Example 1.12.13 $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2}$.

Since

$$\begin{aligned}
\lim_{R\to\infty} \int_0^R \frac{\mathrm{d}x}{1+x^2} &= \lim_{R\to\infty} \Big[\arctan x\Big]_0^R = \lim_{R\to\infty} \arctan R = \frac{\pi}{2} \\
\lim_{r\to-\infty} \int_r^0 \frac{\mathrm{d}x}{1+x^2} &= \lim_{r\to-\infty} \Big[\arctan x\Big]_r^0 = \lim_{r\to-\infty} -\arctan r = \frac{\pi}{2}
\end{aligned}$$

The integral $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2}$ converges and takes the value π.

Example 1.12.14 When does $\int_e^{\infty} \frac{\mathrm{d}x}{x(\log x)^p}$ converge?

For what values of p does $\int_e^{\infty} \frac{\mathrm{d}x}{x(\log x)^p}$ converge?
Solution:

- For $x \geq e$, the denominator $x(\log x)^p$ is never zero. So the integrand is bounded on the entire domain of integration and this integral is improper only because the domain of integration extends to $+\infty$ and we proceed as usual.
- We have

$$\begin{aligned}
\int_e^{\infty} \frac{\mathrm{d}x}{x(\log x)^p} &= \lim_{R\to\infty} \int_e^R \frac{\mathrm{d}x}{x(\log x)^p} && \text{use substitution} \\
&= \lim_{R\to\infty} \int_1^{\log R} \frac{\mathrm{d}u}{u^p} && \text{with } u = \log x, \mathrm{d}u = \frac{\mathrm{d}x}{x} \\
&= \lim_{R\to\infty} \begin{cases} \frac{1}{1-p}\Big[(\log R)^{1-p} - 1\Big] & \text{if } p \neq 1 \\ \log(\log R) & \text{if } p = 1 \end{cases} \\
&= \begin{cases} \text{divergent} & \text{if } p \leq 1 \\ \frac{1}{p-1} & \text{if } p > 1 \end{cases}
\end{aligned}$$

  In this last step we have used similar logic that that used in Example 1.12.8, but with R replaced by $\log R$.

Example 1.12.15 The gamma function.

The gamma function $\Gamma(x)$ is defined by the improper integral

$$\Gamma(t) = \int_0^{\infty} x^{t-1} e^{-x} \mathrm{d}x$$

We shall now compute $\Gamma(n)$ for all natural numbers n.

- To get started, we'll compute

$$\Gamma(1) = \int_0^\infty e^{-x}\mathrm{d}x = \lim_{R\to\infty}\int_0^R e^{-x}\mathrm{d}x = \lim_{R\to\infty}\Big[-e^{-x}\Big]_0^R = 1$$

- Then compute

$$\begin{aligned}\Gamma(2) &= \int_0^\infty xe^{-x}\mathrm{d}x \\ &= \lim_{R\to\infty}\int_0^R xe^{-x}\mathrm{d}x\end{aligned}$$

Use integration by parts with $u = x, \mathrm{d}v = e^{-x}\mathrm{d}x$, so $v = -e^{-x}, \mathrm{d}u = \mathrm{d}x$

$$\begin{aligned}&= \lim_{R\to\infty}\Big[-xe^{-x}\Big|_0^R + \int_0^R e^{-x}\mathrm{d}x\Big] \\ &= \lim_{R\to\infty}\Big[-xe^{-x} - e^{-x}\Big]_0^R \\ &= 1\end{aligned}$$

For the last equality, we used that $\lim_{x\to\infty} xe^{-x} = 0$.

- Now we move on to general n, using the same type of computation as we just used to evaluate $\Gamma(2)$. For any natural number n,

$$\begin{aligned}\Gamma(n+1) &= \int_0^\infty x^n e^{-x}\mathrm{d}x \\ &= \lim_{R\to\infty}\int_0^R x^n e^{-x}\mathrm{d}x\end{aligned}$$

Again integrate by parts with $u = x^n, \mathrm{d}v = e^{-x}\mathrm{d}x$, so $v = -e^{-x}, \mathrm{d}u = nx^{n-1}\mathrm{d}x$

$$\begin{aligned}&= \lim_{R\to\infty}\Big[-x^n e^{-x}\Big|_0^R + \int_0^R nx^{n-1}e^{-x}\mathrm{d}x\Big] \\ &= \lim_{R\to\infty} n\int_0^R x^{n-1}e^{-x}\mathrm{d}x \\ &= n\Gamma(n)\end{aligned}$$

To get to the third row, we used that $\lim_{x\to\infty} x^n e^{-x} = 0$.

- Now that we know $\Gamma(2) = 1$ and $\Gamma(n+1) = n\Gamma(n)$, for all $n \in \mathbb{N}$, we can compute all of the $\Gamma(n)$'s.

$$\Gamma(2) = 1$$

$$\begin{aligned}
\Gamma(3) &= \Gamma(2+1) = 2\Gamma(2) = 2 \cdot 1 \\
\Gamma(4) &= \Gamma(3+1) = 3\Gamma(3) = 3 \cdot 2 \cdot 1 \\
\Gamma(5) &= \Gamma(4+1) = 4\Gamma(4) = 4 \cdot 3 \cdot 2 \cdot 1 \\
&\vdots \\
\Gamma(n) &= (n-1) \cdot (n-2) \cdots 4 \cdot 3 \cdot 2 \cdot 1 = (n-1)!
\end{aligned}$$

That is, the factorial is just [a] the Gamma function shifted by one.

a The Gamma function is far more important than just a generalisation of the factorial. It appears all over mathematics, physics, statistics and beyond. It has all sorts of interesting properties and its definition can be extended from natural numbers n to all numbers excluding $0, -1, -2, -3, \cdots$. For example, one can show that $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}$.

Example 1.12.15

1.12.3 ▸▸ Convergence Tests for Improper Integrals

It is very common to encounter integrals that are too complicated to evaluate explicitly. Numerical approximation schemes, evaluated by computer, are often used instead (see Section 1.11). You want to be sure that at least the integral converges before feeding it into a computer [2]. Fortunately it is usually possible to determine whether or not an improper integral converges even when you cannot evaluate it explicitly.

Remark 1.12.16 For pedagogical purposes, we are going to concentrate on the problem of determining whether or not an integral $\int_a^\infty f(x)\mathrm{d}x$ converges, when $f(x)$ has no singularities for $x \geq a$. Recall that the first step in analyzing any improper integral is to write it as a sum of integrals each of has only a single "source of impropriety" — either a domain of integration that extends to $+\infty$, or a domain of integration that extends to $-\infty$, or an integrand which is singular at one end of the domain of integration. So we are now going to consider only the first of these three possibilities. But the techniques that we are about to see have obvious analogues for the other two possibilities.

Now let's start. Imagine that we have an improper integral $\int_a^\infty f(x)\mathrm{d}x$, that $f(x)$ has no singularities for $x \geq a$ and that $f(x)$ is complicated enough that we cannot evaluate the integral explicitly [3]. The idea is find another improper integral $\int_a^\infty g(x)\mathrm{d}x$

- with $g(x)$ simple enough that we can evaluate the integral $\int_a^\infty g(x)\mathrm{d}x$ explicitly, or at least determine easily whether or not $\int_a^\infty g(x)\mathrm{d}x$ converges, and

2 Applying numerical integration methods to a divergent integral may result in perfectly reasonably looking but very wrong answers.

3 You could, for example, think of something like our running example $\int_a^\infty e^{-t^2}\mathrm{d}t$.

- with $g(x)$ behaving enough like $f(x)$ for large x that the integral $\int_a^\infty f(x)\mathrm{d}x$ converges if and only if $\int_a^\infty g(x)\mathrm{d}x$ converges.

So far, this is a pretty vague strategy. Here is a theorem which starts to make it more precise.

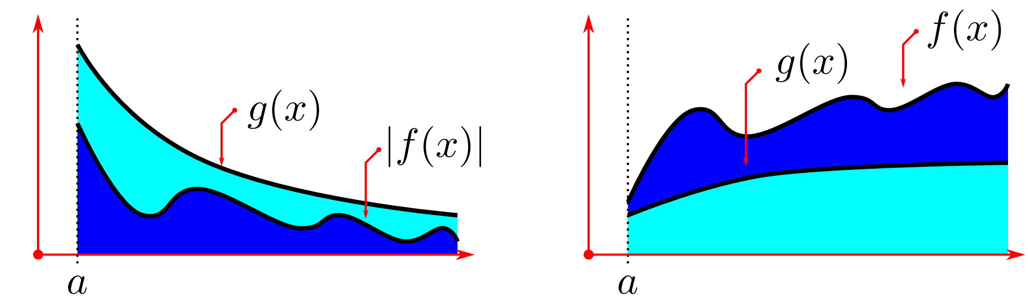
Theorem 1.12.17 Comparison.

Let a be a real number. Let f and g be functions that are defined and continuous for all $x \ge a$ and assume that $g(x) \ge 0$ for all $x \ge a$.

a If $|f(x)| \le g(x)$ for all $x \ge a$ and if $\int_a^\infty g(x)\mathrm{d}x$ converges then $\int_a^\infty f(x)\mathrm{d}x$ also converges.

b If $f(x) \ge g(x)$ for all $x \ge a$ and if $\int_a^\infty g(x)\mathrm{d}x$ diverges then $\int_a^\infty f(x)\mathrm{d}x$ also diverges.

We will not prove this theorem, but, hopefully, the following supporting arguments should at least appear reasonable to you. Consider the figure below:



- If $\int_a^\infty g(x)\mathrm{d}x$ converges, then the area of

$$\left\{ (x,y) \;\middle|\; x \ge a,\ 0 \le y \le g(x) \right\} \text{ is finite.}$$

When $|f(x)| \le g(x)$, the region

$$\left\{ (x,y) \;\middle|\; x \ge a,\ 0 \le y \le |f(x)| \right\} \text{ is contained inside } \left\{ (x,y) \;\middle|\; x \ge a,\ 0 \le y \le g(x)\right\}$$

and so must also have finite area. Consequently the areas of both the regions

$$\left\{ (x,y) \;\middle|\; x \ge a,\ 0 \le y \le f(x) \right\} \text{ and } \left\{ (x,y) \;\middle|\; x \ge a,\ f(x) \le y \le 0\right\}$$

are finite too [4].

4 We have separated the regions in which $f(x)$ is positive and negative, because the integral $\int_a^\infty f(x)\mathrm{d}x$ represents the signed area of the union of $\left\{ (x,y) \;\middle|\; x \ge a,\ 0 \le y \le f(x) \right\}$ and $\left\{ (x,y) \;\middle|\; x \ge a,\ f(x) \le y \le 0 \right\}$.

- If $\int_a^\infty g(x)\mathrm{d}x$ diverges, then the area of

$$\left\{ (x,y) \;\middle|\; x \geq a,\ 0 \leq y \leq g(x) \right\} \text{ is infinite.}$$

When $f(x) \geq g(x)$, the region

$\left\{ (x,y) \;\middle|\; x \geq a,\ 0 \leq y \leq f(x) \right\}$ contains the region $\left\{ (x,y) \;\middle|\; x \geq a,\ 0 \leq y \leq g(x) \right\}$

and so also has infinite area.

Example 1.12.18 $\int_1^\infty e^{-x^2}\mathrm{d}x$.

We cannot evaluate the integral $\int_1^\infty e^{-x^2}\mathrm{d}x$ explicitly [a], however we would still like to understand if it is finite or not — does it converge or diverge?
Solution: We will use Theorem 1.12.17 to answer the question.

- So we want to find another integral that we can compute and that we can compare to $\int_1^\infty e^{-x^2}\mathrm{d}x$. To do so we pick an integrand that looks like e^{-x^2}, but whose indefinite integral we know — such as e^{-x}.

- When $x \geq 1$, we have $x^2 \geq x$ and hence $e^{-x^2} \leq e^{-x}$. Thus we can use Theorem 1.12.17 to compare

$$\int_1^\infty e^{-x^2}\mathrm{d}x \text{ with } \int_1^\infty e^{-x}\mathrm{d}x$$

- The integral

$$\begin{aligned}
\int_1^\infty e^{-x}\mathrm{d}x &= \lim_{R\to\infty} \int_1^R e^{-x}\mathrm{d}x \\
&= \lim_{R\to\infty} \Big[-e^{-x} \Big]_1^R \\
&= \lim_{R\to\infty} \Big[e^{-1} - e^{-R} \Big] = e^{-1}
\end{aligned}$$

converges.

- So, by Theorem 1.12.17, with $a = 1$, $f(x) = e^{-x^2}$ and $g(x) = e^{-x}$, the integral $\int_1^\infty e^{-x^2}\mathrm{d}x$ converges too (it is approximately equal to 0.1394).

a It has been the subject of many remarks and footnotes.

Example 1.12.19 $\int_{1/2}^{\infty} e^{-x^2}\mathrm{d}x$.

Solution:

- The integral $\int_{1/2}^{\infty} e^{-x^2}\mathrm{d}x$ is quite similar to the integral $\int_1^{\infty} e^{-x^2}\mathrm{d}x$ of Example 1.12.18. But we cannot just repeat the argument of Example 1.12.18 because it is not true that $e^{-x^2} \le e^{-x}$ when $0 < x < 1$.
- In fact, for $0 < x < 1$, $x^2 < x$ so that $e^{-x^2} > e^{-x}$.
- However the difference between the current example and Example 1.12.18 is

$$\int_{1/2}^{\infty} e^{-x^2}\mathrm{d}x - \int_1^{\infty} e^{-x^2}\mathrm{d}x = \int_{1/2}^{1} e^{-x^2}\mathrm{d}x$$

  which is clearly a well defined finite number (its actually about 0.286). It is important to note that we are being a little sloppy by taking the difference of two integrals like this — we are assuming that both integrals converge. More on this below.
- So we would expect that $\int_{1/2}^{\infty} e^{-x^2}\mathrm{d}x$ should be the sum of the proper integral integral $\int_{1/2}^{1} e^{-x^2}\mathrm{d}x$ and the convergent integral $\int_1^{\infty} e^{-x^2}\mathrm{d}x$ and so should be a convergent integral. This is indeed the case. The Theorem below provides the justification.

Theorem 1.12.20

Let a and c be real numbers with $a < c$ and let the function $f(x)$ be continuous for all $x \ge a$. Then the improper integral $\int_a^{\infty} f(x)\, \mathrm{d}x$ converges if and only if the improper integral $\int_c^{\infty} f(x)\, \mathrm{d}x$ converges.

Proof. By definition the improper integral $\int_a^{\infty} f(x)\mathrm{d}x$ converges if and only if the limit

$$\begin{aligned}\lim_{R\to\infty} \int_a^R f(x)\mathrm{d}x &= \lim_{R\to\infty} \left[\int_a^c f(x)\mathrm{d}x + \int_c^R f(x)\mathrm{d}x \right] \\ &= \int_a^c f(x)\mathrm{d}x + \lim_{R\to\infty} \int_c^R f(x)\mathrm{d}x\end{aligned}$$

exists and is finite. (Remember that, in computing the limit, $\int_a^c f(x)\mathrm{d}x$ is a finite constant independent of R and so can be pulled out of the limit.) But that is the case if and only if the limit $\lim_{R\to\infty} \int_c^R f(x)\mathrm{d}x$ exists and is finite, which in turn is the case if and only if the integral $\int_c^{\infty} f(x)\mathrm{d}x$ converges.

□

Example 1.12.21 Does $\int_1^\infty \frac{\sqrt{x}}{x^2+x}\mathrm{d}x$ converge?

Does the integral $\int_1^\infty \frac{\sqrt{x}}{x^2+x}\mathrm{d}x$ converge or diverge?
Solution:

- Our first task is to identify the potential sources of impropriety for this integral.
- The domain of integration extends to $+\infty$, but we must also check to see if the integrand contains any singularities. On the domain of integration $x \geq 1$ so the denominator is never zero and the integrand is continuous. So the only problem is at $+\infty$.
- Our second task is to develop some intuition [a]. As the only problem is that the domain of integration extends to infinity, whether or not the integral converges will be determined by the behavior of the integrand for very large x.
- When x is very large, x^2 is much much larger than x (which we can write as $x^2 \gg x$) so that the denominator $x^2 + x \approx x^2$ and the integrand

$$\frac{\sqrt{x}}{x^2+x} \approx \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}$$

- By Example 1.12.8, with $p = \frac{3}{2}$, the integral $\int_1^\infty \frac{\mathrm{d}x}{x^{3/2}}$ converges. So we would expect that $\int_1^\infty \frac{\sqrt{x}}{x^2+x}\mathrm{d}x$ converges too.
- Our final task is to verify that our intuition is correct. To do so, we want to apply part (a) of Theorem 1.12.17 with $f(x) = \frac{\sqrt{x}}{x^2+x}$ and $g(x)$ being $\frac{1}{x^{3/2}}$, or possibly some constant times $\frac{1}{x^{3/2}}$. That is, we need to show that for all $x \geq 1$ (i.e. on the domain of integration)

$$\frac{\sqrt{x}}{x^2+x} \leq \frac{A}{x^{3/2}}$$

  for some constant A. Let's try this.
- Since $x \geq 1$ we know that

$$x^2 + x > x^2$$

  Now take the reciprocal of both sides:

$$\frac{1}{x^2+x} < \frac{1}{x^2}$$

  Multiply both sides by $\sqrt{x}$ (which is always positive, so the sign of the inequality does not change)

$$\frac{\sqrt{x}}{x^2+x} < \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}$$

- So Theorem 1.12.17(a) and Example 1.12.8, with $p = \frac{3}{2}$ do indeed show that the integral $\int_1^\infty \frac{\sqrt{x}}{x^2+x} \mathrm{d}x$ converges.

a This takes practice, practice and more practice. At the risk of alliteration — please perform plenty of practice problems .

Example 1.12.21

Notice that in this last example we managed to show that the integral exists by finding an integrand that behaved the same way for large x. Our intuition then had to be bolstered with some careful inequalities to apply the comparison Theorem 1.12.17. It would be nice to avoid this last step and be able jump from the intuition to the conclusion without messing around with inequalities. Thankfully there is a variant of Theorem 1.12.17 that is often easier to apply and that also fits well with the sort of intuition that we developed to solve Example 1.12.21.

A key phrase in the previous paragraph is "behaves the same way for large x". A good way to formalise this expression — "$f(x)$ behaves like $g(x)$ for large x" — is to require that the limit

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \text{ exists and is a finite nonzero number.}$$

Suppose that this is the case and call the limit $L \neq 0$. Then

- the ratio $\frac{f(x)}{g(x)}$ must approach L as x tends to $+\infty$.
- So when x is very large — say $x > B$, for some big number B — we must have that

$$\frac{1}{2}L \leq \frac{f(x)}{g(x)} \leq 2L \qquad \text{for all } x > B$$

  Equivalently, $f(x)$ lies between $\frac{L}{2}g(x)$ and $2Lg(x)$, for all $x \geq B$.
- Consequently, the integral of $f(x)$ converges if and only if the integral of $g(x)$ converges, by Theorems 1.12.17 and 1.12.20.

These considerations lead to the following variant of Theorem 1.12.17.

Theorem 1.12.22 Limiting comparison.

Let $-\infty < a < \infty$. Let f and g be functions that are defined and continuous for all $x \geq a$ and assume that $g(x) \geq 0$ for all $x \geq a$.

a If $\int_a^\infty g(x)\mathrm{d}x$ converges and the limit

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

exists, then $\int_a^\infty f(x)\mathrm{d}x$ converges.

b If $\int_a^\infty g(x)\mathrm{d}x$ diverges and the limit

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

exists and ***is nonzero***, then $\int_a^\infty f(x)$ diverges.

Note that in (b) the limit must exist and be nonzero, while in (a) we only require that the limit exists (it can be zero).

Here is an example of how Theorem 1.12.22 is used.

Example 1.12.23 $\int_1^\infty \frac{x+\sin x}{e^{-x}+x^2}\mathrm{d}x$.

Does the integral $\displaystyle\int_1^\infty \frac{x+\sin x}{e^{-x}+x^2}\mathrm{d}x$ converge or diverge?

Solution:

- Our first task is to identify the potential sources of impropriety for this integral.
- The domain of integration extends to $+\infty$. On the domain of integration the denominator is never zero so the integrand is continuous. Thus the only problem is at $+\infty$.
- Our second task is to develop some intuition about the behavior of the integrand for very large x. A good way to start is to think about the size of each term when x becomes big.
- When x is very large:
  - $e^{-x} \ll x^2$, so that the denominator $e^{-x}+x^2 \approx x^2$, and
  - $|\sin x| \le 1 \ll x$, so that the numerator $x + \sin x \approx x$, and
  - the integrand $\frac{x+\sin x}{e^{-x}+x^2} \approx \frac{x}{x^2} = \frac{1}{x}$.

  Notice that we are using $A \ll B$ to mean that "A is much much smaller than B". Similarly $A \gg B$ means "A is much much bigger than B". We don't really need to be too precise about its meaning beyond this in the present context.
- Now, since $\int_1^\infty \frac{\mathrm{d}x}{x}$ diverges, we would expect $\int_1^\infty \frac{x+\sin x}{e^{-x}+x^2}\mathrm{d}x$ to diverge too.
- Our final task is to verify that our intuition is correct. To do so, we set

$$f(x) = \frac{x+\sin x}{e^{-x}+x^2} \qquad\qquad g(x) = \frac{1}{x}$$

and compute

$$\begin{aligned}
\lim_{x\to\infty}\frac{f(x)}{g(x)} &= \lim_{x\to\infty}\frac{x+\sin x}{e^{-x}+x^2}\div\frac{1}{x}\\
&= \lim_{x\to\infty}\frac{(1+\sin x/x)x}{(e^{-x}/x^2+1)x^2}\times x\\
&= \lim_{x\to\infty}\frac{1+\sin x/x}{e^{-x}/x^2+1}\\
&= 1
\end{aligned}$$

- Since $\int_1^\infty g(x)\mathrm{d}x = \int_1^\infty \frac{\mathrm{d}x}{x}$ diverges, by Example 1.12.8 with $p=1$, Theorem 1.12.22(b) now tells us that $\int_1^\infty f(x)\mathrm{d}x = \int_1^\infty \frac{x+\sin x}{e^{-x}+x^2}\mathrm{d}x$ diverges too.

Example 1.12.23

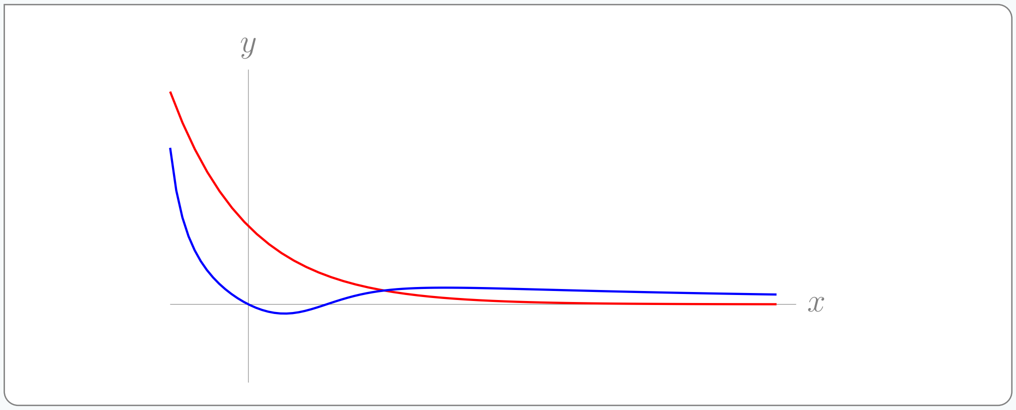
1.12.4 ▸▸ Exercises

Exercises — Stage 1

1. For which values of b is the integral $\displaystyle\int_0^b \frac{1}{x^2-1}\mathrm{d}x$ improper?

2. For which values of b is the integral $\displaystyle\int_0^b \frac{1}{x^2+1}\mathrm{d}x$ improper?

3. Below are the graphs $y=f(x)$ and $y=g(x)$. Suppose $\displaystyle\int_0^\infty f(x)\mathrm{d}x$ converges, and $\displaystyle\int_0^\infty g(x)\mathrm{d}x$ diverges. Assuming the graphs continue on as shown as $x\to\infty$, which graph is $f(x)$, and which is $g(x)$?



4. $\ast$. Decide whether the following statement is true or false. If false, provide a counterexample. If true, provide a brief justification. (Assume that $f(x)$ and $g(x)$ are continuous functions.)
If $\displaystyle\int_1^\infty f(x)\,\mathrm{d}x$ converges and $g(x) \geq f(x) \geq 0$ for all x, then $\displaystyle\int_1^\infty g(x)\,\mathrm{d}x$ converges.

5. Let $f(x) = e^{-x}$ and $g(x) = \dfrac{1}{x+1}$. Note $\int_0^\infty f(x)\mathrm{d}x$ converges while $\int_0^\infty g(x)\mathrm{d}x$ diverges.
For each of the functions $h(x)$ described below, decide whether $\int_0^\infty h(x)\mathrm{d}x$ converges or diverges, or whether there isn't enough information to decide. Justify your decision.

a $h(x)$, continuous and defined for all $x \geq 0$, $h(x) \leq f(x)$.

b $h(x)$, continuous and defined for all $x \geq 0$, $f(x) \leq h(x) \leq g(x)$.

c $h(x)$, continuous and defined for all $x \geq 0$, $-2f(x) \leq h(x) \leq f(x)$.

Exercises — Stage 2

6. $\ast$. Evaluate the integral $\displaystyle\int_0^1 \frac{x^4}{x^5-1}\,\mathrm{d}x$ or state that it diverges.

7. $\ast$. Determine whether the integral $\displaystyle\int_{-2}^2 \frac{1}{(x+1)^{4/3}}\,\mathrm{d}x$ is convergent or divergent. If it is convergent, find its value.

8. $\ast$. Does the improper integral $\displaystyle\int_1^\infty \frac{1}{\sqrt{4x^2-x}}\,\mathrm{d}x$ converge? Justify your answer.

9. $\ast$. Does the integral $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{x^2+\sqrt{x}}$ converge or diverge? Justify your claim.

10. Does the integral $\displaystyle\int_{-\infty}^{\infty} \cos x \,\mathrm{d}x$ converge or diverge? If it converges, evaluate it.

11. Does the integral $\displaystyle\int_{-\infty}^{\infty} \sin x \,\mathrm{d}x$ converge or diverge? If it converges, evaluate it.

12. Evaluate $\displaystyle\int_{10}^{\infty} \frac{x^4 - 5x^3 + 2x - 7}{x^5 + 3x + 8}\mathrm{d}x$, or state that it diverges.

13. Evaluate $\displaystyle\int_{0}^{10} \frac{x - 1}{x^2 - 11x + 10}\mathrm{d}x$, or state that it diverges.

14. $*$. Determine (with justification!) which of the following applies to the integral $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{x^2+1}\mathrm{d}x$:

i $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{x^2+1}\mathrm{d}x$ diverges

ii $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{x^2+1}\mathrm{d}x$ converges but $\displaystyle\int_{-\infty}^{+\infty} \left|\frac{x}{x^2+1}\right| \mathrm{d}x$ diverges

iii $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{x^2+1}\mathrm{d}x$ converges, as does $\displaystyle\int_{-\infty}^{+\infty} \left|\frac{x}{x^2+1}\right| \mathrm{d}x$

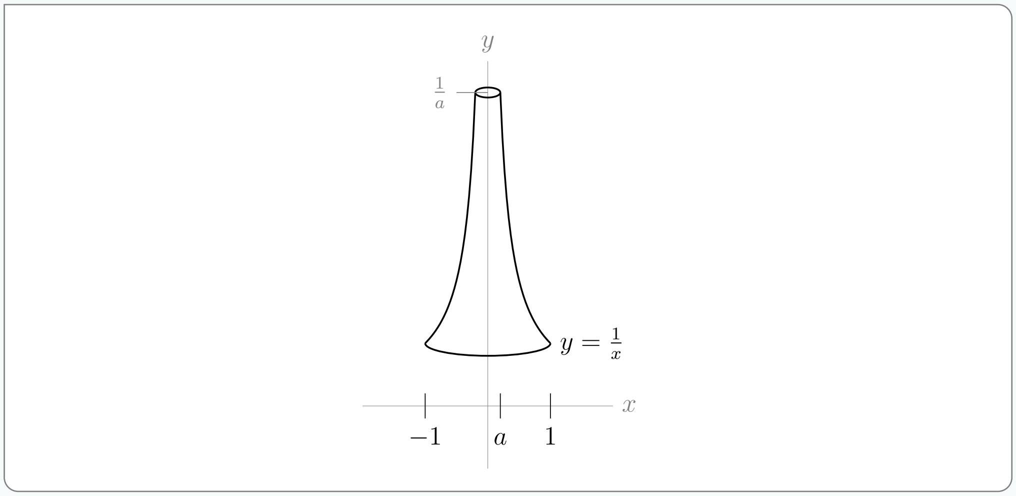
Remark: these options, respectively, are that the integral diverges, converges conditionally, and converges absolutely. You'll see this terminology used for series in Section 3.4.1.

15. $*$. Decide whether $I = \displaystyle\int_{0}^{\infty} \frac{|\sin x|}{x^{3/2} + x^{1/2}}\mathrm{d}x$ converges or diverges. Justify.

16. $*$. Does the integral $\displaystyle\int_{0}^{\infty} \frac{x+1}{x^{1/3}(x^2 + x + 1)} \mathrm{d}x$ converge or diverge?

Exercises — Stage 3

17. We craft a tall, vuvuzela-shaped solid by rotating the line $y = \dfrac{1}{x}$ from $x = a$ to $x = 1$ about the y-axis, where a is some constant between 0 and 1.



True or false: No matter how large a constant M is, there is some value of a that makes a solid with volume larger than M.

18. $*$. What is the largest value of q for which the integral $\displaystyle\int_1^\infty \frac{1}{x^{5q}}\,dx$ diverges?

19. For which values of p does the integral $\displaystyle\int_0^\infty \frac{x}{(x^2+1)^p}dx$ converge?

20. Evaluate $\displaystyle\int_2^\infty \frac{1}{t^4-1}dt$, or state that it diverges.

21. Does the integral $\displaystyle\int_{-5}^5 \left(\frac{1}{\sqrt{|x|}} + \frac{1}{\sqrt{|x-1|}} + \frac{1}{\sqrt{|x-2|}}\right) dx$ converge or diverge?

22. Evaluate $\displaystyle\int_0^\infty e^{-x}\sin x dx$, or state that it diverges.

23. $*$. Is the integral $\displaystyle\int_0^\infty \frac{\sin^4 x}{x^2}\,dx$ convergent or divergent? Explain why.

24. Does the integral $\displaystyle\int_0^\infty \frac{x}{e^x+\sqrt{x}}dx$ converge or diverge?

25. $*$. Let $M_{n,t}$ be the Midpoint Rule approximation for $\displaystyle\int_0^t \frac{e^{-x}}{1+x}dx$ with n equal subintervals. Find a value of t and a value of n such that $M_{n,t}$ differs from $\int_0^\infty \frac{e^{-x}}{1+x}dx$ by at most 10^{-4}. Recall that the error E_n introduced when the

Midpoint Rule is used with n subintervals obeys

$$|E_n| \le \frac{M(b-a)^3}{24n^2}$$

where M is the maximum absolute value of the second derivative of the integrand and a and b are the end points of the interval of integration.

26. Suppose $f(x)$ is continuous for all real numbers, and $\int_1^\infty f(x)\mathrm{d}x$ converges.

a If $f(x)$ is odd, does $\int_{-\infty}^{-1} f(x)\mathrm{d}x$ converge or diverge, or is there not enough information to decide?

b If $f(x)$ is even, does $\int_{-\infty}^{\infty} f(x)\mathrm{d}x$ converge or diverge, or is there not enough information to decide?

27. True or false:
There is some real number x, with $x \ge 1$, such that $\int_0^x \frac{1}{e^t}\mathrm{d}t = 1$.

1.13▲ More Integration Examples

⏩ Exercises

Recall that we are using $\log x$ to denote the logarithm of x with base e. In other courses it is often denoted $\ln x$.

Exercises — Stage 1

1. Match the integration method to a common kind of integrand it's used to antidifferentiate.

(A) $u = f(x)$ substitution	(I)	a function multiplied by its derivative
(B) trigonometric substitution	(II)	a polynomial times an exponential
(C) integration by parts	(III)	a rational function
(D) partial fractions	(IV)	the square root of a quadratic function

Exercises — Stage 2

2. Evaluate $\displaystyle\int_0^{\pi/2} \sin^4 x \cos^5 x \mathrm{d}x.$

3. Evaluate $\displaystyle\int \sqrt{3-5x^2}\mathrm{d}x.$

4. Evaluate $\displaystyle\int_0^{\infty} \frac{x-1}{e^x}\mathrm{d}x.$

5. Evaluate $\displaystyle\int \frac{-2}{3x^2+4x+1}\mathrm{d}x.$

6. Evaluate $\displaystyle\int_1^2 x^2 \log x \mathrm{d}x.$

7. ***.** Evaluate $\displaystyle\int \frac{x}{x^2-3}\,\mathrm{d}x.$

8. ***.** Evaluate the following integrals.

a $\displaystyle\int_0^4 \frac{x}{\sqrt{9+x^2}}\,\mathrm{d}x$

b $\displaystyle\int_0^{\pi/2} \cos^3 x\ \ \sin^2 x\,\mathrm{d}x$

c $\displaystyle\int_1^e x^3 \log x\,\mathrm{d}x$

9. ***.** Evaluate the following integrals.

a $\displaystyle\int_0^{\pi/2} x \sin x\,\mathrm{d}x$

b $\displaystyle\int_0^{\pi/2} \cos^5 x\,\mathrm{d}x$

10. ***.** Evaluate the following integrals.

a $\displaystyle\int_0^2 xe^x\,\mathrm{d}x$

b $\displaystyle\int_0^1 \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x$

c $\displaystyle\int_3^5 \frac{4x}{(x^2-1)(x^2+1)}\,\mathrm{d}x$

11. ***.** Calculate the following integrals.

a $\displaystyle\int_0^3 \sqrt{9-x^2}\,\mathrm{d}x$

b $\displaystyle\int_0^1 \log(1+x^2)\,\mathrm{d}x$

c $\displaystyle\int_3^\infty \frac{x}{(x-1)^2(x-2)}\,\mathrm{d}x$

12. Evaluate $\displaystyle\int \frac{\sin^4\theta - 5\sin^3\theta + 4\sin^2\theta + 10\sin\theta}{\sin^2\theta - 5\sin\theta + 6}\cos\theta\mathrm{d}\theta.$

13. ***.** Evaluate the following integrals. *Show your work.*

a $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2(2x)\cos^3(2x)\ \mathrm{d}x$

b $\displaystyle\int \left(9+x^2\right)^{-\frac{3}{2}}\ \mathrm{d}x$

c $\displaystyle\int \frac{\mathrm{d}x}{(x-1)(x^2+1)}$

d $\displaystyle\int x\arctan x\ \mathrm{d}x$

14. ***.** Evaluate the following integrals.

a $\displaystyle\int_0^{\pi/4} \sin^5(2x)\ \cos(2x)\ \mathrm{d}x$

b $\displaystyle\int \sqrt{4-x^2}\ \mathrm{d}x$

c $\displaystyle\int \frac{x+1}{x^2(x-1)}\ \mathrm{d}x$

15. ***.** Calculate the following integrals.

a $\displaystyle\int_0^\infty e^{-x}\sin(2x)\,\mathrm{d}x$

b $\displaystyle\int_0^{\sqrt{2}} \frac{1}{(2+x^2)^{3/2}}\,\mathrm{d}x$

c $\displaystyle\int_0^1 x\log(1+x^2)\,\mathrm{d}x$

d $\displaystyle\int_3^\infty \frac{1}{(x-1)^2(x-2)}\,\mathrm{d}x$

16. ***.** Evaluate the following integrals.

a $\displaystyle\int x\ \log x\ \mathrm{d}x$

b $\displaystyle\int \frac{(x-1)\,\mathrm{d}x}{x^2+4x+5}$

c $\displaystyle\int \frac{\mathrm{d}x}{x^2-4x+3}$

d $\displaystyle\int \frac{x^2\,\mathrm{d}x}{1+x^6}$

17. $*$. Evaluate the following integrals.

a $\displaystyle\int_0^1 \arctan x\ \mathrm{d}x.$

b $\displaystyle\int \frac{2x-1}{x^2-2x+5}\ \mathrm{d}x.$

18. $*$.

a Evaluate $\displaystyle\int \frac{x^2}{(x^3+1)^{101}}\,\mathrm{d}x.$

b Evaluate $\displaystyle\int \cos^3 x\ \sin^4 x\ \mathrm{d}x.$

19. Evaluate $\displaystyle\int_{\pi/2}^{\pi} \frac{\cos x}{\sqrt{\sin x}}\mathrm{d}x.$

20. $*$. Evaluate the following integrals.

a $\displaystyle\int \frac{e^x}{(e^x+1)(e^x-3)}\,\mathrm{d}x$

b $\displaystyle\int_2^4 \frac{x^2-4x+4}{\sqrt{12+4x-x^2}}\,\mathrm{d}x$

21. $*$. Evaluate these integrals.

a $\displaystyle\int \frac{\sin^3 x}{\cos^3 x}\ \mathrm{d}x$

b $\displaystyle\int_{-2}^{2} \frac{x^4}{x^{10}+16}\ \mathrm{d}x$

22. Evaluate $\displaystyle\int x\sqrt{x-1}\mathrm{d}x.$

23. Evaluate $\displaystyle\int \frac{\sqrt{x^2-2}}{x^2}\mathrm{d}x.$
You may use that $\int \sec x \mathrm{d}x = \log|\sec x + \tan x| + C$.

24. Evaluate $\displaystyle\int_0^{\pi/4} \sec^4 x \tan^5 x\,\mathrm{d}x.$

25. Evaluate $\displaystyle\int \frac{3x^2+4x+6}{(x+1)^3}\,\mathrm{d}x$.

26. Evaluate $\displaystyle\int \frac{1}{x^2+x+1}\,\mathrm{d}x$.

27. Evaluate $\displaystyle\int \sin x\cos x\tan x\mathrm{d}x$.

28. Evaluate $\displaystyle\int \frac{1}{x^3+1}\mathrm{d}x$.

29. Evaluate $\displaystyle\int (3x)^2\arcsin x\mathrm{d}x$.

Exercises — Stage 3

30. Evaluate $\displaystyle\int_0^{\pi/2}\sqrt{\cos t+1}\,\mathrm{d}t$.

31. Evaluate $\displaystyle\int_1^{e}\frac{\log\sqrt{x}}{x}\,\mathrm{d}x$.

32. Evaluate $\displaystyle\int_{0.1}^{0.2}\frac{\tan x}{\log(\cos x)}\,\mathrm{d}x$.

33. $*$. Evaluate these integrals.

a $\displaystyle\int \sin(\log x)\ \mathrm{d}x$

b $\displaystyle\int_0^1 \frac{1}{x^2-5x+6}\ \mathrm{d}x$

34. $*$. Evaluate (with justification).

a $\displaystyle\int_0^3 (x+1)\sqrt{9-x^2}\ \mathrm{d}x$

b $\displaystyle\int \frac{4x+8}{(x-2)(x^2+4)}\ \mathrm{d}x$

c $\displaystyle\int_{-\infty}^{+\infty}\frac{1}{e^x+e^{-x}}\ \mathrm{d}x$

35. Evaluate $\displaystyle\int \sqrt{\frac{x}{1-x}}\mathrm{d}x$.

36. Evaluate $\displaystyle\int_0^1 e^{2x}e^{e^x}\,\mathrm{d}x$.

37. Evaluate $\displaystyle\int \frac{xe^x}{(x+1)^2}\mathrm{d}x$.

38. Evaluate $\displaystyle\int \frac{x\sin x}{\cos^2 x}\,\mathrm{d}x$.
You may use that $\int \sec x \mathrm{d}x = \log|\sec x + \tan x| + C$.

39. Evaluate $\displaystyle\int x(x+a)^n \mathrm{d}x$, where a and n are constants.

40. Evaluate $\displaystyle\int \arctan(x^2)\mathrm{d}x$.

Chapter 2

Applications of Integration

In the previous chapter we defined the definite integral, based on its interpretation as the area of a region in the xy-plane. We also developed a bunch of theory to help us work with integrals. This abstract definition, and the associated theory, turns out to be extremely useful simply because "areas of regions in the xy-plane" appear in a huge number of different settings, many of which seem superficially not to involve "areas of regions in the xy-plane". Here are some examples.

- The work involved in moving a particle or in pumping a fluid out of a reservoir. See section 2.1.
- The average value of a function. See section 2.2.
- The center of mass of an object. See section 2.3.
- The time dependence of temperature. See section 2.4.
- Radiocarbon dating. See section 2.4.

Let us start with the first of these examples.

2.1 ▲ Work

2.1.1 ⏩ Work

While computing areas and volumes are nice mathematical applications of integration we can also use integration to compute quantities of importance in physics and statistics. One such quantity is work. Work is a way of quantifying the amount of energy that is required to act against a force [1]. In SI [2] metric units the force F has units newtons (which are kilogram-metres per second squared), x has units metres and the work W has units joules (which are newton-metres or kilogram-metres squared per second squared).

Definition 2.1.1

The work done by a force $F(x)$ in moving an object from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\mathrm{d}x$$

In particular, if the force is a constant, F, independent of x, the work is $F \cdot (b - a)$.

Here is some motivation for this definition. Consider a particle of mass m moving along the x-axis. Let the position of the particle at time t be $x(t)$. The particle starts at position a at time α, moves to the right, finishing at position $b > a$ at time β. While the particle moves, it is subject to a position-dependent force $F(x)$. Then Newton's law of motion [3] says [4] that force is mass times acceleration

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t) = F\big(x(t)\big)$$

Now consider our definition of work above. It tells us that the work done in moving the particle from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\mathrm{d}x$$

However, we know the position as a function of time, so we can substitute $x = x(t)$, $\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t$ (using Theorem 1.4.6) and rewrite the above integral:

$$W = \int_a^b F(x)\mathrm{d}x = \int_{t=\alpha}^{t=\beta} F(x(t))\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t$$

Using Newton's second law we can rewrite our integrand:

$$= m\int_\alpha^\beta \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t$$

1 For example — if your expensive closed-source textbook has fallen on the floor, work quantifies the amount of energy required to lift the object from the floor acting against the force of gravity.

2 SI is short for “le système international d'unités” which is French for “the international system of units”. It is the most recent internationally sanctioned version of the metric system, published in 1960. It aims to establish sensible units of measurement (no cubic furlongs per hogshead-Fahrenheit). It defines seven base units — metre (length), kilogram (mass), second (time), kelvin (temperature), ampere (electric current), mole (quantity of substance) and candela (luminous intensity). From these one can then establish derived units — such as metres per second for velocity and speed.

3 Specifically, the second of Newton's three law of motion. These were first published in 1687 in his “Philosophiæ Naturalis Principia Mathematica”.

4 It actually says something more graceful in Latin - Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur. Or — The alteration of motion is ever proportional to the motive force impressed; and is made in the line in which that force is impressed. It is amazing what you can find on the internet.

$$= m \int_\alpha^\beta \frac{\mathrm{d}v}{\mathrm{d}t} v(t)\mathrm{d}t \qquad \text{since } v(t) = \frac{\mathrm{d}x}{\mathrm{d}t}$$

$$= m \int_\alpha^\beta \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}v(t)^2\right)\mathrm{d}t$$

What happened here? By the chain rule, for any function $f(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}f(t)^2\right) = f(t)f'(t).$$

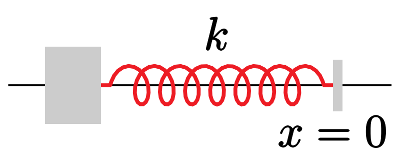
In the above computation we have used this fact with $f(t) = v(t)$. Now using the fundamental theorem of calculus (Theorem 1.3.1 part 2), we have

$$\begin{aligned} W &= m \int_\alpha^\beta \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}v(t)^2\right)\mathrm{d}t \\ &= \frac{1}{2}mv(\beta)^2 - \frac{1}{2}mv(\alpha)^2. \end{aligned}$$

By definition, the function $\frac{1}{2}mv(t)^2$ is the kinetic energy [5] of the particle at time t. So the work W of Definition 2.1.1 is the change in kinetic energy from the time the particle was at $x = a$ to the time it was at $x = b$.

Example 2.1.2 Hooke's Law.

Imagine that a spring lies along the x-axis. The left hand end is fixed to a wall, but the right hand end lies freely at $x = 0$. So the spring is at its "natural length".



- Now suppose that we wish to stretch out the spring so that its right hand end is at $x = L$.
- Hooke's Law [a] says that when a (linear) spring is stretched (or compressed) by x units beyond its natural length, it exerts a force of magnitude kx, where the constant k is the spring constant of that spring.
- In our case, once we have stretched the spring by x units to the right, the spring will be trying to pull back the right hand end by applying a force of magnitude kx directed to the left.

5 This is not a physics text so we will not be too precise. Roughly speaking, kinetic energy is the energy an object possesses due to it being in motion, as opposed to potential energy, which is the energy of the object due to its position in a force field. Leibniz and Bernoulli determined that kinetic energy is proportional to the square of the velocity, while the modern term "kinetic energy" was first used by Lord Kelvin (back while he was still William Thompson).

- For us to continue stretching the spring we will have to apply a compensating force of magnitude kx directed to the right. That is, we have to apply the force $F(x) = +kx$.

- So to stretch a spring by L units from its natural length we have to supply the work

$$W = \int_0^L kx\,\mathrm{d}x = \frac{1}{2}kL^2$$

a Robert Hooke (1635–1703) was an English contemporary of Isaac Newton (1643–1727). It was in a 1676 letter to Hooke that Newton wrote "If I have seen further it is by standing on the shoulders of Giants." There is some thought that this was sarcasm and Newton was actually making fun of Hooke, who had a spinal deformity. However at that time Hooke and Newton were still friends. Several years later they did have a somewhat public falling-out over some of Newton's work on optics.

Example 2.1.2

Example 2.1.3 Spring.

A spring has a natural length of 0.1m. If a 12N force is needed to keep it stretched to a length of 0.12m, how much work is required to stretch it from 0.12m to 0.15m?
Solution: In order to answer this question we will need to determine the spring constant and then integrate the appropriate function.

- Our first task is to determine the spring constant k. We are told that when the spring is stretched to a length of 0.12m, i.e. to a length of $0.12 - 0.1 = 0.02$m beyond its natural length, then the spring generates a force of magnitude 12N.

- Hooke's law states that the force exerted by the spring, when it is stretched by x units, has magnitude kx, so

$$\begin{aligned} 12 &= k \cdot 0.02 = k \cdot \frac{2}{100} && \text{thus} \\ k &= 600. \end{aligned}$$

- So to stretch the spring
  - from a length of 0.12m, i.e. a length of $x = 0.12 - 0.1 = 0.02$m beyond its natural length,
  - to a length of 0.15m, i.e. a length of $x = 0.15 - 0.1 = 0.05$m beyond its natural length,

  takes work

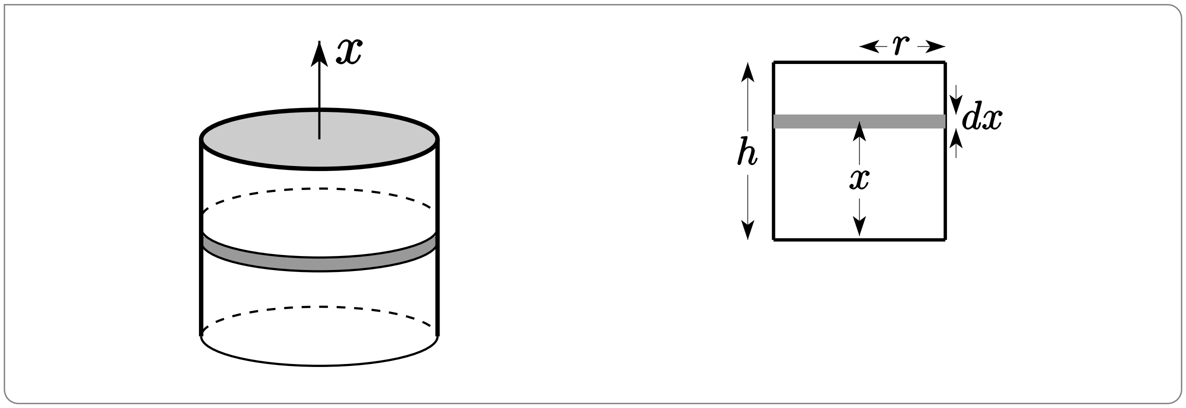
$$W = \int_{0.02}^{0.05} kx\,\mathrm{d}x = \left[\frac{1}{2}kx^2\right]_{0.02}^{0.05}$$

$$= 300\big(0.05^2 - 0.02^2\big)$$
$$= 0.63\text{J}$$

Example 2.1.3

Example 2.1.4 Pumping Out a Reservoir.

A cylindrical reservoir [a] of height h and radius r is filled with a fluid of density ρ. We would like to know how much work is required to pump all of the fluid out the top of the reservoir.



Solution: We are going to tackle this problem by applying the standard integral calculus "slice into small pieces" strategy. This is how we computed areas and volumes — slice the problem into small pieces, work out how much each piece contributes, and then add up the contributions using an integral.

- Start by slicing the reservoir (or rather the fluid inside it) into thin, horizontal, cylindrical pancakes, as in the figure above. We proceed by determining how much work is required to pump out this pancake volume of fluid [b].
- Each pancake is a squat cylinder with thickness $\mathrm{d}x$ and circular cross section of radius r and area πr^2. Hence it has volume $\pi r^2 \mathrm{d}x$ and mass $\rho \times \pi r^2 \mathrm{d}x$.
- Near the surface of the Earth gravity exerts a downward force of mg on a body of mass m. The constant $g = 9.8\text{m}/\text{sec}^2$ is called the *standard acceleration due to gravity* [c]. For us to raise the pancake we have to apply a compensating upward force of mg, which, for our pancake, is

$$F = g\rho \times \pi r^2 \mathrm{d}x$$

- To remove the pancake at height x from the reservoir we need to raise it to height h. So we have to lift it a distance $h - x$ using the force $F = \pi\rho g r^2 \mathrm{d}x$, which takes work $\pi\rho g r^2\,(h-x)\,\mathrm{d}x$.

- The total work to empty the whole reservoir is

$$W = \int_0^h \pi\,\rho g\,r^2(h-x)\mathrm{d}x = \pi\,\rho g\,r^2 \int_0^h (h-x)\mathrm{d}x$$
$$= \pi\,\rho g\,r^2 \left[hx - \frac{x^2}{2}\right]_0^h$$
$$= \frac{\pi}{2}\,\rho g\,r^2 h^2$$

- If we measure lengths in metres and mass in kilograms, then this quantity has units of Joules. If we instead used feet and pounds[d] then this would have units of "foot-pounds". One foot-pound is equal to 1.355817... Joules.

a We could assign units to these measurements — such as metres for the lengths h and r, and kilograms per cubic metre for the density ρ.
b Potential for a bad "work out how much work out" pun here.
c This quantity is not actually constant — it varies slightly across the surface of earth depending on local density, height above sea-level and centrifugal force from the earth's rotation. It is, for example, slightly higher in Oslo and slightly lower in Singapore. It is actually *defined* to be 9.80665 m/sec^2 by the International Organisation for Standardization.
d It is extremely mysterious to the authors why a person would do science or engineering in imperial units. One of the authors still has nightmares about having had to do so as a student.

Example 2.1.4

Example 2.1.5 Escape Velocity.

Suppose that you shoot a probe straight up from the surface of the Earth — at what initial speed must the probe move in order to escape Earth's gravity?

Solution: We determine this by computing how much work must be done in order to escape Earth's gravity. If we assume that all of this work comes from the probe's initial kinetic energy, then we can establish the minimum initial velocity required.

- The work done by gravity when a mass moves from the surface of the Earth to a height h above the surface is

$$W = \int_0^h F(x)\mathrm{d}x$$

where $F(x)$ is the gravitational force acting on the mass at height x above the Earth's surface.

- The gravitational force [a] of the Earth acting on a particle of mass m at a height x above the surface of the Earth is

$$F = -\frac{GMm}{(R+x)^2},$$

where G is the gravitational constant, M is the mass of the Earth and R is the radius of the Earth. Note that $R+x$ is the distance from the object to the centre of the Earth. Additionally, note that this force is negative because gravity acts downward.

- So the work done by gravity on the probe, as it travels from the surface of the Earth to a height h, is

$$\begin{aligned} W &= -\int_0^h \frac{GMm}{(R+x)^2}\mathrm{d}x \\ &= -GMm\int_0^h \frac{1}{(R+x)^2}\mathrm{d}x \end{aligned}$$

A quick application of the substitution rule with $u = R+x$ gives

$$\begin{aligned} &= -GMm\int_{u(0)}^{u(h)} \frac{1}{u^2}\mathrm{d}u \\ &= -GMm\left[-\frac{1}{u}\right]_{u=R}^{u=R+h} \\ &= \frac{GMm}{R+h} - \frac{GMm}{R} \end{aligned}$$

- So if the probe completely escapes the Earth and travels all the way to $h = \infty$, gravity does work

$$\lim_{h\to\infty}\left[\frac{GMm}{R+h} - \frac{GMm}{R}\right] = -\frac{GMm}{R}$$

The minus sign means that gravity has removed energy $\frac{GMm}{R}$ from the probe.

- To finish the problem we need one more assumption. Let us assume that all of this energy comes from the probe's initial kinetic energy and that the probe is not fitted with any sort of rocket engine. Hence the initial kinetic energy $\frac{1}{2}mv^2$ (coming from an initial velocity v) must be at least as large as the work computed above. That is we need

$$\frac{1}{2}mv^2 \geq \frac{GMm}{R} \qquad \text{which rearranges to give}$$

$$v \geq \sqrt{\frac{2GM}{R}}$$

- The right hand side of this inequality, $\sqrt{\frac{2GM}{R}}$, is called the escape velocity.

a Newton published his inverse square law of universal gravitation in his Principia in 1687. His law states that the gravitational force between two masses m_1 and m_2 is $F = -G\frac{m_1 m_2}{r^2}$ where

r is the distance separating the (centres of the) masses and $G = 6.674 \times 10^{-11}\text{Nm}^2/\text{kg}^2$ is the gravitational constant. Notice that r measures the separation between the centres of the masses not the distance between the surfaces of the objects. Also, do not confuse G with g — standard acceleration due to gravity. The first measurement of G was performed by Henry Cavendish in 1798 — the interested reader should look up the "Cavendish experiment" for details of this very impressive work.

Example 2.1.5

Example 2.1.6 Lifting a Cable.

A 10-metre-long cable of mass 5kg is used to lift a bucket of water, with mass 8kg, out of a well. Find the work done.

Solution: Denote by y the height of the bucket above the top of the water in the well. So the bucket is raised from $y = 0$ to $y = 10$. The cable has mass density 0.5kg/m. So when the bucket is at height y,

- the cable that remains to be lifted has mass $0.5(10 - y)$ kg and
- the remaining cable and water is subject to a downward gravitational force of magnitude $\big[0.5(10-y)+8\big]g = \big[13 - \frac{y}{2}\big]g$, where $g = 9.8$ m/sec^2.

So to raise the bucket from height y to height $y + \mathrm{d}y$ we need to apply a compensating upward force of $\big[13 - \frac{y}{2}\big]g$ through distance $\mathrm{d}y$. This takes work $\big[13 - \frac{y}{2}\big]g\mathrm{d}y$. So the total work required is

$$\int_0^{10} \left[13 - \frac{y}{2}\right] g\mathrm{d}y = g\left[13y - \frac{y^2}{4}\right]_0^{10} = \big[130 - 25\big]g = 105g = 1029 \text{ J}$$

2.1.2 ➤ Exercises

Exercises — Stage 1

1. Find the work (in joules) required to lift a 3-gram block of matter a height of 10 centimetres against the force of gravity (with $g = 9.8$ m/sec^2).

2. A rock exerts a force of 1 N on the ground where it sits due to gravity. Use $g = 9.8$ m/sec^2.
What is the mass of the rock?
How much work (in joules) does it take to lift that rock one metre in the air?

3. Consider the equation

$$W = \int_a^b F(x)\,\mathrm{d}x$$

where x is measured in metres and $F(x)$ is measured in kilogram-metres per second squared (newtons).

For some large n, we might approximate

$$W \approx \sum_{i=1}^{n} F(x_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and x_i is some number in the interval $[a+(i-1)\Delta x, a+i\Delta x]$. (This is just the general form of a Riemann sum).

a What are the units of Δx?

b What are the units of $F(x_i)$?

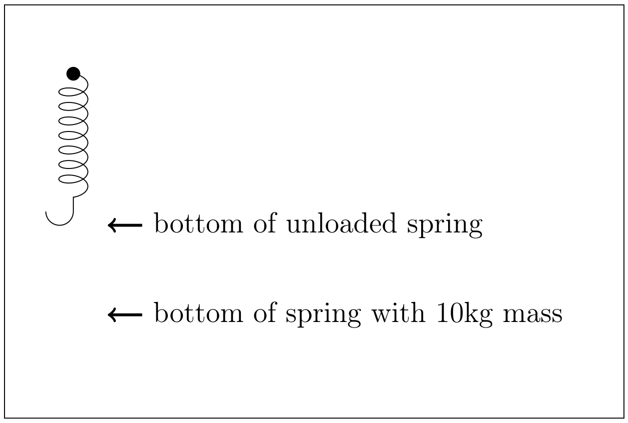
c Using your answers above, what are the units of W?

Remark: we already know the units of W from the text, but the Riemann sum illustrates *why they make sense* arising from this particular integral.

4. Suppose $f(x)$ has units $\dfrac{\text{smoot}}{\text{megaFonzie}}$, and x is measured in barns [a]. What are the units of the quantity $\int_0^1 f(x)\,\mathrm{d}x$?

a For this problem, it doesn't matter what the units measure, but a smoot is a silly measure of length; a megaFonzie is an apocryphal measure of coolness; and a barn is a humorous (but actually used) measure of area. For explanations (and entertainment) see https://en.wikipedia.org/wiki/List_of_humorous_units_of_measurement and https://en.wikipedia.org/wiki/List_of_unusual_units_of_measurement (accessed 27 July 2017).

5. You want to weigh your luggage before a flight. You don't have a scale or balance, but you do have a heavy-duty spring from your local engineering-supply store. You nail it to your wall, marking where the bottom hangs. You hang a one-litre bag of water (with mass one kilogram) from the spring, and observe that the spring stretches 1 cm. Where on the wall should you mark the bottom of the spring corresponding to a hanging mass of 10kg?



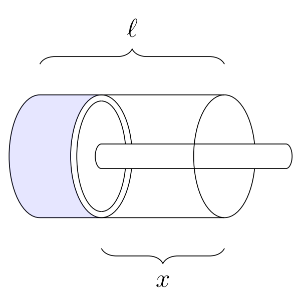
You may assume that the spring obeys Hooke's law.

6. The work done by a force in moving an object from position $x = 1$ to $x = b$ is $W(b) = -b^3 + 6b^2 - 9b + 4$ for any b in $[1, 3]$. At what position x in $[1, 3]$ is the force the strongest?

Exercises — Stage 2 Questions 9 through 16 offer practice on two broad types of calculations covered in the text: lifting things against gravity, and stretching springs. You may make the same physical assumptions as in the text: that is, springs follow Hooke's law, and the acceleration due to gravity is a constant -9.8 metres per second squared.For Questions 18 and 19, use the principle (introduced after Definition 2.1.1 and utilized in Example 2.1.5) that the work done on a particle by a force over a distance is equal to the change in kinetic energy of that particle.

7. ***.** A variable force $F(x) = \frac{a}{\sqrt{x}}$ Newtons moves an object along a straight line when it is a distance of x meters from the origin. If the work done in moving the object from $x = 1$ meters to $x = 16$ meters is 18 joules, what is the value of a? Don't worry about the units of a.

8. A tube of air is fitted with a plunger that compresses the air as it is pushed in. If the natural length of the tube of air is ℓ, when the plunger has been pushed x metres past its natural position, the force exerted by the air is $\frac{c}{\ell - x}$ N, where c is a positive constant (depending on the particulars of the tube of air) and $x < \ell$.



a What are the units of c?

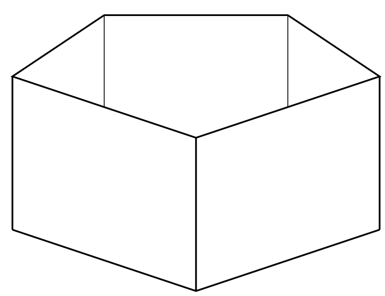
b How much work does it take to push the plunger from 1 metre past its natural position to 1.5 metres past its natural position? (You may assume $\ell > 1.5$.)

9. $*$. Find the work (in joules) required to stretch a string 10 cm beyond equilibrium, if its spring constant is $k = 50$ N/m.

10. $*$. A force of 10 N (newtons) is required to hold a spring stretched 5 cm beyond its natural length. How much work, in joules (J), is done in stretching the spring from its natural length to 50 cm beyond its natural length?

11. $*$. A 5-metre-long cable of mass 8 kg is used to lift a bucket off the ground. How much work is needed to raise the entire cable to height 5 m? Ignore the mass of the bucket and its contents.

12. A tank 1 metre high has pentagonal cross sections of area 3 m^2 and is filled with water. How much work does it take to pump out all the water?
You may assume the density of water is 1 kg per 1000 cm^3.



13. $*$. A sculpture, shaped like a pyramid 3m high sitting on the ground, has been made by stacking smaller and smaller (very thin) iron plates on top of one another. The iron plate at height z m above ground level is a square whose side length is $(3-z)$ m. All of the iron plates started on the floor of a basement 2 m below ground level.
Write down an integral that represents the work, in joules, it took to move all of the iron from its starting position to its present position. Do ***not*** evaluate the integral. (You can use 9.8 m/s^2 for the acceleration due to gravity and 8000 kg/m^3 for the density of iron.)

14. Suppose a spring extends 5 cm past its natural length when one kilogram is hung from its end. How much work is done to extend the spring from 5 cm past its natural length to 7 cm past its natural length?

15. Ten kilograms of firewood are hoisted on a rope up a height of 4 metres to a second-floor deck. If the total work done is 400 joules, what is the mass of the 4 metres of rope?
You may assume that the rope has the same density all the way along.

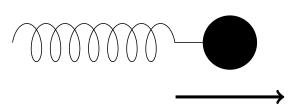
16. A 5 kg weight is attached to the middle of a 10-metre long rope, which dangles out a window. The rope alone has mass 1 kg. How much work does it take to pull the entire rope in through the window, together with the weight?

17. A box is dragged along the floor. Friction exerts a force in the opposite direction of motion from the box, and that force is equal to $\mu \times m \times g$, where μ is a constant, m is the mass of the box and g is the acceleration due to gravity. You may assume $g = 9.8$ m/sec^2.

a How much work is done dragging a box of mass 10 kg along the floor for three metres if $\mu = 0.4$?

b Suppose the box contains a volatile substance that rapidly evaporates. You pull the box at a constant rate of 1 m/sec for three seconds, and the mass of the box at t seconds ($0 \leq t \leq 3$) is $(10 - \sqrt{t})$ kilograms. If $\mu = 0.4$, how much work is done pulling the box for three seconds?

18. A ball of mass 1 kg is attached to a spring, and the spring is attached to a table. The ball moves with some initial velocity, and the spring slows it down. At its farthest, the spring stretches 10 cm past its natural length. If the spring constant is 5 N/m, what was the initial velocity of the ball?



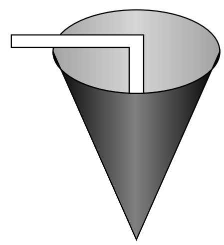
You may assume that the ball starts moving with initial velocity v_0, and that the only force slowing it down is the spring. You may also assume that the spring started out at its natural length, it follows Hooke's law, and when it is stretched its farthest, the velocity of the ball is 0 m/sec.

19. A mild-mannered university professor who is definitely not a spy notices that when their car is on the ground, it is 2 cm shorter than when it is on a jack. (That is: when the car is on a jack, its struts are at their natural length; when on the ground, the weight of the car causes the struts to compress 2 cm.) The university professor calculates that if they were to jump a local neighborhood drawbridge, their car would fall to the ground with a speed of 4 m/sec. If the car can sag 20 cm before important parts scrape the ground, and the car has mass 2000 kg unoccupied (2100 kg with the professor inside), can the professor, who is certainly not involved in international intrigue, safely jump the bridge?
Assume the car falls vertically, the struts obey Hooke's law, and the work

done by the struts is equal to the change in kinetic energy of the car + professor. Use 9.8 m/sec^2 for the acceleration due to gravity.

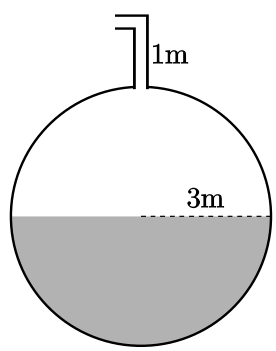
Exercises — Stage 3

20. A disposable paper cup has the shape of a right circular cone with radius 5 cm and height 15 cm, and is completely filled with water. How much work is done sucking all the water out of the cone with a straw?



You may assume that 1 m^3 of water has mass 1000 kilograms, the acceleration due to gravity is -9.8 m/sec^2, and that the water moves as high up as the very top of the cup and no higher.

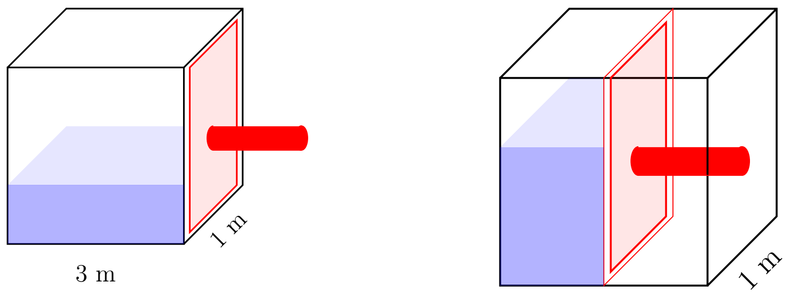
21. $*$. A spherical tank of radius 3 metres is half-full of water. It has a spout of length 1 metre sticking up from the top of the tank. Find the work required to pump all of the water in the tank out the spout. The density of water is 1000 kilograms per cubic metre. The acceleration due to gravity is 9.8 metres per second squared.



22. A 5-metre cable is pulled out of a deep hole, where it was dangling straight down. The cable has density $\rho(x) = (10-x)$ kg/m, where x is the distance from the bottom end of the rope. (So, the bottom of the cable is denser than the top.) How much work is done pulling the cable out of the hole?

23. A rectangular tank is fitted with a plunger that can raise and lower the water level by decreasing and increasing the length of its base, as in the diagrams below. The tank has base width 1 m (which does not change) and contains 3

m^3 of water.



The force of the water acting on any tiny piece of the plunger is PA, where P is the pressure of the water, and $\mathrm{d}A$ is the area of the tiny piece. The pressure varies with the depth of the piece (below the surface of the water). Specifically, $P = cD$, where D is the depth of the tiny piece and c is a constant, in this case $c = 9800$ N/m^3.

a If the length of the base is 3 m, give the force of the water on the entire plunger. (You can do this with an integral: it's the sum of the force on all the tiny pieces of the plunger.)

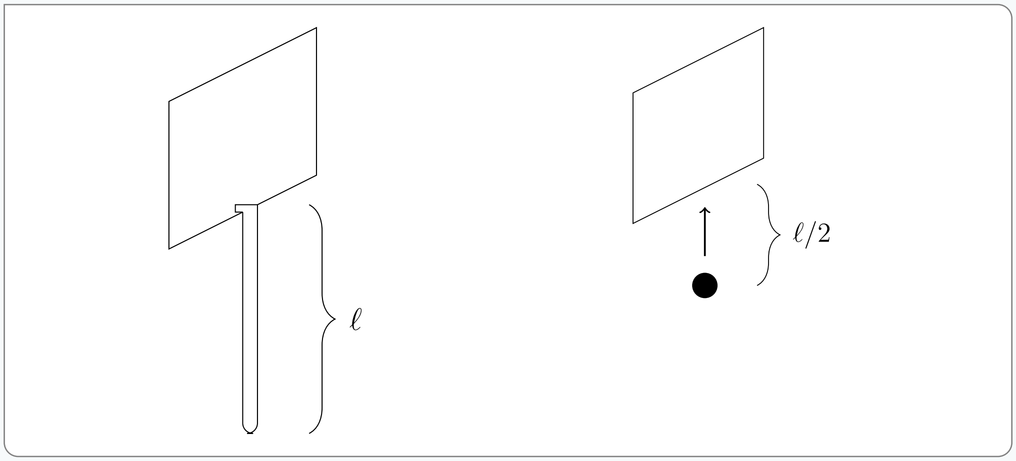
b If the length of the base is x m, give the force of the water on the entire plunger.

c Give the work required to move the plunger in so that the base length changes from 3 m to 1 m.

24. A leaky bucket picks up 5 L of water from a well, but drips out 1 L every ten seconds. If the bucket was hauled up 5 metres at a constant speed of 1 metre every two seconds, how much work was done?
Assume the rope and bucket have negligible mass and one litre of water has 1 kg mass, and use 9.8 m/sec^2 for the acceleration due to gravity.

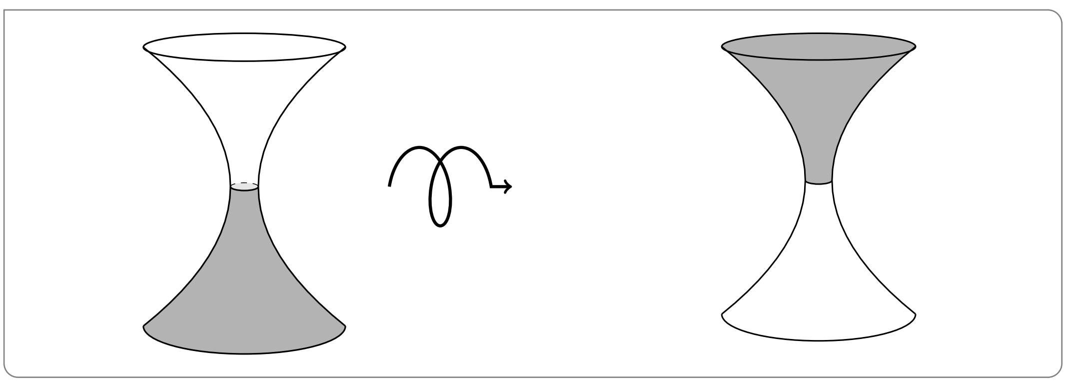
25. The force of gravity between two objects, one of mass m_1 and another of mass m_2, is $F = G\dfrac{m_1m_2}{r^2}$, where r is the distance between them and G is the gravitational constant.
How much work is required to separate the earth and the moon far enough apart that the gravitational attraction between them is negligible?
Assume the mass of the earth is 6×10^{24} kg and the mass of the moon is 7×10^{22} kg, and that they are currently 400 000 km away from each other. Also, assume $G = 6.7 \times 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{sec}^2}$, and the only force acting on the earth and moon is the gravity between them.

26. True or false: the work done pulling up a dangling cable of length ℓ and mass m (with uniform density) is the same as the work done lifting up a ball of mass m a height of $\ell/2$.



27. A tank one metre high is filled with watery mud that has settled to be denser at the bottom than at the top.
At height h metres above the bottom of the tank, the cross-section of the tank has the shape of the finite region bounded by the two curves $y = x^2$ and $y = 2 - h - 3x^2$. At height h metres above the bottom of the tank, the density of the liquid is $1000\sqrt{2-h}$ kilograms per cubic metre.
How much work is done to pump all the liquid out of the tank?
You may assume the acceleration due to gravity is 9.8 m/sec^2.

28. An hourglass is 0.2 m tall and shaped such that that y metres above or below its vertical centre it has a radius of $y^2 + 0.01$ m.
It is exactly half-full of sand, which has mass $M = \frac{1}{7}$ kilograms.
How much work is done on the sand by quickly flipping the hourglass over?



Assume that the work done is only moving against gravity, with $g = 9.8$ m/sec^2, and the sand has uniform density. Also assume that at the instant the hourglass is flipped over, the sand has not yet begun to fall, as in the picture above.

29. Suppose at position x a particle experiences a force of $F(x) = \sqrt{1-x^4}$ N. Approximate the work done moving the particle from $x = 0$ to $x = 1/2$, accurate to within 0.01 J.

2.2▲ Averages

Another frequent [1] application of integration is computing averages and other statistical quantities. We will not spend too much time on this topic — that is best left to a proper course in statistics — however, we will demonstrate the application of integration to the problem of computing averages.

Let us start with the definition [2] of the average of a finite set of numbers.

Definition 2.2.1

The average (mean) of a set of n numbers $y_1, y_2, \cdots, y_n$ is

$$y_{\text{ave}} = \bar{y} = \langle y \rangle = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

The notations y_{ave}, $\bar{y}$ and $\langle y \rangle$ are all commonly used to represent the average.

Now suppose that we want to take the average of a function $f(x)$ with x running continuously from a to b. How do we even define what that means? A natural approach is to

- select, for each natural number n, a sample of n, more or less uniformly distributed, values of x between a and b,
- take the average of the values of f at the selected points,
- and then take the limit as n tends to infinity.

Unsurprisingly, this process looks very much like how we computed areas and volumes previously. So let's get to it.

1 Awful pun. The two main approaches to statistics are frequentism and Bayesianism; the latter named after Bayes' Theorem which is, in turn, named for Reverend Thomas Bayes. While this (both the approaches to statistics and their history and naming) is a very interesting and quite philosophical topic, it is beyond the scope of this course. The interested reader has plenty of interesting reading here to interest them.

2 We are being a little loose here with the distinction between mean and average. To be much more pedantic — the average is the arithmetic mean. Other interesting "means" are the geometric and harmonic means:

$$\begin{aligned}
\text{arithmetic mean} &= \frac{1}{n}\left(y_1 + y_2 + \cdots + y_n\right) \\
\text{geometric mean} &= \left(y_1 \cdot y_2 \cdots y_n\right)^{\frac{1}{n}} \\
\text{harmonic mean} &= \left[\frac{1}{n}\left(\frac{1}{y_1} + \frac{1}{y_2} + \cdots \frac{1}{y_n}\right)\right]^{-1}
\end{aligned}$$

All of these quantities, along with the median and mode, are ways to measure the typical value of a set of numbers. They all have advantages and disadvantages — another interesting topic beyond the scope of this course, but plenty of fodder for the interested reader and their favourite search engine. But let us put pedantry (and beyond-the-scope-of-the-course-reading) aside and just use the terms average and mean interchangeably for our purposes here.

- First fix any natural number n.
- Subdivide the interval $a \le x \le b$ into n equal subintervals, each of width $\Delta x = \frac{b-a}{n}$.
- The subinterval number i runs from x_{i-1} to x_i with $x_i = a + i\frac{b-a}{n}$.
- Select, for each $1 \le i \le n$, one value of x from subinterval number i and call it x_i^*. So $x_{i-1} \le x_i^* \le x_i$.
- The average value of f at the selected points is

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i^*) = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta x \qquad \text{since } \Delta x = \frac{b-a}{n}$$

giving us a Riemann sum.

Now when we take the limit $n \to \infty$ we get exactly $\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$. That's why we define

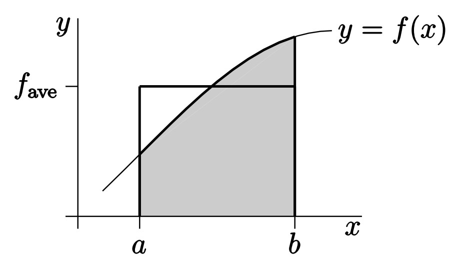
Definition 2.2.2

Let $f(x)$ be an integrable function defined on the interval $a \le x \le b$. The average value of f on that interval is

$$f_{\text{ave}} = \bar{f} = \langle f \rangle = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$$

Consider the case when $f(x)$ is positive. Then rewriting Definition 2.2.2 as

$f_{\text{ave}}\ (b-a) = \int_a^b f(x)\mathrm{d}x$



gives us a link between the average value and the area under the curve. The right-hand side is the area of the region

$$\left\{(x,y) \,\middle|\, a \le x \le b,\ 0 \le y \le f(x)\ \right\}$$

while the left-hand side can be seen as the area of a rectangle of width $b-a$ and height f_{ave}. Since these areas must be the same, we interpret f_{ave} as the height of the rectangle which has the same width and the same area as $\left\{(x,y) \,\middle|\, a \le x \le b,\ 0 \le y \le f(x)\ \right\}$.

Let us start with a couple of simple examples and then work our way up to harder ones.

Example 2.2.3 An easy warm-up.

Let $f(x) = x$ and $g(x) = x^2$ and compute their average values over $1 \le x \le 5$.
Solution: We can just plug things into the definition.

$$\begin{aligned}
f_{\text{ave}} &= \frac{1}{5-1}\int_1^5 x\,\mathrm{d}x \\
&= \frac{1}{4}\left[\frac{x^2}{2}\right]_1^5 \\
&= \frac{1}{8}(25-1) = \frac{24}{8} \\
&= 3
\end{aligned}$$

as we might expect. And then

$$\begin{aligned}
g_{\text{ave}} &= \frac{1}{5-1}\int_1^5 x^2\,\mathrm{d}x \\
&= \frac{1}{4}\left[\frac{x^3}{3}\right]_1^5 \\
&= \frac{1}{12}(125-1) = \frac{124}{12} \\
&= \frac{31}{3}
\end{aligned}$$

Something a little more trigonometric

Example 2.2.4 Average of sine.

Find the average value of $\sin(x)$ over $0 \le x \le \frac{\pi}{2}$.
Solution: Again, we just need the definition.

$$\begin{aligned}
\text{average} &= \frac{1}{\frac{\pi}{2}-0}\int_0^{\frac{\pi}{2}} \sin(x)\,\mathrm{d}x \\
&= \frac{2}{\pi}\cdot\Big[-\cos(x)\Big]_0^{\frac{\pi}{2}} \\
&= \frac{2}{\pi}(-\cos(\frac{\pi}{2}) + \cos(0)) \\
&= \frac{2}{\pi}.
\end{aligned}$$

We could keep going… But better to do some more substantial examples.

Example 2.2.5 Average velocity.

Let $x(t)$ be the position at time t of a car moving along the x-axis. The velocity of the car at time t is the derivative $v(t) = x'(t)$. The average velocity of the car over the time interval $a \le t \le b$ is

$$\begin{aligned} v_{\text{ave}} &= \frac{1}{b-a}\int_a^b v(t)\mathrm{d}t \\ &= \frac{1}{b-a}\int_a^b x'(t)\mathrm{d}t \\ &= \frac{x(b)-x(a)}{b-a} && \text{by the fundamental theorem of calculus.} \end{aligned}$$

The numerator in this formula is just the displacement (net distance travelled — if $x'(t) \ge 0$, it's the distance travelled) between time a and time b and the denominator is just the time it took.
Notice that this is exactly the formula we used way back at the start of your *differential* calculus class to help introduce the idea of the derivative. Of course this is a very circuitous way to get to this formula — but it is reassuring that we get the same answer.

A very physics example.

Example 2.2.6 Peak vs RMS voltage.

When you plug a light bulb into a socket [a] and turn it on, it is subjected to a voltage

$$V(t) = V_0 \sin(\omega t - \delta)$$

where

- $V_0 = 170$ volts,
- $\omega = 2\pi \times 60$ (which corresponds to 60 cycles per second [b]) and
- the constant δ is an (unimportant) phase. It just shifts the time at which the voltage is zero

The voltage V_0 is the "peak voltage" — the maximum value the voltage takes over time. More typically we quote the "root mean square" voltage [c] (or RMS-voltage). In this example we explain the difference, but to simplify the calculations, let us simplify the voltage function and just use

$$V(t) = V_0 \sin(t)$$

Since the voltage is a sine-function, it takes both positive and negative values. If we take its simple average over 1 period then we get

$$V_{\text{ave}} = \frac{1}{2\pi - 0}\int_0^{2\pi} V_0 \sin(t)\mathrm{d}t$$

$$
\begin{aligned}
&= \frac{V_0}{2\pi}\Big[-\cos(t)\Big]_0^{2\pi} \\
&= \frac{V_0}{2\pi}\left(-\cos(2\pi)+\cos 0\right) = \frac{V_0}{2\pi}(-1+1) \\
&= 0
\end{aligned}
$$

This is clearly not a good indication of the typical voltage.
What we actually want here is a measure of how far the voltage is from zero. Now we could do this by taking the average of $|V(t)|$, but this is a little harder to work with. Instead we take the average of the square [d] of the voltage (so it is always positive) and then take the square root at the end. That is

$$
\begin{aligned}
V_{\mathrm{rms}} &= \sqrt{\frac{1}{2\pi-0}\int_0^{2\pi} V(t)^2 \mathrm{d}t} \\
&= \sqrt{\frac{1}{2\pi}\int_0^{2\pi} V_0^2\sin^2(t)\mathrm{d}t} \\
&= \sqrt{\frac{V_0^2}{2\pi}\int_0^{2\pi} \sin^2(t)\mathrm{d}t}
\end{aligned}
$$

This is called the "root mean square" voltage.
Though we do know how to integrate sine and cosine, we don't (yet) know how to integrate their squares. A quick look at double-angle formulas [e] gives us a way to eliminate the square:

$$\cos(2\theta) = 1 - 2\sin^2\theta \implies \sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

Using this we manipulate our integrand a little more:

$$
\begin{aligned}
V_{\mathrm{rms}} &= \sqrt{\frac{V_0^2}{2\pi}\int_0^{2\pi} \frac{1}{2}(1-\cos(2t))\mathrm{d}t} \\
&= \sqrt{\frac{V_0^2}{4\pi}\left[t - \frac{1}{2}\sin(2t)\right]_0^{2\pi}} \\
&= \sqrt{\frac{V_0^2}{4\pi}\left(2\pi - \frac{1}{2}\sin(4\pi) - 0 + \frac{1}{2}\sin(0)\right)} \\
&= \sqrt{\frac{V_0^2}{4\pi}\cdot 2\pi} \\
&= \frac{V_0}{\sqrt{2}}
\end{aligned}
$$

So if the peak voltage is 170 volts then the RMS voltage is $\frac{170}{\sqrt{2}} \approx 120.2$.

a A normal household socket delivers alternating current, rather than the direct current USB supplies. At the risk of yet another "the interested reader" suggestion — the how and why household plugs supply AC current is another worthwhile and interesting digression from studying integration. The interested reader should look up the "War of Currents". The diligent and interested reader should bookmark this, finish the section and come back to it later.

b Some countries supply power at 50 cycles per second. Japan actually supplies both — 50 cycles in the east of the country and 60 in the west.

c This example was written in North America where the standard voltage supplied to homes is 120 volts. Most of the rest of the world supplies homes with 240 volts. The main reason for this difference is the development of the light bulb. The USA electrified earlier when the best voltage for bulb technology was 110 volts. As time went on, bulb technology improved and countries that electrified later took advantage of this (and the cheaper transmission costs that come with higher voltage) and standardised at 240 volts. So many digressions in this section!

d For a finite set of numbers one can compute the "quadratic mean" which is another way to generalise the notion of the average:

$$\text{quadratic mean} = \sqrt{\frac{1}{n}\left(y_1^2 + y_2^2 + \cdots + y_n^2\right)}$$

e A quick glance at Appendix A.14 will refresh your memory.

Example 2.2.6

Continuing this very physics example:

Example 2.2.7 Peak vs RMS voltage — continued.

Let us take our same light bulb with voltage (after it is plugged in) given by

$$V(t) = V_0 \sin(\omega t - \delta)$$

where

- V_0 is the peak voltage,
- $\omega = 2\pi \times 60$, and
- the constant δ is an (unimportant) phase.

If the light bulb is "100 watts", then what is its resistance?
To answer this question we need the following facts from physics.

- If the light bulb has resistance R ohms, this causes, by Ohm's law, a current of

  $$I(t) = \frac{1}{R}V(t)$$

  (amps) to flow through the light bulb.
- The current I is the number of units of charge moving through the bulb per unit time.

- The voltage is the energy required to move one unit of charge through the bulb.
- The power is the energy used by the bulb per unit time and is measured in watts.

So the power is the product of the current times the voltage and, so

$$P(t) = I(t)V(t) = \frac{V(t)^2}{R} = \frac{V_0^2}{R}\sin^2(\omega t - \delta)$$

The average power used over the time interval $a \le t \le b$ is

$$P_{\text{ave}} = \frac{1}{b-a}\int_a^b P(t)\mathrm{d}t = \frac{V_0^2}{R(b-a)}\int_a^b \sin^2(\omega t - \delta)\mathrm{d}t$$

Notice that this is almost exactly the form we had in the previous example when computing the root mean square voltage.
Again we simplify the integrand using the identity

$$\cos(2\theta) = 1 - 2\sin^2\theta \implies \sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

So

$$\begin{aligned}
P_{\text{ave}} &= \frac{1}{b-a}\int_a^b P(t)\mathrm{d}t = \frac{V_0^2}{2R(b-a)}\int_a^b \big[1 - \cos(2\omega t - 2\delta)\big]\mathrm{d}t \\
&= \frac{V_0^2}{2R(b-a)}\left[t - \frac{\sin(2\omega t - 2\delta)}{2\omega}\right]_a^b \\
&= \frac{V_0^2}{2R(b-a)}\left[b - a - \frac{\sin(2\omega b - 2\delta)}{2\omega} + \frac{\sin(2\omega a - 2\delta)}{2\omega}\right] \\
&= \frac{V_0^2}{2R} - \frac{V_0^2}{4\omega R(b-a)}\big[\sin(2\omega b - 2\delta) - \sin(2\omega a - 2\delta)\big]
\end{aligned}$$

In the limit as the length of the time interval $b-a$ tends to infinity, this converges to $\frac{V_0^2}{2R}$. The resistance R of a "100 watt bulb" obeys

$$\frac{V_0^2}{2R} = 100 \qquad \text{so that} \qquad R = \frac{V_0^2}{200}.$$

We finish this example off with two side remarks.

- If we translate the peak voltage to the root mean square voltage using

  $$V_0 = V_{\text{rms}} \cdot \sqrt{2}$$

  then we have

  $$P = \frac{V_{\text{rms}}^2}{R}$$

- If we were using direct voltage rather than alternating current then the computation is much simpler. The voltage and current are constants, so

$$\begin{aligned} P &= V \cdot I && \text{but } I = V/R \text{ by Ohm's law} \\ &= \frac{V^2}{R} \end{aligned}$$

So if we have a direct current giving voltage equal to the root mean square voltage, then we would expend the same power.

Example 2.2.7

2.2.1 ▸▸ Optional — Return to the mean value theorem

Here is another application of the Definition 2.2.2 of the average value of a function on an interval. The following theorem can be thought of as an analogue of the mean value theorem (which was covered in your differential calculus class) but for integrals. The theorem says that a continuous function $f(x)$ must be exactly equal to its average value for some x. For example, if you went for a drive along the x-axis and you were at $x(a)$ at time a and at $x(b)$ at time b, then your velocity $x'(t)$ had to be exactly your average velocity $\frac{x(b)-x(a)}{b-a}$ at some time t between a and b. In particular, if your average velocity was greater than the speed limit, you were definitely speeding at some point during the trip. This is, of course, no great surprise[3].

Theorem 2.2.8 Mean Value Theorem for Integrals.

Let $f(x)$ be a continuous function on the interval $a \le x \le b$. Then there is some c obeying $a < c < b$ such that

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = f(c) \qquad \text{or} \qquad \int_a^b f(x)\mathrm{d}x = f(c)\,(b-a)$$

Proof. We will apply the mean value theorem (Theorem 2.13.4 in the CLP-1 text) to the function

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

3 There are many unsurprising things that are true, but there are also many unsurprising things that surprisingly turn out to be false. Mathematicians like to prove things - surprising or not.

By the part 1 of the fundamental theorem of calculus (Theorem 1.3.1), $F'(x) = f(x)$, so the mean value theorem says that there is a $a < c < b$ with

$$\begin{aligned} f(c) = F'(c) = \frac{F(b) - F(a)}{b-a} &= \frac{1}{b-a}\left\{\int_a^b f(t)\,\mathrm{d}t - \int_a^a f(t)\,\mathrm{d}t\right\} \\ &= \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \end{aligned}$$

$\square$

In the next section, we will encounter an application in which we want to take the average value of a function $f(x)$, but in doing so we want some values of x to count more than other values of x. That is, we want to weight some x's more than other x's. To do so, we choose a "weight function" $w(x) \geq 0$ with $w(x)$ larger for more important x's. Then we define the weighted average of f as follows.

Definition 2.2.9

Let $f(x)$ and $w(x)$ be integrable functions defined on the interval $a \leq x \leq b$ with $w(x) \geq 0$ for all $a \leq x \leq b$ and with $\int_a^b w(x)\,\mathrm{d}x > 0$. The average value of f on that interval, weighted by w, is

$$\frac{\int_a^b f(x)\,w(x)\,\mathrm{d}x}{\int_a^b w(x)\,\mathrm{d}x}$$

We typically refer to this simply as the weighted average of f.

Here are a few remarks concerning this definition.

- The definition has been rigged so that, if $f(x) = 1$ for all x, then the weighted average of f is 1, no matter what weight function $w(x)$ is used.
- If the weight function $w(x) = C$ for some constant $C > 0$ then the weighted average
$$\frac{\int_a^b f(x)\,w(x)\,\mathrm{d}x}{\int_a^b w(x)\,\mathrm{d}x} = \frac{\int_a^b f(x)\,C\,\mathrm{d}x}{\int_a^b C\,\mathrm{d}x} = \frac{\int_a^b f(x)\,\mathrm{d}x}{b-a}$$
is just the usual average.
- For any function $w(x) \geq 0$ and any $a < b$, we have $\int_a^b w(x)\,\mathrm{d}x \geq 0$. But for the definition of weighted average to make sense, we need to be able to divide by $\int_a^b w(x)\,\mathrm{d}x$. So we need $\int_a^b w(x)\,\mathrm{d}x \neq 0$.

The next theorem says that a continuous function $f(x)$ must be equal to its weighted average at some point x.

Theorem 2.2.10 Mean Value Theorem for Weighted Integrals.

Let $f(x)$ and $w(x)$ be continuous functions on the interval $a \le x \le b$. Assume that $w(x) > 0$ for all $a < x < b$. Then there is some c obeying $a < c < b$ such that

$$\frac{\int_a^b f(x)\, w(x)\, \mathrm{d}x}{\int_a^b w(x)\, \mathrm{d}x} = f(c) \qquad \text{or} \qquad \int_a^b f(x)\, w(x)\, \mathrm{d}x = f(c) \int_a^b w(x)\, \mathrm{d}x$$

Proof. We will apply the generalised mean value theorem (Theorem 3.4.38 in the CLP-1 text) to

$$F(x) = \int_a^x f(t)\, w(t) \mathrm{d}t \qquad G(x) = \int_a^x w(t) \mathrm{d}t$$

By the part 1 of the fundamental theorem of calculus (Theorem 1.3.1), $F'(x) = f(x)w(x)$ and $G'(x) = w(x)$, so the generalised mean value theorem says that there is a $a < c < b$ with

$$\begin{aligned} f(c) &= \frac{F'(c)}{G'(c)} = \frac{F(b) - F(a)}{G(b) - G(a)} = \frac{\int_a^b f(t)\, w(t) \mathrm{d}t - \int_a^a f(t) w(t)\, \mathrm{d}t}{\int_a^b w(t) \mathrm{d}t - \int_a^a w(t) \mathrm{d}t} \\ &= \frac{\int_a^b f(t)\, w(t)\, \mathrm{d}t}{\int_a^b w(t)\, \mathrm{d}t} \end{aligned}$$

$\square$

Example 2.2.11

In this example, we will take a number of weighted averages of the simple function $f(x) = x$ over the simple interval $a = 1 \le x \le 2 = b$. As x increases from 1 to 2, the function $f(x)$ increases linearly from 1 to 2. So it is no shock that the ordinary average of f is exactly its middle value:

$$\frac{1}{b-a} \int_a^b f(t)\, \mathrm{d}t = \frac{1}{2-1} \int_1^2 t\, \mathrm{d}t = \frac{3}{2}$$

Pick any natural number $N \ge 1$ and consider the weight function $w_N(x) = x^N$. Note that $w_N(x)$ increases as x increases. So $w_N(x)$ weights bigger x's more than it weights smaller x's. In particular w_N weights the point $x = 2$ by a factor of 2^N (which is greater than 1 and grows to infinity as N grows to infinity) more than it weights the point $x = 1$. The weighted average of f is

$$\frac{\int_a^b f(t)\,w_N(t)\,\mathrm{d}t}{\int_a^b w_N(t)\,\mathrm{d}t} = \frac{\int_1^2 t^{N+1}\,\mathrm{d}t}{\int_1^2 t^N\,\mathrm{d}t} = \frac{\frac{2^{N+2}-1}{N+2}}{\frac{2^{N+1}-1}{N+1}} = \frac{N+1}{N+2}\,\frac{2^{N+2}-1}{2^{N+1}-1}$$

$$= \begin{cases} \frac{2\times 7}{3\times 3} = 1.555 & \text{if } N=1 \\ \frac{3\times 15}{4\times 7} = 1.607 & \text{if } N=2 \\ \frac{4\times 31}{5\times 15} = 1.653 & \text{if } N=3 \\ \frac{5\times 63}{6\times 31} = 1.694 & \text{if } N=4 \\ 1.889 & \text{if } N=16 \\ 1.992 & \text{if } N=256 \end{cases}$$

As we would expect, the w_N-weighted average is between 1.5 (which is the ordinary, unweighted, average) and 2 (which is the biggest value of f in the interval) and grows as N grows. The limit as $N \to \infty$ of the w_N-weighted average is

$$\begin{aligned} \lim_{N\to\infty} \frac{N+1}{N+2}\,\frac{2^{N+2}-1}{2^{N+1}-1} &= \lim_{N\to\infty} \frac{N+2-1}{N+2}\,\frac{2^{N+2}-2+1}{2^{N+1}-1} \\ &= \lim_{N\to\infty} \left[1 - \frac{1}{N+2}\right]\left[2 + \frac{1}{2^{N+1}-1}\right] \\ &= 2 \end{aligned}$$

Example 2.2.11

Example 2.2.12

Here is an example which shows what can go wrong with Theorem 2.2.10 if we allow the weight function $w(x)$ to change sign. Let $a = -0.99$ and $b = 1$. Let

$$w(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$$f(x) = \begin{cases} x & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Then

$$\int_a^b f(x)\,w(x)\,\mathrm{d}x = \int_0^1 x\,\mathrm{d}x = \frac{1}{2}$$

$$\int_a^b w(x)\,\mathrm{d}x = \int_0^1 \mathrm{d}x - \int_{-0.99}^0 \mathrm{d}x = 1 - 0.99 = 0.01$$

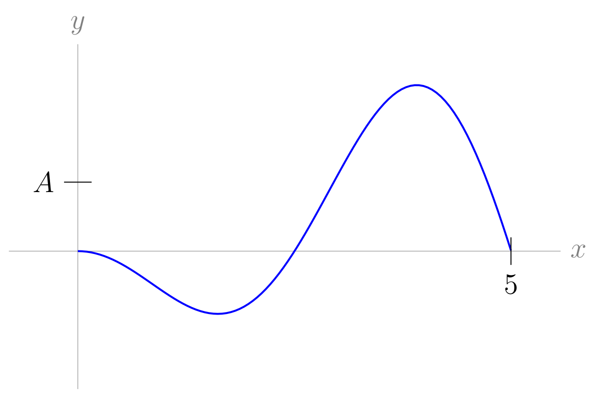
As c runs from a to b, $f(c)\int_a^b w(x)\,\mathrm{d}x = 0.01 f(c)$ runs from 0 to 0.01 and, in particular, never takes a value anywhere near $\int_a^b f(x)\,w(x)\,\mathrm{d}x = \frac{1}{2}$. There is no c value which works.

2.2.2 ▸▸ Exercises

Recall that we are using $\log x$ to denote the logarithm of x with base e. In other courses it is often denoted $\ln x$.

Exercises — Stage 1

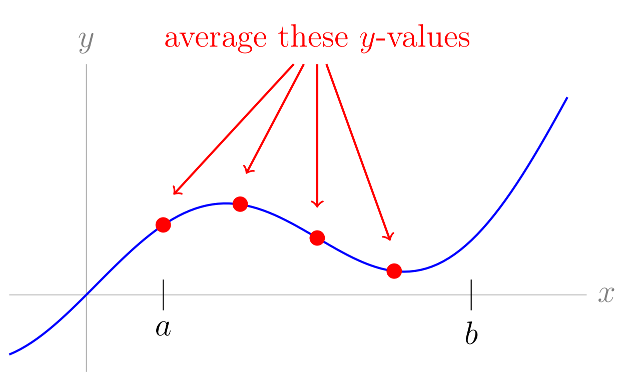
1. Below is the graph of a function $y = f(x)$. Its average value on the interval $[0, 5]$ is A. Draw a rectangle on the graph with area $\int_0^5 f(x)\,\mathrm{d}x$.



2. Suppose a car travels for 5 hours in a straight line, with an average velocity of 100 kph. How far did the car travel?

3. A force $F(x)$ acts on an object from position $x = a$ metres to position $x = b$ metres, for a total of W joules of work. What was the average force on the object?

4. Suppose we want to approximate the average value of the function $f(x)$ on the interval $[a, b]$. To do this, we cut the interval $[a, b]$ into n pieces, then take n samples by finding the function's output at the left endpoint of each piece, starting with a. Then, we average those n samples. (In the example below, $n = 4$.)



a Using n samples, what is the distance between two consecutive sample points x_i and x_{i+1}?

b Assuming $n \geq 4$, what is the x-coordinate of the fourth sample?

c Assuming $n \geq 4$, what is the y-value of the fourth sample?

d Write the approximation of the average value of $f(x)$ over the interval $[a, b]$ using sigma notation.

5. Suppose $f(x)$ and $g(x)$ are functions that are defined for all numbers in the interval $[0, 10]$.

a If $f(x) \leq g(x)$ for all x in $[0, 10]$, then is the average value of $f(x)$ is less than or equal to the average value of $g(x)$ on the interval $[0, 10]$, or is there not enough information to tell?

b Suppose $f(x) \leq g(x)$ for all x in $[0.01, 10]$. Is the average value of $f(x)$ less than or equal to the average value of $g(x)$ over the interval $[0, 10]$, or is there not enough information to tell?

6. Suppose f is an odd function, defined for all real numbers. What is the average of f on the interval $[-10, 10]$?

Exercises — Stage 2 For Questions 16 through 18, let the root mean square of $f(x)$ on $[a, b]$ be $\sqrt{\frac{1}{b-a}\int_a^b f^2(x)\,\mathrm{d}x}$. This is the formula used in Example 2.2.6 in the text.

7. $*$. Find the average value of $f(x) = \sin(5x) + 1$ over the interval $-\pi/2 \leq x \leq \pi/2$.

8. $*$. Find the average value of the function $y = x^2 \log x$ on the interval $1 \leq x \leq e$.

9. $*$. Find the average value of the function $f(x) = 3\cos^3 x + 2\cos^2 x$ on the interval $0 \leq x \leq \frac{\pi}{2}$.

10. $*$. Let k be a positive constant. Find the average value of the function $f(x) = \sin(kx)$ on the interval $0 \leq x \leq \pi/k$.

11. $*$. The temperature in Celsius in a 3 m long rod at a point x metres from the left end of the rod is given by the function $T(x) = \frac{80}{16-x^2}$. Determine the average temperature in the rod.

12. $*$. What is the average value of the function $f(x) = \dfrac{\log x}{x}$ on the interval $[1, e]$?

13. $*$. Find the average value of $f(x) = \cos^2(x)$ over $0 \leq x \leq 2\pi$.

14. The carbon dioxide concentration in the air at a particular location over one year is approximated by $C(t) = 400 + 50\cos\left(\frac{t}{12}\pi\right) + 200\cos\left(\frac{t}{4380}\pi\right)$ parts per million, where t is measured in hours.

a What is the average carbon dioxide concentration for that location for that year?

b What is the average over the first day?

c Suppose measurements were only made at noon every day: that is, when $t = 12 + 24n$, where n is any whole number between 0 and 364. Then the daily variation would cease: $50\cos\left(\frac{(12+24n)}{12}\pi\right) = 50\cos(\pi + 2\pi n) = 50\cos\pi = -50$. So, the approximation for the concentration of carbon dioxide in the atmosphere might be given as

$$N(t) = 350 + 200\cos\left(\frac{t}{4380}\pi\right) \qquad \text{ppm}$$

What is the relative error in the yearly average concentration of carbon dioxide involved in using $N(t)$, instead of $C(t)$?

You may assume a day has exactly 24 hours, and a year has exactly 8760 hours.

15. Let S be the solid formed by rotating the parabola $y = x^2$ from $x = 0$ to $x = 2$ about the x-axis.

a What is the average area of the circular cross-sections of S? Call this value A.

b What is the volume of S?

c What is the volume of a cylinder with circular cross-sectional area A and length 2?

16. Let $f(x) = x$.

a Calculate the average of $f(x)$ over $[-3, 3]$.

b Calculate the root mean square of $f(x)$ over $[-3, 3]$.

17. Calculate the root mean square of $f(x) = \tan x$ over $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

18. A force acts on a spring, and the spring stretches and contracts. The distance beyond its natural length at time t is $f(t) = \sin(t\pi)$ cm, where t is measured in seconds. The spring constant is 3 N/cm.

a What is the force exerted by the spring at time t, if it obeys Hooke's law?

b Find the average of the force exerted by the spring from $t = 0$ to $t = 6$.

c Find the root mean square of the force exerted by the spring from $t = 0$ to $t = 6$.

Exercises — Stage 3

19. ***.** A car travels two hours without stopping. The driver records the car's speed every 20 minutes, as indicated in the table below:

time in hours	0	1/3	2/3	1	4/3	5/3	2
speed in km/hr	50	70	80	55	60	80	40

a Use the trapezoidal rule to estimate the total distance traveled in the two hours.

b Use the answer to part (a) to estimate the average speed of the car during this period.

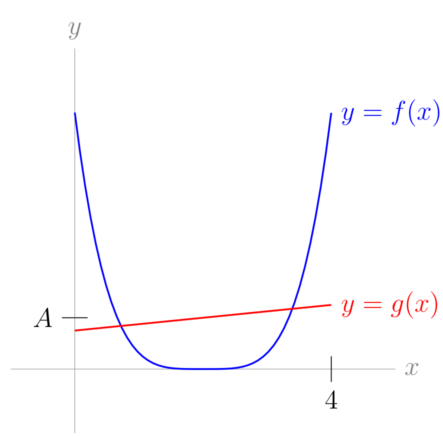
20. Let $s(t) = e^t$.

a Find the average of $s(t)$ on the interval $[0, 1]$. Call this quantity A.

b For any point t, the difference between $s(t)$ and A is $s(t) - A$. Find the average value of $s(t) - A$ on the interval $[0, 1]$.

c For any point t, the absolute difference between $s(t)$ and A is $|s(t) - A|$. Find the average value of $|s(t) - A|$ on the interval $[0, 1]$.

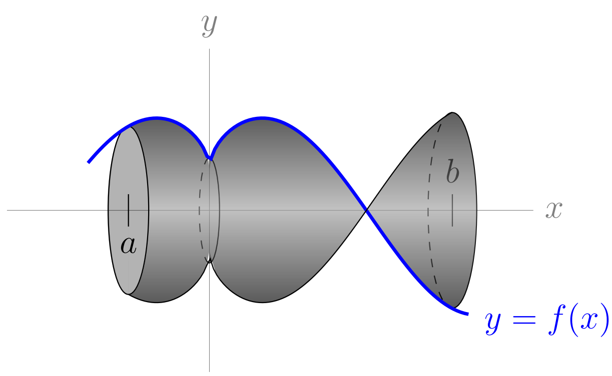
21. Consider the two functions $f(x)$ and $g(x)$ below, both of which have average A on $[0, 4]$.



a Which function has a larger average on $[0, 4]$: $f(x) - A$ or $g(x) - A$?

b Which function has a larger average on $[0, 4]$: $|f(x) - A|$ or $|g(x) - A|$?

22. Suppose the root mean square of a function $f(x)$ on the interval $[a, b]$ is R. What is the volume of the solid formed by rotating the portion of $f(x)$ from a to b about the x-axis?



As in Example 2.2.6, let the root mean square of $f(x)$ on $[a, b]$ be $\sqrt{\frac{1}{b-a}\int_a^b f^2(x)\,\mathrm{d}x}$.

23. Suppose $f(x) = ax^2 + bx + c$, and the average value of $f(x)$ on the interval $[0, 1]$ is the same as the average of $f(0)$ and $f(1)$. What is a?

24. Suppose $f(x) = ax^2 + bx + c$, and the average value of $f(x)$ on the interval $[s, t]$ is the same as the average of $f(s)$ and $f(t)$. Is it possible that $a \neq 0$? That is — does the result of Question 23 generalize?

25. Let $f(x)$ be a function defined for all numbers in the interval $[a, b]$, with average value A over that interval. What is the average of $f(a+b-x)$ over the interval $[a, b]$?

26. Suppose $f(t)$ is a continuous function, and $A(x)$ is the average of $f(t)$ on the interval from 0 to x.

a What is the average of $f(t)$ on $[a, b]$, where $a < b$? Give your answer in terms of A.

b What is $f(t)$? Again, give your answer in terms of A.

27.

a Find a function $f(x)$ with average 0 over $[-1, 1]$ but $f(x) \neq 0$ for all x in $[-1, 1]$, or show that no such function exists.

b Find a *continuous* function $f(x)$ with average 0 over $[-1, 1]$ but $f(x) \neq 0$ for all x in $[-1, 1]$, or show that no such function exists.

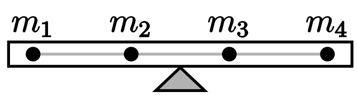
28. Suppose $f(x)$ is a positive, continuous function with $\lim_{x\to\infty} f(x) = 0$, and let $A(x)$ be the average of $f(x)$ on $[0, x]$.
True or false: $\lim_{x\to\infty} A(x) = 0$.

29. Let $A(x)$ be the average of the function $f(t) = e^{-t^2}$ on the interval $[0, x]$. What is $\lim_{x\to\infty} A(x)$?

2.3 ▲ Centre of Mass and Torque

2.3.1 ▶▶ Centre of Mass

If you support a body at its center of mass (in a uniform gravitational field) it balances perfectly. That's the definition of the center of mass of the body.



If the body consists of a finite number of masses $m_1, \cdots, m_n$ attached to an infinitely strong, weightless (idealized) rod with mass number i attached at position x_i, then the center of mass is at the (weighted) average value of x:

Equation 2.3.1 Centre of mass (discrete masses).

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

The denominator $m = \sum_{i=1}^{n} m_i$ is the total mass of the body.

This formula for the center of mass is derived in the following (optional) section. See equation (2.3.14).

For many (but certainly not all) purposes an (extended rigid) body acts like a point particle located at its center of mass. For example it is very common to treat the Earth as a point particle. Here is a more detailed example in which we think of a body as being made up of a number of component parts and compute the center of mass of the body as a whole by using the center of masses of the component parts. Suppose that we have a dumbbell which consists of

- a left end made up of particles of masses $m_{l,1}, \cdots, m_{l,3}$ located at $x_{l,1}, \cdots, x_{l,3}$ and

- a right end made up of particles of masses $m_{r,1}, \cdots, m_{r,4}$ located at $x_{r,1}, \cdots, x_{r,4}$ and
- an infinitely strong, weightless (idealized) rod joining all of the particles.

Then the mass and center of mass of the left end are

$$M_l = m_{l,1} + \cdots + m_{l,3} \qquad \bar{X}_l = \frac{m_{l,1}x_{l,1} + \cdots + m_{l,3}x_{l,3}}{M_l}$$

and the mass and center of mass of the right end are

$$M_r = m_{r,1} + \cdots + m_{r,4} \qquad \bar{X}_r = \frac{m_{r,1}x_{r,1} + \cdots + m_{r,4}x_{r,4}}{M_r}$$

The mass and center of mass of the entire dumbbell are

$$\begin{aligned} M &= m_{l,1} + \cdots + m_{l,3} \;+\; m_{r,1} + \cdots + m_{r,4} \\ &= M_l + M_r \\ \bar{x} &= \frac{m_{l,1}x_{l,1} + \cdots + m_{l,3}x_{l,3} \;+\; m_{r,1}x_{r,1} + \cdots + m_{r,4}x_{r,4}}{M} \\ &= \frac{M_l\bar{X}_l + M_r\bar{X}_r}{M_r + M_l} \end{aligned}$$

So we can compute the center of mass of the entire dumbbell by treating it as being made up of two point particles, one of mass M_l located at the centre of mass of the left end, and one of mass M_r located at the center of mass of the right end.

Example 2.3.2 Work and Centre of Mass.

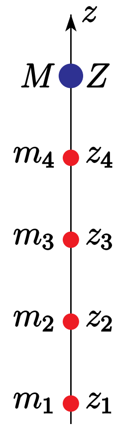
Here is another example in which an extended body acts like a point particle located at its centre of mass. Imagine that there are a finite number of masses $m_1, \cdots, m_n$ arrayed along a (vertical) z-axis with mass number i attached at height z_i. Note that the total mass of the array is $M = \sum_{i=1}^n m_i$ and that the centre of mass of the array is at height

$$\bar{z} = \frac{\sum_{i=1}^n m_i z_i}{\sum_{i=1}^n m_i} = \frac{1}{M}\sum_{i=1}^n m_i z_i$$

Now suppose that we lift all of the masses, against gravity, to height Z. So after the lift there is a total mass M located at height Z. The i^{th} mass is subject to a downward gravitational force of $m_i g$. So to lift the i^{th} mass we need to apply a compensating upward force of $m_i g$ through a distance of $Z - z_i$. This takes work $m_i g(Z - z_i)$. So the total work required to lift all n masses is

$$\begin{aligned} \text{Work} &= \sum_{i=1}^n m_i g(Z - z_i) \\ &= gZ\sum_{i=1}^n m_i - g\sum_{i=1}^n m_i z_i \end{aligned}$$

$$= gZM - gM\bar{z}$$
$$= Mg(Z - \bar{z})$$



So the work required to lift the array of n particles is identical to the work required to lift a single particle, whose mass, M, is the total mass of the array, from height $\bar{z}$, the centre of mass of the array, to height Z.

Example 2.3.2

Example 2.3.3 Example 2.3.2, continued.

Imagine, as in Example 2.3.2, that there are a finite number of masses $m_1, \cdots, m_n$ arrayed along a (vertical) z-axis with mass number i attached at height z_i. Again, the total mass and centre of mass of the array are

$$M = \sum_{i=1}^{n} m_i \qquad \bar{z} = \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i$$

Now suppose that we lift, for each $1 \le i \le n$, mass number i, against gravity, from its initial height z_i to a final height Z_i. So after the lift we have a new array of masses with total mass and centre of mass

$$M = \sum_{i=1}^{n} m_i \qquad \bar{Z} = \frac{\sum_{i=1}^{n} m_i Z_i}{\sum_{i=1}^{n} m_i} = \frac{1}{M}\sum_{i=1}^{n} m_i Z_i$$

To lift the i^{th} mass took work $m_i g(Z_i - z_i)$. So the total work required to lift all n masses was

$$\begin{aligned} \text{Work} &= \sum_{i=1}^{n} m_i g(Z_i - z_i) \\ &= g\sum_{i=1}^{n} m_i Z_i - g\sum_{i=1}^{n} m_i z_i \end{aligned}$$

$$= gM\bar{Z} - gM\bar{z} = Mg(\bar{Z} - \bar{z})$$

So the work required to lift the array of n particles is identical to the work required to lift a single particle, whose mass, M, is the total mass of the array, from height $\bar{z}$, the initial centre of mass of the array, to height $\bar{Z}$, the final centre of mass of the array.

Example 2.3.3

Now we'll extend the above ideas to cover more general classes of bodies. If the body consists of mass distributed continuously along a straight line, say with mass density $\rho(x)$kg/m and with x running from a to b, rather than consisting of a finite number of point masses, the formula for the center of mass becomes

Equation 2.3.4 Centre of mass (continuous mass).

$$\bar{x} = \frac{\int_a^b x\ \rho(x)\,\mathrm{d}x}{\int_a^b \rho(x)\,\mathrm{d}x}$$

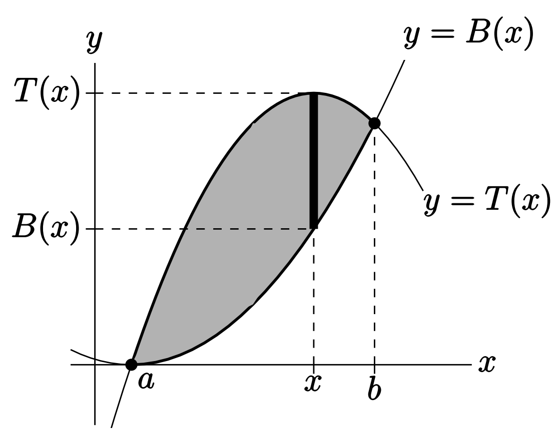
Think of $\rho(x)\,\mathrm{d}x$ as the mass of the "almost point particle" between x and $x + \mathrm{d}x$.

If the body is a two dimensional object, like a metal plate, lying in the xy-plane, its center of mass is a point $(\bar{x}, \bar{y})$ with $\bar{x}$ being the (weighted) average value of the x-coordinate over the body and $\bar{y}$ being the (weighted) average value of the y-coordinate over the body. To be concrete, suppose the body fills the region

$$\left\{\ (x,y)\ \middle|\ a \le x \le b,\ \ B(x) \le y \le T(x)\ \right\}$$

in the xy-plane. For simplicity, we will assume that the density of the body is a constant, say ρ. When the density is constant, the center of mass is also called the *centroid* and is thought of as the geometric center of the body.

To find the centroid of the body, we use our standard "slicing" strategy. We slice the body into thin vertical strips, as illustrated in the figure below.



Here is a detailed description of a generic strip.

- The strip has width $\mathrm{d}x$.
- Each point of the strip has essentially the same x-coordinate. Call it x.
- The top of the strip is at $y = T(x)$ and the bottom of the strip is at $y = B(x)$.
- So the strip has
  - height $T(x) - B(x)$
  - area $[T(x) - B(x)]\,\mathrm{d}x$
  - mass $\rho[T(x) - B(x)]\,\mathrm{d}x$
  - centroid, i.e. middle point, $\big(x\,,\,\frac{B(x)+T(x)}{2}\big)$.

In computing the centroid of the entire body, we may treat each strip as a single particle of mass $\rho[T(x) - B(x)]\,\mathrm{d}x$ located at $\big(x\,,\,\frac{B(x)+T(x)}{2}\big)$. So:

Equation 2.3.5 Centroid of object with constant density.

The mass of the entire body bounded by curves $T(x)$ above and $B(x)$ below is

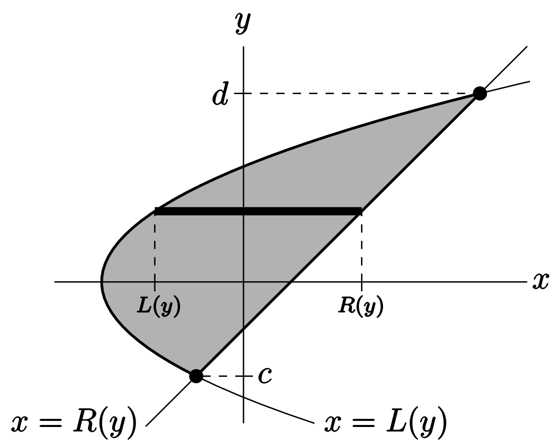
$$M = \rho\int_a^b [T(x) - B(x)]\,\mathrm{d}x = \rho A \tag{a}$$

where $A = \int_a^b [T(x) - B(x)]\,\mathrm{d}x$ is the area of the region. The coordinates of the centroid are

$$\bar{x} = \frac{\int_a^b x\ \overbrace{\rho[T(x) - B(x)]\,\mathrm{d}x}^{\text{mass of slice}}}{M} = \frac{\int_a^b x[T(x) - B(x)]\,\mathrm{d}x}{A} \tag{b}$$

$$\bar{y} = \frac{\int_a^b \overbrace{\frac{B(x)+T(x)}{2}}^{\text{average } y \text{ on slice}}\ \overbrace{\rho[T(x) - B(x)]\,\mathrm{d}x}^{\text{mass of slice}}}{M} = \frac{\int_a^b [T(x)^2 - B(x)^2]\,\mathrm{d}x}{2A} \tag{c}$$

We can of course also slice up the body using horizontal slices.



If the body has constant density ρ and fills the region

$$\left\{ (x, y) \;\middle|\; L(y) \le x \le R(y),\ c \le y \le d \right\}$$

then the same computation as above gives:

Equation 2.3.6 Centroid of object with constant density.

The mass of the entire body bounded by curves $L(y)$ to the left and $R(y)$ to the right is

$$M = \rho \int_c^d [R(y) - L(y)]\,\mathrm{d}y = \rho A \tag{a}$$

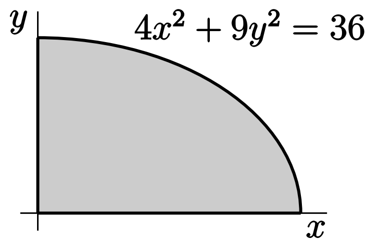
where $A = \int_c^d [R(y) - L(y)]\,\mathrm{d}y$ is the area of the region, and gives the coordinates of the centroid to be

$$\bar{x} = \frac{\int_c^d \overbrace{\frac{R(y)+L(y)}{2}}^{\text{average } x \text{ on slice}} \overbrace{\rho[R(y) - L(y)]\,\mathrm{d}y}^{\text{mass of slice}}}{M} = \frac{\int_c^d [R(y)^2 - L(y)^2]\,\mathrm{d}y}{2A} \tag{b}$$

$$\bar{y} = \frac{\int_c^d y \overbrace{\rho[R(y) - L(y)]\,\mathrm{d}y}^{\text{mass of slice}}}{M} = \frac{\int_c^d y[R(y) - L(y)]\,\mathrm{d}y}{A} \tag{c}$$

Example 2.3.7 Centroid of a quarter ellipse.

Find the x-coordinate of the centroid (centre of gravity) of the plane region R that lies in the first quadrant $x \ge 0$, $y \ge 0$ and inside the ellipse $4x^2 + 9y^2 = 36$. (The area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab square units.)



Solution: In standard form $4x^2 + 9y^2 = 36$ is $\frac{x^2}{9} + \frac{y^2}{4} = 1$. So, on R, x runs from 0 to 3 and R has area $A = \frac{1}{4}\pi \times 3 \times 2 = \frac{3}{2}\pi$. For each fixed x, between 0 and 3, y runs from 0 to $2\sqrt{1 - \frac{x^2}{9}}$. So, applying (2.3.5.b) with $a = 0$, $b = 3$, $T(x) = 2\sqrt{1 - \frac{x^2}{9}}$ and $B(x) = 0$,

$$\bar{x} = \frac{1}{A}\int_0^3 x\,T(x)\,\mathrm{d}x = \frac{1}{A}\int_0^3 x\,2\sqrt{1 - \frac{x^2}{9}}\,\mathrm{d}x = \frac{4}{3\pi}\int_0^3 x\sqrt{1 - \frac{x^2}{9}}\,\mathrm{d}x$$

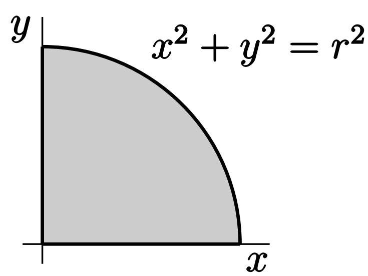
Sub in $u = 1 - \frac{x^2}{9}$, $\mathrm{d}u = -\frac{2}{9}x\,\mathrm{d}x$.

$$\bar{x} = -\frac{9}{2}\frac{4}{3\pi}\int_1^0 \sqrt{u}\,\mathrm{d}u = -\frac{9}{2}\frac{4}{3\pi}\left[\frac{u^{3/2}}{3/2}\right]_1^0 = -\frac{9}{2}\frac{4}{3\pi}\left[-\frac{2}{3}\right] = \frac{4}{\pi}$$

Example 2.3.7

Example 2.3.8 Centroid of a quarter disk.

Find the centroid of the quarter circular disk $x \geq 0$, $y \geq 0$, $x^2 + y^2 \leq r^2$.



Solution: By symmetry, $\bar{x} = \bar{y}$. The area of the quarter disk is $A = \frac{1}{4}\pi r^2$. By (2.3.5.b) with $a = 0$, $b = r$, $T(x) = \sqrt{r^2 - x^2}$ and $B(x) = 0$,

$$\bar{x} = \frac{1}{A}\int_0^r x\sqrt{r^2 - x^2}\,\mathrm{d}x$$

To evaluate the integral, sub in $u = r^2 - x^2$, $\mathrm{d}u = -2x\,\mathrm{d}x$.

$$\int_0^r x\sqrt{r^2 - x^2}\,\mathrm{d}x = \int_{r^2}^0 \sqrt{u}\,\frac{\mathrm{d}u}{-2} = -\frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_{r^2}^0 = \frac{r^3}{3} \qquad (*)$$

So

$$\bar{x} = \frac{4}{\pi r^2}\left[\frac{r^3}{3}\right] = \frac{4r}{3\pi}$$

As we observed above, we should have $\bar{x} = \bar{y}$. But, just for practice, let's compute $\bar{y}$ by the integral formula (2.3.5.c), again with $a = 0$, $b = r$, $T(x) = \sqrt{r^2 - x^2}$ and $B(x) = 0$,

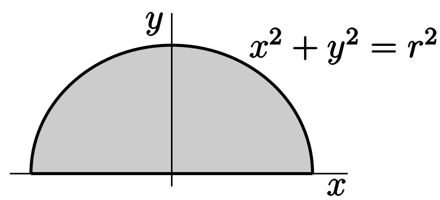
$$\begin{aligned} \bar{y} &= \frac{1}{2A}\int_0^r \left(\sqrt{r^2-x^2}\right)^2 \mathrm{d}x & &= \frac{2}{\pi r^2}\int_0^r \left(r^2 - x^2\right)\mathrm{d}x \\ &= \frac{2}{\pi r^2}\left[r^2 x - \frac{x^3}{3}\right]_0^r & &= \frac{2}{\pi r^2}\frac{2r^3}{3} \\ &= \frac{4r}{3\pi} \end{aligned}$$

as expected.

Example 2.3.8

Example 2.3.9 Centroid of a half disc.

Find the centroid of the half circular disk $y \geq 0$, $x^2 + y^2 \leq r^2$.

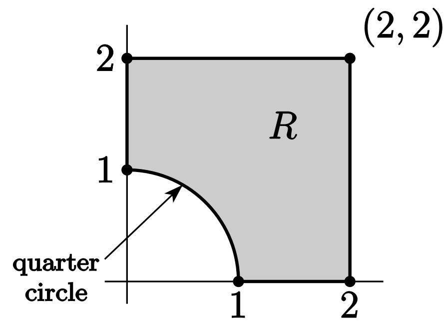


Solution: Once again, we have a symmetry — namely the half disk is symmetric about the y-axis. So the centroid lies on the y-axis and $\bar{x} = 0$. The area of the half disk is $A = \frac{1}{2}\pi r^2$. By (2.3.5.c), with $a = -r$, $b = r$, $T(x) = \sqrt{r^2 - x^2}$ and $B(x) = 0$,

$$\begin{aligned} \bar{y} &= \frac{1}{2A}\int_{-r}^r \left(\sqrt{r^2-x^2}\right)^2 \mathrm{d}x & &= \frac{1}{\pi r^2}\int_{-r}^r \left(r^2 - x^2\right)\mathrm{d}x \\ &= \frac{2}{\pi r^2}\int_0^r \left(r^2 - x^2\right)\mathrm{d}x & &\text{since the integrand is even} \\ &= \frac{2}{\pi r^2}\left[r^2 x - \frac{x^3}{3}\right]_0^r \\ &= \frac{4r}{3\pi} \end{aligned}$$

Example 2.3.10 Another centroid.

Find the centroid of the region R in the diagram.



Solution: By symmetry, $\bar{x} = \bar{y}$. The region R is a 2×2 square with one quarter of a circle of radius 1 removed and so has area $2 \times 2 - \frac{1}{4}\pi = \frac{16-\pi}{4}$. The top of R is $y = T(x) = 2$. The bottom is $y = B(x)$ with $B(x) = \sqrt{1-x^2}$ when $0 \le x \le 1$ and $B(x) = 0$ when $1 \le x \le 2$. So

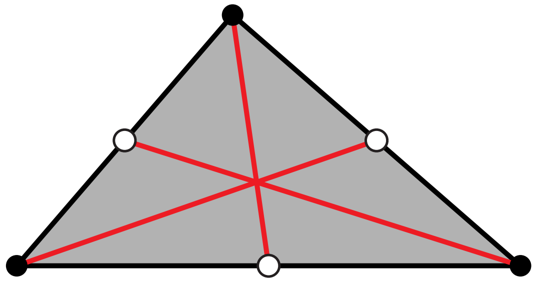
$$\begin{aligned}
\bar{y} = \bar{x} &= \frac{1}{A}\left[\int_0^1 x[2-\sqrt{1-x^2}]\,\mathrm{d}x + \int_1^2 x[2-0]\,\mathrm{d}x\right] \\
&= \frac{4}{16-\pi}\left[x^2\Big|_0^1 + x^2\Big|_1^2 - \int_0^1 x\sqrt{1-x^2}\,\mathrm{d}x\right]
\end{aligned}$$

Now we can make use of the starred equation in Example 2.3.8 with $r = 1$ to obtain

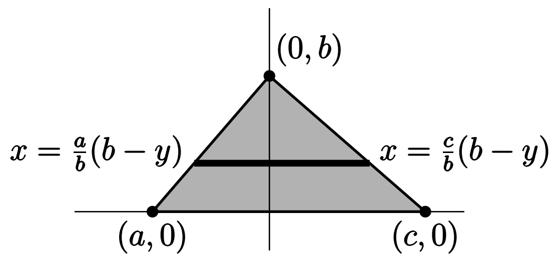
$$\begin{aligned}
&= \frac{4}{16-\pi}\left[4 - \frac{1}{3}\right] \\
&= \frac{44}{48-3\pi}
\end{aligned}$$

Example 2.3.11 Centroid of a triangle and its medians.

Prove that the centroid of any triangle is located at the point of intersection of the medians. A median of a triangle is a line segment joining a vertex to the midpoint of the opposite side.



Solution: Choose a coordinate system so that the vertices of the triangle are located at $(a, 0)$, $(0, b)$ and $(c, 0)$. (In the figure below, a is negative.)



The line joining $(a, 0)$ and $(0, b)$ has equation $bx + ay = ab$. (Check that $(a, 0)$ and $(0, b)$ both really are on this line.) The line joining $(c, 0)$ and $(0, b)$ has equation $bx + cy = bc$. (Check that $(c, 0)$ and $(0, b)$ both really are on this line.) Hence for each fixed y between 0 and b, x runs from $a - \frac{a}{b}y$ to $c - \frac{c}{b}y$.

We'll use horizontal strips to compute $\bar{x}$ and $\bar{y}$. We could just apply equation (2.3.6) with $c = 0$, $d = b$, $R(y) = \frac{c}{b}(b-y)$ (which is gotten by solving $bx + cy = bc$ for x) and $L(y) = \frac{a}{b}(b-y)$ (which is gotten by solving $bx + ay = ab$ for x).

But rather than memorizing or looking up those formulae, we'll derive them for this example. So consider a thin strip at height y as illustrated in the figure above.

- The strip has length

$$\ell(y) = \left[\frac{c}{b}(b-y) - \frac{a}{b}(b-y)\right] = \frac{c-a}{b}(b-y)$$

- The strip has width $\mathrm{d}y$.
- On this strip, y has average value y.
- On this strip, x has average value $\frac{1}{2}\left[\frac{a}{b}(b-y) + \frac{c}{b}(b-y)\right] = \frac{a+c}{2b}(b-y)$.

As the area of the triangle is $A = \frac{1}{2}(c-a)b$,

$$\bar{y} = \frac{1}{A}\int_0^b y\ \ell(y)\mathrm{d}y = \frac{2}{(c-a)b}\int_0^b y\frac{c-a}{b}(b-y)\mathrm{d}y$$

$$
\begin{aligned}
&= \frac{2}{b^2}\int_0^b (by-y^2)\mathrm{d}y = \frac{2}{b^2}\Big(b\frac{b^2}{2}-\frac{b^3}{3}\Big)\\
&= \frac{2}{b^2}\frac{b^3}{6} = \frac{b}{3}\\
\bar{x} &= \frac{1}{A}\int_0^b \frac{a+c}{2b}(b-y)\ \ell(y)\mathrm{d}y\\
&= \frac{2}{(c-a)b}\int_0^b \frac{a+c}{2b}(b-y)\frac{c-a}{b}(b-y)\mathrm{d}y\\
&= \frac{a+c}{b^3}\int_0^b (y-b)^2\mathrm{d}y\\
&= \frac{a+c}{b^3}\Big[\frac{1}{3}(y-b)^3\Big]_0^b = \frac{a+c}{b^3}\,\frac{b^3}{3} = \frac{a+c}{3}
\end{aligned}
$$

We have found that the centroid of the triangle is at $(\bar{x},\bar{y}) = \big(\frac{a+c}{3},\frac{b}{3}\big)$. We shall now show that this point lies on all three medians.

- One vertex is at $(a,0)$. The opposite side runs from $(0,b)$ and $(c,0)$ and so has midpoint $\frac{1}{2}(c,b)$. The line from $(a,0)$ to $\frac{1}{2}(c,b)$ has slope $\frac{b/2}{c/2-a} = \frac{b}{c-2a}$ and so has equation $y = \frac{b}{c-2a}(x-a)$. As $\frac{b}{c-2a}(\bar{x}-a) = \frac{b}{c-2a}\big(\frac{a+c}{3}-a\big) = \frac{1}{3}\frac{b}{c-2a}(c+a-3a) = \frac{b}{3} = \bar{y}$, the centroid does indeed lie on this median. In this computation we have implicitly assumed that $c \neq 2a$ so that the denominator $c-2a \neq 0$. In the event that $c = 2a$, the median runs from $(a,0)$ to $\big(a,\frac{b}{2}\big)$ and so has equation $x = a$. When $c = 2a$ we also have $\bar{x} = \frac{a+c}{3} = a$, so that the centroid still lies on the median.

- Another vertex is at $(c,0)$. The opposite side runs from $(a,0)$ and $(0,b)$ and so has midpoint $\frac{1}{2}(a,b)$. The line from $(c,0)$ to $\frac{1}{2}(a,b)$ has slope $\frac{b/2}{a/2-c} = \frac{b}{a-2c}$ and so has equation $y = \frac{b}{a-2c}(x-c)$. As $\frac{b}{a-2c}(\bar{x}-c) = \frac{b}{a-2c}\big(\frac{a+c}{3}-c\big) = \frac{1}{3}\frac{b}{a-2c}(a+c-3c) = \frac{b}{3} = \bar{y}$, the centroid does indeed lie on this median. In this computation we have implicitly assumed that $a \neq 2c$ so that the denominator $a-2c \neq 0$. In the event that $a = 2c$, the median runs from $(c,0)$ to $\big(c,\frac{b}{2}\big)$ and so has equation $x = c$. When $a = 2c$ we also have $\bar{x} = \frac{a+c}{3} = c$, so that the centroid still lies on the median.

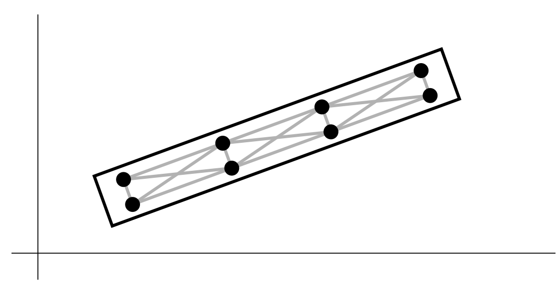
- The third vertex is at $(0,b)$. The opposite side runs from $(a,0)$ and $(c,0)$ and so has midpoint $\big(\frac{a+c}{2},0\big)$. The line from $(0,b)$ to $\big(\frac{a+c}{2},0\big)$ has slope $\frac{-b}{(a+c)/2} = -\frac{2b}{a+c}$ and so has equation $y = b - \frac{2b}{a+c}x$. As $b - \frac{2b}{a+c}\bar{x} = b - \frac{2b}{a+c}\frac{a+c}{3} = \frac{b}{3} = \bar{y}$, the centroid does indeed lie on this median. This time, we have implicitly assumed that $a+c \neq 0$. In the event that $a+c = 0$, the median runs from $(0,b)$ to $(0,0)$ and so has equation $x = 0$. When $a+c = 0$ we also have $\bar{x} = \frac{a+c}{3} = 0$, so that the centroid still lies on the median.

Example 2.3.11

2.3.2 ▸▸ Optional — Torque

Newton's law of motion says that the position $x(t)$ of a single particle moving under the influence of a force F obeys $mx''(t) = F$. Similarly, the positions $x_i(t)$, $1 \le i \le n$, of a set of particles moving under the influence of forces F_i obey $mx''_i(t) = F_i$, $1 \le i \le n$. Often systems of interest consist of some small number of rigid bodies. Suppose that we are interested in the motion of a single rigid body, say a piece of wood. The piece of wood is made up of a huge number of atoms. So the system of equations determining the motion of all of the individual atoms in the piece of wood is huge. On the other hand, because the piece of wood is rigid, its configuration is completely determined by the position of, for example, its centre of mass and its orientation. (Rather than get into what is precisely meant by "orientation", let's just say that it is certainly determined by, for example, the positions of a few of the corners of the piece of wood). It is possible to extract from the huge system of equations that determine the motion of all of the individual atoms, a small system of equations that determine the motion of the centre of mass and the orientation. We can avoid some vector analysis, that is beyond the scope of this course, by assuming that our rigid body is moving in two rather than three dimensions.

So, imagine a piece of wood moving in the xy-plane.



Furthermore, imagine that the piece of wood consists of a huge number of particles joined by a huge number of weightless but very strong steel rods. The steel rod joining particle number one to particle number two just represents a force acting between particles number one and two. Suppose that

- there are n particles, with particle number i having mass m_i
- at time t, particle number i has x-coordinate $x_i(t)$ and y-coordinate $y_i(t)$
- at time t, the external force (gravity and the like) acting on particle number i has x-coordinate $H_i(t)$ and y-coordinate $V_i(t)$. Here H stands for horizontal and V stands for vertical.
- at time t, the force acting on particle number i, due to the steel rod joining particle number i to particle number j has x-coordinate $H_{i,j}(t)$ and y-coordinate $V_{i,j}(t)$. If there is no steel rod joining particles number i and j, just set $H_{i,j}(t) = V_{i,j}(t) = 0$. In particular, $H_{i,i}(t) = V_{i,i}(t) = 0$.

The only assumptions that we shall make about the steel rod forces are

(A1) for each $i \neq j$, $H_{i,j}(t) = -H_{j,i}(t)$ and $V_{i,j}(t) = -V_{j,i}(t)$. In words, the steel rod joining particles i and j applies equal and opposite forces to particles i and j.

(A2) for each $i \neq j$, there is a function $M_{i,j}(t)$ such that $H_{i,j}(t) = M_{i,j}(t)\big[x_i(t) - x_j(t)\big]$ and $V_{i,j}(t) = M_{i,j}(t)\big[y_i(t) - y_j(t)\big]$. In words, the force due to the rod joining particles i and j acts parallel to the line joining particles i and j. For (A1) to be true, we need $M_{i,j}(t) = M_{j,i}(t)$.

Newton's law of motion, applied to particle number i, now tells us that

$$m_i x_i''(t) = H_i(t) + \sum_{j=1}^{n} H_{i,j}(t) \tag{X_i}$$

$$m_i y_i''(t) = V_i(t) + \sum_{j=1}^{n} V_{i,j}(t) \tag{Y_i}$$

Adding up all of the equations (X_i), for $i = 1,\ 2,\ 3,\ \cdots,\ n$ and adding up all of the equations (Y_i), for $i = 1,\ 2,\ 3,\ \cdots,\ n$ gives

$$\sum_{i=1}^{n} m_i x_i''(t) = \sum_{i=1}^{n} H_i(t) + \sum_{1\le i,j\le n} H_{i,j}(t) \tag{$\Sigma_i X_i$}$$

$$\sum_{i=1}^{n} m_i y_i''(t) = \sum_{i=1}^{n} V_i(t) + \sum_{1\le i,j\le n} V_{i,j}(t) \tag{$\Sigma_i Y_i$}$$

The sum $\sum_{1\le i,j\le n} H_{i,j}(t)$ contains $H_{1,2}(t)$ exactly once and it also contains $H_{2,1}(t)$ exactly once and these two terms cancel exactly, by assumption (A1). In this way, all terms in $\sum_{1\le i,j\le n} H_{i,j}(t)$ with $i \neq j$ exactly cancel. All terms with $i = j$ are assumed to be zero. So $\sum_{1\le i,j\le n} H_{i,j}(t) = 0$. Similarly, $\sum_{1\le i,j\le n} V_{i,j}(t) = 0$, so the equations $(\Sigma_i X_i)$ and $(\Sigma_i Y_i)$ simplify to

$$\sum_{i=1}^{n} m_i x_i''(t) = \sum_{i=1}^{n} H_i(t) \tag{$\Sigma_i X_i$}$$

$$\sum_{i=1}^{n} m_i y_i''(t) = \sum_{i=1}^{n} V_i(t) \tag{$\Sigma_i Y_i$}$$

Denote by

$$M = \sum_{i=1}^{n} m_i$$

the total mass of the system, by

$$X(t) = \frac{1}{M}\sum_{i=1}^{n} m_i x_i(t) \qquad \text{and} \qquad Y(t) = \frac{1}{M}\sum_{i=1}^{n} m_i y_i(t)$$

the x- and y-coordinates of the centre of mass of the system at time t and by

$$H(t) = \sum_{i=1}^{n} H_i(t) \qquad \text{and} \qquad V(t) = \sum_{i=1}^{n} V_i(t)$$

the x- and y-coordinates of the total external force acting on the system at time t. In this notation, the equations $(\Sigma_i X_i)$ and $(\Sigma_i Y_i)$ are

Equation 2.3.12 Rectilinear motion of centre of mass.

$$MX''(t) = H(t) \qquad MY''(t) = V(t)$$

So the centre of mass of the system moves just like a single particle of mass M subject to the total external force.

Now multiply equation (Y_i) by $x_i(t)$, subtract from it equation (X_i) multiplied by $y_i(t)$, and sum over i. This gives the equation $\sum_i \big[x_i(t)\,(Y_i) - y_i(t)\,(X_i)\big]$:

$$\begin{aligned}\sum_{i=1}^{n} m_i\big[x_i(t)y_i''(t) - y_i(t)x_i''(t)\big] &= \sum_{i=1}^{n}\big[x_i(t)V_i(t) - y_i(t)H_i(t)\big]\\ &\quad + \sum_{1\le i,j\le n}\big[x_i(t)V_{i,j}(t) - y_i(t)H_{i,j}(t)\big]\end{aligned}$$

By the assumption (A2)

$$\begin{aligned}x_1(t)V_{1,2}(t) - y_1(t)H_{1,2}(t) &= x_1(t)M_{1,2}(t)\big[y_1(t)-y_2(t)\big] - y_1(t)M_{1,2}(t)\big[x_1(t)-x_2(t)\big]\\ &= M_{1,2}(t)\big[y_1(t)x_2(t) - x_1(t)y_2(t)\big]\\ x_2(t)V_{2,1}(t) - y_2(t)H_{2,1}(t) &= x_2(t)M_{2,1}(t)\big[y_2(t)-y_1(t)\big] - y_2(t)M_{2,1}(t)\big[x_2(t)-x_1(t)\big]\\ &= M_{2,1}(t)\big[-y_1(t)x_2(t) + x_1(t)y_2(t)\big]\\ &= M_{1,2}(t)\big[-y_1(t)x_2(t) + x_1(t)y_2(t)\big]\end{aligned}$$

So the $i=1$, $j=2$ term in $\sum_{1\le i,j\le n}\big[x_i(t)V_{i,j}(t)-y_i(t)H_{i,j}(t)\big]$ exactly cancels the $i=2$, $j=1$ term. In this way all of the terms in $\sum_{1\le i,j\le n}\big[x_i(t)V_{i,j}(t)-y_i(t)H_{i,j}(t)\big]$ with $i\ne j$ cancel. Each term with $i=j$ is exactly zero. So $\sum_{1\le i,j\le n}\big[x_i(t)V_{i,j}(t)-y_i(t)H_{i,j}(t)\big]=0$ and

$$\sum_{i=1}^{n} m_i\big[x_i(t)y_i''(t) - y_i(t)x_i''(t)\big] = \sum_{i=1}^{n}\big[x_i(t)V_i(t) - y_i(t)H_i(t)\big]$$

Define

$$\begin{aligned}L(t) &= \sum_{i=1}^{n} m_i\big[x_i(t)y_i'(t) - y_i(t)x_i'(t)\big]\\ T(t) &= \sum_{i=1}^{n}\big[x_i(t)V_i(t) - y_i(t)H_i(t)\big]\end{aligned}$$

In this notation

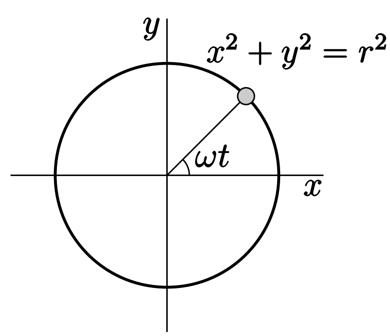
Equation 2.3.13 Rotational motion of centre of mass.

$$\frac{\mathrm{d}}{\mathrm{d}t} L(t) = T(t)$$

- Equation (2.3.13) plays the role of Newton's law of motion for rotational motion.
- $T(t)$ is called the torque and plays the role of "rotational force".
- $L(t)$ is called the angular momentum (about the origin) and is a measure of the rate at which the piece of wood is rotating.
  - For example, if a particle of mass m is traveling in a circle of radius r, centred on the origin, at ω radians per unit time, then $x(t) = r\cos(\omega t)$, $y(t) = r\sin(\omega t)$ and

    $$\begin{aligned} m\big[x(t)y'(t) - y(t)x'(t)\big] &= m\big[r\cos(\omega t)\ r\omega\cos(\omega t) - r\sin(\omega t)\ \big(-r\omega\sin(\omega t)\big)\big] \\ &= mr^2\ \omega \end{aligned}$$

    is proportional to ω, which is the rate of rotation about the origin.



In any event, in order for the piece of wood to remain stationary, that is to have $x_i(t)$ and $y_i(t)$ be constant for all $1 \le i \le n$, we need to have

$$X''(y) = Y''(t) = L(t) = 0$$

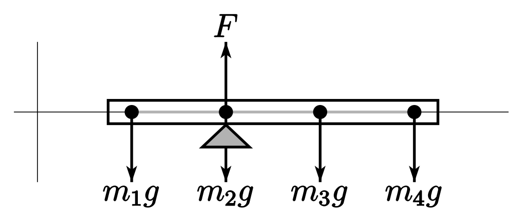
and then equations (2.3.12) and (2.3.13) force

$$H(t) = V(t) = T(t) = 0$$

Now suppose that the piece of wood is a seesaw that is long and thin and is lying on the x-axis, supported on a fulcrum at $x = p$. Then every $y_i = 0$ and the torque simplifies to $T(t) = \sum_{i=1}^{n} x_i(t)V_i(t)$. The forces consist of

- gravity, $m_i g$, acting downwards on particle number i, for each $1 \le i \le n$ and the

- force F imposed by the fulcrum that is pushing straight up on the particle at $x = p$.



So

- The net vertical force is $V(t) = F - \sum_{i=1}^{n} m_i g = F - Mg$. If the seesaw is to remain stationary, this must be zero so that $F = Mg$.
- The total torque (about the origin) is

$$T = Fp - \sum_{i=1}^{n} m_i g x_i = Mgp - \sum_{i=1}^{n} m_i g x_i$$

If the seesaw is to remain stationary, this must also be zero and the fulcrum must be placed at

Equation 2.3.14 Placement of fulcrum.

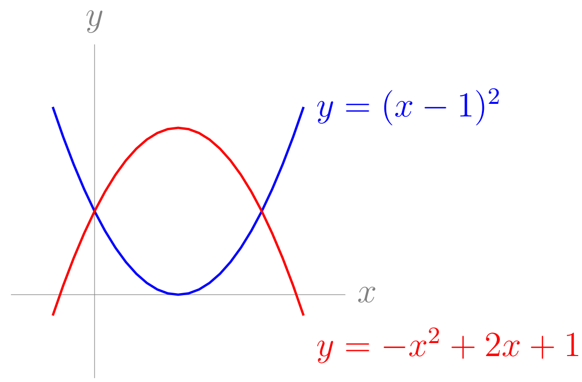
$$p = \frac{1}{M} \sum_{i=1}^{n} m_i x_i$$

which is the centre of mass of the piece of wood.

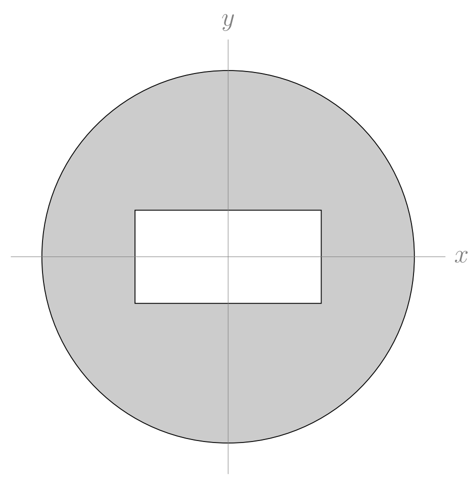
2.3.3 ▸▸ Exercises

Exercises — Stage 1 In Questions 8 through 10, you will derive the formulas for the centre of mass of a rod of variable density, and the centroid of a two-dimensional region using vertical slices (Equations 2.3.4 and 2.3.5 in the text). Knowing the equations by heart will allow you to answer many questions in this section; understanding where they came from will you allow to generalize their ideas to answer even more questions.

1. Using symmetry, find the centroid of the finite region between the curves $y = (x - 1)^2$ and $y = -x^2 + 2x + 1$.



2. Using symmetry, find the centroid of the region inside the unit circle, and outside a rectangle centred at the origin with width 1 and height 0.5.



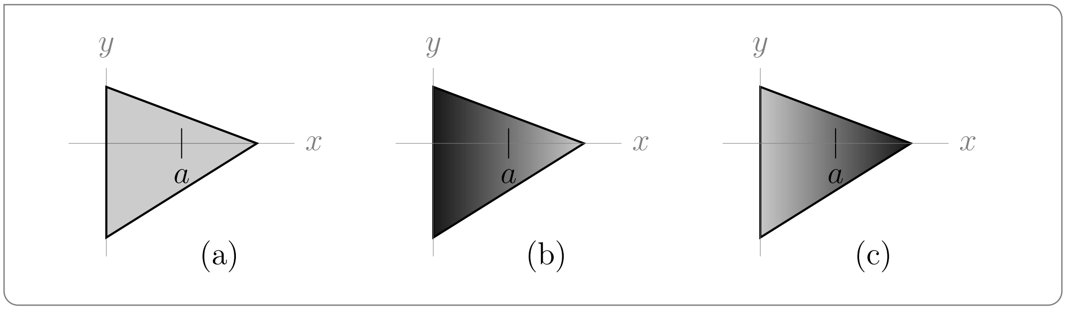
3. A long, straight, thin rod has a number of weights attached along it. True or false: if it balances at position x, then the mass to the right of x is the same as the mass to the left of x.

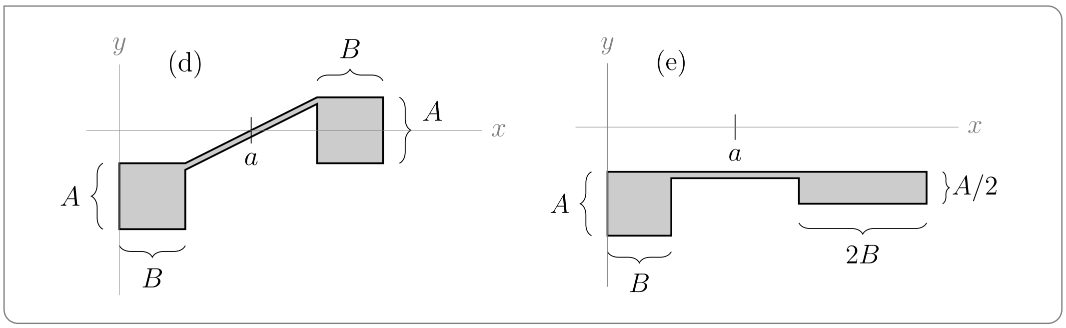
4. A straight rod with negligible mass has the following weights attached to it:

- A weight of mass 1 kg, 1m from the left end,
- a weight of mass 2 kg, 3m from the left end,
- a weight of mass 2 kg, 4m from the left end, and
- a weight of mass 1 kg, 6m from the left end.

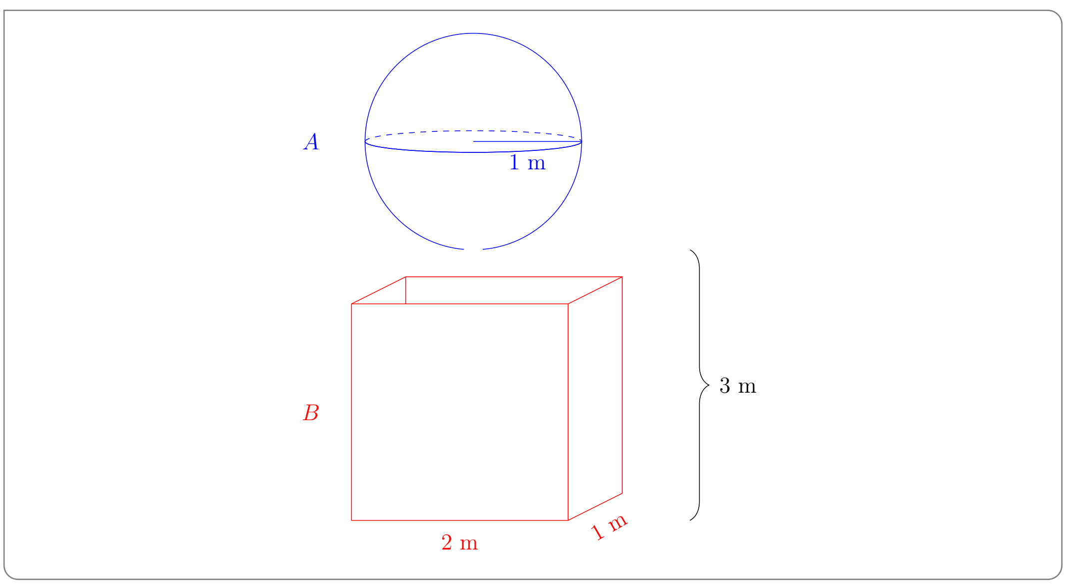
Where is the centre of mass of the weighted rod?

5. For each picture below, determine whether the centre of mass is to the left of, to the right of, or along the line $x = a$, or whether there is not enough information to tell. The shading of a region indicates density: darker shading corresponds to a denser area.





6. Tank A is spherical, of radius 1 metre, and filled completely with water. The bottom of tank A is three metres above the ground, where Tank B sits. Tank B is tall and rectangular, with base dimensions 2 metres by 1 metre, and empty. Calculate the work done by gravity to drain all the water from Tank A to Tank B by modelling the situation as a point mass, of the same mass as the water, being moved from the height of the centre of mass of A to the height of the centre of mass of the water after it has been moved to B.



You may use 1000 kg/m^3 for the density of water, and $g = 9.8$ m/sec^2 for the acceleration due to gravity.

7. Let S be the region bounded above by $y = \frac{1}{x}$ and and below by the x-axis, $1 \le x \le 3$. Let R be a rod with density $\rho(x) = \frac{1}{x}$ at position x, $1 \le x \le 3$.

a What is the area of a thin slice of S at position x with width dx?

b What is the mass of a small piece of R at position x with length dx?

c What is the total area of S?

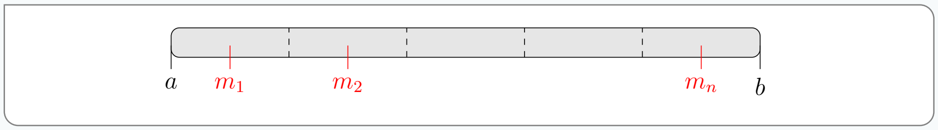
d What is the total mass of R?

e What is the x-coordinate of the centroid of S?

f What is the centre of mass of R?

8. Suppose R is a straight, thin rod with density $\rho(x)$ at a position x. Let the left endpoint of R lie at $x = a$, and the right endpoint lie at $x = b$.

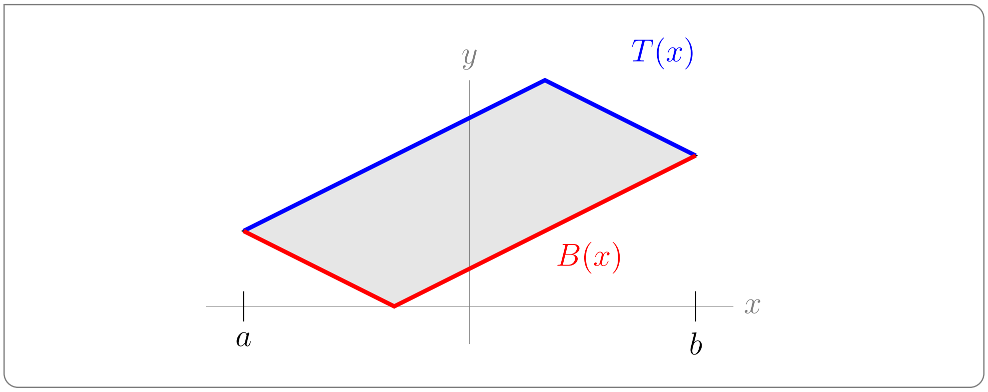
a To approximate the centre of mass of R, imagine chopping it into n pieces of equal length, and approximating the mass of each piece using the density at its midpoint. Give your approximation for the centre of mass in sigma notation.



b Take the limit as n goes to infinity of your approximation in part (a), and express the result using a definite integral.

9. Suppose S is a two-dimensional object and at (horizontal) position x its height is $T(x) - B(x)$. Its leftmost point is at position $x = a$, and its rightmost point is at position $x = b$.
To approximate the x-coordinate of the centroid of S, we imagine it as a straight, thin rod R, where the mass of R from $a \leq x \leq b$ is equal to the area of S from $a \leq x \leq b$.

a If S is the sheet shown below, sketch R as a rod with the same horizontal length, shaded darker when R is denser, and lighter when R is less dense.



b If we cut S into strips of very small width $\mathrm{d}x$, what is the area of the strip at position x?

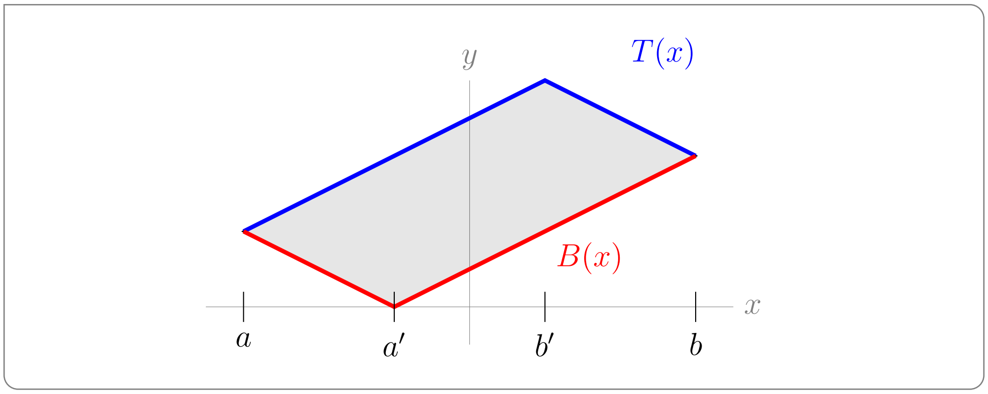
c Using your answer from (b), what is the density $\rho(x)$ of R at position x?

d Using your result from Question 8(b), give the x-coordinate of the centroid of S. Your answer will be in terms of a, b, $T(X)$, and $B(x)$.

10. Suppose S is flat sheet with uniform density, and at (horizontal) position x its height is $T(x)-B(x)$. Its leftmost point is at position $x=a$, and its rightmost point is at position $x=b$.
To approximate the y-coordinate of the centroid of S, we imagine it as a straight, thin, vertical rod R. We slice S into thin, vertical strips, and model these as weights on R with:

- position y on R, where y is the centre of mass of the strip, and
- mass in R equal to the area of the strip in S.

a If S is the sheet shown below, slice it into a number of vertical pieces of equal length, approximated by rectangles. For each rectangle, mark its centre of mass. Sketch R as a rod with the same vertical height, with weights corresponding to the slices you made of S.



b Imagine a thin strip of S at position x, with thickness $\mathrm{d}x$. What is the area of the strip? What is the y-value of its centre of mass?

c Recall the centre of mass of a rod with n weights of mass M_i at position y_i is given by

$$\frac{\sum_{i=1}^{n}(M_i \times y_i)}{\sum_{i=1}^{n} M_i}$$

Considering the limit of this formula as n goes to infinity, give the y-coordinate of the centre of mass of S.

11. **$*$.** Express the x-coordinate of the centroid of the triangle with vertices $(-1,-3)$, $(-1,3)$, and $(0,0)$ in terms of a definite integral. ***Do not evaluate the integral.***

Exercises — Stage 2 Use Equations 2.3.4 and 2.3.5 to find centroids and centres of mass in Questions 12 through 23.

12. A long, thin rod extends from $x = 0$ to $x = 7$ metres, and its density at position x is given by $\rho(x) = x$ kg/m. Where is the centre of mass of the rod?

13. A long, thin rod extends from $x = -3$ to $x = 10$ metres, and its density at position x is given by $\rho(x) = \frac{1}{1+x^2}$ kg/m. Where is the centre of mass of the rod?

14. **$*$.** Find the y-coordinate of the centroid of the region bounded by the curves $y = 1$, $y = -e^x$, $x = 0$ and $x = 1$. You may use the fact that the area of this region equals e.

15. **$*$.** Consider the region bounded by $y = \frac{1}{\sqrt{16-x^2}}$, $y = 0$, $x = 0$ and $x = 2$.

    a Sketch this region.

    b Find the y-coordinate of the centroid of this region.

16. **$*$.** Find the centroid of the finite region bounded by $y = \sin(x)$, $y = \cos(x)$, $x = 0$, and $x = \pi/4$.

17. **$*$.** Let A denote the area of the plane region bounded by $x = 0$, $x = 1$, $y = 0$ and $y = \dfrac{k}{\sqrt{1+x^2}}$, where k is a positive constant.

    a Find the coordinates of the centroid of this region in terms of k and A.

    b For what value of k is the centroid on the line $y = x$?

18. **$*$.** The region R is the portion of the plane which is above the curve $y = x^2 - 3x$ and below the curve $y = x - x^2$.

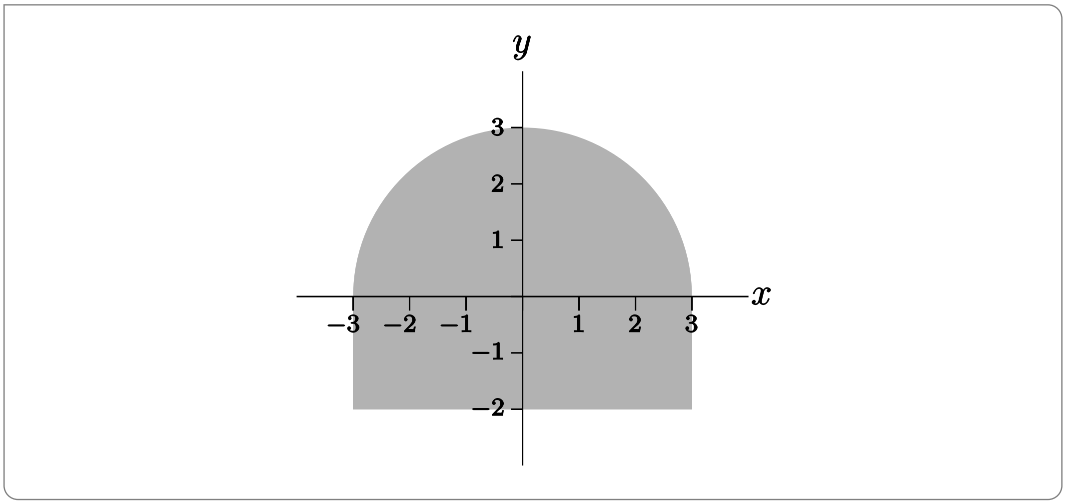
    a Sketch the region R

    b Find the area of R.

    c Find the x coordinate of the centroid of R.

19. **$*$.** Let R be the region where $0 \le x \le 1$ and $0 \le y \le \frac{1}{1+x^2}$. Find the x-coordinate of the centroid of R.

20. **$*$.** Find the centroid of the region below, which consists of a semicircle of radius 3 on top of a rectangle of width 6 and height 2.



21. $*$. Let D be the region below the graph of the curve $y = \sqrt{9 - 4x^2}$ and above the x-axis.

a Using an appropriate integral, find the *area* of the region D; simplify your answer completely.

b Find the *centre of mass* of the region D; simplify your answer completely. (Assume it has constant density ρ.)

22. The finite region S is bounded by the lines $y = \arcsin x$, $y = \arcsin(2 - x)$, and $y = -\frac{\pi}{2}$. Find the centroid of S.

23. Calculate the centroid of the figure bounded by the curves $y = e^x$, $y = 3(x - 1)$, $y = 0$, $x = 0$, and $x = 2$.

Exercises — Stage 3

24. $*$. Find the *y-coordinate* of the centre of mass of the (infinite) region lying to the right of the line $x = 1$, above the x-axis, and below the graph of $y = 8/x^3$.

25. $*$. Let A be the region to the right of the y-axis that is bounded by the graphs of $y = x^2$ and $y = 6 - x$.

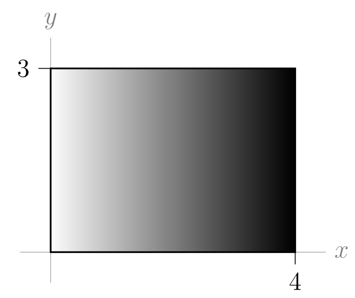
a Find the centroid of A, assuming it has constant density $\rho = 1$. The area of A is $\dfrac{22}{3}$ (you don't have to show this).

b Write down an expression, using horizontal slices (disks), for the volume obtained when the region A is rotated around the y-axis. Do not evaluate any integrals; simply write down an expression for the volume.

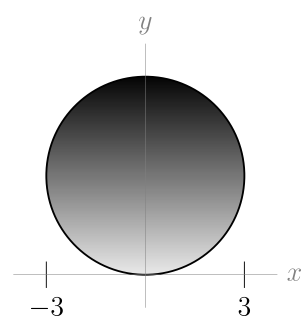
26. $*$. (a) Find the y-coordinate of the centroid of the region bounded by $y = e^x$, $x = 0$, $x = 1$, and $y = -1$.
(b) Calculate the volume of the solid generated by rotating the region from

part (a) about the line $y = -1$.

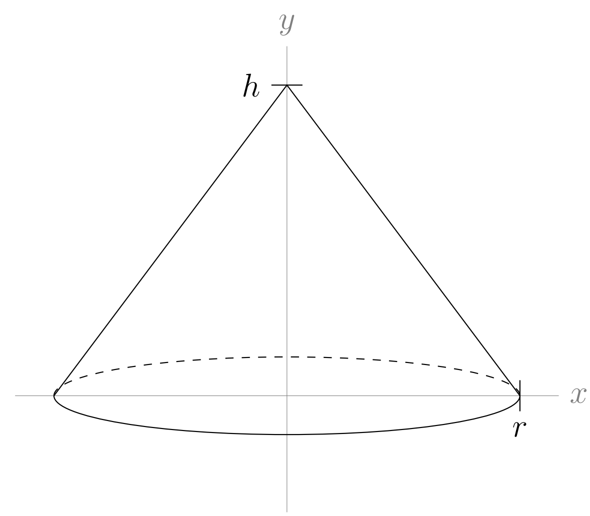
27. Suppose a rectangle has width 4 m, height 3 m, and its density x metres from its left edge is x^2 kg/m^2. Find the centre of mass of the rectangle.



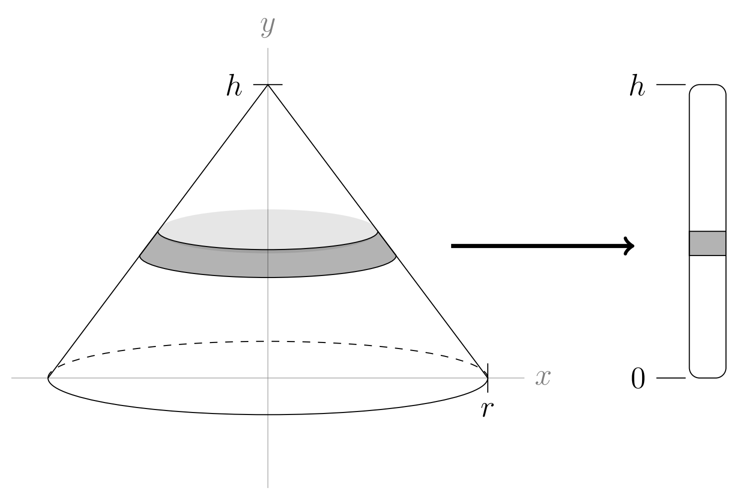
28. Suppose a circle of radius 3 m has density $(2+y)$ kg/m^2 at any point y metres above its bottom. Find the centre of mass of the circle.



29. A right circular cone of uniform density has base radius r m and height h m. We want to find its centre of mass. By symmetry, we know that the centre of mass will occur somewhere along the straight vertical line through the tip of the cone and the centre of its base. The only question is the *height* of the centre of mass.



We will model the cone as a rod R with height h, such that the mass of the section of the rod from position a to position b is the same as the volume of the cone from height a to height b. (You can imagine that the cone is an umbrella, and we've closed it up to look like a cane. [a])

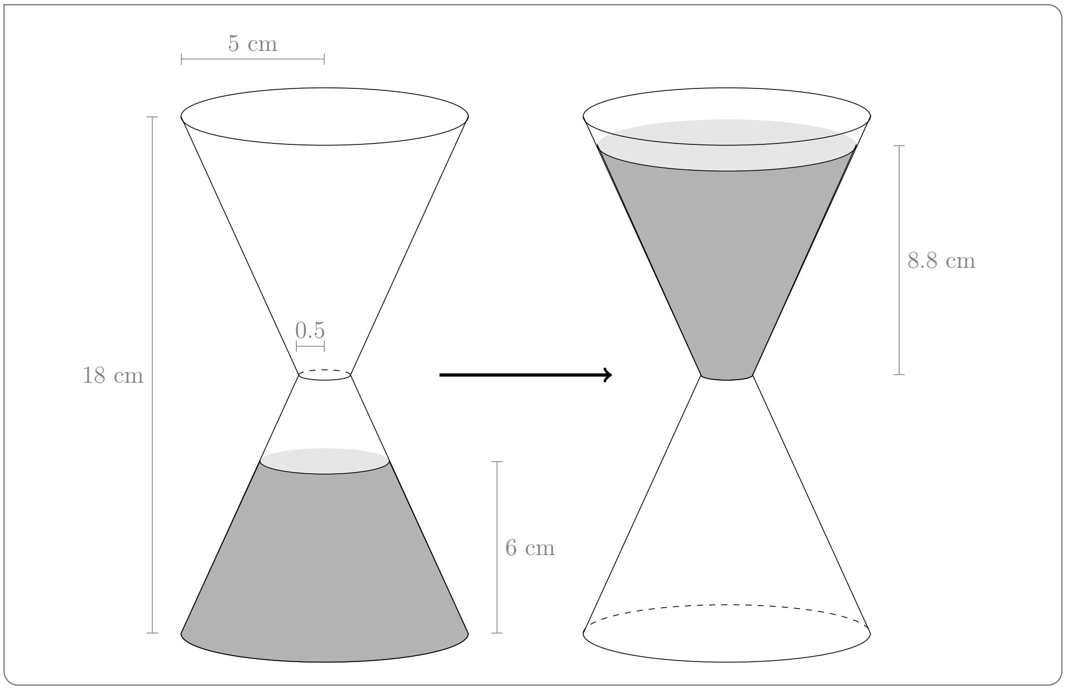


a Using this model, calculate how high above the base of the cone its centre of mass is.

b If we cut off the top $h-k$ metres of the cone (leaving an object of height k), how high above the base is the new centre of mass?

a This analogy isn't exact: if the cone were an umbrella, closing it would move the outside fabric vertically. A more accurate, but less familiar, image might be vacuum-wrapping an umbrella, watching it shrivel towards the middle but not move vertically.

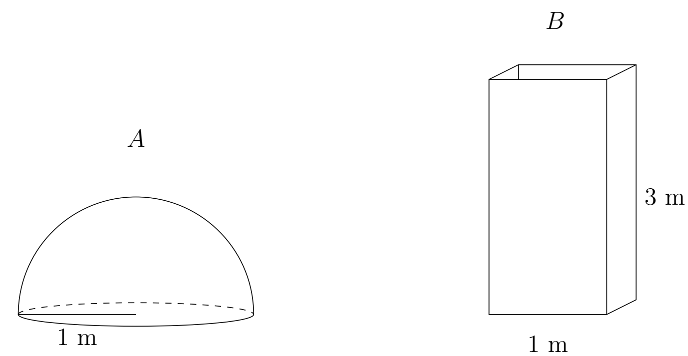
30. An hourglass is shaped like two identical truncated cones attached together. Their base radius is 5 cm, the height of the entire hourglass is 18 cm, and the radius at the thinnest point is .5 cm. The hourglass contains sand that fills up the bottom 6 cm when it's settled, with mass 600 grams and uniform density. We want to know the work done flipping the hourglass smoothly, so the sand settles into a truncated, inverted-cone shape before it starts to fall down.



Using the methods of Section 2.1 to calculate the work done would be quite tedious. Instead, we will model the sand as a point of mass 0.6 kg, being lifted from the centre of mass of its original position to the centre of mass of its upturned position. Using the results of Question 29, how much work was done on the sand?

To simplify your calculation, you may assume that the height of the upturned sand (that is, the distance from the skinniest part of the hourglass to the top of the sand) is 8.8 cm. (Actually, it's $\sqrt[3]{937} - 1 \approx 8.7854$ cm.) So, the top 0.2 cm of the hourglass is empty.

31. Tank A is in the shape of half a sphere of radius 1 metre, with its flat face resting on the ground, and is completely filled with water. Tank B is empty and rectangular, with a square base of side length 1 m and a height of 3 m.



a To pump the water from Tank A to Tank B, we need to pump all the water from Tank A to a height of 3 m. How much work is done to pump all the water from Tank A to a height of 3 m? You may model the water as a point mass, originally situated at the centre of mass of the full Tank A.

b Suppose we could move the water from Tank A directly to its final position in Tank B without going over the top of Tank B. (For example, maybe tank A is elastic, and Tank B is just Tank A after being smooshed into a different form.) How much work is done pumping the water? (That is, how much work is done moving a point mass from the centre of mass of Tank A to the centre of mass of Tank B?)

c What percentage of work from part (a) was “wasted” by pumping the water over the top of Tank B, instead of moving it directly to its final position?

You may assume that the only work done is against the acceleration due to gravity, $g = 9.8$ m/sec^2, and that the density of water is 1000 kg/m^3. Remark: the answer from (b) is what you might think of as the *net* work involved in pumping the water from Tank A to Tank B. When work gets “wasted,” the pump does some work pumping water up, then gravity does equal and opposite work bringing the water back down.

32. Let R be the region bounded above by $y = 2x\sin(x^2)$ and below by the x-axis, $0 \le x \le \sqrt{\frac{\pi}{2}}$. Give an approximation of the x-value of the centroid of R with error no more than $\frac{1}{100}$.

You may assume without proof that $\left|\frac{\mathrm{d}^4}{\mathrm{d}x^4}\left\{2x^2\sin(x^2)\right\}\right| \le 415$ over the interval $\left[0, \sqrt{\frac{\pi}{2}}\right]$.

2.4 ▲ Separable Differential Equations

A differential equation is an equation for an unknown function that involves the derivative of the unknown function. Differential equations play a central role in modelling a huge number of different phenomena. Here is a table giving a bunch of named differential equations and what they are used for. It is far from complete.

Newton's Law of Motion	describes motion of particles
Maxwell's equations	describes electromagnetic radiation
Navier-Stokes equations	describes fluid motion
Heat equation	describes heat flow
Wave equation	describes wave motion
Schrödinger equation	describes atoms, molecules and crystals
Stress-strain equations	describes elastic materials
Black-Scholes models	used for pricing financial options
Predator-prey equations	describes ecosystem populations
Einstein's equations	connects gravity and geometry
Ludwig-Jones-Holling's equation	models spruce budworm/Balsam fir ecosystem
Zeeman's model	models heart beats and nerve impulses
Sherman-Rinzel-Keizer model	for electrical activity in Pancreatic β-cells
Hodgkin-Huxley equations	models nerve action potentials

We are just going to scratch the surface of the study of differential equations. Most universities offer half a dozen different undergraduate courses on various aspects of differential equations. We will just look at one special, but important, type of equation.

2.4.1 ▶▶ Separate and integrate

Definition 2.4.1

A *separable differential equation* is an equation for a function $y(x)$ of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x) = f(x)\ g\big(y(x)\big)$$

We'll start by developing a recipe for solving separable differential equations. Then we'll look at many examples. Usually one suppresses the argument of $y(x)$ and writes the equation [1]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)\ g(y)$$

1 Look at the right hand side of the equation. The x-dependence is separated from the y-dependence. That's the reason for the name "separable".

and solves such an equation by cross multiplying/dividing to get all of the y's, including the $\mathrm{d}y$ on one side of the equation and all of the x's, including the $\mathrm{d}x$, on the other side of the equation.

$$\frac{\mathrm{d}y}{g(y)} = f(x)\,\mathrm{d}x$$

(We are of course assuming that $g(y)$ is nonzero.) Then you integrate both sides

$$\int \frac{\mathrm{d}y}{g(y)} = \int f(x)\,\mathrm{d}x \tag{$\star$}$$

This looks illegal, and indeed is illegal — $\frac{\mathrm{d}y}{\mathrm{d}x}$ is not a fraction. But we'll now see that the answer is still correct. This procedure is simply a mnemonic device to help you remember that answer ($\star$).

- Our goal is to find all functions $y(x)$ that obey $\frac{\mathrm{d}y}{\mathrm{d}x}(x) = f(x)\ g\big(y(x)\big)$.
- Assuming that g is nonzero,

$$\begin{aligned} y'(x) = f(x)\ g(y(x)) \iff \frac{y'(x)}{g(y(x))} = f(x) &\iff \int \frac{y'(x)}{g(y(x))}\,\mathrm{d}x = \int f(x)\,\mathrm{d}x \\ &\iff \int \frac{\mathrm{d}y}{g(y)}\bigg|_{y=y(x)} = \int f(x)\,\mathrm{d}x \\ &\ \text{with the substitution } y = y(x), \mathrm{d}y = y'(x)\,\mathrm{d}x \end{aligned}$$

- That's our answer ($\star$) again.

Let $G(y)$ be an antiderivative of $\frac{1}{g(y)}$ (i.e. $G'(y) = \frac{1}{g(y)}$) and $F(x)$ be an antiderivative of $f(x)$ (i.e. $F'(x) = f(x)$). If we reinstate the argument of y, ($\star$) is

$$G\big(y(x)\big) = F(x) + C \tag{2.4.1}$$

Observe that the solution equation (2.4.1) contains an arbitrary constant, C. The value of this arbitrary constant *can not* be determined by the differential equation. You need additional data to determine it. Often this data consists of the value of the unknown function for one value of x. That is, often the problem you have to solve is of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x) = f(x)\ g\big(y(x)\big) \qquad y(x_0) = y_0$$

where $f(x)$ and $g(y)$ are given functions and x_0 and y_0 are given numbers. This type of problem is called an "initial value problem". It is solved by first using the method above to find the general solution to the differential equation, including the arbitrary constant C, and then using the "initial condition" $y(x_0) = y_0$ to determine the value of C. We'll see examples of this shortly.

Example 2.4.2 A separable warm-up.

The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xe^{-y}$$

is separable, and we now find all of its solutions by using our mnemonic device. We start by cross-multiplying so as to move all y's to the left hand side and all x's to the right hand side.

$$e^y\,\mathrm{d}y = x\,\mathrm{d}x$$

Then we integrate both sides

$$\int e^y \mathrm{d}y = \int x \mathrm{d}x \iff e^y = \frac{x^2}{2} + C$$

The C on the right hand side contains both the arbitrary constant for the indefinite integral $\int e^y \mathrm{d}y$ and the arbitrary constant for the indefinite integral $\int x\mathrm{d}x$. Finally, we solve for y, which is really a function of x.

$$y(x) = \log\left(\frac{x^2}{2} + C\right)$$

Recall that we are using log to refer to the natural (base e) logarithm.

Note that C is an arbitrary constant. It can take any value. It cannot be determined by the differential equation itself. In applications C is usually determined by a requirement that y take some prescribed value (determined by the application) when x is some prescribed value. For example, suppose that we wish to find a function $y(x)$ that obeys both

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xe^{-y} \qquad \text{and} \qquad y(0) = 1$$

We know that, to have $\frac{\mathrm{d}y}{\mathrm{d}x} = xe^{-y}$ satisfied, we must have $y(x) = \log\left(\frac{x^2}{2} + C\right)$, for some constant C. To also have $y(0) = 1$, we must have

$$1 = y(0) = \log\left(\frac{x^2}{2} + C\right)\bigg|_{x=0} = \log C \iff \log C = 1 \iff C = e$$

So our final solution is $y(x) = \log\left(\frac{x^2}{2} + e\right)$.

Example 2.4.3 A little more warm-up.

Let a and b be any two constants. We'll now solve the family of differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(y - b)$$

using our mnemonic device.

$$\begin{aligned}\frac{\mathrm{d}y}{y-b} = a\,\mathrm{d}x &\implies \int \frac{\mathrm{d}y}{y-b} = \int a\,\mathrm{d}x \\ &\implies \log|y-b| = ax + c \implies |y-b| = e^{ax+c} = e^c e^{ax} \\ &\implies y - b = Ce^{ax}\end{aligned}$$

where C is either $+e^c$ or $-e^c$. Note that as c runs over all real numbers, $+e^c$ runs over all strictly positive real numbers and $-e^c$ runs over all strictly negative real numbers. So, so far, C can be any real number except 0. But we were a bit sloppy here. We implicitly assumed that $y - b$ was nonzero, so that we could divide it across. None-the-less, the constant function $y = b$, which corresponds to $C = 0$, is a perfectly good solution — when y is the constant function $y = b$, both $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $a(y-b)$ are zero. So the general solution to $\frac{\mathrm{d}y}{\mathrm{d}x} = a(y-b)$ is $y(x) = Ce^{ax} + b$, where the constant C can be any real number. Note that when $y(x) = Ce^{ax} + b$ we have $y(0) = C + b$. So $C = y(0) - b$ and the general solution is

$$y(x) = \{y(0) - b\}\, e^{ax} + b$$

Example 2.4.3

This is worth stating as a theorem.

Theorem 2.4.4

Let a and b be constants. The differentiable function $y(x)$ obeys the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(y-b)$$

if and only if

$$y(x) = \{y(0) - b\}\, e^{ax} + b$$

Example 2.4.5 Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = y^2$.

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = y^2$
Solution: When $y \neq 0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 \implies \frac{\mathrm{d}y}{y^2} = \mathrm{d}x \implies \frac{y^{-1}}{-1} = x + C \implies y = -\frac{1}{x+C}$$

When $y = 0$, this computation breaks down because $\frac{\mathrm{d}y}{y^2}$ contains a division by 0. We can check if the function $y(x) = 0$ satisfies the differential equation by just subbing it in:

$$y(x) = 0 \implies y'(x) = 0,\ y(x)^2 = 0 \implies y'(x) = y(x)^2$$

So $y(x) = 0$ is a solution and the full solution is

$$y(x) = 0 \text{ or } y(x) = -\frac{1}{x+C}, \text{ for any constant } C$$

Example 2.4.5

Example 2.4.6 A falling raindrop.

When a raindrop falls it increases in size so that its mass $m(t)$, is a function of time t. The rate of growth of mass, i.e. $\frac{dm}{dt}$, is $km(t)$ for some positive constant k. According to Newton's law of motion, $\frac{\mathrm{d}}{\mathrm{d}t}(mv) = gm$, where v is the velocity of the raindrop (with v being positive for downward motion) and g is the acceleration due to gravity. Find the terminal velocity, $\lim_{t\to\infty} v(t)$, of a raindrop.

Solution: In this problem we have two unknown functions, $m(t)$ and $v(t)$, and two differential equations, $\frac{dm}{dt} = km$ and $\frac{\mathrm{d}}{\mathrm{d}t}(mv) = gm$. The first differential equation, $\frac{dm}{dt} = km$, involves only $m(t)$, not $v(t)$, so we use it to determine $m(t)$. By Theorem 2.4.4, with $b = 0$, $a = k$, y replaced by m and x replaced by t,

$$\frac{\mathrm{d}m}{\mathrm{d}t} = km \implies m(t) = m(0)e^{kt}$$

Now that we know $m(t)$ (except for the value of the constant $m(0)$), we can substitute it into the second differential equation, which we can then use to determine the remaining unknown function $v(t)$. Observe that the second equation, $\frac{\mathrm{d}}{\mathrm{d}t}(mv) = gm(t) = gm(0)e^{kt}$ tells that the derivative of the function $y(t) = m(t)v(t)$ is $gm(0)e^{kt}$. So $y(t)$ is just an antiderivative of $gm(0)e^{kt}$.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = gm(t) = gm(0)e^{kt} \qquad \implies y(t) = \int gm(0)e^{kt}\,\mathrm{d}t = gm(0)\frac{e^{kt}}{k} + C$$

Now that we know $y(t) = m(t)v(t) = m(0)e^{kt}v(t)$, we can get $v(t)$ just by dividing out the $m(0)e^{kt}$.

$$\begin{aligned} y(t) = gm(0)\frac{e^{kt}}{k} + C &\implies m(0)e^{kt}v(t) = gm(0)\frac{e^{kt}}{k} + C \\ &\implies v(t) = \frac{g}{k} + \frac{C}{m(0)e^{kt}} \end{aligned}$$

Our solution, $v(t)$, contains two arbitrary constants, namely C and $m(0)$. They will be determined by, for example, the mass and velocity at time $t = 0$. But since we are only interested in the terminal velocity $\lim_{t\to\infty} v(t)$, we don't need to know C and $m(0)$. Since $k > 0$, $\lim_{t\to\infty} \frac{C}{e^{kt}} = 0$ and the terminal velocity $\lim_{t\to\infty} v(t) = \frac{g}{k}$.

Example 2.4.7 Intravenous glucose.

A glucose solution is administered intravenously into the bloodstream at a constant rate r. As the glucose is added, it is converted into other substances at a rate that is proportional to the concentration at that time. The concentration, $C(t)$, of the glucose in the bloodstream at time t obeys the differential equation

$$\frac{\mathrm{d}C}{\mathrm{d}t} = r - kC$$

where k is a positive constant of proportionality.

a Express $C(t)$ in terms of k and $C(0)$.

b Find $\lim\limits_{t\to\infty} C(t)$.

Solution: (a) Since $r - kC = -k\big(C - \frac{r}{k}\big)$ the given equation is

$$\frac{\mathrm{d}C}{\mathrm{d}t} = -k\big(C - \frac{r}{k}\big)$$

which is of the form solved in Theorem 2.4.4 with $a = -k$ and $b = \frac{r}{k}$. So the solution is

$$C(t) = \frac{r}{k} + \Big(C(0) - \frac{r}{k}\Big)e^{-kt}$$

For any $k > 0$, $\lim\limits_{t\to\infty} e^{-kt} = 0$. Consequently, for any $C(0)$ and any $k > 0$, $\lim\limits_{t\to\infty} C(t) = \frac{r}{k},$. We could have predicted this limit without solving for $C(t)$. If we assume that $C(t)$ approaches some equilibrium value C_e as t approaches infinity, then taking the limits of both sides of $\frac{dC}{dt} = r - kC$ as $t \to \infty$ gives

$$0 = r - kC_e \implies C_e = \frac{r}{k}$$

2.4.2 ➧ Optional — Carbon Dating

Scientists can determine the age of objects containing organic material by a method called *carbon dating* or *radiocarbon dating* [2]. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, ^{14}C, with a half-life of about 5730 years. Vegetation absorbs carbon dioxide from the atmosphere through photosynthesis and animals acquire ^{14}C by eating plants. When a plant or animal dies, it stops replacing its carbon and the amount of ^{14}C begins to decrease through radioactive decay. Therefore the level of radioactivity also decreases. More

2 Willard Libby, of Chicago University was awarded the Nobel Prize in Chemistry in 1960, for developing radiocarbon dating.

precisely, let $Q(t)$ denote the amount of ^{14}C in the plant or animal t years after it dies. The number of radioactive decays per unit time, at time t, is proportional to the amount of ^{14}C present at time t, which is $Q(t)$. Thus

Equation 2.4.8 Radioactive decay.

$$\frac{\mathrm{d}Q}{\mathrm{d}t}(t) = -kQ(t)$$

Here k is a constant of proportionality that is determined by the half-life. We shall explain what half-life is, and also determine the value of k, in Example 2.4.9, below.

Before we do so, let's think about the sign in equation (2.4.8).

- Recall that $Q(t)$ denotes a quantity, namely the amount of ^{14}C present at time t. There cannot be a negative amount of ^{14}C. Nor can this quantity be zero. (We would not use carbon dating when there is no ^{14}C present.) Consequently, $Q(t) > 0$.
- As the time t increases, $Q(t)$ decreases, because ^{14}C is being continuously converted into ^{14}N by radioactive decay [3]. Thus $\frac{\mathrm{d}Q}{\mathrm{d}t}(t) < 0$.
- The signs $Q(t) > 0$ and $\frac{\mathrm{d}Q}{\mathrm{d}t}(t) < 0$ are consistent with (2.4.8) provided the constant of proportionality $k > 0$.
- In (2.4.8), we chose to call the constant of proportionality "$-k$". We did so in order to make $k > 0$. We could just as well have chosen to call the constant of proportionality "K". That is, we could have replaced equation (2.4.8) by $\frac{\mathrm{d}Q}{\mathrm{d}t}(t) = KQ(t)$. The constant of proportionality K would have to be negative, (and K and k would be related by $K = -k$).

Example 2.4.9 Half life and the constant k.

In this example, we determine the value of the constant of proportionality k in (2.4.8) that corresponds to the half-life of ^{14}C, which is 5730 years.

- Imagine that some plant or animal contains a quantity Q_0 of ^{14}C at its time of death. Let's choose the zero point of time $t = 0$ to be the instant that the plant or animal died.
- Denote by $Q(t)$ the amount of ^{14}C in the plant or animal t years after it died. Then $Q(t)$ must obey both (2.4.8) and $Q(0) = Q_0$.
- Theorem 2.4.4, with $b = 0$ and $a = -k$, then tells us that $Q(t) = Q_0 e^{-kt}$ for all $t \geq 0$.

3 The precise transition is $^{14}C \to {}^{14}N + e^- + \bar{\nu}_e$ where e^- is an electron and $\bar{\nu}_e$ is an electron neutrino.

- By definition, the half-life of ^{14}C is the length of time that it takes for half of the ^{14}C to decay. That is, the half-life $t_{1/2}$ is determined by

$$\begin{aligned} Q(t_{1/2}) &= \frac{1}{2}Q(0) = \frac{1}{2}Q_0 && \text{but we know that } Q(t) = Q_0e^{-kt} \\ Q_0e^{-kt_{1/2}} &= \frac{1}{2}Q_0 && \text{now cancel } Q_0 \\ e^{-kt_{1/2}} &= \frac{1}{2} \end{aligned}$$

Taking the logarithm of both sides gives

$$-kt_{1/2} = \log\frac{1}{2} = -\log 2 \implies k = \frac{\log 2}{t_{1/2}}$$

Recall that, in this text, we use $\log x$ to indicate the natural logarithm. That is,

$$\log x = \log_e x = \log x$$

We are told that, for ^{14}C, the half-life $t_{1/2} = 5730$, so

$$k = \frac{\log 2}{5730} = 0.000121 \qquad \text{to 6 decimal places}$$

Example 2.4.9

From the work in the above example we have accumulated enough new facts to make a corollary to Theorem 2.4.4.

Corollary 2.4.10

The function $Q(t)$ satisfies the equation

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -kQ(t)$$

if and only if

$$Q(t) = Q(0)\,e^{-kt}$$

The half-life is defined to be the time $t_{1/2}$ which obeys

$$Q\big(t_{1/2}\big) = \frac{1}{2}\,Q(0)$$

The half-life is related to the constant k by

$$t_{1/2} = \frac{\log 2}{k}$$

Now here is a typical problem that is solved using Corollary 2.4.10.

Example 2.4.11 The age of a piece of parchment.

A particular piece of parchment contains about 64% as much ^{14}C as plants do today. Estimate the age of the parchment.

Solution: Let $Q(t)$ denote the amount of ^{14}C in the parchment t years after it was first created. By equation (2.4.8) and Example 2.4.9

$$\frac{\mathrm{d}Q}{\mathrm{d}t}(t) = -kQ(t) \qquad \text{with } k = \frac{\log 2}{5730} = 0.000121$$

By Corollary 2.4.10

$$Q(t) = Q(0)\, e^{-kt}$$

The time at which $Q(t)$ reaches $0.64\, Q(0)$ is determined by

$$\begin{aligned} Q(t) &= 0.64\, Q(0) && \text{but } Q(t) = Q(0)\, e^{-kt} \\ Q(0)\, e^{-kt} &= 0.64\, Q(0) && \text{cancel } Q(0) \\ e^{-kt} &= 0.64 && \text{take logarithms} \\ -kt &= \log 0.64 \\ t &= \frac{\log 0.64}{-k} = \frac{\log 0.64}{-0.000121} = 3700 && \text{to 2 significant digits} \end{aligned}$$

That is, the parchment [a] is about 37 centuries old.

[a] The British Museum has an Egyptian mathematical text from the seventeenth century B.C.

We have stated that the half-life of ^{14}C is 5730 years. How can this be determined? We can explain this using the following example.

Example 2.4.12 Half life of implausium.

A scientist in a B-grade science fiction film is studying a sample of the rare and fictitious element, implausium. With great effort he has produced a sample of pure implausium. The next day — 17 hours later — he comes back to his lab and discovers that his sample is now only 37% pure. What is the half-life of the element?

Solution: We can again set up our problem using Corollary 2.4.10. Let $Q(t)$ denote the quantity of implausium at time t, measured in hours. Then we know

$$Q(t) = Q(0) \cdot e^{-kt}$$

We also know that

$$Q(17) = 0.37Q(0).$$

That enables us to determine k via

$$\begin{aligned} Q(17) = 0.37Q(0) &= Q(0)e^{-17k} && \text{divide both sides by } Q(0) \\ 0.37 &= e^{-17k} \end{aligned}$$

and so

$$k = -\frac{\log 0.37}{17} = 0.05849$$

We can then convert this to the half life using Corollary 2.4.10:

$$t_{1/2} = \frac{\log 2}{k} \approx 11.85 \text{ hours}$$

While this example is entirely fictitious, one really can use this approach to measure the half-life of materials.

Example 2.4.12

2.4.3 ▸▸ Optional — Newton's Law of Cooling

Newton's law of cooling says:

- The rate of change of temperature of an object is proportional to the difference in temperature between the object and its surroundings. The temperature of the surroundings is sometimes called the ambient temperature.

If we denote by $T(t)$ the temperature of the object at time t and by A the temperature of its surroundings, Newton's law of cooling says that there is some constant of proportionality, K, such that

Equation 2.4.13 Newton's law of cooling.

$$\frac{\mathrm{d}T}{\mathrm{d}t}(t) = K\big[T(t) - A\big]$$

This mathematical model of temperature change works well when studying a small object in a large, fixed temperature, environment. For example, a hot cup of coffee in a large room [4]. Let's start by thinking a little about the sign of the constant of proportionality. At any time t, there are three possibilities.

4 It does not work so well when the object is of a similar size to its surroundings since the temperature of the surroundings will rise as the object cools. It also fails when there are phase transitions involved — for example, an ice-cube melting in a warm room does not obey Newton's law of cooling.

- If $T(t) > A$, that is, if the body is warmer than its surroundings, we would expect heat to flow from the body into its surroundings and so we would expect the body to cool off so that $\frac{\mathrm{d}T}{\mathrm{d}t}(t) < 0$. For this expectation to be consistent with (2.4.13), we need $K < 0$.
- If $T(t) < A$, that is the body is cooler than its surroundings, we would expect heat to flow from the surroundings into the body and so we would expect the body to warm up so that $\frac{\mathrm{d}T}{\mathrm{d}t}(t) > 0$. For this expectation to be consistent with (2.4.13), we again need $K < 0$.
- Finally if $T(t) = A$, that is the body and its environment have the same temperature, we would not expect any heat to flow between the two and so we would expect that $\frac{\mathrm{d}T}{\mathrm{d}t}(t) = 0$. This does not impose any condition on K.

In conclusion, we would expect $K < 0$. Of course, we could have chosen to call the constant of proportionality $-k$, rather than K. Then the differential equation would be $\frac{\mathrm{d}T}{\mathrm{d}t} = -k\big(T - A\big)$ and we would expect $k > 0$.

Example 2.4.14 Warming iced tea.

The temperature of a glass of iced tea is initially 5°. After 5 minutes, the tea has heated to 10° in a room where the air temperature is 30°.

a Determine the temperature as a function of time.

b What is the temperature after 10 minutes?

c Determine when the tea will reach a temperature of 20°.

Solution: (a)

- Denote by $T(t)$ the temperature of the tea t minutes after it was removed from the fridge, and let $A = 30$ be the ambient temperature.
- By Newton's law of cooling,
$$\frac{\mathrm{d}T}{\mathrm{d}t} = K(T - A) = K(T - 30)$$
for some, as yet unknown, constant of proportionality K.
- By Theorem 2.4.4 with $a = K$ and $b = 30$,
$$T(t) = [T(0) - 30]\, e^{Kt} + 30 = 30 - 25e^{Kt}$$
since the initial temperature $T(0) = 5$.
- This solution is not complete because it still contains an unknown constant, namely K. We have not yet used the given data that $T(5) = 10$. We can use it to determine K. At $t = 5$,
$$T(5) = 30 - 25e^{5K} = 10 \implies e^{5K} = \frac{20}{25} \implies 5K = \log\frac{20}{25}$$

$$\implies K = \frac{1}{5}\log\frac{4}{5} = -0.044629$$

to six decimal places.

(b) To find the temperature at 10 minutes we can just use the solution we have determined above.

$$\begin{aligned} T(10) &= 30 - 25e^{10K} \\ &= 30 - 25e^{10\times\frac{1}{5}\log\frac{4}{5}} \\ &= 30 - 25e^{2\log\frac{4}{5}} = 30 - 25e^{\log\frac{16}{25}} \\ &= 30 - 16 = 14^\circ \end{aligned}$$

(c) The temperature is 20° when

$$\begin{aligned} 30 - 25e^{Kt} = 20 &\implies e^{Kt} = \frac{10}{25} \implies Kt = \log\frac{10}{25} \\ &\implies t = \frac{1}{K}\log\frac{2}{5} = 20.5 \text{ min} \end{aligned}$$

to one decimal place.

Example 2.4.14

Example 2.4.15 Temperature back in time.

A dead body is discovered at 3:45pm in a room where the temperature is 20°C. At that time the temperature of the body 1s 27°C. Two hours later, at 5:45pm, the temperature of the body is 25.3°C. What was the time of death? Note that the normal (adult human) body temperature is 37°C.

Solution: We will assume that the body's temperature obeys Newton's law of cooling.

- Denote by $T(t)$ the temperature of the body at time t, with $t = 0$ corresponding to 3:45pm. We wish to find the time of death — call it t_d.
- There is a lot of data in the statement of the problem. We are told
  1. the ambient temperature: $A = 20$
  2. the temperature of the body when discovered: $T(0) = 27$
  3. the temperature of the body 2 hours later: $T(2) = 25.3$
  4. assuming the person was a healthy adult right up until he died, the temperature at the time of death: $T(t_d) = 37$.
- Theorem 2.4.4 with $a = K$ and $b = A = 20$

$$T(t) = [T(0) - A]\, e^{Kt} + A = 20 + 7e^{Kt}$$

Two unknowns remain, K and t_d.

- We can find the first, K, by using the condition (3), which says $T(2) = 25.3$.

$$\begin{aligned} 25.3 = T(2) = 20 + 7e^{2K} &\implies 7e^{2K} = 5.3 \implies 2K = \log\left(\tfrac{5.3}{7}\right) \\ &\implies K = \tfrac{1}{2}\log\left(\tfrac{5.3}{7}\right) = -0.139 \end{aligned}$$

- Finally, t_d is determined by the condition (4).

$$\begin{aligned} 37 = T(t_d) = 20 + 7e^{-0.139t_d} &\implies e^{-0.139t_d} = \tfrac{17}{7} \\ &\implies -0.139t_d = \log\left(\tfrac{17}{7}\right) \\ &\implies t_d = -\tfrac{1}{0.139}\log\left(\tfrac{17}{7}\right) = -6.38 \end{aligned}$$

to two decimal places. Now 6.38 hours is 6 hours and $0.38 \times 60 = 23$ minutes. So the time of death was 6 hours and 23 minutes before 3:45pm, which is 9:22am.

Example 2.4.15

A slightly tricky example — we need to determine the ambient temperature from three measurements at different times.

Example 2.4.16 Finding the ambient temperature.

A glass of room-temperature water is carried out onto a balcony from an apartment where the temperature is 22°C. After one minute the water has temperature 26°C and after two minutes it has temperature 28°C. What is the outdoor temperature?

Solution: We will assume that the temperature of the thermometer obeys Newton's law of cooling.

- Let A be the outdoor temperature and $T(t)$ be the temperature of the water t minutes after it is taken outside.
- By Newton's law of cooling,

$$T(t) = A + \big(T(0) - A\big)e^{Kt}$$

Theorem 2.4.4 with $a = K$ and $b = A$. Notice there are 3 unknowns here — A, $T(0)$ and K — so we need three pieces of information to find them all.

- We are told $T(0) = 22$, so

$$T(t) = A + \big(22 - A\big)e^{Kt}.$$

- We are also told $T(1) = 26$, which gives

$$\begin{aligned} 26 &= A + \big(22 - A\big)e^{K} && \text{rearrange things} \\ e^K &= \frac{26 - A}{22 - A} \end{aligned}$$

- Finally, $T(2) = 28$, so

$$\begin{aligned} 28 &= A + \big(22 - A\big)e^{2K} && \text{rearrange} \\ e^{2K} &= \frac{28 - A}{22 - A} && \text{but } e^K = \frac{26 - A}{22 - A}\text{, so} \\ \left(\frac{26 - A}{22 - A}\right)^2 &= \frac{28 - A}{22 - A} && \text{multiply through by } (22 - A)^2 \\ (26 - A)^2 &= (28 - A)(22 - A) \end{aligned}$$

We can expand out both sides and collect up terms to get

$$\begin{aligned} \underbrace{26^2}_{=676} - 52A + A^2 &= \underbrace{28 \times 22}_{=616} - 50A + A^2 \\ 60 &= 2A \\ 30 &= A \end{aligned}$$

So the temperature outside is 30°.

Example 2.4.16

2.4.4 ➡ Optional — Population Growth

Suppose that we wish to predict the size $P(t)$ of a population as a function of the time t. In the most naive model of population growth, each couple produces β offspring (for some constant β) and then dies. Thus over the course of one generation $\beta \frac{P(t)}{2}$ children are produced and $P(t)$ parents die so that the size of the population grows from $P(t)$ to

$$P(t + t_g) = \underbrace{P(t) + \beta \frac{P(t)}{2}}_{\text{parents+offspring}} - \underbrace{P(t)}_{\text{parents die}} = \frac{\beta}{2} P(t)$$

where t_g denotes the lifespan of one generation. The rate of change of the size of the population per unit time is

$$\frac{P(t + t_g) - P(t)}{t_g} = \frac{1}{t_g}\Big[\frac{\beta}{2} P(t) - P(t)\Big] = bP(t)$$

where $b = \frac{\beta - 2}{2t_g}$ is the net birthrate per member of the population per unit time. If we approximate

$$\frac{P(t + t_g) - P(t)}{t_g} \approx \frac{\mathrm{d}P}{\mathrm{d}t}(t)$$

we get the differential equation

Equation 2.4.17 Population growth.

$$\frac{\mathrm{d}P}{\mathrm{d}t} = bP(t)$$

By Corollary 2.4.10, with $-k$ replaced by b,

Equation 2.4.18 Malthusian growth model.

$$P(t) = P(0) \cdot e^{bt}$$

This is called the Malthusian [5] growth model. It is, of course, very simplistic. One of its main characteristics is that, since $P(t+T) = P(0) \cdot e^{b(t+T)} = P(t) \cdot e^{bT}$, every time you *add* T to the time, the population size is *multiplied* by e^{bT}. In particular, the population size doubles every $\frac{\log 2}{b}$ units of time. The Malthusian growth model can be a reasonably good model only when the population size is very small compared to its environment [6]. A more sophisticated model of population growth, that takes into account the "carrying capacity of the environment" is considered below.

Example 2.4.19 A rough estimate of the earth's population.

In 1927 the population of the world was about 2 billion. In 1974 it was about 4 billion. Estimate when it reached 6 billion. What will the population of the world be in 2100, assuming the Malthusian growth model?

Solution: We follow our usual pattern for dealing with such problems.

- Let $P(t)$ be the world's population, in billions, t years after 1927. Note that 1974 corresponds to $t = 1974 - 1927 = 47$.
- We are assuming that $P(t)$ obeys equation (2.4.17). So, by (2.4.18)

$$P(t) = P(0) \cdot e^{bt}$$

  Notice that there are 2 unknowns here — b and $P(0)$ — so we need two pieces of information to find them.
- We are told $P(0) = 2$, so

$$P(t) = 2 \cdot e^{bt}$$

- We are also told $P(47) = 4$, which gives

$$4 = 2 \cdot e^{47b} \qquad \text{clean up}$$

5 This is named after Rev. Thomas Robert Malthus. He described this model in a 1798 paper called "An essay on the principle of population".

6 That is, the population has plenty of food and space to grow.

$$\begin{aligned} e^{47b} &= 2 && \text{take the log and clean up} \\ b &= \frac{\log 2}{47} = 0.0147 && \text{to 3 decimal places} \end{aligned}$$

- We now know $P(t)$ completely, so we can easily determine the predicted population [a] in 2100, i.e. at $t = 2100 - 1927 = 173$.

$$P(173) = 2e^{173b} = 2e^{173\times 0.0147} = 12.7 \text{ billion}$$

- Finally, our crude model predicts that the population is 6 billion at the time t that obeys

$$\begin{aligned} P(t) = 2e^{bt} &= 6 && \text{clean up} \\ e^{bt} &= 3 && \text{take the log and clean up} \\ t &= \frac{\log 3}{b} = 47\frac{\log 3}{\log 2} = 74.5 \end{aligned}$$

which corresponds [b] to the middle of 2001.

a The *2015 Revision of World Population*, a publication of the United Nations, predicts that the world's population in 2100 will be about 11 billion. But "about" covers a pretty large range. They give an 80% confidence interval running from 10 billion to 12.5 billion.
b The world population really reached 6 billion in about 1999.

Example 2.4.19

Logistic growth adds one more wrinkle to the simple population model. It assumes that the population only has access to limited resources. As the size of the population grows the amount of food available to each member decreases. This in turn causes the net birth rate b to decrease. In the logistic growth model $b = b_0\left(1 - \frac{P}{K}\right)$, where K is called the carrying capacity of the environment, so that

$$P'(t) = b_0\left(1 - \frac{P(t)}{K}\right)P(t)$$

This is a separable differential equation and we can solve it explicitly. We shall do so shortly. See Example 2.4.20, below. But, before doing that, we'll see what we can learn about the behaviour of solutions to differential equations like this without finding formulae for the solutions. It turns out that we can learn a lot just by watching the sign of $P'(t)$. For concreteness, we'll look at solutions of the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t}(t) = \big(6000 - 3P(t)\big)\,P(t)$$

We'll sketch the graphs of four functions $P(t)$ that obey this equation.

- For the first function, $P(0) = 0$.

- For the second function, $P(0) = 1000$.
- For the third function, $P(0) = 2000$.
- For the fourth function, $P(0) = 3000$.

The sketches will be based on the observation that $(6000 - 3P)\, P = 3(2000 - P)\, P$

- is zero for $P = 0,\ 2000$,
- is strictly positive for $0 < P < 2000$ and
- is strictly negative for $P > 2000$.

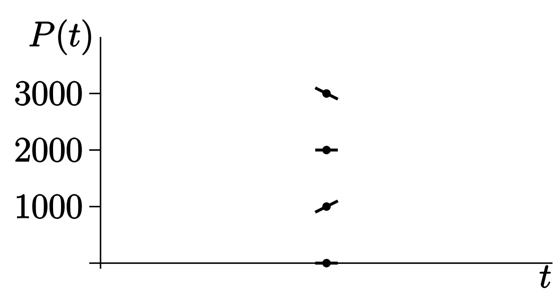
Consequently

$$\frac{\mathrm{d}P}{\mathrm{d}t}(t) \begin{cases} = 0 & \text{if } P(t) = 0 \\ > 0 & \text{if } 0 < P(t) < 2000 \\ = 0 & \text{if } P(t) = 2000 \\ < 0 & \text{if } P(t) > 2000 \end{cases}$$

Thus if $P(t)$ is some function that obeys $\frac{\mathrm{d}P}{\mathrm{d}t}(t) = \big(6000 - 3P(t)\big)P(t)$, then as the graph of $P(t)$ passes through the point $\big(t, P(t)\big)$

$$\text{the graph has} \begin{cases} \text{slope zero,} & \text{i.e. is horizontal,} & \text{if } P(t) = 0 \\ \text{positive slope,} & \text{i.e. is increasing,} & \text{if } 0 < P(t) < 2000 \\ \text{slope zero,} & \text{i.e. is horizontal,} & \text{if } P(t) = 2000 \\ \text{negative slope,} & \text{i.e. is decreasing,} & \text{if } 0 < P(t) < 2000 \end{cases}$$

as illustrated in the figure

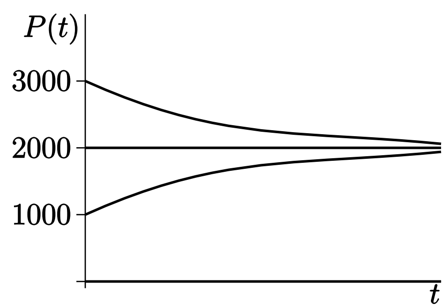


As a result,

- if $P(0) = 0$, the graph starts out horizontally. In other words, as t starts to increase, $P(t)$ remains at zero, so the slope of the graph remains at zero. The population size remains zero for all time. As a check, observe that the function $P(t) = 0$ obeys $\frac{\mathrm{d}P}{\mathrm{d}t}(t) = \big(6000 - 3P(t)\big)P(t)$ for all t.

- Similarly, if $P(0) = 2000$, the graph again starts out horizontally. So $P(t)$ remains at 2000 and the slope remains at zero. The population size remains 2000 for all time. Again, the function $P(t) = 2000$ obeys $\frac{\mathrm{d}P}{\mathrm{d}t}(t) = \big(6000 - 3P(t)\big)P(t)$ for all t.
- If $P(0) = 1000$, the graph starts out with positive slope. So $P(t)$ increases with t. As $P(t)$ increases towards 2000, the slope $\big(6000 - 3P(t)\big)P(t)$, while remaining positive, gets closer and closer to zero. As the graph approaches height 2000, it becomes more and more horizontal. The graph cannot actually cross from below 2000 to above 2000, because to do so it would have to have strictly positive slope for some value of P above 2000, which is not allowed.
- If $P(0) = 3000$, the graph starts out with negative slope. So $P(t)$ decreases with t. As $P(t)$ decreases towards 2000, the slope $\big(6000 - 3P(t)\big)P(t)$, while remaining negative, gets closer and closer to zero. As the graph approaches height 2000, it becomes more and more horizontal. The graph cannot actually cross from above 2000 to below 2000, because to do so it would have to have negative slope for some value of P below 2000, which is not allowed.

These curves are sketched in the figure below. We conclude that for any initial population size $P(0)$, except $P(0) = 0$, the population size approaches 2000 as $t \to \infty$.



Now we'll do an example in which we explicitly solve the logistic growth equation.

Example 2.4.20 Population predictions using logistic growth.

In 1986, the population of the world was 5 billion and was increasing at a rate of 2% per year. Using the logistic growth model with an assumed maximum population of 100 billion, predict the population of the world in the years 2000, 2100 and 2500.

Solution: Let $y(t)$ be the population of the world, in billions of people, at time $1986+t$. The logistic growth model assumes

$$y' = ay(K - y)$$

where K is the carrying capacity and $a = \frac{b_0}{K}$.
First we'll determine the values of the constants a and K from the given data.

- We know that, if at time zero the population is below K, then as time increases the population increases, approaching the limit K as t tends to infinity. So in this problem K is the maximum population. That is, $K = 100$.

- We are also told that, at time zero, the percentage rate of change of population, $100\frac{y'}{y}$, is 2, so that, at time zero, $\frac{y'}{y} = 0.02$. But, from the differential equation, $\frac{y'}{y} = a(K - y)$. Hence at time zero, $0.02 = a(100 - 5)$, so that $a = \frac{2}{9500}$.

We now know a and K and can solve the (separable) differential equation

$$\begin{aligned}\frac{\mathrm{d}y}{dt} = ay(K-y) &\implies \frac{\mathrm{d}y}{y(K-y)} = a\,\mathrm{d}t\\ &\implies \int \frac{1}{K}\Big[\frac{1}{y} - \frac{1}{y-K}\Big]\mathrm{d}y = \int a\,\mathrm{d}t\\ &\implies \frac{1}{K}[\log|y| - \log|y-K|] = at + C\\ &\implies \log\frac{|y|}{|y-K|} = aKt + CK \implies \left|\frac{y}{y-K}\right| = De^{aKt}\end{aligned}$$

with $D = e^{CK}$. We know that y remains between 0 and K, so that $\left|\frac{y}{y-K}\right| = \frac{y}{K-y}$ and our solution obeys

$$\frac{y}{K-y} = De^{aKt}$$

At this stage, we know the values of the constants a and K, but not the value of the constant D. We are given that at $t = 0$, $y = 5$. Subbing in this, and the values of K and a,

$$\frac{5}{100-5} = De^{0} \implies D = \frac{5}{95}$$

So the solution obeys the algebraic equation

$$\frac{y}{100-y} = \frac{5}{95}e^{2t/95}$$

which we can solve to get y as a function of t.

$$\begin{aligned}y = (100-y)\frac{5}{95}e^{2t/95} &\implies 95y = (500-5y)e^{2t/95}\\ &\implies \big(95 + 5e^{2t/95}\big)y = 500e^{2t/95}\\ &\implies y = \frac{500e^{2t/95}}{95+5e^{2t/95}} = \frac{100e^{2t/95}}{19+e^{2t/95}} = \frac{100}{1+19e^{-2t/95}}\end{aligned}$$

Finally,

- In the year 2000, $t = 14$ and $y = \frac{100}{1+19e^{-28/95}} \approx 6.6$ billion.
- In the year 2100, $t = 114$ and $y = \frac{100}{1+19e^{-228/95}} \approx 36.7$ billion.
- In the year 2200, $t = 514$ and $y = \frac{100}{1+19e^{-1028/95}} \approx 100$ billion.

Example 2.4.20

2.4.5 ▸▸ Optional — Mixing Problems

Example 2.4.21 Dissolving salt.

At time $t = 0$, where t is measured in minutes, a tank with a 5-litre capacity contains 3 litres of water in which 1 kg of salt is dissolved. Fresh water enters the tank at a rate of 2 litres per minute and the fully mixed solution leaks out of the tank at the *varying* rate of $2t$ litres per minute.

a Determine the volume of solution $V(t)$ in the tank at time t.

b Determine the amount of salt $Q(t)$ in solution when the amount of water in the tank is at maximum.

Solution*[a]*: (a) The rate of change of the volume in the tank, at time t, is $2 - 2t$, because water is entering at a rate 2 and solution is leaking out at a rate $2t$. Thus

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 2 - 2t \implies \mathrm{d}V = (2-2t)\,\mathrm{d}t \implies V = \int (2-2t)\,\mathrm{d}t = 2t - t^2 + C$$

at least until $V(t)$ reaches either the capacity of the tank or zero. When $t = 0$, $V = 3$ so $C = 3$ and $V(t) = 3+2t-t^2$. Observe that $V(t)$ is at a maximum when $\frac{\mathrm{d}V}{\mathrm{d}t} = 2-2t = 0$, or $t = 1$.

(b) In the very short time interval from time t to time $t + \mathrm{d}t$, $2t\,\mathrm{d}t$ litres of brine leaves the tank. That is, the fraction $\frac{2t\,\mathrm{d}t}{V(t)}$ of the total salt in the tank, namely $Q(t)\frac{2t\,\mathrm{d}t}{V(t)}$ kilograms, leaves. Thus salt is leaving the tank at the rate

$$\frac{Q(t)\frac{2t\,\mathrm{d}t}{V(t)}}{\mathrm{d}t} = \frac{2tQ(t)}{V(t)} = \frac{2tQ(t)}{3+2t-t^2} \text{ kilograms per minute}$$

so

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{2tQ(t)}{3+2t-t^2} \implies \frac{dQ}{Q} &= -\frac{2t}{3+2t-t^2}\mathrm{d}t = -\frac{2t}{(3-t)(1+t)}\mathrm{d}t \\ &= \left[\frac{3/2}{t-3} + \frac{1/2}{t+1}\right]\mathrm{d}t \\ \implies \log Q &= \frac{3}{2}\log|t-3| + \frac{1}{2}\log|t+1| + C\end{aligned}$$

We are interested in the time interval $0 \le t \le 1$. In this time interval $|t-3| = 3-t$ and $|t+1| = t+1$ so

$$\log Q = \frac{3}{2}\log(3-t) + \frac{1}{2}\log(t+1) + C$$

At $t = 0$, Q is 1 so

$$\log 1 = \frac{3}{2}\log(3-0) + \frac{1}{2}\log(0+1) + C$$

$$\implies C = \log 1 - \frac{3}{2}\log 3 - \frac{1}{2}\log 1 = -\frac{3}{2}\log 3$$

At $t = 1$

$$\begin{aligned}\log Q &= \frac{3}{2}\log(3-1) + \frac{1}{2}\log(1+1) - \frac{3}{2}\log 3 \\ &= 2\log 2 - \frac{3}{2}\log 3 = \log 4 - \log 3^{\frac{3}{2}}\end{aligned}$$

so $Q = \frac{4}{3^{\frac{3}{2}}}$.

a No pun intended (sorry).

Example 2.4.21

Example 2.4.22 Mixing brines.

A tank contains 1500 liters of brine with a concentration of 0.3 kg of salt per liter. Another brine solution, this with a concentration of 0.1 kg of salt per liter is poured into the tank at a rate of 20 li/min. At the same time, 20 li/min of the solution in the tank, which is stirred continuously, is drained from the tank.

a How many kilograms of salt will remain in the tank after half an hour?

b How long will it take to reduce the concentration to 0.2 kg/li?

Solution: Denote by $Q(t)$ the amount of salt in the tank at time t. In a very short time interval $\mathrm{d}t$, the incoming solution adds $20\,\mathrm{d}t$ liters of a solution carrying 0.1 kg/li. So the incoming solution adds $0.1 \times 20\,\mathrm{d}t = 2\,\mathrm{d}t$ kg of salt. In the same time interval $20\,\mathrm{d}t$ liters is drained from the tank. The concentration of the drained brine is $\frac{Q(t)}{1500}$. So $\frac{Q(t)}{1500}20\,\mathrm{d}t$ kg were removed. All together, the change in the salt content of the tank during the short time interval is

$$dQ = 2\,\mathrm{d}t - \frac{Q(t)}{1500}20\,\mathrm{d}t = \left(2 - \frac{Q(t)}{75}\right)\mathrm{d}t$$

The rate of change of salt content per unit time is

$$\frac{dQ}{dt} = 2 - \frac{Q(t)}{75} = -\frac{1}{75}\big(Q(t) - 150\big)$$

The solution of this equation is

$$Q(t) = \big\{Q(0) - 150\big\}e^{-t/75} + 150$$

by Theorem 2.4.4, with $a = -\frac{1}{75}$ and $b = 150$. At time 0, $Q(0) = 1500 \times 0.3 = 450$. So

$$Q(t) = 150 + 300e^{-t/75}$$

(a) At $t = 30$

$$Q(30) = 150 + 300e^{-30/75} = 351.1 \text{ kg}$$

(b) $Q(t) = 0.2 \times 1500 = 300$ kg is achieved when

$$150 + 300e^{-t/75} = 300 \implies 300e^{-t/75} = 150 \implies e^{-t/75} = 0.5$$
$$\implies -\frac{t}{75} = \log(0.5) \implies t = -75\log(0.5) = 51.99 \text{ min}$$

Example 2.4.22

2.4.6 ▸▸ Optional — Interest on Investments

Suppose that you deposit $\$P$ in a bank account at time $t = 0$. The account pays $r\%$ interest per year compounded n times per year.

- The first interest payment is made at time $t = \frac{1}{n}$. Because the balance in the account during the time interval $0 < t < \frac{1}{n}$ is $\$P$ and interest is being paid for $\left(\frac{1}{n}\right)^{\text{th}}$ of a year, that first interest payment is $\frac{1}{n} \times \frac{r}{100} \times P$. After the first interest payment, the balance in the account is $P + \frac{1}{n} \times \frac{r}{100} \times P = \left(1 + \frac{r}{100n}\right)P$.
- The second interest payment is made at time $t = \frac{2}{n}$. Because the balance in the account during the time interval $\frac{1}{n} < t < \frac{2}{n}$ is $\left(1 + \frac{r}{100n}\right)P$ and interest is being paid for $\left(\frac{1}{n}\right)^{\text{th}}$ of a year, the second interest payment is $\frac{1}{n} \times \frac{r}{100} \times \left(1 + \frac{r}{100n}\right)P$. After the second interest payment, the balance in the account is $\left(1 + \frac{r}{100n}\right)P + \frac{1}{n} \times \frac{r}{100} \times \left(1 + \frac{r}{100n}\right)P = \left(1 + \frac{r}{100n}\right)^2 P$.
- And so on.

In general, at time $t = \frac{m}{n}$ (just after the m^{th} interest payment), the balance in the account is

Equation 2.4.23 Discrete compounding interest.

$$B(t) = \left(1 + \frac{r}{100n}\right)^m P = \left(1 + \frac{r}{100n}\right)^{nt} P$$

Three common values of n are 1 (interest is paid once a year), 12 (i.e. interest is paid once a month) and 365 (i.e. interest is paid daily). The limit $n \to \infty$ is called continuous compounding [7]. Under continuous compounding, the balance at time t is

$$B(t) = \lim_{n\to\infty} \left(1 + \frac{r}{100n}\right)^{nt} P$$

7 There are banks that advertise continuous compounding. You can find some by googling "interest is compounded continuously and paid"

You may have already seen the limit

Equation 2.4.24 A useful limit.

$$\lim_{x\to 0}(1+x)^{a/x} = e^a$$

If so, you can evaluate $B(t)$ by applying 2.4.24 with $x = \frac{r}{100n}$ and $a = \frac{rt}{100}$ (so that $\frac{a}{x} = nt$). As $n \to \infty$, $x \to 0$ so that

Equation 2.4.25 Continuous compound interest.

$$B(t) = \lim_{n\to\infty}\left(1+\frac{r}{100n}\right)^{nt} P = \lim_{x\to 0}(1+x)^{a/x}P = e^aP = e^{rt/100}P$$

If you haven't seen (2.4.24) before, that's OK. In the following example, we rederive (2.4.25) using a differential equation instead of (2.4.24).

Example 2.4.26 Computing my future bank balance.

Suppose, again, that you deposit $\$P$ in a bank account at time $t = 0$, and that the account pays $r\%$ interest per year compounded n times per year, and denote by $B(t)$ the balance at time t. Suppose that you have just received an interest payment at time t. Then the next interest payment will be made at time $t+\frac{1}{n}$ and will be $\frac{1}{n}\times\frac{r}{100}\times B(t) = \frac{r}{100n}B(t)$. So, calling $\frac{1}{n} = h$,

$$B(t+h) = B(t) + \frac{r}{100}B(t)h \qquad \text{or} \qquad \frac{B(t+h)-B(t)}{h} = \frac{r}{100}B(t)$$

To get continuous compounding we take the limit $n \to \infty$ or, equivalently, $h \to 0$. This gives

$$\lim_{h\to 0}\frac{B(t+h)-B(t)}{h} = \frac{r}{100}B(t) \qquad \text{or} \qquad \frac{\mathrm{d}B}{\mathrm{d}t}(t) = \frac{r}{100}B(t)$$

By Theorem 2.4.4, with $a = \frac{r}{100}$ and $b = 0$, (or Corollary 2.4.10 with $k = -\frac{r}{100}$),

$$B(t) = e^{rt/100}B(0) = e^{rt/100}P$$

once again.

Example 2.4.27 Double your money.

a A bank advertises that it compounds interest continuously and that it will double your money in ten years. What is the annual interest rate?

b A bank advertises that it compounds monthly and that it will double your money in ten years. What is the annual interest rate?

Solution: (a) Let the interest rate be $r\%$ per year. If you start with $\$P$, then after t years, you have $Pe^{rt/100}$, under continuous compounding. This was equation (2.4.25). After 10 years you have $Pe^{r/10}$. This is supposed to be $2P$, so

$$\begin{aligned} Pe^{r/10} = 2P \quad &\implies \quad e^{r/10} = 2 \\ &\implies \quad \frac{r}{10} = \log 2 \quad \implies \quad r = 10 \log 2 = 6.93\% \end{aligned}$$

(b) Let the interest rate be $r\%$ per year. If you start with $\$P$, then after t years, you have $P\big(1 + \frac{r}{100\times12}\big)^{12t}$, under monthly compounding. This was (2.4.23). After 10 years you have $P\big(1 + \frac{r}{100\times12}\big)^{120}$. This is supposed to be $2P$, so

$$\begin{aligned} P\Big(1 + \frac{r}{100 \times 12}\Big)^{120} = 2P \qquad &\implies \quad \Big(1 + \frac{r}{1200}\Big)^{120} = 2 \\ &\implies \quad 1 + \frac{r}{1200} = 2^{1/120} \\ &\implies \quad \frac{r}{1200} = 2^{1/120} - 1 \\ &\implies \quad r = 1200\big(2^{1/120} - 1\big) = 6.95\% \end{aligned}$$

Example 2.4.28 Pension planning.

A 25 year old graduate of UBC is given \$50,000 which is invested at 5% per year compounded continuously. The graduate also intends to deposit money continuously at the rate of \$2000 per year.

a Find a differential equation that $A(t)$ obeys, assuming that the interest rate remains 5%.

b Determine the amount of money in the account when the graduate is 65.

c At age 65, the graduate will start withdrawing money continuously at the rate of W dollars per year. If the money must last until the person is 85, what is the largest possible value of W?

Solution: (a) Let's consider what happens to A over a very short time interval from time t to time $t + \Delta t$. At time t the account balance is $A(t)$. During the (really short) specified time interval the balance remains very close to $A(t)$ and so earns interest of $\frac{5}{100} \times \Delta t \times A(t)$. During the same time interval, the graduate also deposits an additional \$$2000\Delta t$. So

$$\begin{aligned} A(t+\Delta t) &\approx A(t) + 0.05A(t)\Delta t + 2000\Delta t \\ \implies \frac{A(t+\Delta t) - A(t)}{\Delta t} &\approx 0.05A(t) + 2000 \end{aligned}$$

In the limit $\Delta t \to 0$, the approximation becomes exact and we get

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 0.05A + 2000$$

(b) The amount of money at time t obeys

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 0.05A(t) + 2{,}000 = 0.05\big(A(t) + 40{,}000\big)$$

So by Theorem 2.4.4 (with $a = 0.05$ and $b = -40{,}000$),

$$A(t) = \big(A(0) + 40{,}000\big)e^{0.05t} - 40{,}000$$

At time 0 (when the graduate is 25), $A(0) = 50{,}000$, so the amount of money at time t is

$$A(t) = 90{,}000\, e^{0.05t} - 40,000$$

In particular, when the graduate is 65 years old, $t = 40$ and

$$A(40) = 90{,}000\, e^{0.05\times 40} - 40,000 = \$625{,}015.05$$

(c) When the graduate stops depositing money and instead starts withdrawing money at a rate W, the equation for A becomes

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 0.05A - W = 0.05(A - 20W)$$

assuming that the interest rate remains 5%. This time, Theorem 2.4.4 (with $a = 0.05$ and $b = 20W$) gives

$$A(t) = \big(A(0) - 20W\big)e^{0.05t} + 20W$$

If we now reset our clock so that $t = 0$ when the graduate is 65, $A(0) = 625,015.05$. So the amount of money at time t is

$$A(t) = 20W + e^{0.05t}(625,015.05 - 20W)$$

We want the account to be depleted when the graduate is 85. So, we want $A(20) = 0$. This is the case if

$$\begin{aligned} 20W + e^{0.05\times 20}(625,015.05 - 20W) &= 0 \\ \implies 20W + e(625,015.05 - 20W) &= 0 \\ \implies 20(e-1)W &= 625,015.05e \\ \implies W = \frac{625,015.05e}{20(e-1)} &= \$49,437.96 \end{aligned}$$

Example 2.4.28

2.4.7 ▸▸ Exercises

Recall that we are using $\log x$ to denote the logarithm of x with base e. In other courses it is often denoted $\ln x$.

Exercises — Stage 1

1. Below are pairs of functions $y = f(x)$ and differential equations. For each pair, decide whether the function is a solution of the differential equation.

	function	*differential equation*
(a)	$y = 5(e^x - 3x^2 - 6x - 6)$	$\frac{dy}{dx} = y + 15x^2$
(b)	$y = \frac{-2}{x^2+1}$	$y'(x) = xy^2$
(c)	$y = x^{3/2} + x$	$\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} = y$

2. Following Definition 2.4.1, a separable differential equation has the form

$$\frac{dy}{dx}(x) = f(x)\ g\big(y(x)\big).$$

Show that each of the following equations can be written in this form, identifying $f(x)$ and $g(y)$.

a $3y\frac{dy}{dx} = x \sin y$

b $\frac{dy}{dx} = e^{x+y}$

c $\frac{dy}{dx} + 1 = x$

d $\left(\frac{dy}{dx}\right)^2 - 2x\frac{dy}{dx} + x^2 = 0$

3. Suppose we have the following functions:

- y is a differentiable function of x
- f is a function of x, with $\int f(x)\,dx = F(x)$
- g is a nonzero function of y, with $\int \frac{1}{g(y)}\,dy = G(y) = G(y(x))$.

In the work below, we set up a solution to the separable differential equation

$$\frac{dy}{dx} = f(x)g(y) = f(x)g(y(x)))$$

without using the mnemonic of Equation ($\star$).
By deleting some portion of our work, we can create the solution as it would look using the mnemonic. What portion can be deleted?
Remark: the purpose of this exercise is to illuminate what, exactly, the mnemonic is a shortcut for. Despite its peculiar look, it agrees with what we already know about integration.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y(x))$$

Since $g(y(x))$ is a nonzero function, we can divide both sides by it.

$$\frac{1}{g(y(x))} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

If these functions of x are the same, then they have the same antiderivative with respect to x.

$$\int \frac{1}{g(y(x))} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int f(x)\,\mathrm{d}x$$

The left integral is in the correct form for a change of variables to y. To make this easier to see, we'll use a u-substitution, since it's a little more familiar than a y-substitution. If $u = y$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}$, so $\mathrm{d}u = \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x$.

$$\int \frac{1}{g(u)}\,\mathrm{d}u = \int f(x)\,\mathrm{d}x$$

Since u was just the same as y, again for cosmetic reasons, we can swap it back. (Formally, you could have skipped the step above–we just included it to be extra clear that we're not using any integration techniques we haven't seen before.)

$$\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$$

We're given the antiderivatives in question.

$$\begin{aligned} G(y) + C_1 &= F(x) + C_2 \\ G(y) &= F(x) + (C_2 - C_1) \end{aligned}$$

where C_1 and C_2 are arbitrary constants. Then also $C_2 - C_1$ is an arbitrary constant, so we might as well call it C.

$$G(y) = F(x) + C$$

4. Suppose $y = f(x)$ is a solution to the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = xy$. True or false: $f(x) + C$ is also a solution, for any constant C.

5. Suppose a function $y = f(x)$ satisfies $|y| = Cx$, for some constant $C > 0$.

a What is the largest possible domain of $f(x)$, given the information at hand?

b Give an example of function $y = f(x)$ with the following properties, or show that none exists:

- $|y| = Cx$,
- $\frac{\mathrm{d}y}{\mathrm{d}x}$ exists for all $x > 0$, and
- $y > 0$ for some values of x, and $y < 0$ for others.

6. Express the following sentence [a] as a differential equation. You don't have to solve the equation.

> About 0.3 percent of the total quantity of morphine in the bloodstream is eliminated every minute.

a The sentence is paraphrased from the Pharmakokinetics website of Université de Lausanne, Elimination Kinetics. The half-life of morphine is given on the same website. Accessed 12 August 2017.

7. Suppose a particular change is occurring in a language, from an old form to a new form. [a] Let $p(t)$ be the proportion (measured as a number between 0, meaning none, and 1, meaning all) of the time that speakers use the new form. Piotrowski's law [b] predicts the following.

> Use of the new form over time spreads at a rate that is proportional to the product of the proportion of the new form and the proportion of the old form.

Express this as a differential equation. You do not need to solve the differential equation.

a An example is the change in German from "wollt" to "wollst" for the second-person conjugation of the verb "wollen." This example is provided by the site Laws in Quantitative Linguistics, Change in Language, accessed 18 August 2017.

b Piotrowski's law is paraphrased from the page Piotrowski-Gesetz on Glottopedia, accessed 18 August 2017. According to this source, the law was based on work by the married couple R. G. Piotrowski and A. A. Piotrowskaja, later generalized by G. Altmann.

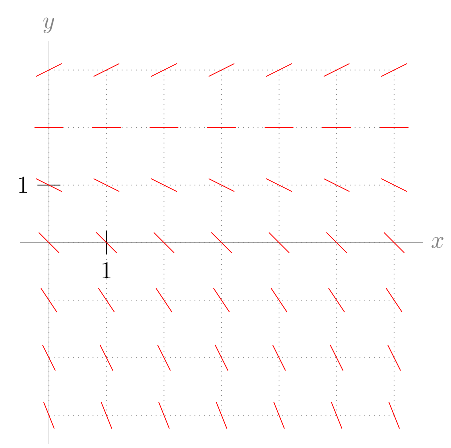
8. Consider the differential equation $y' = \frac{y}{2} - 1$.

a When $y = 0$, what is y'?

b When $y = 2$, what is y'?

c When $y = 3$, what is y'?

d On the axes below, interpret the marks we have made, and use them to sketch a possible solution to the differential equation.



9. Consider the differential equation $y' = y - \frac{x}{2}$.

a If $y(1) = 0$, what is $y'(1)$?

b If $y(1) = 2$, what is $y'(1)$?

c If $y(1) = -2$, what is $y'(1)$?

d Draw a sketch similar to that of Question 8(d) showing the derivatives of y at the points with integer values for x in $[0, 6]$ and y in $[-3, 3]$.

e Sketch a possible graph of y.

Exercises — Stage 2

10. $*$. Find the solution to the separable initial value problem:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{e^y}, \qquad\qquad y(0) = \log 2$$

Express your solution explicitly as $y = y(x)$.

11. $*$. Find the solution $y(x)$ of $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{xy}{x^2+1}$, $\quad y(0) = 3$.

12. $*$. Solve the differential equation $y'(t) = e^{\frac{y}{3}} \cos t$. You should express the solution $y(t)$ in terms of t explicitly.

13. $*$. Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xe^{x^2 - \log(y^2)}$$

14. $*$. Let $y = y(x)$. Find the general solution of the differential equation $y' = xe^y$.

15. $*$. Find the solution to the differential equation $\dfrac{yy'}{e^x - 2x} = \dfrac{1}{y}$ that satisfies $y(0) = 3$. Solve completely for y as a function of x.

16. $*$. Find the function $y = f(x)$ that satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -xy^3 \qquad \text{and} \qquad f(0) = -\frac{1}{4}$$

17. $*$. Find the function $y = y(x)$ that satisfies $y(1) = 4$ and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{15x^2 + 4x + 3}{y}$$

18. $*$. Find the solution $y(x)$ of $y' = x^3 y$ with $y(0) = 1$.

19. $*$. Find the solution of the initial value problem

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = y^2 \qquad y(1) = -1$$

20. $*$. A function $f(x)$ is always positive, has $f(0) = e$ and satisfies $f'(x) = x\,f(x)$ for all x. Find this function.

21. $*$. Solve the following initial value problem:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{(x^2 + x)y} \qquad y(1) = 2$$

22. $*$. Find the solution of the differential equation $\dfrac{1 + \sqrt{y^2 - 4}}{\tan x}y' = \dfrac{\sec x}{y}$ that satisfies $y(0) = 2$. You don't have to solve for y in terms of x.

23. $*$. The fish population in a lake is attacked by a disease at time $t = 0$, with the result that the size $P(t)$ of the population at time $t \geq 0$ satisfies

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -k\sqrt{P}$$

where k is a positive constant. If there were initially 90,000 fish in the lake and 40,000 were left after 6 weeks, when will the fish population be reduced to 10,000?

24. $*$. An object of mass m is projected straight upward at time $t = 0$ with initial speed v_0. While it is going up, the only forces acting on it are gravity (assumed constant) and a drag force proportional to the square of the object's speed $v(t)$. It follows that the differential equation of motion is

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -(mg + kv^2)$$

where g and k are positive constants. At what time does the object reach its highest point?

25. $*$. A motor boat is traveling with a velocity of 40 ft/sec when its motor shuts off at time $t = 0$. Thereafter, its deceleration due to water resistance is given by

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -k\,v^2$$

where k is a positive constant. After 10 seconds, the boat's velocity is 20 ft/sec.

a What is the value of k?

b When will the boat's velocity be 5 ft/sec?

26. $*$. Consider the initial value problem $\frac{\mathrm{d}x}{\mathrm{d}t} = k(3-x)(2-x)$, $x(0) = 1$, where k is a positive constant. (This kind of problem occurs in the analysis of certain chemical reactions.)

a Solve the initial value problem. That is, find x as a function of t.

b What value will $x(t)$ approach as t approaches $+\infty$.

27. $*$. The quantity $P = P(t)$, which is a function of time t, satisfies the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 4P - P^2$$

and the initial condition $P(0) = 2$.

a Solve this equation for $P(t)$.

b What is P when $t = 0.5$? What is the limiting value of P as t becomes large?

28. $*$. An object moving in a fluid has an initial velocity v of 400 m/min. The velocity is decreasing at a rate proportional to the square of the velocity. After 1 minute the velocity is 200 m/min.

a Give a differential equation for the velocity $v = v(t)$ where t is time.

b Solve this differential equation.

c When will the object be moving at 50 m/min?

Exercises — Stage 3

29. $*$. An investor places some money in a mutual fund where the interest is compounded continuously and where the interest rate fluctuates between 4% and 8%. Assume that the amount of money $B = B(t)$ in the account in dollars after t years satisfies the differential equation

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \big(0.06 + 0.02 \sin t\big) B$$

a Solve this differential equation for B as a function of t.

b If the initial investment is \$1000, what will the balance be at the end of two years?

30. $*$. An endowment is an investment account in which the balance ideally remains constant and withdrawals are made on the interest earned by the account. Such an account may be modeled by the initial value problem $B'(t) = aB - m$ for $t \geq 0$, with $B(0) = B_0$. The constant a reflects the annual interest rate, m is the annual rate of withdrawal, and B_0 is the initial balance in the account.

a Solve the initial value problem with $a = 0.02$ and $B(0) = B_0 = \$30,000$. Note that your answer depends on the constant m.

b If $a = 0.02$ and $B(0) = B_0 = \$30,000$, what is the annual withdrawal rate m that ensures a constant balance in the account?

31. $*$. A certain continuous function $y = y(x)$ satisfies the integral equation

$$y(x) = 3 + \int_0^x \big(y(t)^2 - 3y(t) + 2\big) \sin t \,\mathrm{d}t \qquad (*)$$

for all x in some open interval containing 0. Find $y(x)$ and the largest interval for which (∗) holds.

32. **∗.** A cylindrical water tank, of radius 3 meters and height 6 meters, is full of water when its bottom is punctured. Water drains out through a hole of radius 1 centimeter. If

- $h(t)$ is the height of the water in the tank at time t (in meters) and
- $v(t)$ is the velocity of the escaping water at time t (in meters per second) then
- Torricelli's law states that $v(t) = \sqrt{2gh(t)}$ where $g = 9.8$ m/sec^2. Determine how long it takes for the tank to empty.

33. **∗.** A spherical tank of radius 6 feet is full of mercury when a circular hole of radius 1 inch is opened in the bottom. How long will it take for all of the mercury to drain from the tank?
Use the value $g = 32$ feet/sec^2. Also use Torricelli's law, which states when the height of mercury in the tank is h, the speed of the mercury escaping from the tank is $v = \sqrt{2gh}$.

34. **∗.** Consider the equation

$$f(x) = 3 + \int_0^x \big(f(t) - 1\big)\big(f(t) - 2\big)\mathrm{d}t$$

a What is $f(0)$?

b Find the differential equation satisfied by $f(x)$.

c Solve the initial value problem determined in (a) and (b).

35. **∗.** A tank 2 m tall is to be made with circular cross–sections with radius $r = y^p$. Here y measures the vertical distance from the bottom of the tank and p is a positive constant to be determined. You may assume that when the tank drains, it obeys Torricelli's law, that is

$$A(y)\frac{\mathrm{d}y}{\mathrm{d}t} = -c\sqrt{y}$$

for some constant c where $A(y)$ is the cross–sectional area of the tank at height y. It is desired that the tank be constructed so that the top half ($y = 2$ to $y = 1$) takes exactly the same amount of time to drain as the bottom half ($y = 1$ to $y = 0$). Determine the value of p so that the tank has this property. *Note:* it is not possible or necessary to find c for this question.

36. Suppose $f(t)$ is a continuous, differentiable function and the root mean square of $f(t)$ on $[a,x]$ is equal to the average of $f(t)$ on $[a,x]$ for all x. That is,

$$\frac{1}{x-a}\int_a^x f(t)\,\mathrm{d}(t) = \sqrt{\frac{1}{x-a}\int_a^x f^2(t)\,\mathrm{d}t} \qquad (*)$$

You may assume $x > a$.

a Guess a function $f(t)$ for which the average of $f(t)$ is the same as the root mean square of $f(t)$ on any interval.

b Differentiate both sides of the given equation.

c Simplify your answer from (b) by using Equation $(*)$ to replace all terms containing $\int_a^x f^2(t)\,\mathrm{d}t$ with terms containing $\int_a^x f(t)\,\mathrm{d}t$.

d Let $Y(x) = \int_a^x f(t)\,\mathrm{d}t$, so the equation from (c) becomes a differential equation. Find all functions that satisfy it.

e What is $f(t)$?

37. Find the function $y(x)$ such that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{2}{y^3}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}$$

and if $x = -\frac{1}{16}\log 3$, then $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 3$.
You do not need to solve for y explicitly.

Chapter 3

SEQUENCE AND SERIES

You have probably learned about Taylor polynomials [1] and, in particular, that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + E_n(x)$$

where $E_n(x)$ is the error introduced when you approximate e^x by its Taylor polynomial of degree n. You may have even seen a formula for $E_n(x)$. We are now going to ask what happens as n goes to infinity? Does the error go to zero, giving an exact formula for e^x? We shall later see that it does and that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

At this point we haven't defined, or developed any understanding of, this infinite sum. How do we compute the sum of an infinite number of terms? Indeed, when does a sum of an infinite number of terms even make sense? Clearly we need to build up foundations to deal with these ideas. Along the way we shall also see other functions for which the corresponding error obeys $\lim\limits_{n\to\infty} E_n(x) = 0$ for some values of x and not for other values of x.

To motivate the next section, consider using the above formula with $x = 1$ to compute the number e:

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}$$

As we stated above, we don't yet understand what to make of this infinite number of terms, but we might try to sneak up on it by thinking about what happens as we take more and more terms.

1 term	$1 = 1$

1 Now would be an excellent time to quickly read over your notes on the topic.

2 terms	$1 + 1 = 2$
3 terms	$1 + 1 + \frac{1}{2} = 2.5$
4 terms	$1 + 1 + \frac{1}{2} + \frac{1}{6} = 2.666666\ldots$
5 terms	$1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2.708333\ldots$
6 terms	$1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = 2.716666\ldots$

By looking at the infinite sum in this way, we naturally obtain a sequence of numbers

$$\{\,1\,,\,2\,,\,2.5\,,\,2.666666\,,\cdots\,,\,2.708333\,,\cdots\,,\,2.716666\,,\cdots\,,\,\cdots\,\}.$$

The key to understanding the original infinite sum is to understand the behaviour of this sequence of numbers — in particularly, what do the numbers do as we go further and further? Does it settle down [2] to a given limit?

3.1 Sequences

3.1.1 Sequences

In the discussion above we used the term "sequence" without giving it a precise mathematical meaning. Let us rectify this now.

Definition 3.1.1

A sequence is a list of infinitely [a] many numbers with a specified order. It is denoted

$$\big\{a_1,\ a_2,\ a_3,\ \cdots,\ a_n,\ \cdots\big\} \quad \text{or} \quad \big\{a_n\big\} \quad \text{or} \quad \big\{a_n\big\}_{n=1}^{\infty}$$

a For the more pedantic reader, here we mean a countably infinite list of numbers. The interested (pedantic or otherwise) reader should look up countable and uncountable sets.

We will often specify a sequence by writing it more explicitly, like

$$\Big\{a_n = f(n)\Big\}_{n=1}^{\infty}$$

where $f(n)$ is some function from the natural numbers to the real numbers.

2 You will notice a great deal of similarity between the results of the next section and "limits at infinity" which was covered last term.

Example 3.1.2 Three sequences and another one.

Here are three sequences.

$$\left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \cdots,\ \frac{1}{n},\ \cdots\right\} \qquad \text{or} \qquad \left\{a_n = \frac{1}{n}\right\}_{n=1}^{\infty}$$
$$\left\{1,\ 2,\ 3,\ \cdots,\ n,\ \cdots\right\} \qquad \text{or} \qquad \left\{a_n = n\right\}_{n=1}^{\infty}$$
$$\left\{1,\ -1,\ 1,\ -1,\ \cdots,\ (-1)^{n-1},\ \cdots\right\} \qquad \text{or} \qquad \left\{a_n = (-1)^{n-1}\right\}_{n=1}^{\infty}$$

It is not necessary that there be a simple explicit formula for the n^{th} term of a sequence. For example the decimal digits of π is a perfectly good sequence

$$\left\{3,\ 1,\ 4,\ 1,\ 5,\ 9,\ 2,\ 6,\ 5,\ 3,\ 5,\ 8,\ 9,\ 7,\ 9,\ 3,\ 2,\ 3,\ 8,\ 4,\ 6,\ 2,\ 6,\ 4,\ \cdots\ \right\}$$

but there is no simple formula [a] for the n^{th} digit.

a There is, however, a remarkable result due to Bailey, Borwein and Plouffe that can be used to compute the n^{th} binary digit of π (i.e. writing π in base 2 rather than base 10) without having to work out the preceding digits.

Our primary concern with sequences will be the behaviour of a_n as n tends to infinity and, in particular, whether or not a_n "settles down" to some value as n tends to infinity.

Definition 3.1.3

A sequence $\left\{a_n\right\}_{n=1}^{\infty}$ is said to converge to the limit A if a_n approaches A as n tends to infinity. If so, we write

$$\lim_{n\to\infty} a_n = A \qquad \text{or} \qquad a_n \to A \text{ as } n \to \infty$$

A sequence is said to converge if it converges to some limit. Otherwise it is said to diverge.

The reader should immediately recognise the similarity with limits at infinity

$$\lim_{x\to\infty} f(x) = L \qquad \text{if} \qquad f(x) \to L \text{ as } x \to \infty$$

Example 3.1.4 Convergence in Example 3.1.2.

Three of the four sequences in Example 3.1.2 diverge:

- The sequence $\left\{a_n = n\right\}_{n=1}^{\infty}$ diverges because a_n grows without bound, rather than approaching some finite value, as n tends to infinity.
- The sequence $\left\{a_n = (-1)^{n-1}\right\}_{n=1}^{\infty}$ diverges because a_n oscillates between $+1$ and

-1 rather than approaching a single value as n tends to infinity.

- The sequence of the decimal digits of π also diverges, though the proof that this is the case is a bit beyond us right now [a] .

The other sequence in Example 3.1.2 has $a_n = \frac{1}{n}$. As n tends to infinity, $\frac{1}{n}$ tends to zero. So

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

a If the digits of π were to converge, then π would have to be a rational number. The irrationality of π (that it cannot be written as a fraction) was first proved by Lambert in 1761. Niven's 1947 proof is more accessible and we invite the interested reader to use their favourite search engine to find step-by-step guides to that proof.

Example 3.1.4

Example 3.1.5 $\lim_{n\to\infty} \frac{n}{2n+1}$.

Here is a little less trivial example. To study the behaviour of $\frac{n}{2n+1}$ as $n \to \infty$, it is a good idea to write it as

$$\frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$$

As $n \to \infty$, the $\frac{1}{n}$ in the denominator tends to zero, so that the denominator $2 + \frac{1}{n}$ tends to 2 and $\frac{1}{2+\frac{1}{n}}$ tends to $\frac{1}{2}$. So

$$\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2}$$

Notice that in this last example, we are really using techniques that we used before to study infinite limits like $\lim_{x\to\infty} f(x)$. This experience can be easily transferred to dealing with $\lim_{n\to\infty} a_n$ limits by using the following result.

Theorem 3.1.6

If

$$\lim_{x\to\infty} f(x) = L$$

and if $a_n = f(n)$ for all positive integers n, then

$$\lim_{n\to\infty} a_n = L$$

Example 3.1.7 $\lim_{n\to\infty} e^{-n}$.

Set $f(x) = e^{-x}$. Then $e^{-n} = f(n)$ and

$$\text{since} \quad \lim_{x\to\infty} e^{-x} = 0 \qquad \text{we know that} \qquad \lim_{n\to\infty} e^{-n} = 0$$

The bulk of the rules for the arithmetic of limits of functions that you already know also apply to the limits of sequences. That is, the rules you learned to work with limits such as $\lim_{x\to\infty} f(x)$ also apply to limits like $\lim_{n\to\infty} a_n$.

Theorem 3.1.8 Arithmetic of limits.

Let A, B and C be real numbers and let the two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ converge to A and B respectively. That is, assume that

$$\lim_{n\to\infty} a_n = A \qquad\qquad \lim_{n\to\infty} b_n = B$$

Then the following limits hold.

a $\lim_{n\to\infty} \big[a_n + b_n\big] = A + B$

(The limit of the sum is the sum of the limits.)

b $\lim_{n\to\infty} \big[a_n - b_n\big] = A - B$

(The limit of the difference is the difference of the limits.)

c $\lim_{n\to\infty} Ca_n = CA$.

d $\lim_{n\to\infty} a_n\, b_n = A\, B$

(The limit of the product is the product of the limits.)

e If $B \neq 0$ then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$

(The limit of the quotient is the quotient of the limits *provided* the limit of the denominator is not zero.)

We use these rules to evaluate limits of more complicated sequences in terms of the limits of simpler sequences — just as we did for limits of functions.

Example 3.1.9 Arithmetic of limits.

Combining Examples 3.1.5 and 3.1.7,

$$\begin{aligned}\lim_{n\to\infty}\left[\frac{n}{2n+1}+7e^{-n}\right] &= \lim_{n\to\infty}\frac{n}{2n+1}+\lim_{n\to\infty}7e^{-n} && \text{by Theorem 3.1.8(a)}\\ &= \lim_{n\to\infty}\frac{n}{2n+1}+7\lim_{n\to\infty}e^{-n} && \text{by Theorem 3.1.8(c)}\end{aligned}$$

and then using Examples 3.1.5 and 3.1.7

$$\begin{aligned}&= \frac{1}{2}+7\cdot 0\\ &= \frac{1}{2}\end{aligned}$$

There is also a squeeze theorem for sequences.

Theorem 3.1.10 Squeeze theorem.

If $a_n \le c_n \le b_n$ for all natural numbers n, and if

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$$

then

$$\lim_{n\to\infty} c_n = L$$

Example 3.1.11 A simple squeeze.

In this example we use the squeeze theorem to evaluate

$$\lim_{n\to\infty}\left[1+\frac{\pi_n}{n}\right]$$

where π_n is the n^{th} decimal digit of π. That is,

$$\pi_1 = 3 \quad \pi_2 = 1 \quad \pi_3 = 4 \quad \pi_4 = 1 \quad \pi_5 = 5 \quad \pi_6 = 9 \quad \cdots$$

We do not have a simple formula for π_n. But we do know that

$$0 \le \pi_n \le 9 \implies 0 \le \frac{\pi_n}{n} \le \frac{9}{n} \implies 1 \le 1+\frac{\pi_n}{n} \le 1+\frac{9}{n}$$

and we also know that

$$\lim_{n\to\infty} 1 = 1 \qquad \lim_{n\to\infty}\left[1+\frac{9}{n}\right] = 1$$

So the squeeze theorem with $a_n = 1$, $b_n = 1 + \frac{\pi_n}{n}$, and $c_n = 1 + \frac{9}{n}$ gives

$$\lim_{n\to\infty} \left[1 + \frac{\pi_n}{n}\right] = 1$$

Example 3.1.11

Finally, recall that we can compute the limit of the composition of two functions using continuity. In the same way, we have the following result:

Theorem 3.1.12 Continuous functions of limits.

If $\lim_{n\to\infty} a_n = L$ and if the function $g(x)$ is continuous at L, then

$$\lim_{n\to\infty} g(a_n) = g(L)$$

Example 3.1.13 $\lim_{n\to\infty} \sin \frac{\pi n}{2n+1}$.

Write $\sin \frac{\pi n}{2n+1} = g\left(\frac{n}{2n+1}\right)$ with $g(x) = \sin(\pi x)$. We saw, in Example 3.1.5 that

$$\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$$

Since $g(x) = \sin(\pi x)$ is continuous at $x = \frac{1}{2}$, which is the limit of $\frac{n}{2n+1}$, we have

$$\lim_{n\to\infty} \sin \frac{\pi n}{2n+1} = \lim_{n\to\infty} g\left(\frac{n}{2n+1}\right) = g\left(\frac{1}{2}\right) = \sin \frac{\pi}{2} = 1$$

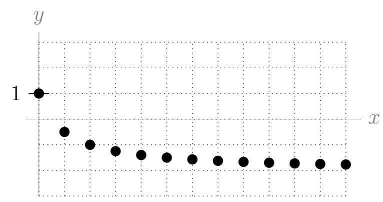
With this introduction to sequences and some tools to determine their limits, we can now return to the problem of understanding infinite sums.

3.1.2 ⏩ Exercises

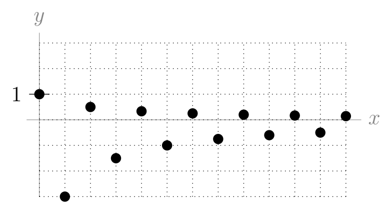
Exercises — Stage 1

1. Assuming the sequences continue as shown, estimate the limit of each sequence from its graph.

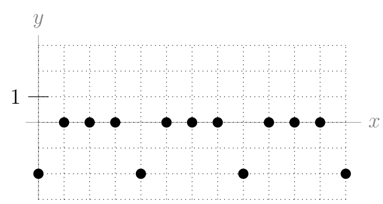
(a)



(b)



(c)



2. Suppose a_n and b_n are sequences, and $a_n = b_n$ for all $n \geq 100$, but $a_n \neq b_n$ for $n < 100$.
True or false: $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n$.

3. Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$, be sequences with $\lim\limits_{n\to\infty} a_n = A$, $\lim\limits_{n\to\infty} b_n = B$, and $\lim\limits_{n\to\infty} c_n = C$. Assume A, B, and C are nonzero real numbers.
Evaluate the limits of the following sequences.

a $\dfrac{a_n - b_n}{c_n}$

b $\dfrac{c_n}{n}$

c $\dfrac{a_{2n+5}}{b_n}$

4. Give an example of a sequence $\{a_n\}_{n=1}^{\infty}$ with the following properties:

- $a_n > 1000$ for all $n \leq 1000$,
- $a_{n+1} < a_n$ for all n, and
- $\lim\limits_{n\to\infty} a_n = -2$

5. Give an example of a sequence $\{a_n\}_{n=1}^{\infty}$ with the following properties:

- $a_n > 0$ for all even n,
- $a_n < 0$ for all odd n,
- $\lim_{n\to\infty} a_n$ does not exist.

6. Give an example of a sequence $\{a_n\}_{n=1}^{\infty}$ with the following properties:

- $a_n > 0$ for all even n,
- $a_n < 0$ for all odd n,
- $\lim_{n\to\infty} a_n$ exists.

7. The limits of the sequences below can be evaluated using the squeeze theorem. For each sequence, choose an upper bounding sequence and lower bounding sequence that will work with the squeeze theorem.

a $a_n = \dfrac{\sin n}{n}$

b $b_n = \dfrac{n^2}{e^n(7 + \sin n - 5\cos n)}$

c $c_n = (-n)^{-n}$

8. Below is a list of sequences, and a list of functions.

a Match each sequence a_n to any and all functions $f(x)$ such that $f(n) = a_n$ for all whole numbers n.

b Match each sequence a_n to any and all functions $f(x)$ such that $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$.

$$a_n = 1 + \frac{1}{n} \qquad\qquad f(x) = \cos(\pi x)$$

$$b_n = 1 + \frac{1}{|n|} \qquad\qquad g(x) = \frac{\cos(\pi x)}{x}$$

$$c_n = e^{-n} \qquad\qquad h(x) = \begin{cases} \frac{x+1}{x} & x \text{ is a whole number} \\ 1 & \text{else} \end{cases}$$

$$d_n = (-1)^n \qquad\qquad i(x) = \begin{cases} \frac{x+1}{x} & x \text{ is a whole number} \\ 0 & \text{else} \end{cases}$$

$$e_n = \frac{(-1)^n}{n} \qquad j(x) = \frac{1}{e^x}$$

9. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence defined by $a_n = \cos n$.

a Give three different whole numbers n that are within 0.1 of an odd integer multiple of π, and find the corresponding values of a_n.

b Give three different whole numbers n such that a_n is close to 0. Justify your answers.

Remark: this demonstrates intuitively, though not rigorously, why $\lim\limits_{n\to\infty} \cos n$ is undefined. We consistently find terms in the series that are close to -1, and also consistently find terms in the series that are close to 1. Contrast this to a series like $\big\{\cos(2\pi n)\big\}$, whose terms are always 1, and whose limit therefore is 1. It is possible to turn the ideas of this question into a rigorous proof that $\lim\limits_{n\to\infty} \cos n$ is undefined. See the solution.

Exercises — Stage 2

10. Determine the limits of the following sequences.

a $a_n = \dfrac{3n^2 - 2n + 5}{4n + 3}$

b $b_n = \dfrac{3n^2 - 2n + 5}{4n^2 + 3}$

c $c_n = \dfrac{3n^2 - 2n + 5}{4n^3 + 3}$

11. Determine the limit of the sequence $a_n = \dfrac{4n^3 - 21}{n^e + \frac{1}{n}}$.

12. Determine the limit of the sequence $b_n = \dfrac{\sqrt[4]{n} + 1}{\sqrt{9n + 3}}$.

13. Determine the limit of the sequence $c_n = \dfrac{\cos(n + n^2)}{n}$.

14. Determine the limit of the sequence $a_n = \dfrac{n^{\sin n}}{n^2}$.

15. Determine the limit of the sequence $d_n = e^{-1/n}$.

16. Determine the limit of the sequence $a_n = \dfrac{1 + 3\sin(n^2) - 2\sin n}{n}$.

17. Determine the limit of the sequence $b_n = \dfrac{e^n}{2^n + n^2}$.

18. $*$. Find the limit, if it exists, of the sequence $\{a_k\}$, where

$$a_k = \frac{k! \sin^3 k}{(k+1)!}$$

19. $*$. Consider the sequence $\left\{(-1)^n \sin\left(\frac{1}{n}\right)\right\}$. State whether this sequence converges or diverges, and if it converges give its limit.

20. $*$. Evaluate $\displaystyle\lim_{n\to\infty} \left[\frac{6n^2+5n}{n^2+1} + 3\cos(1/n^2)\right]$.

Exercises — Stage 3

21. $*$. Find the limit of the sequence $\left\{\log\left(\sin\dfrac{1}{n}\right) + \log(2n)\right\}$.

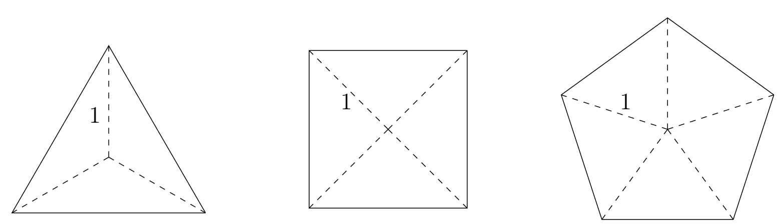
22. Evaluate $\displaystyle\lim_{n\to\infty} \left[\sqrt{n^2+5n} - \sqrt{n^2-5n}\right]$.

23. Evaluate $\displaystyle\lim_{n\to\infty} \left[\sqrt{n^2+5n} - \sqrt{2n^2-5}\right]$.

24. Evaluate the limit of the sequence $\left\{n\left[\left(2+\frac{1}{n}\right)^{100} - 2^{100}\right]\right\}_{n=1}^{\infty}$.

25. Write a sequence $\{a_n\}_{n=1}^{\infty}$ whose limit is $f'(a)$ for a function $f(x)$ that is differentiable at the point a.
Your answer will depend on f and a.

26. Let $\{A_n\}_{n=3}^{\infty}$ be the area of a regular polygon with n sides, with the distance from the centroid of the polygon to each corner equal to 1.



$A(3) = \frac{3\sqrt{3}}{4}$ $\quad$ $A(4) = 2$ $\quad$ $A(5) = 2.5\sin(0.4\pi)$

a By dividing the polygon into n triangles, give a formula for A_n.

b What is $\lim_{n\to\infty} A_n$?

27. Suppose we define a sequence $\{f_n\}$, which depends on some constant x, as the following:

$$f_n(x) = \begin{cases} 1 & n \le x < n+1 \\ 0 & \text{else} \end{cases}$$

For a fixed constant $x \ge 1$, $\{f_n\}$ is the sequence $\{0, 0, 0, \ldots, 0, 1, 0, \ldots, 0, 0, 0, \ldots\}$. The sole nonzero element comes in position k, where k is what we get when we round x *down* to a whole number. If $x < 1$, then the sequence consists of all zeroes.
Since we can plug in different values of x, we can think of $f_n(x)$ as a function of sequences: a different x gives you a different sequence. On the other hand, if we imagine fixing n, then $f_n(x)$ is just a function, where $f_n(x)$ gives the nth term in the sequence corresponding to x.

a Sketch the curve $y = f_2(x)$.

b Sketch the curve $y = f_3(x)$.

c Define $A_n = \int_0^\infty f_n(x)\,\mathrm{d}x$. Give a simple description of the sequence $\{A_n\}_{n=1}^\infty$.

d Evaluate $\lim_{n\to\infty} A_n$.

e Evaluate $\lim_{n\to\infty} f_n(x)$ for a constant x, and call the result $g(x)$.

f Evaluate $\displaystyle\int_0^\infty g(x)\,\mathrm{d}x$.

28. Determine the limit of the sequence $b_n = \left(1 + \dfrac{3}{n} + \dfrac{5}{n^2}\right)^n$.

29. A sequence $\left\{a_n\right\}_{n=1}^\infty$ of real numbers satisfies the recursion relation $a_{n+1} = \dfrac{a_n + 8}{3}$ for $n \ge 1$.

a Suppose $a_1 = 4$. What is $\lim_{n\to\infty} a_n$?

b Find x if $x = \dfrac{x+8}{3}$.

c Suppose $a_1 = 1$. Show that $\lim_{n\to\infty} a_n = L$, where L is the solution to equation above.

30. Zipf's Law applied to word frequency can be phrased as follows:

> The most-used word in a language is used n times as frequently as the n-th most word used in a language.

a Suppose the sequence $\{w_1, w_2, w_3, \ldots\}$ is a list of all words in a language, where w_n is the word that is the nth most frequently used. Let f_n be the frequency of word w_n. Is $\{f_1, f_2, f_3, \ldots\}$ an increasing sequence or a decreasing sequence?

b Give a general formula for f_n, treating f_1 as a constant.

c Suppose in a language, w_1 (the most frequently used word) has frequency 6%. If the language follows Zipf's Law, then what frequency does w_3 have?

d Suppose $f_6 = 0.3\%$ for a language following Zipf's law. What is f_{10}?

e The word "the" is the most-used word in contemporary American English. In a collection of about 450 million words, "the" appeared 22,038,615 times. The second-most used word is "be," followed by "and." About how many usages of these words do you expect in the same collection of 450 million words?

Sources:

- Zipf's word frequency law in natural language: A critical review and future directions, Steven T. Piantadosi. Psychon Bull Rev. 2014 Oct; 21(5): 1112–1130. Accessed online 11 October 2017.
- Word Frequency Data. Accessed online 11 October 2017.

3.2▲ Series

3.2.1 ▶▶ Series

A series is a sum

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

of infinitely many terms. In summation notation, it is written

$$\sum_{n=1}^{\infty} a_n$$

You already have a lot of experience with series, though you might not realise it. When you write a number using its decimal expansion you are really expressing it as a series. Perhaps the simplest example of this is the decimal expansion of $\frac{1}{3}$:

$$\frac{1}{3} = 0.3333\cdots$$

Recall that the expansion written in this way actually means

$$0.333333\cdots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots = \sum_{n=1}^{\infty} \frac{3}{10^n}$$

The summation index n is of course a dummy index. You can use any symbol you like (within reason) for the summation index.

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = \sum_{i=1}^{\infty} \frac{3}{10^i} = \sum_{j=1}^{\infty} \frac{3}{10^j} = \sum_{\ell=1}^{\infty} \frac{3}{10^\ell}$$

A series can be expressed using summation notation in many different ways. For example the following expressions all represent the same series:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{3}{10^n} &= \overbrace{\frac{3}{10}}^{n=1} + \overbrace{\frac{3}{100}}^{n=2} + \overbrace{\frac{3}{1000}}^{n=3} + \cdots \\
\sum_{j=2}^{\infty} \frac{3}{10^{j-1}} &= \overbrace{\frac{3}{10}}^{j=2} + \overbrace{\frac{3}{100}}^{j=3} + \overbrace{\frac{3}{1000}}^{j=4} + \cdots \\
\sum_{\ell=0}^{\infty} \frac{3}{10^{\ell+1}} &= \overbrace{\frac{3}{10}}^{\ell=0} + \overbrace{\frac{3}{100}}^{\ell=1} + \overbrace{\frac{3}{1000}}^{\ell=3} + \cdots \\
\frac{3}{10} + \sum_{n=2}^{\infty} \frac{3}{10^n} &= \frac{3}{10} + \overbrace{\frac{3}{100}}^{n=2} + \overbrace{\frac{3}{1000}}^{n=3} + \cdots
\end{aligned}$$

We can get from the first line to the second line by substituting $n = j - 1$ — don't forget to also change the limits of summation (so that $n = 1$ becomes $j - 1 = 1$ which is rewritten as $j = 2$). To get from the first line to the third line, substitute $n = \ell + 1$ everywhere, including in the limits of summation (so that $n = 1$ becomes $\ell + 1 = 1$ which is rewritten as $\ell = 0$).

Whenever you are in doubt as to what series a summation notation expression represents, it is a good habit to write out the first few terms, just as we did above.

Of course, at this point, it is not clear whether the sum of infinitely many terms adds up to a finite number or not. In order to make sense of this we will recast the problem in terms of the convergence of sequences (hence the discussion of the previous section). Before we proceed more formally let us illustrate the basic idea with a few simple examples.

Example 3.2.1 $\displaystyle\sum_{n=1}^{\infty} \frac{3}{10^n}$.

As we have just seen above the series $\sum_{n=1}^{\infty} \frac{3}{10^n}$ is

$$\sum_{n=1}^{\infty} \frac{3}{10^n} = \overbrace{\frac{3}{10}}^{n=1} + \overbrace{\frac{3}{100}}^{n=2} + \overbrace{\frac{3}{1000}}^{n=3} + \cdots$$

Notice that the n^{th} term in that sum is

$$3 \times 10^{-n} = 0.\overbrace{00\cdots0}^{n-1 \text{ zeroes}}3$$

So the sum of the first 5, 10, 15 and 20 terms in that series are

$$\sum_{n=1}^{5} \frac{3}{10^n} = 0.33333 \qquad \sum_{n=1}^{10} \frac{3}{10^n} = 0.3333333333$$
$$\sum_{n=1}^{15} \frac{3}{10^n} = 0.333333333333333 \qquad \sum_{n=1}^{20} \frac{3}{10^n} = 0.33333333333333333333$$

It sure looks like that, as we add more and more terms, we get closer and closer to $0.\dot{3} = \frac{1}{3}$. So it is very reasonable [a] to define $\sum_{n=1}^{\infty} \frac{3}{10^n}$ to be $\frac{1}{3}$.

a Of course we are free to define the series to be whatever we want. The hard part is defining it to be something that makes sense and doesn't lead to contradictions. We'll get to a more systematic definition shortly.

Example 3.2.2 $\displaystyle\sum_{n=1}^{\infty} 1$ and $\sum_{n=1}^{\infty}(-1)^n$.

Every term in the series $\sum_{n=1}^{\infty} 1$ is exactly 1. So the sum of the first N terms is exactly N. As we add more and more terms this grows unboundedly. So it is very reasonable to say that the series $\sum_{n=1}^{\infty} 1$ diverges.
The series

$$\sum_{n=1}^{\infty}(-1)^n = \overbrace{(-1)}^{n=1} + \overbrace{1}^{n=2} + \overbrace{(-1)}^{n=3} + \overbrace{1}^{n=4} + \overbrace{(-1)}^{n=5} + \cdots$$

So the sum of the first N terms is 0 if N is even and -1 if N is odd. As we add more and more terms from the series, the sum alternates between 0 and -1 for ever and ever. So the sum of all infinitely many terms does not make any sense and it is again reasonable to say that the series $\sum_{n=1}^{\infty}(-1)^n$ diverges.

Example 3.2.2

In the above examples we have tried to understand the series by examining the sum of the first few terms and then extrapolating as we add in more and more terms. That is, we tried to sneak up on the infinite sum by looking at the limit of (partial) sums of the first few terms. This approach can be made into a more formal rigorous definition. More precisely, to define what is meant by the infinite sum $\sum_{n=1}^{\infty} a_n$, we approximate it by the sum of its first N terms and then take the limit as N tends to infinity.

Definition 3.2.3

The N^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ is the sum of its first N terms

$$S_N = \sum_{n=1}^{N} a_n.$$

The partial sums form a sequence $\{S_N\}_{N=1}^{\infty}$. If this sequence of partial sums converges $S_N \to S$ as $N \to \infty$ then we say that the series $\sum_{n=1}^{\infty} a_n$ converges to S and we write

$$\sum_{n=1}^{\infty} a_n = S$$

If the sequence of partial sums diverges, we say that the series diverges.

Example 3.2.4 Geometric Series.

Let a and r be any two fixed real numbers with $a \neq 0$. The series

$$a + ar + ar^2 + \cdots + ar^n + \cdots = \sum_{n=0}^{\infty} ar^n$$

is called the geometric series with first term a and ratio r.
Notice that we have chosen to start the summation index at $n = 0$. That's fine. The first [a] term is the $n = 0$ term, which is $ar^0 = a$. The second term is the $n = 1$ term, which is $ar^1 = ar$. And so on. We could have also written the series $\sum_{n=1}^{\infty} ar^{n-1}$. That's exactly the same series — the first term is $ar^{n-1}\big|_{n=1} = ar^{1-1} = a$, the second term is $ar^{n-1}\big|_{n=2} = ar^{2-1} = ar$, and so on [b]. Regardless of how we write the geometric series, a is the first term and r is the ratio between successive terms.
Geometric series have the extremely useful property that there is a very simple formula for their partial sums. Denote the partial sum by

$$S_N = \sum_{n=0}^{N} ar^n = a + ar + ar^2 + \cdots + ar^N.$$

The secret to evaluating this sum is to see what happens when we multiply it by r:

$$\begin{aligned} rS_N &= r\big(a + ar + ar^2 + \cdots + ar^N\big) \\ &= ar + ar^2 + ar^3 + \cdots + ar^{N+1} \end{aligned}$$

Notice that this is almost the same [c] as S_N. The only differences are that the first term, a, is missing and one additional term, ar^{N+1}, has been tacked on the end. So

$$\begin{aligned} S_N &= a + ar + ar^2 + \cdots + ar^N \\ rS_N &= \qquad ar + ar^2 + \cdots + ar^N + ar^{N+1} \end{aligned}$$

Hence taking the difference of these expressions cancels almost all the terms:

$$(1-r)S_N = a - ar^{N+1} = a(1 - r^{N+1})$$

Provided $r \neq 1$ we can divide both side by $1 - r$ to isolate S_N:

$$S_N = a \cdot \frac{1 - r^{N+1}}{1 - r}.$$

On the other hand, if $r = 1$, then

$$S_N = \underbrace{a + a + \cdots + a}_{N+1 \text{ terms}} = a(N+1)$$

So in summary:

$$S_N = \begin{cases} a\frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

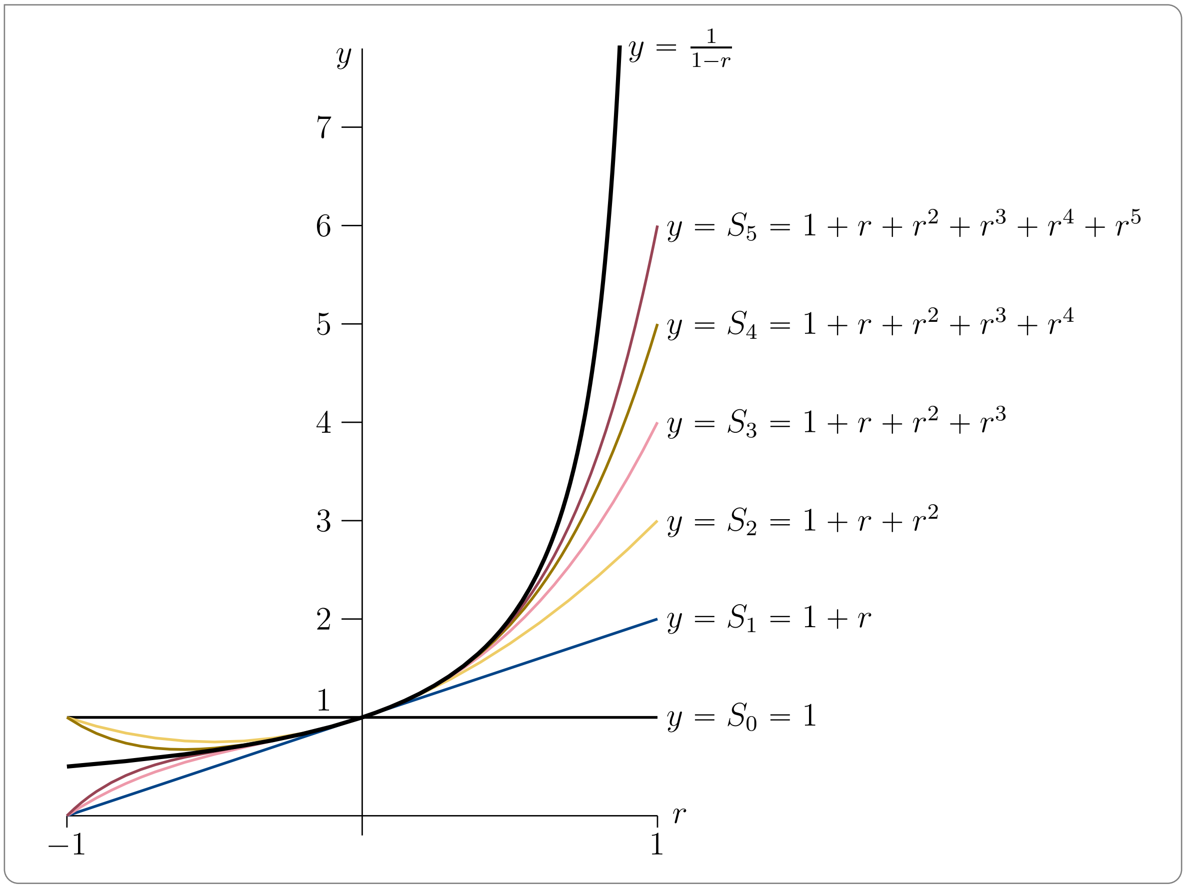
Now that we have this expression we can determine whether or not the series converges. If $|r| < 1$, then r^{N+1} tends to zero as $N \to \infty$, so that S_N converges to $\frac{a}{1-r}$ as $N \to \infty$ and

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{provided } |r| < 1.$$

On the other hand if $|r| \geq 1$, S_N diverges. To understand this divergence, consider the following 4 cases:

- If $r > 1$, then r^N grows to ∞ as $N \to \infty$.
- If $r < -1$, then the magnitude of r^N grows to ∞, and the sign of r^N oscillates between $+$ and $-$, as $N \to \infty$.
- If $r = +1$, then $N + 1$ grows to ∞ as $N \to \infty$.
- If $r = -1$, then r^N just oscillates between $+1$ and -1 as $N \to \infty$.

In each case the sequence of partial sums does not converge and so the series does not converge.
Here are some sketches of the graphs of $\frac{1}{1-r}$ and S_N, $0 \le N \le 5$, for $a = 1$ and $-1 \le r < 1$.



In these sketches we see that

- when $0 < r < 1$, and also when $-1 < r < 0$ with N odd, we have $S_N = \frac{1-r^{N+1}}{1-r} < \frac{1}{1-r}$. On the other hand, when $-1 < r < 0$ with N even, we have $S_N = \frac{1-r^{N+1}}{1-r} > \frac{1}{1-r}$.
- When $0 < |r| < 1$, $S_N = \frac{1-r^{N+1}}{1-r}$ gets closer and closer to $\frac{1}{1-r}$ as N increases.
- When $r = -1$, S_N just alternates between 0, when N is odd, and 1, when N is even.

a It is actually quite common in computer science to think of 0 as the first integer. In that context, the set of natural numbers is defined to contain 0: $\mathbb{N} = \{0, 1, 2, \cdots\}$ while the notation $\mathbb{Z}^+ = \{1, 2, 3, \cdots\}$ is used to denote the (strictly) positive integers. Remember that in this text, as is more standard in mathematics, we define the set of natural numbers to be the set of (strictly) positive integers.

b This reminds the authors of the paradox of Hilbert's hotel. The hotel with an infinite number of rooms is completely full, but can always accommodate one more guest. The interested reader should use their favourite search engine to find more information on this.

c One can find similar properties of other special series, that allow us, with some work, to cancel many terms in the partial sums. We will shortly see a good example of this. The interested reader should look up "creative telescoping" to see how this idea might be used more generally, though it is somewhat beyond this course.

Example 3.2.4

We should summarise the results in the previous example in a lemma.

Lemma 3.2.5 Geometric series.

Let a and r be real numbers and let $N \geq 0$ be an integer then

$$\sum_{n=0}^{N} ar^n = \begin{cases} a\dfrac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}.$$

Further, if $|r| < 1$ then

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

Now that we know how to handle geometric series let's return to Example 3.2.1.

Example 3.2.6 Decimal Expansions.

The decimal expansion

$$0.3333\cdots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots = \sum_{n=1}^{\infty} \frac{3}{10^n}$$

is a geometric series with the first term $a = \frac{3}{10}$ and the ratio $r = \frac{1}{10}$. So, by Lemma 3.2.5,

$$0.3333\cdots = \sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \frac{\frac{3}{10}}{\frac{9}{10}} = \frac{1}{3}$$

just as we would have expected.

We can push this idea further. Consider the repeating decimal expansion:

$$0.16161616\cdots = \frac{16}{100} + \frac{16}{10000} + \frac{16}{1000000} + \cdots$$

This is another geometric series with the first term $a = \frac{16}{100}$ and the ratio $r = \frac{1}{100}$. So, by Lemma 3.2.5,

$$0.16161616\cdots = \sum_{n=1}^{\infty} \frac{16}{100^n} = \frac{\frac{16}{100}}{1-\frac{1}{100}} = \frac{\frac{16}{100}}{\frac{99}{100}} = \frac{16}{99}$$

again, as expected. In this way any periodic decimal expansion converges to a ratio of two integers — that is, to a rational number [a].

Here is another more complicated example.

$$\begin{aligned} 0.1234343434\cdots &= \frac{12}{100} + \frac{34}{10000} + \frac{34}{1000000} + \cdots \\ &= \frac{12}{100} + \sum_{n=2}^{\infty} \frac{34}{100^n} \end{aligned}$$

Now apply Lemma 3.2.5 with $a = \frac{34}{100^2}$ and $r = \frac{1}{100}$

$$\begin{aligned} &= \frac{12}{100} + \frac{34}{10000}\frac{1}{1-\frac{1}{100}} \\ &= \frac{12}{100} + \frac{34}{10000}\frac{100}{99} \\ &= \frac{1222}{9900} \end{aligned}$$

a We have included a (more) formal proof of this fact in the optional §3.7 at the end of this chapter. Proving that a repeating decimal expansion gives a rational number isn't too hard. Proving the converse — that every rational number has a repeating decimal expansion is a little trickier, but we also do that in the same optional section.

Example 3.2.6

Typically, it is quite difficult to write down a neat closed form expression for the partial sums of a series. Geometric series are very notable exceptions to this. Another family of series for which we can write down partial sums is called "telescoping series". These series have the desirable property that many of the terms in the sum cancel each other out rendering the partial sums quite simple.

Example 3.2.7 Telescoping Series.

In this example, we are going to study the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. This is a rather artificial series [a] that has been rigged to illustrate a phenomenon called "telescoping". Notice that the n^{th} term can be rewritten as

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

and so we have

$$a_n = b_n - b_{n+1} \qquad \text{where } b_n = \frac{1}{n}.$$

Because of this we get big cancellations when we add terms together. This allows us to get a simple formula for the partial sums of this series.

$$\begin{aligned} S_N &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{N\cdot(N+1)} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \end{aligned}$$

The second term of each bracket exactly cancels the first term of the following bracket. So the sum "telescopes" leaving just

$$S_N = 1 - \frac{1}{N+1}$$

and we can now easily compute

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(1 - \frac{1}{N+1}\right) = 1$$

a Well... this sort of series does show up when you start to look at the Maclaurin polynomial of functions like $(1-x)\log(1-x)$. So it is not *totally* artificial. At any rate, it illustrates the basic idea of telescoping very nicely, and the idea of "creative telescoping" turns out to be extremely useful in the study of series — though it is well beyond the scope of this course.

Example 3.2.7

More generally, if we can write

$$a_n = b_n - b_{n+1}$$

for some other known sequence b_n, then the series telescopes and we can compute partial sums using

$$\begin{aligned} \sum_{n=1}^{N} a_n &= \sum_{n=1}^{N}(b_n - b_{n+1}) \\ &= \sum_{n=1}^{N} b_n - \sum_{n=1}^{N} b_{n+1} \\ &= b_1 - b_{N+1}. \end{aligned}$$

and hence

$$\sum_{n=1}^{\infty} a_n = b_1 - \lim_{N\to\infty} b_{N+1}$$

provided this limit exists.Often $\lim_{N\to\infty} b_{N+1} = 0$ and then $\sum_{n=1}^{\infty} a_n = b_1$. But this does not always happen. Here is an example.

Example 3.2.8 A Divergent Telescoping Series.

In this example, we are going to study the series $\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right)$. Let's start by just writing out the first few terms.

$$\begin{aligned}\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right) &= \overbrace{\log\left(1+\frac{1}{1}\right)}^{n=1} + \overbrace{\log\left(1+\frac{1}{2}\right)}^{n=2} + \overbrace{\log\left(1+\frac{1}{3}\right)}^{n=3} \\ &\qquad + \overbrace{\log\left(1+\frac{1}{4}\right)}^{n=4} + \cdots \\ &= \log(2) + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) \\ &\qquad + \log\left(\frac{5}{4}\right) + \cdots\end{aligned}$$

This is pretty suggestive since

$$\log(2) + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \log\left(\frac{5}{4}\right) = \log\left(2 \times \frac{3}{2} \times \frac{4}{3} \times \frac{5}{4}\right) = \log(5)$$

So let's try using this idea to compute the partial sum S_N:

$$\begin{aligned}S_N &= \sum_{n=1}^{N} \log\left(1+\frac{1}{n}\right) \\ &= \overbrace{\log\left(1+\frac{1}{1}\right)}^{n=1} + \overbrace{\log\left(1+\frac{1}{2}\right)}^{n=2} + \overbrace{\log\left(1+\frac{1}{3}\right)}^{n=3} + \cdots \\ &\qquad + \overbrace{\log\left(1+\frac{1}{N-1}\right)}^{n=N-1} + \overbrace{\log\left(1+\frac{1}{N}\right)}^{n=N} \\ &= \log(2) + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \cdots + \log\left(\frac{N}{N-1}\right) + \log\left(\frac{N+1}{N}\right) \\ &= \log\left(2 \times \frac{3}{2} \times \frac{4}{3} \times \cdots \times \frac{N}{N-1} \times \frac{N+1}{N}\right) \\ &= \log(N+1)\end{aligned}$$

Uh oh!

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \log(N+1) = +\infty$$

This telescoping series diverges! There is an important lesson here. Telescoping series *can diverge*. They do not always converge to b_1.

As was the case for limits, differentiation and antidifferentiation, we can compute more complicated series in terms of simpler ones by understanding how series interact with the usual operations of arithmetic. It is, perhaps, not so surprising that there are simple rules for addition and subtraction of series and for multiplication of a series by a constant. Unfortunately there are no simple general rules for computing products or ratios of series.

Theorem 3.2.9 Arithmetic of series.

Let A, B and C be real numbers and let the two series $\sum_{n=1}^\infty a_n$ and $\sum_{n=1}^\infty b_n$ converge to S and T respectively. That is, assume that

$$\sum_{n=1}^\infty a_n = S \qquad\qquad \sum_{n=1}^\infty b_n = T$$

Then the following hold.

a $\displaystyle\sum_{n=1}^\infty \big[a_n + b_n\big] = S+T$ and $\displaystyle\sum_{n=1}^\infty \big[a_n - b_n\big] = S-T$

b $\displaystyle\sum_{n=1}^\infty C a_n = CS.$

Example 3.2.10 $\sum_{n=1}^\infty \left(\frac{1}{7^n} + \frac{2}{n(n+1)}\right)$.

As a simple example of how we use the arithmetic of series Theorem 3.2.9, consider

$$\sum_{n=1}^\infty \left[\frac{1}{7^n} + \frac{2}{n(n+1)}\right]$$

We recognize that we know how to compute parts of this sum. We know that

$$\sum_{n=1}^\infty \frac{1}{7^n} = \frac{\frac{1}{7}}{1-\frac{1}{7}} = \frac{1}{6}$$

because it is a geometric series (Example 3.2.4 and Lemma 3.2.5) with first term $a = \frac{1}{7}$ and ratio $r = \frac{1}{7}$. And we know that

$$\sum_{n=1}^\infty \frac{1}{n(n+1)} = 1$$

by Example 3.2.7. We can now use Theorem 3.2.9 to build the specified "complicated" series out of these two "simple" pieces.

$$\sum_{n=1}^\infty \left[\frac{1}{7^n} + \frac{2}{n(n+1)}\right] = \sum_{n=1}^\infty \frac{1}{7^n} + \sum_{n=1}^\infty \frac{2}{n(n+1)} \qquad \text{by Theorem 3.2.9(a)}$$

$$= \sum_{n=1}^{\infty} \frac{1}{7^n} + 2\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \qquad \text{by Theorem 3.2.9(b)}$$
$$= \frac{1}{6} + 2 \cdot 1 = \frac{13}{6}$$

Example 3.2.10

3.2.2 ▸▸ Exercises

Exercises — Stage 1

1. Write out the first five partial sums corresponding to the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

You don't need to simplify the terms.

2. Every student who comes to class brings their instructor cookies, and leaves them on the instructor's desk. Let C_k be the total number of cookies on the instructor's desk after the kth student comes.
If $C_{11} = 20$, and $C_{10} = 17$, how many cookies did the 11th student bring to class?

3. Suppose the sequence of partial sums of the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is $\{S_N\} = \left\{\frac{N}{N+1}\right\}$.

a What is $\{a_n\}$?

b What is $\displaystyle\lim_{n\to\infty} a_n$?

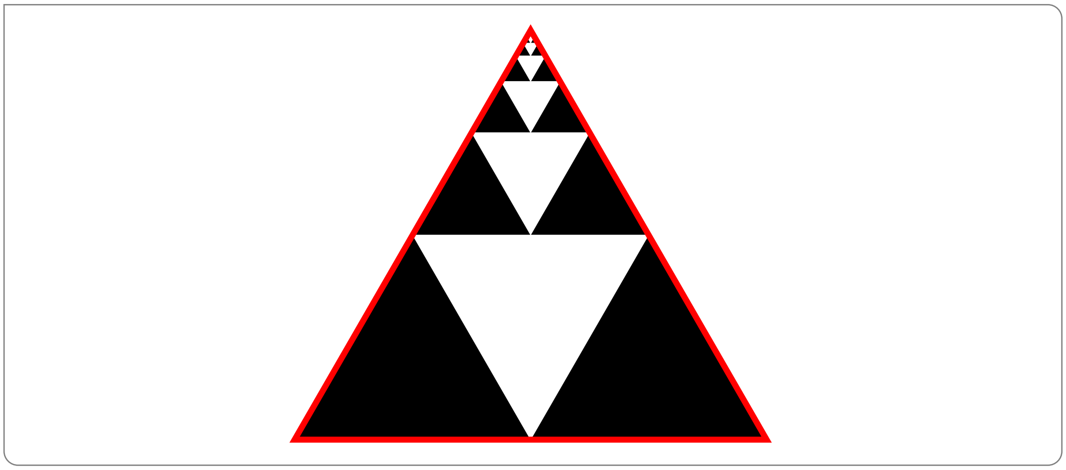
c Evaluate $\displaystyle\sum_{n=1}^{\infty} a_n$.

4. Suppose the sequence of partial sums of the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is $\{S_N\} = \left\{(-1)^N + \frac{1}{N}\right\}$.
What is $\{a_n\}$?

5. Let $f(N)$ be a formula for the Nth partial sum of $\sum_{n=1}^{\infty} a_n$. (That is, $f(N) = S_N$.) If $f'(N) < 0$ for all $N > 1$, what does that say about a_n?

Questions 6 through 8 invite you to explore geometric sums in a geometric way. This is complementary to than the algebraic method discussed in the text.

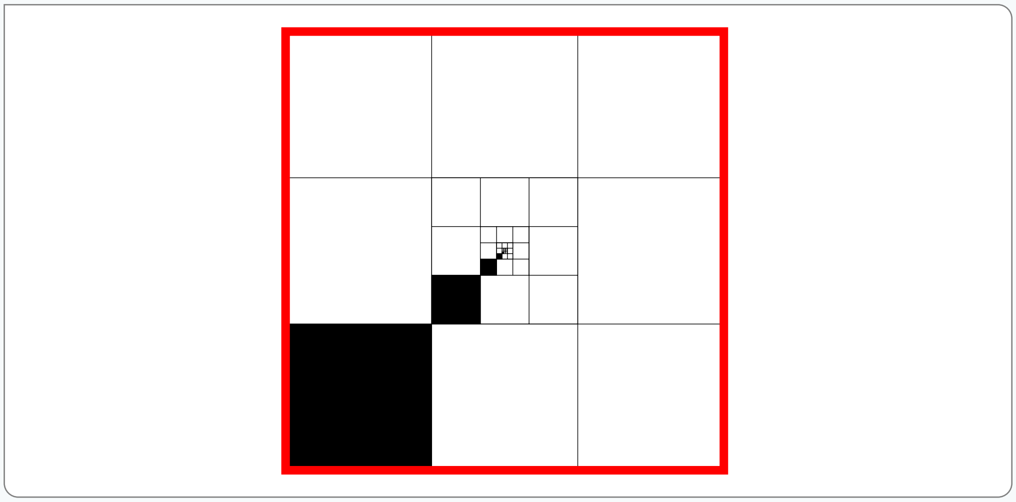
6. Suppose the triangle outlined in red in the picture below has area one.



a Express the combined area of the black triangles as a series, assuming the pattern continues forever.

b Evaluate the series using the picture (*not* the formula from your book).

7. Suppose the square outlined in red in the picture below has area one.



a Express the combined area of the black squares as a series, assuming the pattern continues forever.

b Evaluate the series using the picture (*not* the formula from your book).

8. In the style of Questions 6 and 7, draw a picture that represents $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^n}$ as an area.

9. Evaluate $\displaystyle\sum_{n=0}^{100}\frac{1}{5^n}$.

10. Every student who comes to class brings their instructor cookies, and leaves them on the instructor's desk. Let C_k be the total number of cookies on the instructor's desk after the kth student comes.
If $C_{20}=53$, and $C_{10}=17$, what does $C_{20}-C_{10}=36$ represent?

11. Evaluate $\displaystyle\sum_{n=50}^{100}\frac{1}{5^n}$. (Note the starting index.)

12.

a Starting on day $d=1$, every day you give your friend $\$\frac{1}{d+1}$, and they give $\$\frac{1}{d}$ back to you. After a long time, how much money have you gained by this arrangement?

b Evaluate $\displaystyle\sum_{d=1}^{\infty}\left(\frac{1}{d}-\frac{1}{(d+1)}\right)$.

c Starting on day $d=1$, every day your friend gives you $\$(d+1)$, and they take $\$(d+2)$ from you. After a long time, how much money have you gained by this arrangement?

d Evaluate $\displaystyle\sum_{d=1}^{\infty}((d+1)-(d+2))$.

13. Suppose $\displaystyle\sum_{n=1}^{\infty}a_n=A$, $\displaystyle\sum_{n=1}^{\infty}b_n=B$, and $\displaystyle\sum_{n=1}^{\infty}c_n=C$.

Evaluate $\displaystyle\sum_{n=1}^{\infty}(a_n+b_n+c_{n+1})$.

14. Suppose $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B \neq 0$, and $\sum_{n=1}^{\infty} c_n = C$.

True or false: $\sum_{n=1}^{\infty} \left(\frac{a_n}{b_n} + c_n\right) = \frac{A}{B} + C$.

Exercises — Stage 2

15. $*$. To what value does the series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots$ converge?

16. $*$. Evaluate $\sum_{k=7}^{\infty} \frac{1}{8^k}$

17. $*$. Show that the series $\sum_{k=1}^{\infty} \left(\frac{6}{k^2} - \frac{6}{(k+1)^2}\right)$ converges and find its limit.

18. $*$. Find the sum of the convergent series $\sum_{n=3}^{\infty} \left(\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+1}\right)\right)$.

19. $*$. The n^{th} partial sum of a series $\sum_{n=1}^{\infty} a_n$ is known to have the formula $s_n = \frac{1+3n}{5+4n}$.

a (a) Find an expression for a_n, valid for $n \geq 2$.

b (b) Show that the series $\sum_{n=1}^{\infty} a_n$ converges and find its value.

20. $*$. Find the sum of the series $\sum_{n=2}^{\infty} \frac{3 \cdot 4^{n+1}}{8 \cdot 5^n}$. Simplify your answer completely.

21. $*$. Relate the number $0.2\bar{3} = 0.233333\ldots$ to the sum of a geometric series, and use that to represent it as a rational number (a fraction or combination of fractions, with no decimals).

22. $*$. Express $2.656565\ldots$ as a rational number, i.e. in the form p/q where p and q are integers.

23. $*$. Express the decimal $0.\overline{321} = 0.321321321\ldots$ as a fraction.

24. ***.** Find the *value* of the convergent series

$$\sum_{n=2}^{\infty}\left(\frac{2^{n+1}}{3^n}+\frac{1}{2n-1}-\frac{1}{2n+1}\right)$$

Simplify your answer completely.

25. ***.** Evaluate

$$\sum_{n=1}^{\infty}\left[\left(\frac{1}{3}\right)^n+\left(-\frac{2}{5}\right)^{n-1}\right]$$

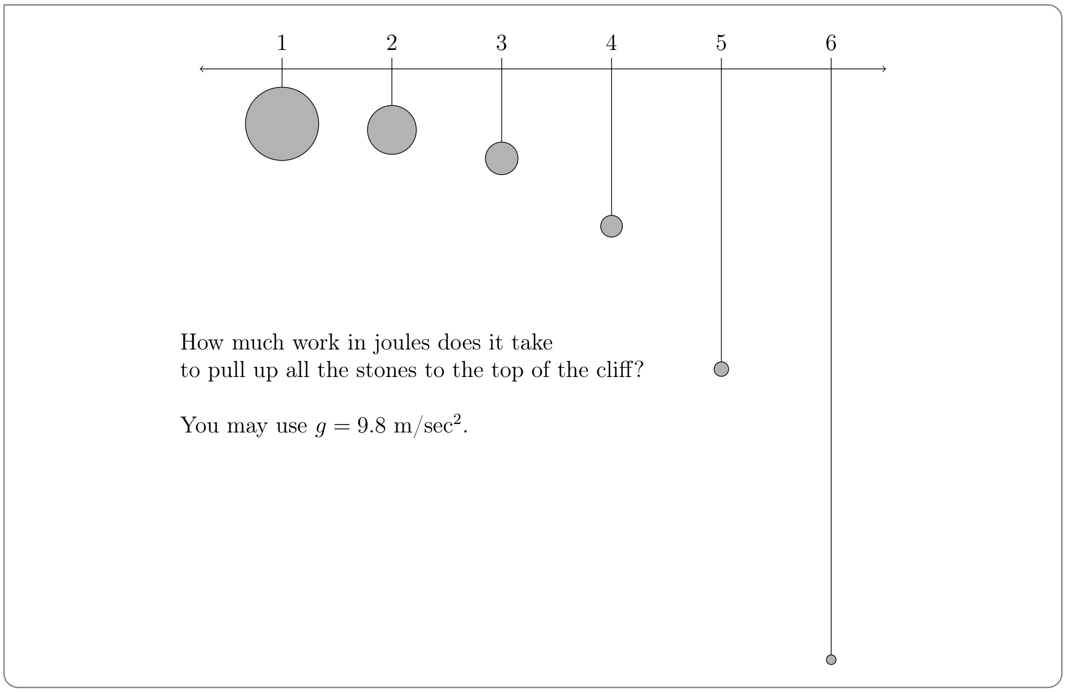
26. ***.** Find the sum of the series $\displaystyle\sum_{n=0}^{\infty}\frac{1+3^{n+1}}{4^n}$.

27. Evaluate $\displaystyle\sum_{n=5}^{\infty}\log\left(\frac{n-3}{n}\right)$.

28. Evaluate $\displaystyle\sum_{n=2}^{\infty}\left(\frac{2}{n}-\frac{1}{n+1}-\frac{1}{n-1}\right)$.

Exercises — Stage 3

29. An infinitely long, flat cliff has stones hanging off it, attached to thin wire of negligible mass. Starting at position $x=1$, every metre (at position x, where x is some whole number) the stone has mass $\frac{1}{4^x}$ kg and is hanging 2^x metres below the top of the cliff.

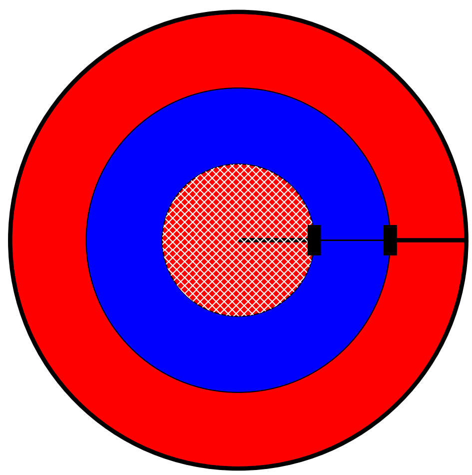


30. Find the combined volume of an infinite collection of spheres, where for each whole number $n = 1, 2, 3, \ldots$ there is exactly one sphere of radius $\dfrac{1}{\pi^n}$.

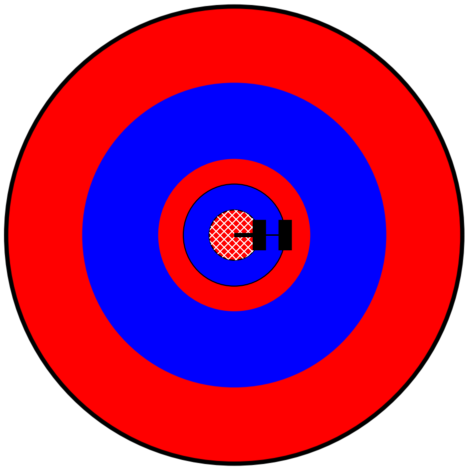
31. Evaluate $\displaystyle\sum_{n=3}^{\infty}\left(\frac{\sin^2 n}{2^n}+\frac{\cos^2(n+1)}{2^{n+1}}\right)$.

32. Suppose a series $\displaystyle\sum_{n=1}^{\infty} a_n$ has sequence of partial sums $\{S_N\}$, and the series $\displaystyle\sum_{N=1}^{\infty} S_N$ has sequence of partial sums $\{\mathbb{S}_M\} = \left\{\displaystyle\sum_{N=1}^{M} S_N\right\}$.
If $\mathbb{S}_M = \dfrac{M+1}{M}$, what is a_n?

33. Create a bullseye using the following method:
Starting with a red circle of area 1, divide the radius into thirds, creating two rings and a circle. Colour the middle ring blue.
Continue the pattern with the inside circle: divide its radius into thirds, and colour the middle ring blue.

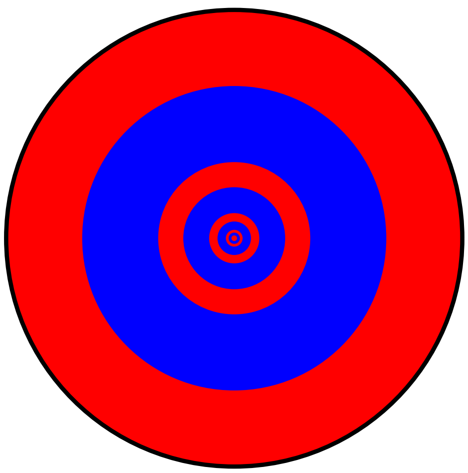


Step 1



Step 2

Continue in this way indefinitely: dividing the radius of the innermost circle into thirds, creating two rings and another circle, and colouring the middle ring blue.



What is the area of the red portion?

3.3▲ Convergence Tests

It is very common to encounter series for which it is difficult, or even virtually impossible, to determine the sum exactly. Often you try to evaluate the sum approximately by truncating it, i.e. having the index run only up to some finite N, rather than infinity.

But there is no point in doing so if the series diverges[1,2]. So you like to at least know if the series converges or diverges. Furthermore you would also like to know what error is introduced when you approximate $\sum_{n=1}^{\infty} a_n$ by the "truncated series" $\sum_{n=1}^{N} a_n$. That's called the truncation error. There are a number of "convergence tests" to help you with this.

3.3.1 ▸▸ The Divergence Test

Our first test is very easy to apply, but it is also rarely useful. It just allows us to quickly reject some "trivially divergent" series. It is based on the observation that

- by definition, a series $\sum_{n=1}^{\infty} a_n$ converges to S when the partial sums $S_N = \sum_{n=1}^{N} a_n$ converge to S.
- Then, as $N \to \infty$, we have $S_N \to S$ and, because $N - 1 \to \infty$ too, we also have $S_{N-1} \to S$.
- So $a_N = S_N - S_{N-1} \to S - S = 0$.

This tells us that, if we already know that a given series $\sum a_n$ is convergent, then the n^{th} term of the series, a_n, must converge to 0 as n tends to infinity. In this form, the test is not so useful. However the contrapositive [3] of the statement is a useful test for *divergence*.

1 The authors should be a little more careful making such a blanket statement. While it is true that it is not wise to approximate a divergent series by taking N terms with N large, there are cases when one can get a very good approximation by taking N terms with N small! For example, the Taylor remainder theorem shows us that when the n^{th} derivative of a function $f(x)$ grows very quickly with n, Taylor polynomials of degree N, with N large, can give bad approximations of $f(x)$, while the Taylor polynomials of degree one or two can still provide very good approximations of $f(x)$ when x is very small. As an example of this, one of the triumphs of quantum electrodynamics, namely the computation of the anomalous magnetic moment of the electron, depends on precisely this. A number of important quantities were predicted using the first few terms of divergent power series. When those quantities were measured experimentally, the predictions turned out to be incredibly accurate.

2 The field of asymptotic analysis often makes use of the first few terms of divergent series to generate approximate solutions to problems; this, along with numerical computations, is one of the most important techniques in applied mathematics. Indeed, there is a whole wonderful book (which, unfortunately, is too advanced for most Calculus 2 students) devoted to playing with divergent series called, unsurprisingly, "Divergent Series" by G.H. Hardy. This is not to be confused with the "Divergent" series by V. Roth set in a post-apocalyptic dystopian Chicago. That latter series diverges quite dramatically from mathematical topics, while the former does not have a film adaptation (yet).

3 We have discussed the contrapositive a few times in the CLP notes, but it doesn't hurt to discuss it again here (or for the reader to quickly look up the relevant footnote in Section 1.3 of the CLP-1 text). At any rate, given a statement of the form "If A is true, then B is true" the contrapositive is "If B is not true, then A is not true". The two statements in quotation marks are logically equivalent — if one is true, then so is the other. In the present context we have "If ($\sum a_n$ converges) then (a_n converges to 0)." The contrapositive of this statement is then "If (a_n does not converge to 0) then ($\sum a_n$ does not converge)."

Theorem 3.3.1 Divergence Test.

If the sequence $\{a_n\}_{n=1}^{\infty}$ fails to converge to zero as $n \to \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example 3.3.2 A simple divergence.

Let $a_n = \frac{n}{n+1}$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \neq 0$$

So the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

Warning 3.3.3

The divergence test is a "one way test". It tells us that if $\lim_{n\to\infty} a_n$ is nonzero, or fails to exist, then the series $\sum_{n=1}^{\infty} a_n$ diverges. But it tells us *absolutely nothing* when $\lim_{n\to\infty} a_n = 0$. In particular, it is perfectly possible for a series $\sum_{n=1}^{\infty} a_n$ to *diverge* even though $\lim_{n\to\infty} a_n = 0$. An example is $\sum_{n=1}^{\infty} \frac{1}{n}$. We'll show in Example 3.3.6, below, that it diverges.

Now while convergence or divergence of series like $\sum_{n=1}^{\infty} \frac{1}{n}$ can be determined using some clever tricks — see the optional §3.3.9 —, it would be much better to have methods that are more systematic and rely less on being sneaky. Over the next subsections we will discuss several methods for testing series for convergence.

Note that while these tests will tell us whether or not a series converges, they do not (except in rare cases) tell us what the series adds up to. For example, the test we will see in the next subsection tells us quite immediately that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges. However it does not tell us its value [4].

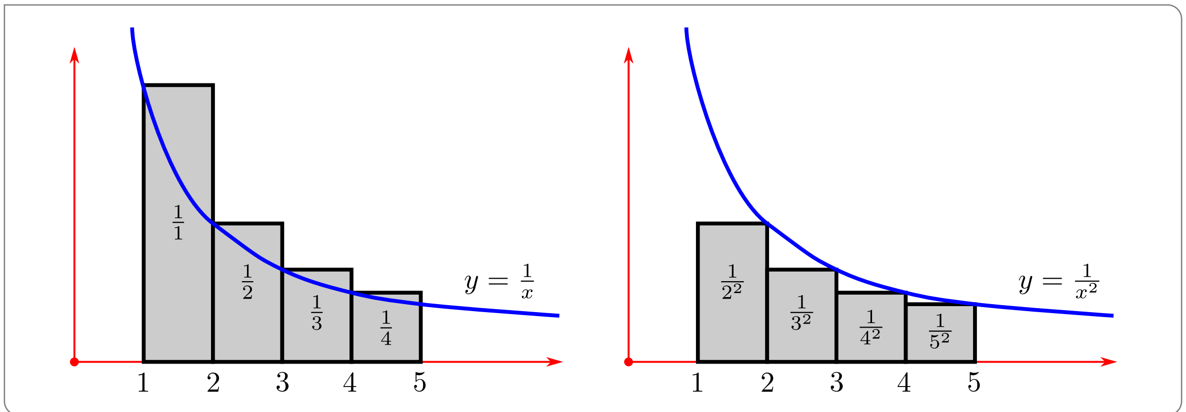
4 This series converges to Apéry's constant $1.2020569031\ldots$. The constant is named for Roger Apéry (1916–1994) who proved that this number must be irrational. This number appears in many contexts including the following cute fact — the reciprocal of Apéry's constant gives the probability that three positive integers, chosen at random, do not share a common prime factor.

3.3.2 ➞ The Integral Test

In the integral test, we think of a series $\sum_{n=1}^{\infty} a_n$, that we cannot evaluate explicitly, as the area of a union of rectangles, with a_n representing the area of a rectangle of width one and height a_n. Then we compare that area with the area represented by an integral, that we can evaluate explicitly, much as we did in Theorem 1.12.17, the comparison test for improper integrals. We'll start with a simple example, to illustrate the idea. Then we'll move on to a formulation of the test in general.

Example 3.3.4 Convergence of the harmonic series.

Visualise the terms of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ as a bar graph — each term is a rectangle of height $\frac{1}{n}$ and width 1. The limit of the series is then the limiting area of this union of rectangles. Consider the sketch on the left below.



It shows that the area of the shaded columns, $\sum_{n=1}^{4} \frac{1}{n}$, is bigger than the area under the curve $y = \frac{1}{x}$ with $1 \le x \le 5$. That is

$$\sum_{n=1}^{4} \frac{1}{n} \ge \int_1^5 \frac{1}{x} \mathrm{d}x$$

If we were to continue drawing the columns all the way out to infinity, then we would have

$$\sum_{n=1}^{\infty} \frac{1}{n} \ge \int_1^{\infty} \frac{1}{x} \mathrm{d}x$$

We are able to compute this improper integral exactly:

$$\int_1^{\infty} \frac{1}{x} \mathrm{d}x = \lim_{R\to\infty} \Big[\log|x| \Big]_1^R = +\infty$$

That is the area under the curve diverges to $+\infty$ and so the area represented by the columns must also diverge to $+\infty$.
It should be clear that the above argument can be quite easily generalised. For example the same argument holds *mutatis mutandis* [a] for the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Indeed we see from the sketch on the right above that

$$\sum_{n=2}^{N} \frac{1}{n^2} \leq \int_1^N \frac{1}{x^2} \mathrm{d}x$$

and hence

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \leq \int_1^{\infty} \frac{1}{x^2} \mathrm{d}x$$

This last improper integral is easy to evaluate:

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2} \mathrm{d}x &= \lim_{R \to \infty} \left[-\frac{1}{x} \right]_1^R \\ &= \lim_{R \to \infty} \left(\frac{1}{1} - \frac{1}{R} \right) = 1 \end{aligned}$$

Thus we know that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \leq 2.$$

and so the series must converge.

a Latin for "Once the necessary changes are made". This phrase still gets used a little, but these days mathematicians tend to write something equivalent in English. Indeed, English is pretty much the *lingua franca* for mathematical publishing. *Quidquid erit.*

Example 3.3.4

The above arguments are formalised in the following theorem.

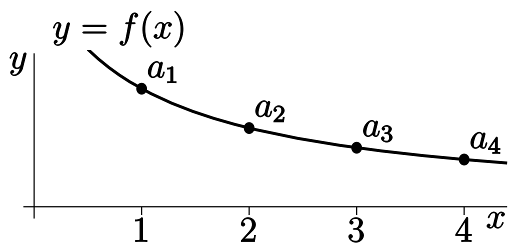
Theorem 3.3.5 The Integral Test.

Let N_0 be any natural number. If $f(x)$ is a function which is defined and continuous for all $x \geq N_0$ and which obeys

i $f(x) \geq 0$ for all $x \geq N_0$ and

ii $f(x)$ decreases as x increases and

iii $f(n) = a_n$ for all $n \geq N_0$.



Then

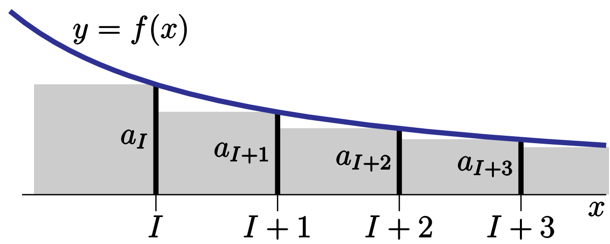
$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_{N_0}^{\infty} f(x)\,\mathrm{d}x \text{ converges}$$

Furthermore, when the series converges, the truncation error

$$\left|\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n\right| \leq \int_{N}^{\infty} f(x)\,\mathrm{d}x \qquad \text{for all } N \geq N_0$$

Proof. Let I be any fixed integer with $I > N_0$. Then

- $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=I}^{\infty} a_n$ converges — removing a fixed finite number of terms from a series cannot impact whether or not it converges.
- Since $a_n \geq 0$ for all $n \geq I > N_0$, the sequence of partial sums $s_\ell = \sum_{n=I}^{\ell} a_n$ obeys $s_{\ell+1} = s_\ell + a_{n+1} \geq s_\ell$. That is, s_ℓ increases as ℓ increases.
- So $\{s_\ell\}$ must either converge to some finite number or increase to infinity. That is, either $\sum_{n=I}^{\infty} a_n$ converges to a finite number or it is $+\infty$.



Look at the figure above. The shaded area in the figure is $\sum_{n=I}^{\infty} a_n$ because

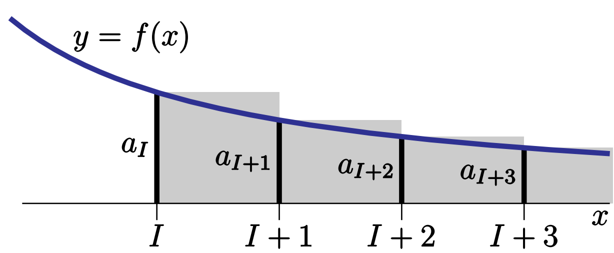
- the first shaded rectangle has height a_I and width 1, and hence area a_I and
- the second shaded rectangle has height a_{I+1} and width 1, and hence area a_{I+1}, and so on

This shaded area is smaller than the area under the curve $y = f(x)$ for $I - 1 \leq x < \infty$. So

$$\sum_{n=I}^{\infty} a_n \leq \int_{I-1}^{\infty} f(x)\ \mathrm{d}x$$

and, if the integral is finite, the sum $\sum_{n=I}^{\infty} a_n$ is finite too. Furthermore, the desired bound on the truncation error is just the special case of this inequality with $I = N + 1$:

$$\sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n = \sum_{n=N+1}^{\infty} a_n \leq \int_{N}^{\infty} f(x)\ \mathrm{d}x$$



For the "divergence case" look at the figure above. The (new) shaded area in the figure is again $\sum_{n=I}^{\infty} a_n$ because

- the first shaded rectangle has height a_I and width 1, and hence area a_I and
- the second shaded rectangle has height a_{I+1} and width 1, and hence area a_{I+1}, and so on

This time the shaded area is larger than the area under the curve $y = f(x)$ for $I \leq x < \infty$. So

$$\sum_{n=I}^{\infty} a_n \geq \int_{I}^{\infty} f(x)\ \mathrm{d}x$$

and, if the integral is infinite, the sum $\sum_{n=I}^{\infty} a_n$ is infinite too.

□

Now that we have the integral test, it is straightforward to determine for which values of p the series [5]

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges.

Example 3.3.6 The p test: $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$.

Let $p > 0$. We'll now use the integral test to determine whether or not the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (which is sometimes called the p-series) converges.

- To do so, we need a function $f(x)$ that obeys $f(n) = a_n = \frac{1}{n^p}$ for all n bigger than some N_0. Certainly $f(x) = \frac{1}{x^p}$ obeys $f(n) = \frac{1}{n^p}$ for all $n \geq 1$. So let's pick this f and try $N_0 = 1$. (We can always increase N_0 later if we need to.)
- This function also obeys the other two conditions of Theorem 3.3.5:
  - i $f(x) > 0$ for all $x \geq N_0 = 1$ and
  - ii $f(x)$ decreases as x increases because $f'(x) = -p\frac{1}{x^{p+1}} < 0$ for all $x \geq N_0 = 1$.
- So the integral test tells us that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if the integral $\int_1^{\infty} \frac{\mathrm{d}x}{x^p}$ converges.
- We have already seen, in Example 1.12.8, that the integral $\int_1^{\infty} \frac{\mathrm{d}x}{x^p}$ converges if and only if $p > 1$.

So we conclude that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. This is sometimes called the p-test.

- In particular, the series $\sum_{n=1}^{\infty} \frac{1}{n}$, which is called the harmonic series, has $p = 1$ and so diverges. As we add more and more terms of this series together, the terms we add, namely $\frac{1}{n}$, get smaller and smaller and tend to zero, but they tend to zero so slowly that the full sum is still infinite.
- On the other hand, the series $\sum_{n=1}^{\infty} \frac{1}{n^{1.000001}}$ has $p = 1.000001 > 1$ and so converges. This time as we add more and more terms of this series together, the terms we add, namely $\frac{1}{n^{1.000001}}$, tend to zero (just) fast enough that the full sum is finite. Mind you, for this example, the convergence takes place very slowly — you have to take a huge number of terms to get a decent approximation to the full sum. If

5 This series, viewed as a function of p, is called the Riemann zeta function, $\zeta(p)$, or the Euler-Riemann zeta function. It is extremely important because of its connections to prime numbers (among many other things). Indeed Euler proved that $\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} = \prod_{\mathrm{P\ prime}} (1 - \mathrm{P}^{-p})^{-1}$. Riemann showed the connections between the zeros of this function (over complex numbers p) and the distribution of prime numbers. Arguably the most famous unsolved problem in mathematics, the Riemann hypothesis, concerns the locations of zeros of this function.

we approximate $\sum_{n=1}^{\infty} \frac{1}{n^{1.000001}}$ by the truncated series $\sum_{n=1}^{N} \frac{1}{n^{1.000001}}$, we make an error of at most

$$\begin{aligned} \int_N^{\infty} \frac{dx}{x^{1.000001}} &= \lim_{R\to\infty} \int_N^R \frac{dx}{x^{1.000001}} \\ &= \lim_{R\to\infty} -\frac{1}{0.000001}\left[\frac{1}{R^{0.000001}} - \frac{1}{N^{0.000001}}\right] \\ &= \frac{10^6}{N^{0.000001}} \end{aligned}$$

This does tend to zero as $N \to \infty$, but really slowly.

Example 3.3.6

We now know that the dividing line between convergence and divergence of $\sum_{n=1}^{\infty} \frac{1}{n^p}$ occurs at $p = 1$. We can dig a little deeper and ask ourselves how much more quickly than $\frac{1}{n}$ the n^{th} term needs to shrink in order for the series to converge. We know that for large x, the function $\log x$ is smaller than x^a for any positive a — you can convince yourself of this with a quick application of L'Hôpital's rule. So it is not unreasonable to ask whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n \log n}$$

converges. Notice that we sum from $n = 2$ because when $n = 1, n \log n = 0$. And we don't need to stop there [6]. We can analyse the convergence of this sum with any power of $\log n$.

Example 3.3.7 $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$.

Let $p > 0$. We'll now use the integral test to determine whether or not the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges.

- As in the last example, we start by choosing a function that obeys $f(n) = a_n = \frac{1}{n(\log n)^p}$ for all n bigger than some N_0. Certainly $f(x) = \frac{1}{x(\log x)^p}$ obeys $f(n) = \frac{1}{n(\log n)^p}$ for all $n \geq 2$. So let's use that f and try $N_0 = 2$.
- Now let's check the other two conditions of Theorem 3.3.5:

  i Both x and $\log x$ are positive for all $x > 1$, so $f(x) > 0$ for all $x \geq N_0 = 2$.

  ii As x increases both x and $\log x$ increase and so $x(\log x)^p$ increases and $f(x)$ decreases.

6 We could go even further and see what happens if we include powers of $\log(\log(n))$ and other more exotic slow growing functions.

- So the integral test tells us that the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if and only if the integral $\int_2^\infty \frac{\mathrm{d}x}{x(\log x)^p}$ converges.
- To test the convergence of the integral, we make the substitution $u = \log x$, $\mathrm{d}u = \frac{\mathrm{d}x}{x}$.

$$\int_2^R \frac{\mathrm{d}x}{x(\log x)^p} = \int_{\log 2}^{\log R} \frac{\mathrm{d}u}{u^p}$$

  We already know that the integral $\int_1^\infty \frac{\mathrm{d}u}{u^p}$, and hence the integral $\int_2^R \frac{\mathrm{d}x}{x(\log x)^p}$, converges if and only if $p > 1$.

So we conclude that $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if and only if $p > 1$.

Example 3.3.7

3.3.3 ►► The Comparison Test

Our next convergence test is the comparison test. It is much like the comparison test for improper integrals (see Theorem 1.12.17) and is true for much the same reasons. The rough idea is quite simple. A sum of larger terms must be bigger than a sum of smaller terms. So if we know the big sum converges, then the small sum must converge too. On the other hand, if we know the small sum diverges, then the big sum must also diverge. Formalising this idea gives the following theorem.

Theorem 3.3.8 The Comparison Test.

Let N_0 be a natural number and let $K > 0$.

a If $|a_n| \le K c_n$ for all $n \ge N_0$ and $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

b If $a_n \ge K d_n \ge 0$ for all $n \ge N_0$ and $\sum_{n=0}^{\infty} d_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.

"Proof". We will not prove this theorem here. We'll just observe that it is very reasonable. That's why there are quotation marks around "Proof". For an actual proof see the optional section 3.3.10.

a If $\sum_{n=0}^{\infty} c_n$ converges to a finite number and if the terms in $\sum_{n=0}^{\infty} a_n$ are smaller than the terms in $\sum\limits_{n=0}^{\infty} c_n$, then it is no surprise that $\sum\limits_{n=0}^{\infty} a_n$ converges too.

b If $\sum\limits_{n=0}^{\infty} d_n$ diverges (i.e. adds up to ∞) and if the terms in $\sum\limits_{n=0}^{\infty} a_n$ are larger than the terms in $\sum\limits_{n=0}^{\infty} d_n$, then of course $\sum\limits_{n=0}^{\infty} a_n$ adds up to ∞, and so diverges, too.

$\square$

The comparison test for series is also used in much the same way as is the comparison test for improper integrals. Of course, one needs a good series to compare against, and often the series $\sum n^{-p}$ (from Example 3.3.6), for some $p > 0$, turns out to be just what is needed.

Example 3.3.9 $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+3}$.

We could determine whether or not the series $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+3}$ converges by applying the integral test. But it is not worth the effort [a]. Whether or not any series converges is determined by the behaviour of the summand [b] for very large n. So the first step in tackling such a problem is to develop some intuition about the behaviour of a_n when n is very large.

- *Step 1: Develop intuition.* In this case, when n is very large [c] $n^2 \gg 2n \gg 3$ so that $\frac{1}{n^2+2n+3} \approx \frac{1}{n^2}$. We already know, from Example 3.3.6, that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. So $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which has $p = 2$, converges, and we would expect that $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+3}$ converges too.
- *Step 2: Verify intuition.* We can use the comparison test to confirm that this is indeed the case. For any $n \geq 1$, $n^2 + 2n + 3 > n^2$, so that $\frac{1}{n^2+2n+3} \leq \frac{1}{n^2}$. So the comparison test, Theorem 3.3.8, with $a_n = \frac{1}{n^2+2n+3}$ and $c_n = \frac{1}{n^2}$, tells us that $\sum_{n=1}^{\infty} \frac{1}{n^2+2n+3}$ converges.

a Go back and quickly scan Theorem 3.3.5; to apply it we need to show that $\frac{1}{n^2+2n+3}$ is positive and decreasing (it is), and then we need to integrate $\int \frac{1}{x^2+2x+3}\mathrm{d}x$. To do that we reread the notes on partial fractions, then rewrite $x^2+2x+3 = (x+1)^2+2$ and so $\int_1^{\infty} \frac{1}{x^2+2x+3}\mathrm{d}x = \int_1^{\infty} \frac{1}{(x+1)^2+2}\mathrm{d}x \cdots$ and then arctangent appears, etc etc. Urgh. Okay — let's go back to the text now and see how to avoid this.

b To understand this consider any series $\sum_{n=1}^{\infty} a_n$. We can always cut such a series into two parts — pick some huge number like 10^6. Then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{10^6} a_n + \sum_{n=10^6+1}^{\infty} a_n$. The first sum, though it could be humongous, is finite. So the left hand side, $\sum_{n=1}^{\infty} a_n$, is a well-defined finite number if and only if $\sum_{n=10^6+1}^{\infty} a_n$, is a well-defined finite number. The convergence or divergence of the series is determined by the second sum, which only contains a_n for "large" n.

c The symbol "$\gg$" means "much larger than". Similarly, the symbol "$\ll$" means "much less than". Good shorthand symbols can be quite expressive.

Of course the previous example was "rigged" to give an easy application of the comparison test. It is often relatively easy, using arguments like those in Example 3.3.9, to find a "simple" series $\sum_{n=1}^{\infty} b_n$ with b_n almost the same as a_n when n is large. However it is pretty rare that $a_n \le b_n$ for all n. It is much more common that $a_n \le K b_n$ for some constant K. This is enough to allow application of the comparison test. Here is an example.

Example 3.3.10 $\sum_{n=1}^{\infty} \frac{n+\cos n}{n^3-1/3}$.

As in the previous example, the first step is to develop some intuition about the behaviour of a_n when n is very large.

- *Step 1: Develop intuition.* When n is very large,
  - $n \gg |\cos n|$ so that the numerator $n + \cos n \approx n$ and
  - $n^3 \gg \frac{1}{3}$ so that the denominator $n^3 - \frac{1}{3} \approx n^3$.

  So when n is very large

  $$a_n = \frac{n+\cos n}{n^3 - \frac{1}{3}} \approx \frac{n}{n^3} = \frac{1}{n^2}$$

  We already know from Example 3.3.6, with $p = 2$, that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so we would expect that $\sum_{n=1}^{\infty} \frac{n+\cos n}{n^3-\frac{1}{3}}$ converges too.

- *Step 2: Verify intuition.* We can use the comparison test to confirm that this is indeed the case. To do so we need to find a constant K such that $|a_n| = \frac{|n+\cos n|}{n^3-1/3} = \frac{n+\cos n}{n^3-1/3}$ is smaller than $\frac{K}{n^2}$ for all n. A good way [a] to do that is to factor the dominant term (in this case n) out of the numerator and also factor the dominant term (in this case n^3) out of the denominator.

  $$a_n = \frac{n+\cos n}{n^3 - \frac{1}{3}} = \frac{n}{n^3}\,\frac{1+\frac{\cos n}{n}}{1-\frac{1}{3n^3}} = \frac{1}{n^2}\,\frac{1+\frac{\cos n}{n}}{1-\frac{1}{3n^3}}$$

  So now we need to find a constant K such that $\frac{1+\frac{(\cos n)}{n}}{1-\frac{1}{3n^3}}$ is smaller than K for all $n \ge 1$.

  - First consider the numerator $1 + (\cos n)\frac{1}{n}$. For all $n \ge 1$
    - $\frac{1}{n} \le 1$ and
    - $|\cos n| \le 1$

    So the numerator $1 + (\cos n)\frac{1}{n}$ is always smaller than $1 + (1)\frac{1}{1} = 2$.
  - Next consider the denominator $1 - \frac{1}{3n^3}$.
    - When $n \ge 1$, $\frac{1}{3n^3}$ lies between $\frac{1}{3}$ and 0 so that
    - $1 - \frac{1}{3n^3}$ is between $\frac{2}{3}$ and 1 and consequently

- $\frac{1}{1-\frac{1}{3n^3}}$ is between $\frac{3}{2}$ and 1.

◦ As the numerator $1+(\cos n)\frac{1}{n}$ is always smaller than 2 and $\frac{1}{1-\frac{1}{3n^3}}$ is always smaller than $\frac{3}{2}$, the fraction

$$\frac{1+\frac{\cos n}{n}}{1-\frac{1}{3n^3}} \le 2\left(\frac{3}{2}\right) = 3$$

We now know that

$$|a_n| = \frac{1}{n^2}\,\frac{1+\frac{2}{n}}{1-\frac{1}{3n^3}} \le \frac{3}{n^2}$$

and, since we know $\sum_{n=1}^{\infty} n^{-2}$ converges, the comparison test tells us that $\sum_{n=1}^{\infty}\frac{n+\cos n}{n^3-1/3}$ converges.

a This is very similar to how we computed limits at infinity way way back near the beginning of CLP-1.

Example 3.3.10

The last example was actually a relatively simple application of the comparison theorem — finding a suitable constant K can be *really* tedious [7]. Fortunately, there is a variant of the comparison test that completely eliminates the need to explicitly find K.

The idea behind this isn't too complicated. We have already seen that the convergence or divergence of a series depends not on its first few terms, but just on what happens when n is really large. Consequently, if we can work out how the series terms behave for really big n then we can work out if the series converges. So instead of comparing the terms of our series for all n, just compare them when n is big.

Theorem 3.3.11 Limit Comparison Theorem.

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with $b_n > 0$ for all n. Assume that

$$\lim_{n\to\infty}\frac{a_n}{b_n} = L$$

exists.

a If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges too.

b If $L \ne 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges too.

In particular, if $L \ne 0$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

7 Really, really tedious. And you thought some of those partial fractions computations were bad ...

Proof. (a) Because we are told that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, we know that,

- when n is large, $\frac{a_n}{b_n}$ is very close to L, so that $\left|\frac{a_n}{b_n}\right|$ is very close to $|L|$.
- In particular, there is some natural number N_0 so that $\left|\frac{a_n}{b_n}\right| \le |L| + 1$, for all $n \ge N_0$, and hence
- $|a_n| \le K b_n$ with $K = |L| + 1$, for all $n \ge N_0$.
- The comparison Theorem 3.3.8 now implies that $\sum_{n=1}^{\infty} a_n$ converges.

(b) Let's suppose that $L > 0$. (If $L < 0$, just replace a_n with $-a_n$.) Because we are told that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, we know that,

- when n is large, $\frac{a_n}{b_n}$ is very close to L.
- In particular, there is some natural number N so that $\frac{a_n}{b_n} \ge \frac{L}{2}$, and hence
- $a_n \ge K b_n$ with $K = \frac{L}{2} > 0$, for all $n \ge N$.
- The comparison Theorem 3.3.8 now implies that $\sum_{n=1}^{\infty} a_n$ diverges.

$\square$

The next two examples illustrate how much of an improvement the above theorem is over the straight comparison test (though of course, we needed the comparison test to develop the limit comparison test).

Example 3.3.12 $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2-2n+3}$.

Set $a_n = \frac{\sqrt{n+1}}{n^2-2n+3}$. We first try to develop some intuition about the behaviour of a_n for large n and then we confirm that our intuition was correct.

- *Step 1: Develop intuition.* When $n \gg 1$, the numerator $\sqrt{n+1} \approx \sqrt{n}$, and the denominator $n^2 - 2n + 3 \approx n^2$ so that $a_n \approx \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$ and it looks like our series should converge by Example 3.3.6 with $p = \frac{3}{2}$.
- *Step 2: Verify intuition.* To confirm our intuition we set $b_n = \frac{1}{n^{3/2}}$ and compute the limit
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n+1}}{n^2-2n+3}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n^{3/2}\sqrt{n+1}}{n^2-2n+3}$$
Again it is a good idea to factor the dominant term out of the numerator and the dominant term out of the denominator.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2\sqrt{1+\frac{1}{n}}}{n^2\left(1-\frac{2}{n}+\frac{3}{n^2}\right)} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n}}}{1-\frac{2}{n}+\frac{3}{n^2}} = 1$$

We already know that the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by Example 3.3.6 with $p = \frac{3}{2}$. So our series converges by the limit comparison test, Theorem 3.3.11.

Example 3.3.12

Example 3.3.13 $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2-2n+3}$, again.

We can also try to deal with the series of Example 3.3.12, using the comparison test directly. But that requires us to find K so that

$$\frac{\sqrt{n+1}}{n^2-2n+3} \le \frac{K}{n^{3/2}}$$

We might do this by examining the numerator and denominator separately:

- The numerator isn't too bad since for all $n \ge 1$:

$$\begin{aligned} n+1 &\le 2n \qquad \text{and so} \\ \sqrt{n+1} &\le \sqrt{2n} \end{aligned}$$

- The denominator is quite a bit more tricky, since we need a *lower* bound, rather than an upper bound, and we cannot just write $|n^2 - 2n + 3| \ge n^2$, which is false. Instead we have to make a more careful argument. In particular, we'd like to find N_0 and K' so that $n^2 - 2n + 3 \ge K'n^2$, i.e. $\frac{1}{n^2-2n+3} \le \frac{1}{K'n^2}$ for all $n \ge N_0$. For $n \ge 4$, we have $2n = \frac{1}{2}4n \le \frac{1}{2}n \cdot n = \frac{1}{2}n^2$. So for $n \ge 4$,

$$n^2 - 2n + 3 \ge n^2 - \frac{1}{2}n^2 + 3 \ge \frac{1}{2}n^2$$

Putting the numerator and denominator back together we have

$$\frac{\sqrt{n+1}}{n^2-2n+3} \le \frac{\sqrt{2n}}{n^2/2} = 2\sqrt{2}\frac{1}{n^{3/2}} \qquad \text{for all } n \ge 4$$

and the comparison test then tells us that our series converges. It is pretty clear that the approach of Example 3.3.12 was much more straightforward.

3.3.4 ▸▸ The Alternating Series Test

When the signs of successive terms in a series alternate between $+$ and $-$, like for example in $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, the series is called an *alternating series.* More generally, the series

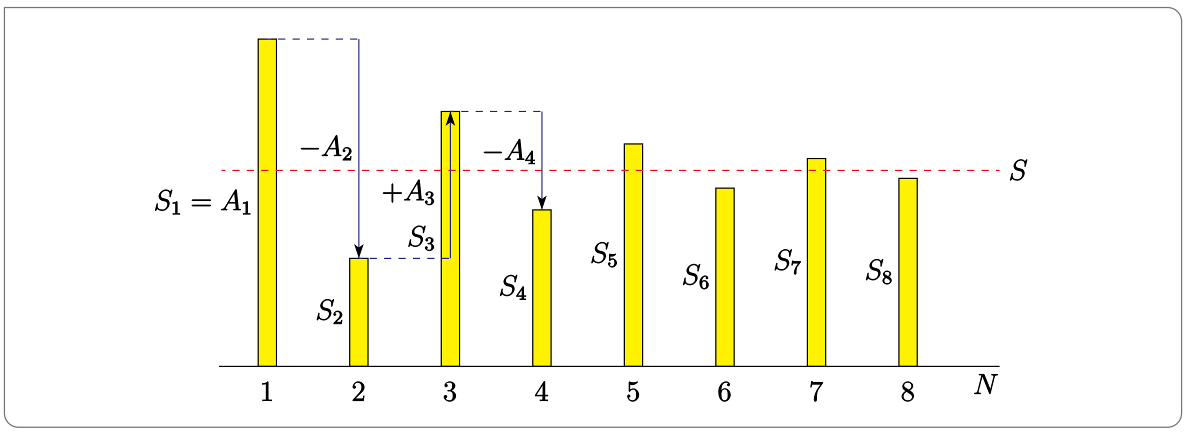
$$A_1 - A_2 + A_3 - A_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} A_n$$

is alternating if every $A_n \geq 0$. Often (but not always) the terms in alternating series get successively smaller. That is, then $A_1 \geq A_2 \geq A_3 \geq \cdots$. In this case:

- The first partial sum is $S_1 = A_1$.
- The second partial sum, $S_2 = A_1 - A_2$, is smaller than S_1 by A_2.
- The third partial sum, $S_3 = S_2 + A_3$, is bigger than S_2 by A_3, but because $A_3 \leq A_2$, S_3 remains smaller than S_1. See the figure below.
- The fourth partial sum, $S_4 = S_3 - A_4$, is smaller than S_3 by A_4, but because $A_4 \leq A_3$, S_4 remains bigger than S_2. Again, see the figure below.
- And so on.

So the successive partial sums oscillate, but with ever decreasing amplitude. If, in addition, A_n tends to 0 as n tends to ∞, the amplitude of oscillation tends to zero and the sequence $S_1, S_2, S_3, \cdots$ converges to some limit S.

This is illustrated in the figure



Here is a convergence test for alternating series that exploits this structure, and that is really easy to apply.

Theorem 3.3.14 Alternating Series Test.

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

i $A_n \geq 0$ for all $n \geq 1$ and

ii $A_{n+1} \leq A_n$ for all $n \geq 1$ (i.e. the sequence is monotone decreasing) and

iii $\lim_{n\to\infty} A_n = 0$.

Then

$$A_1 - A_2 + A_3 - A_4 + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}A_n = S$$

converges and, for each natural number N, $S - S_N$ is between 0 and (the first dropped term) $(-1)^N A_{N+1}$. Here S_N is, as previously, the N^{th} partial sum $\sum\limits_{n=1}^{N}(-1)^{n-1}A_n$.

"Proof". We shall only give part of the proof here. For the rest of the proof see the optional section 3.3.10. We shall fix any natural number N and concentrate on the last statement, which gives a bound on the truncation error (which is the error introduced when you approximate the full series by the partial sum S_N)

$$\begin{aligned} E_N = S - S_N &= \sum_{n=N+1}^{\infty}(-1)^{n-1}A_n \\ &= (-1)^N\Big[A_{N+1} - A_{N+2} + A_{N+3} - A_{N+4} + \cdots\Big] \end{aligned}$$

This is of course another series. We're going to study the partial sums

$$S_{N,\ell} = \sum_{n=N+1}^{\ell}(-1)^{n-1}A_n = (-1)^N\sum_{m=1}^{\ell-N}(-1)^{m-1}A_{N+m}$$

for that series.

- If $\ell' > N+1$, with $\ell' - N$ even,

$$\begin{aligned} (-1)^N S_{N,\ell'} &= \overbrace{(A_{N+1} - A_{N+2})}^{\geq 0} + \overbrace{(A_{N+3} - A_{N+4})}^{\geq 0} + \cdots \\ &\qquad + \overbrace{(A_{\ell'-1} - A_{\ell'})}^{\geq 0} \\ &\geq 0 \end{aligned}$$

and

$$(-1)^N S_{N,\ell'+1} = \overbrace{(-1)^N S_{N,\ell'}}^{\geq 0} + \overbrace{A_{\ell'+1}}^{\geq 0} \geq 0$$

This tells us that $(-1)^N S_{N,\ell} \geq 0$ for all $\ell > N+1$, both even and odd.

- Similarly, if $\ell' > N+1$, with $\ell' - N$ odd,

$$(-1)^N S_{N,\ell'} = A_{N+1} - \overbrace{(A_{N+2} - A_{N+3})}^{\geq 0} - \overbrace{(A_{N+4} - A_{N+5})}^{\geq 0} - \cdots$$

$$- \overbrace{(A_{\ell'-1} - A_{\ell'})}^{\geq 0}$$

$$\leq A_{N+1}$$

$$(-1)^N S_{N,\ell'+1} = \overbrace{(-1)^N S_{N,\ell'}}^{\leq A_{N+1}} - \overbrace{A_{\ell'+1}}^{\geq 0} \leq A_{N+1}$$

This tells us that $(-1)^N S_{N,\ell} \leq A_{N+1}$ for all for all $\ell > N+1$, both even and odd.

So we now know that $S_{N,\ell}$ lies between its first term, $(-1)^N A_{N+1}$, and 0 for all $\ell > N+1$. While we are not going to prove it here (see the optional section 3.3.10), this implies that, since $A_{N+1} \to 0$ as $N \to \infty$, the series converges and that

$$S - S_N = \lim_{\ell\to\infty} S_{N,\ell}$$

lies between $(-1)^N A_{N+1}$ and 0.

□

Example 3.3.15 Convergence of the alternating harmonic series.

We have already seen, in Example 3.3.6, that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. On the other hand, the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ converges by the alternating series test with $A_n = \frac{1}{n}$. Note that

i $A_n = \frac{1}{n} \geq 0$ for all $n \geq 1$, so that $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ really is an alternating series, and

ii $A_n = \frac{1}{n}$ decreases as n increases, and

iii $\lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$.

so that all of the hypotheses of the alternating series test, i.e. of Theorem 3.3.14, are satisfied. We shall see, in Example 3.5.20, that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log 2.$$

Example 3.3.16 e.

You may already know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. In any event, we shall prove this in Example 3.6.5, below. In particular

$$\frac{1}{e} = e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$$

is an alternating series and satisfies all of the conditions of the alternating series test, Theorem 3.3.14a:

i The terms in the series alternate in sign.

ii The magnitude of the n^{th} term in the series decreases monotonically as n increases.

iii The n^{th} term in the series converges to zero as $n \to \infty$.

So the alternating series test guarantees that, if we approximate, for example,

$$\frac{1}{e} \approx \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!}$$

then the error in this approximation lies between 0 and the next term in the series, which is $\frac{1}{10!}$. That is

$$\begin{aligned}\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} &\le \frac{1}{e} \\ &\le \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \frac{1}{10!}\end{aligned}$$

so that

$$\begin{aligned}\frac{1}{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \frac{1}{10!}} &\le e \\ &\le \frac{1}{\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!}}\end{aligned}$$

which, to seven decimal places says

$$2.7182816 \le e \le 2.7182837$$

(To seven decimal places $e = 2.7182818$.)
The alternating series test tells us that, for any natural number N, the error that we make when we approximate $\frac{1}{e}$ by the partial sum $S_N = \sum_{n=0}^{N} \frac{(-1)^n}{n!}$ has magnitude no larger than $\frac{1}{(N+1)!}$. This tends to zero spectacularly quickly as N increases, simply because $(N+1)!$ increases spectacularly quickly as N increases [a]. For example $20! \approx 2.4 \times 10^{27}$.

a The interested reader may wish to check out "Stirling's approximation", which says that $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

Example 3.3.16

Example 3.3.17 Computing $\log \frac{11}{10}$.

We will shortly see, in Example 3.5.20, that if $-1 < x \leq 1$, then

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

Suppose that we have to compute $\log \frac{11}{10}$ to within an accuracy of 10^{-12}. Since $\frac{11}{10} = 1 + \frac{1}{10}$, we can get $\log \frac{11}{10}$ by evaluating $\log(1+x)$ at $x = \frac{1}{10}$, so that

$$\begin{aligned}\log \frac{11}{10} &= \log\left(1 + \frac{1}{10}\right) = \frac{1}{10} - \frac{1}{2\times 10^2} + \frac{1}{3\times 10^3} - \frac{1}{4\times 10^4} + \cdots \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n\times 10^n}\end{aligned}$$

By the alternating series test, this series converges. Also by the alternating series test, approximating $\log \frac{11}{10}$ by throwing away all but the first N terms

$$\begin{aligned}\log \frac{11}{10} &\approx \frac{1}{10} - \frac{1}{2\times 10^2} + \frac{1}{3\times 10^3} - \frac{1}{4\times 10^4} + \cdots + (-1)^{N-1}\frac{1}{N\times 10^N} \\ &= \sum_{n=1}^{N}(-1)^{n-1}\frac{1}{n\times 10^n}\end{aligned}$$

introduces an error whose magnitude is no more than the magnitude of the first term that we threw away.

$$\text{error} \leq \frac{1}{(N+1)\times 10^{N+1}}$$

To achieve an error that is no more than 10^{-12}, we have to choose N so that

$$\frac{1}{(N+1)\times 10^{N+1}} \leq 10^{-12}$$

The best way to do so is simply to guess — we are not going to be able to manipulate the inequality $\frac{1}{(N+1)\times 10^{N+1}} \leq \frac{1}{10^{12}}$ into the form $N \leq \cdots$, and even if we could, it would not be worth the effort. We need to choose N so that the denominator $(N+1)\times 10^{N+1}$ is at least 10^{12}. That is easy, because the denominator contains the factor 10^{N+1} which is at least 10^{12} whenever $N+1 \geq 12$, i.e. whenever $N \geq 11$. So we will achieve an error of less than 10^{-12} if we choose $N = 11$.

$$\left.\frac{1}{(N+1)\times 10^{N+1}}\right|_{N=11} = \frac{1}{12\times 10^{12}} < \frac{1}{10^{12}}$$

This is not the smallest possible choice of N, but in practice that just doesn't matter — your computer is not going to care whether or not you ask it to compute a few extra terms. If you really need the smallest N that obeys $\frac{1}{(N+1)\times 10^{N+1}} \le \frac{1}{10^{12}}$, you can next just try $N = 10$, then $N = 9$, and so on.

$$\left.\frac{1}{(N+1)\times 10^{N+1}}\right|_{N=11} = \frac{1}{12\times 10^{12}} < \frac{1}{10^{12}}$$
$$\left.\frac{1}{(N+1)\times 10^{N+1}}\right|_{N=10} = \frac{1}{11\times 10^{11}} < \frac{1}{10\times 10^{11}} = \frac{1}{10^{12}}$$
$$\left.\frac{1}{(N+1)\times 10^{N+1}}\right|_{N=9} = \frac{1}{10\times 10^{10}} = \frac{1}{10^{11}} > \frac{1}{10^{12}}$$

So in this problem, the smallest acceptable $N = 10$.

Example 3.3.17

3.3.5 ➤ The Ratio Test

The idea behind the ratio test comes from a reexamination of the geometric series. Recall that the geometric series

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} ar^n$$

converges when $|r| < 1$ and diverges otherwise. So the convergence of this series is completely determined by the number r. This number is just the ratio of successive terms — that is $r = a_{n+1}/a_n$.

In general the ratio of successive terms of a series, $\frac{a_{n+1}}{a_n}$, is not constant, but depends on n. However, as we have noted above, the convergence of a series $\sum a_n$ is determined by the behaviour of its terms when n is large. In this way, the behaviour of this ratio when n is small tells us nothing about the convergence of the series, but the limit of the ratio as $n \to \infty$ does. This is the basis of the ratio test.

Theorem 3.3.18 Ratio Test.

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

a If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

b If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = +\infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Warning 3.3.19

Beware that the ratio test provides absolutely no conclusion about the convergence or divergence of the series $\sum\limits_{n=1}^{\infty} a_n$ if $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$. See Example 3.3.22, below.

Proof. (a) Pick any number R obeying $L < R < 1$. We are assuming that $\left|\frac{a_{n+1}}{a_n}\right|$ approaches L as $n \to \infty$. In particular there must be some natural number M so that $\left|\frac{a_{n+1}}{a_n}\right| \le R$ for all $n \ge M$. So $|a_{n+1}| \le R|a_n|$ for all $n \ge M$. In particular

$$\begin{aligned}
|a_{M+1}| &\le R\,|a_M| & \\
|a_{M+2}| &\le R\,|a_{M+1}| & \le R^2\,|a_M| \\
|a_{M+3}| &\le R\,|a_{M+2}| & \le R^3\,|a_M| \\
&\vdots & \\
|a_{M+\ell}| &\le R^\ell\,|a_M| &
\end{aligned}$$

for all $\ell \ge 0$. The series $\sum_{\ell=0}^{\infty} R^\ell\,|a_M|$ is a geometric series with ratio R smaller than one in magnitude and so converges. Consequently, by the comparison test with a_n replaced by $A_\ell = a_{n+\ell}$ and c_n replaced by $C_\ell = R^\ell\,|a_M|$, the series $\sum\limits_{\ell=1}^{\infty} a_{M+\ell} = \sum\limits_{n=M+1}^{\infty} a_n$ converges. So the series $\sum\limits_{n=1}^{\infty} a_n$ converges too.

(b) We are assuming that $\left|\frac{a_{n+1}}{a_n}\right|$ approaches $L > 1$ as $n \to \infty$. In particular there must be some natural number $M > N$ so that $\left|\frac{a_{n+1}}{a_n}\right| \ge 1$ for all $n \ge M$. So $|a_{n+1}| \ge |a_n|$ for all $n \ge M$. That is, $|a_n|$ increases as n increases as long as $n \ge M$. So $|a_n| \ge |a_M|$ for all $n \ge M$ and a_n cannot converge to zero as $n \to \infty$. So the series diverges by the divergence test.

□

Example 3.3.20 $\sum_{n=0}^{\infty} anx^{n-1}$.

Fix any two nonzero real numbers a and x. We have already seen in Example 3.2.4 and Lemma 3.2.5 — we have just renamed r to x — that the geometric series $\sum_{n=0}^{\infty} ax^n$ converges when $|x| < 1$ and diverges when $|x| \ge 1$. We are now going to consider a new series, constructed by differentiating [a] each term in the geometric series $\sum_{n=0}^{\infty} ax^n$. This new series is

$$\sum_{n=0}^{\infty} a_n \qquad \text{with} \qquad a_n = a\,n\,x^{n-1}$$

Let's apply the ratio test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{a\,(n+1)\,x^n}{a\,n\,x^{n-1}}\right| = \frac{n+1}{n}|x| = \left(1+\frac{1}{n}\right)|x| \to L = |x| \quad \text{as } n \to \infty$$

The ratio test now tells us that the series $\sum_{n=0}^{\infty} a\,n\,x^{n-1}$ converges if $|x| < 1$ and diverges if $|x| > 1$. It says nothing about the cases $x = \pm 1$. But in both of those cases $a_n = a\,n\,(\pm 1)^n$ does not converge to zero as $n \to \infty$ and the series diverges by the divergence test.

a We shall see later, in Theorem 3.5.13, that the function $\sum_{n=0}^{\infty} anx^{n-1}$ is indeed the derivative of the function $\sum_{n=0}^{\infty} ax^n$. Of course, such a statement only makes sense where these series converge — how can you differentiate a divergent series? (This is not an allusion to a popular series of dystopian novels.) Actually, there is quite a bit of interesting and useful mathematics involving divergent series, but it is well beyond the scope of this course.

Example 3.3.20

Notice that in the above example, we had to apply another convergence test in addition to the ratio test. This will be commonplace when we reach power series and Taylor series — the ratio test will tell us something like

The series converges for $|x| < R$ and diverges for $|x| > R$.

Of course, we will still have to to determine what happens when $x = +R, -R$. To determine convergence or divergence in those cases we will need to use one of the other tests we have seen.

Example 3.3.21 $\sum_{n=0}^{\infty} \frac{a}{n+1} X^{n+1}$.

Once again, fix any two nonzero real numbers a and X. We again start with the geometric series $\sum_{n=0}^{\infty} ax^n$ but this time we construct a new series by integrating [a] each term, ax^n, from $x = 0$ to $x = X$ giving $\frac{a}{n+1}X^{n+1}$. The resulting new series is

$$\sum_{n=0}^{\infty} a_n \qquad \text{with } a_n = \frac{a}{n+1}X^{n+1}$$

To apply the ratio test we need to compute

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{a}{n+2}X^{n+2}}{\frac{a}{n+1}X^{n+1}}\right| = \frac{n+1}{n+2}|X| = \frac{1+\frac{1}{n}}{1+\frac{2}{n}}|X| \to L = |X| \quad \text{as } n \to \infty$$

The ratio test now tells us that the series $\sum_{n=0}^{\infty} \frac{a}{n+1}X^{n+1}$ converges if $|X| < 1$ and diverges if $|X| > 1$. It says nothing about the cases $X = \pm 1$.
If $X = 1$, the series reduces to

$$\sum_{n=0}^{\infty} \frac{a}{n+1}X^{n+1}\bigg|_{X=1} = \sum_{n=0}^{\infty} \frac{a}{n+1} = a\sum_{m=1}^{\infty} \frac{1}{m} \qquad \text{with } m = n+1$$

which is just a times the harmonic series, which we know diverges, by Example 3.3.6. If $X = -1$, the series reduces to

$$\sum_{n=0}^{\infty} \frac{a}{n+1} X^{n+1} \Big|_{X=-1} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{a}{n+1}$$

which converges by the alternating series test. See Example 3.3.15.
In conclusion, the series $\sum_{n=0}^{\infty} \frac{a}{n+1} X^{n+1}$ converges if and only if $-1 \le X < 1$.

[a] We shall also see later, in Theorem 3.5.13, that the function $\sum_{n=0}^{\infty} \frac{a}{n+1} x^{n+1}$ is indeed an antiderivative of the function $\sum_{n=0}^{\infty} a x^n$.

Example 3.3.21

The ratio test is often quite easy to apply, but one must always be careful when the limit of the ratio is 1. The next example illustrates this.

Example 3.3.22 $L = 1$.

In this example, we are going to see three different series that all have $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$. One is going to diverge and the other two are going to converge.

- The first series is the harmonic series

$$\sum_{n=1}^{\infty} a_n \qquad \text{with } a_n = \frac{1}{n}$$

We have already seen, in Example 3.3.6, that this series diverges. It has

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to L = 1 \quad \text{as } n \to \infty$$

- The second series is the alternating harmonic series

$$\sum_{n=1}^{\infty} a_n \qquad \text{with } a_n = (-1)^{n-1}\frac{1}{n}$$

We have already seen, in Example 3.3.15, that this series converges. But it also has

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^n \frac{1}{n+1}}{(-1)^{n-1}\frac{1}{n}}\right| = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to L = 1 \quad \text{as } n \to \infty$$

- The third series is

$$\sum_{n=1}^{\infty} a_n \qquad \text{with } a_n = \frac{1}{n^2}$$

We have already seen, in Example 3.3.6 with $p = 2$, that this series converges. But it also has

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}}\right| = \frac{n^2}{(n+1)^2} = \frac{1}{(1+\frac{1}{n})^2} \to L = 1 \quad \text{as } n \to \infty$$

Example 3.3.22

Let's do a somewhat artificial example that forces us to combine a few of the techniques we have seen.

Example 3.3.23 $\sum_{n=1}^{\infty} \frac{(-3)^n\sqrt{n+1}}{2n+3}x^n$.

Again, the convergence of this series will depend on x.

- Let us start with the ratio test — so we compute

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(-3)^{n+1}\sqrt{n+2}(2n+3)x^{n+1}}{(-3)^n\sqrt{n+1}(2n+5)x^n}\right| \\ &= |-3| \cdot \frac{\sqrt{n+2}}{\sqrt{n+1}} \cdot \frac{2n+3}{2n+5} \cdot |x|\end{aligned}$$

  So in the limit as $n \to \infty$ we are left with

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 3|x|$$

- The ratio test then tells us that if $3|x| > 1$ the series diverges, while when $3|x| < 1$ the series converges.
- This leaves us with the cases $x = +\frac{1}{3}$ and $-\frac{1}{3}$.
- Setting $x = \frac{1}{3}$ gives the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n+1}}{2n+3}$$

  The fact that the terms alternate here suggests that we use the alternating series test. That will show that this series converges provided $\frac{\sqrt{n+1}}{2n+3}$ decreases as n increases. So we define the function

$$f(t) = \frac{\sqrt{t+1}}{2t+3}$$

  (which is constructed by replacing the n in $\frac{\sqrt{n+1}}{2n+3}$ with t) and verify that $f(t)$ is a decreasing function of t. To prove that, it suffices to show its derivative is negative when $t \geq 1$:

$$f'(t) = \frac{(2t+3)\cdot\frac{1}{2}\cdot(t+1)^{-1/2} - 2\sqrt{t+1}}{(2t+3)^2}$$

$$= \frac{(2t+3) - 4(t+1)}{2\sqrt{t+1}(2t+3)^2}$$
$$= \frac{-2t-1}{2\sqrt{t+1}(2t+3)^2}$$

When $t \geq 1$ this is negative and so $f(t)$ is a decreasing function. Thus we can apply the alternating series test to show that the series converges when $x = \frac{1}{3}$.

- When $x = -\frac{1}{3}$ the series becomes

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{2n+3}.$$

Notice that when n is large, the summand is approximately $\frac{\sqrt{n}}{2n}$ which suggests that the series will diverge by comparison with $\sum n^{-1/2}$. To formalise this, we can use the limit comparison theorem:

$$\lim_{n\to\infty} \frac{\sqrt{n+1}}{2n+3} \frac{1}{n^{-1/2}} = \lim_{n\to\infty} \frac{\sqrt{n} \cdot \sqrt{1+1/n}}{n(2+3/n)} \cdot n^{1/2}$$
$$= \lim_{n\to\infty} \frac{n \cdot \sqrt{1+1/n}}{n(2+3/n)}$$
$$= \frac{1}{2}$$

So since this ratio has a finite limit and the series $\sum n^{-1/2}$ diverges, we know that our series also diverges.

So in summary the series converges when $-\frac{1}{3} < x \leq \frac{1}{3}$ and diverges otherwise.

Example 3.3.23

3.3.6 ▸▸ Convergence Test List

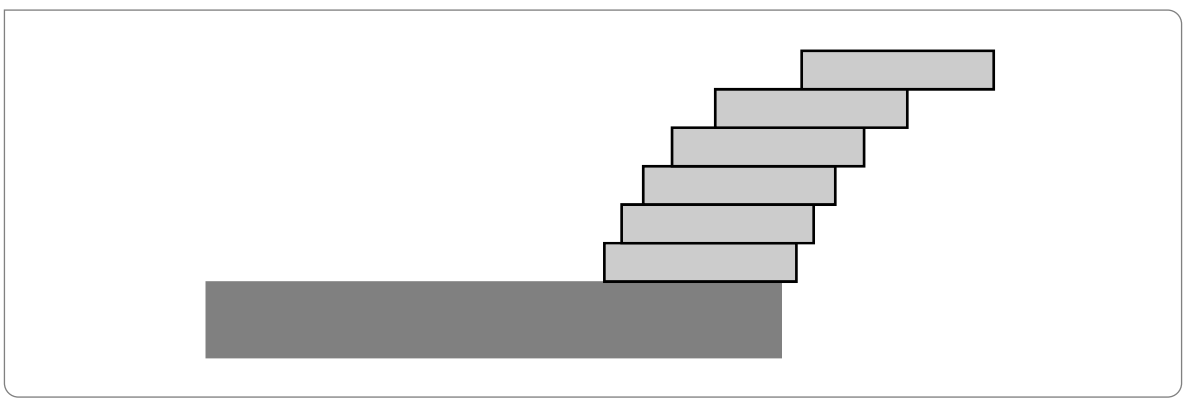
We now have half a dozen convergence tests:

- *Divergence Test*
  - works well when the n^{th} term in the series *fails* to converge to zero as n tends to infinity
- *Alternating Series Test*
  - works well when successive terms in the series alternate in sign
  - don't forget to check that successive terms decrease in magnitude and tend to zero as n tends to infinity

- *Integral Test*
  - works well when, if you substitute x for n in the n^{th} term you get a function, $f(x)$, that you can integrate
  - don't forget to check that $f(x) \geq 0$ and that $f(x)$ decreases as x increases
- *Ratio Test*
  - works well when $\frac{a_{n+1}}{a_n}$ simplifies enough that you can easily compute $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$
  - this often happens when a_n contains powers, like 7^n, or factorials, like $n!$
  - don't forget that $L = 1$ tells you nothing about the convergence/divergence of the series
- *Comparison Test and Limit Comparison Test*
  - works well when, for very large n, the n^{th} term a_n is approximately the same as a simpler term b_n (see Example 3.3.10) and it is easy to determine whether or not $\sum_{n=1}^{\infty} b_n$ converges
  - don't forget to check that $b_n \geq 0$
  - usually the Limit Comparison Test is easier to apply than the Comparison Test

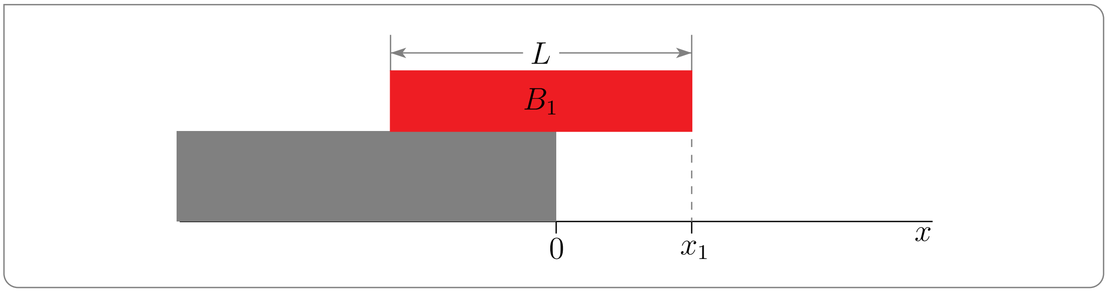
3.3.7 ➤ Optional — The Leaning Tower of Books

Imagine that you are about to stack a bunch of identical books on a table. But you don't want to just stack them exactly vertically. You want to built a "leaning tower of books" that overhangs the edge of the table as much as possible.



How big an overhang can you get? The answer to that question, which we'll now derive, uses a series!

- Let's start by just putting book #1 on the table. It's the red book labelled "B_1" in the figure below.



Use a horizontal x-axis with $x = 0$ corresponding to the right hand edge of the table. Imagine that we have placed book #1 so that its right hand edge overhangs the end of the table by a distance x_1.

- In order for the book to not topple off of the table, we need its centre of mass to lie above the table. That is, we need the x-coordinate of the centre mass of B_1, which we shall denote $\bar{X}(B_1)$, to obey

$$\bar{X}(B_1) \le 0$$

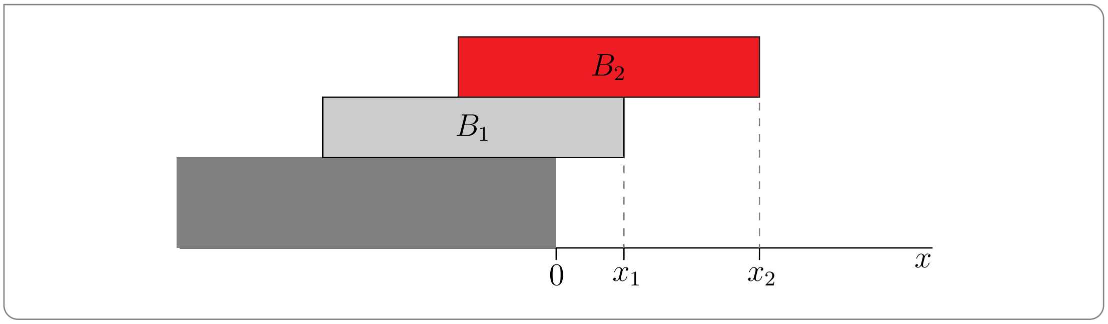
Assuming that our books have uniform density and are of length L, $\bar{X}(B_1)$ will be exactly half way between the right hand end of the book, which is at $x = x_1$, and the left hand end of the book, which is at $x = x_1 - L$. So

$$\bar{X}(B_1) = \frac{1}{2}x_1 + \frac{1}{2}(x_1 - L) = x_1 - \frac{L}{2}$$

Thus book #1 does not topple off of the table provided

$$x_1 \le \frac{L}{2}$$

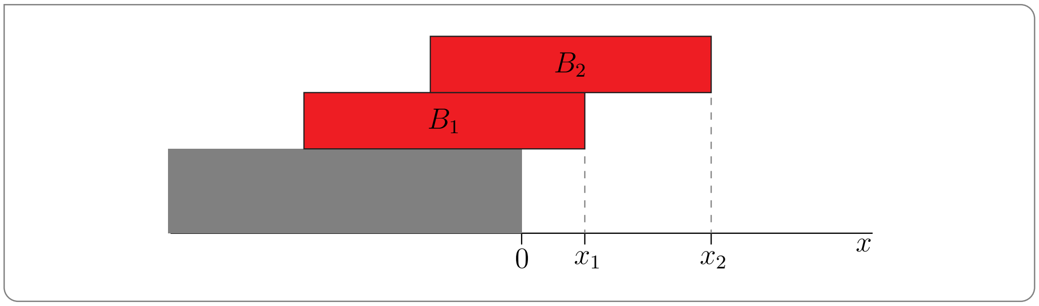
- Now let's put books #1 and #2 on the table, with the right hand edge of book #1 at $x = x_1$ and the right hand edge of book #2 at $x = x_2$, as in the figure below.



- In order for book #2 to not topple off of book #1, we need the centre of mass of book #2 to lie above book #1. That is, we need the x-coordinate of the centre mass of B_2, which is $\bar{X}(B_2) = x_2 - \frac{L}{2}$, to obey

$$\bar{X}(B_2) \le x_1 \iff x_2 - \frac{L}{2} \le x_1 \iff x_2 \le x_1 + \frac{L}{2}$$

- Assuming that book #2 does not topple off of book #1, we still need to arrange that the pair of books does not topple off of the table. Think of the pair of books as the combined red object in the figure



In order for the combined red object to not topple off of the table, we need the centre of mass of the combined red object to lie above the table. That is, we need the x-coordinate of the centre mass of the combined red object, which we shall denote $\bar{X}(B_1 \cup B_2)$, to obey

$$\bar{X}(B_1 \cup B_2) \le 0$$

The centre of mass of the combined red object is the weighted average [8] of the centres of mass of B_1 and B_2. As B_1 and B_2 have the same weight,

$$\begin{aligned}\bar{X}(B_1 \cup B_2) &= \frac{1}{2}\bar{X}(B_1) + \frac{1}{2}\bar{X}(B_2) = \frac{1}{2}\Big(x_1 - \frac{L}{2}\Big) + \frac{1}{2}\Big(x_2 - \frac{L}{2}\Big)\\ &= \frac{1}{2}(x_1 + x_2) - \frac{L}{2}\end{aligned}$$

and the combined red object does not topple off of the table if

$$\bar{X}(B_1 \cup B_2) = \frac{1}{2}(x_1 + x_2) - \frac{L}{2} \le 0 \iff x_1 + x_2 \le L$$

In conclusion, our two-book tower survives if

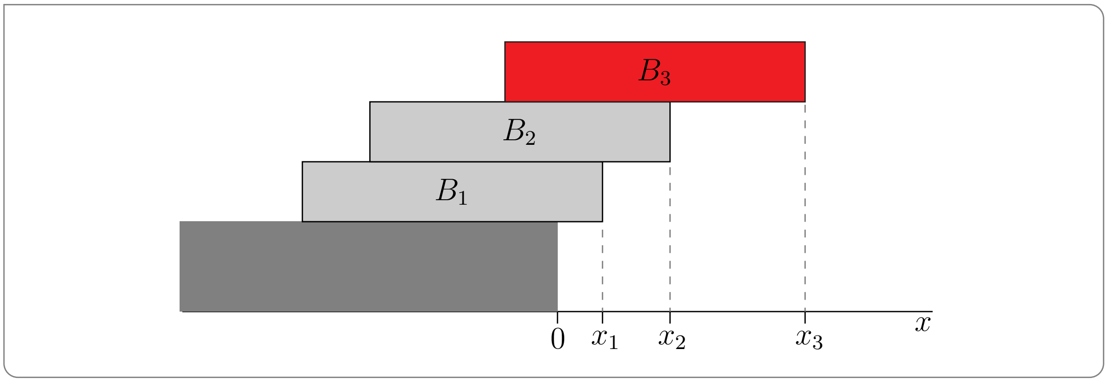
$$x_2 \le x_1 + \frac{L}{2} \qquad \text{and} \qquad x_1 + x_2 \le L$$

In particular we may choose x_1 and x_2 to satisfy $x_2 = x_1 + \frac{L}{2}$ and $x_1 + x_2 = L$. Then, substituting $x_2 = x_1 + \frac{L}{2}$ into $x_1 + x_2 = L$ gives

$$x_1 + \Big(x_1 + \frac{L}{2}\Big) = L \iff 2x_1 = \frac{L}{2} \iff x_1 = \frac{L}{2}\Big(\frac{1}{2}\Big), \quad x_2 = \frac{L}{2}\Big(1 + \frac{1}{2}\Big)$$

- Before considering the general "n-book tower", let's now put books #1, #2 and #3 on the table, with the right hand edge of book #1 at $x = x_1$, the right hand edge of book #2 at $x = x_2$, and the right hand edge of book #3 at $x = x_3$, as in the figure below.

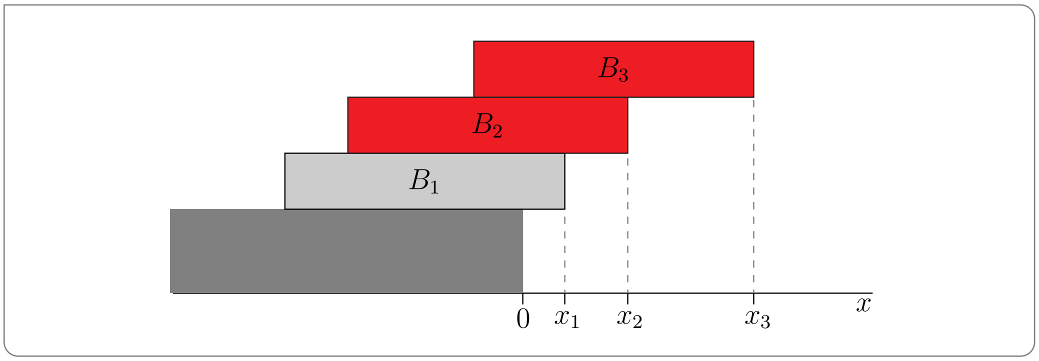
8 It might be a good idea to review the beginning of §2.3 at this point.



- In order for book #3 to not topple off of book #2, we need the centre of mass of book #3 to lie above book #2. That is, we need the x-coordinate of the centre mass of B_3, which is $\bar{X}(B_3) = x_3 - \frac{L}{2}$, to obey

$$\bar{X}(B_3) \le x_2 \iff x_3 - \frac{L}{2} \le x_2 \iff x_3 \le x_2 + \frac{L}{2}$$

- Assuming that book #3 does not topple off of book #2, we still need to arrange that the pair of books, book #2 plus book #3 (the red object in the figure below), does not topple off of book #1.



In order for this combined red object to not topple off of book #1, we need the x-coordinate of its centre mass, which we denote $\bar{X}(B_2 \cup B_3)$, to obey

$$\bar{X}(B_2 \cup B_3) \le x_1$$

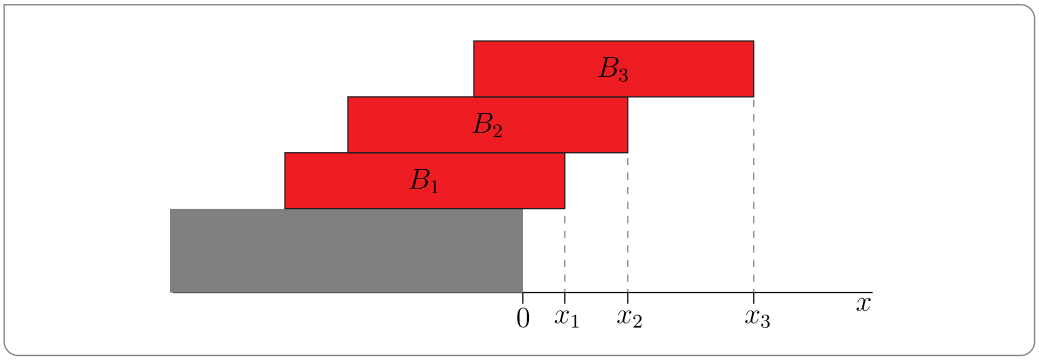
The centre of mass of the combined red object is the weighted average of the centre of masses of B_2 and B_3. As B_2 and B_3 have the same weight,

$$\begin{aligned}\bar{X}(B_2 \cup B_3) &= \frac{1}{2}\bar{X}(B_2) + \frac{1}{2}\bar{X}(B_3) = \frac{1}{2}\Big(x_2 - \frac{L}{2}\Big) + \frac{1}{2}\Big(x_3 - \frac{L}{2}\Big) \\ &= \frac{1}{2}(x_2 + x_3) - \frac{L}{2}\end{aligned}$$

and the combined red object does not topple off of book #1 if

$$\frac{1}{2}(x_2 + x_3) - \frac{L}{2} \le x_1 \iff x_2 + x_3 \le 2x_1 + L$$

- Assuming that book #3 does not topple off of book #2, and also that the combined book #2 plus book #3 does not topple off of book #1, we still need to arrange that the whole tower of books, book #1 plus book #2 plus book #3 (the red object in the figure below), does not topple off of the table.



In order for this combined red object to not topple off of the table, we need the x-coordinate of its centre mass, which we denote $\bar{X}(B_1 \cup B_2 \cup B_3)$, to obey

$$\bar{X}(B_1 \cup B_2 \cup B_3) \le 0$$

The centre of mass of the combined red object is the weighted average of the centre of masses of B_1 and B_2 and B_3. As they all have the same weight,

$$\begin{aligned}\bar{X}(B_1 \cup B_2 \cup B_3) &= \frac{1}{3}\bar{X}(B_1) + \frac{1}{3}\bar{X}(B_2) + \frac{1}{3}\bar{X}(B_3) \\ &= \frac{1}{3}\Big(x_1 - \frac{L}{2}\Big) + \frac{1}{3}\Big(x_2 - \frac{L}{2}\Big) + \frac{1}{3}\Big(x_3 - \frac{L}{2}\Big) \\ &= \frac{1}{3}(x_1 + x_2 + x_3) - \frac{L}{2}\end{aligned}$$

and the combined red object does not topple off of the table if

$$\frac{1}{3}(x_1 + x_2 + x_3) - \frac{L}{2} \le 0 \iff x_1 + x_2 + x_3 \le \frac{3L}{2}$$

In conclusion, our three-book tower survives if

$$x_3 \le x_2 + \frac{L}{2} \qquad \text{and} \qquad x_2 + x_3 \le 2x_1 + L \qquad \text{and} \qquad x_1 + x_2 + x_3 \le \frac{3L}{2}$$

In particular, we may choose x_1, x_2 and x_3 to satisfy

$$\begin{aligned}x_1 + x_2 + x_3 &= \frac{3L}{2} \qquad \text{and} \\ x_2 + x_3 &= 2x_1 + L \qquad \text{and} \\ x_3 &= \frac{L}{2} + x_2\end{aligned}$$

Substituting the second equation into the first gives

$$3x_1 + L = \frac{3L}{2} \implies x_1 = \frac{L}{2}\Big(\frac{1}{3}\Big)$$

Next substituting the third equation into the second, and then using the formula above for x_1, gives

$$2x_2 + \frac{L}{2} = 2x_1 + L = \frac{L}{3} + L \implies x_2 = \frac{L}{2}\Big(\frac{1}{2} + \frac{1}{3}\Big)$$

and finally

$$x_3 = \frac{L}{2} + x_2 = \frac{L}{2}\Big(1 + \frac{1}{2} + \frac{1}{3}\Big)$$

- We are finally ready for the general "n-book tower". Stack n books on the table, with book B_1 on the bottom and book B_n at the top, and with the right hand edge of book $\#j$ at $x = x_j$. The same centre of mass considerations as above show that the tower survives if

$$\begin{aligned}
\bar{X}(B_n) &\le x_{n-1} & x_n - \frac{L}{2} &\le x_{n-1}\\
\bar{X}(B_{n-1}\cup B_n) &\le x_{n-2} & \frac{1}{2}(x_{n-1}+x_n) - \frac{L}{2} &\le x_{n-2}\\
&\vdots & &\vdots\\
\bar{X}(B_3\cup\cdots\cup B_n) &\le x_2 & \frac{1}{n-2}(x_3+\cdots+x_n) - \frac{L}{2} &\le x_2\\
\bar{X}(B_2\cup B_3\cup\cdots\cup B_n) &\le x_1 & \frac{1}{n-1}(x_2+x_3+\cdots+x_n) - \frac{L}{2} &\le x_1\\
\bar{X}(B_1\cup B_2\cup B_3\cup\cdots\cup B_n) &\le 0 & \frac{1}{n}(x_1+x_2+x_3+\cdots+x_n) - \frac{L}{2} &\le 0
\end{aligned}$$

In particular, we may choose the x_j's to obey

$$\begin{aligned}
\frac{1}{n}(x_1+x_2+x_3+\cdots+x_n) &= \frac{L}{2}\\
\frac{1}{n-1}(x_2+x_3+\cdots+x_n) &= \frac{L}{2} + x_1\\
\frac{1}{n-2}(x_3+\cdots+x_n) &= \frac{L}{2} + x_2\\
&\vdots\\
\frac{1}{2}(x_{n-1}+x_n) &= \frac{L}{2} + x_{n-2}\\
x_n &= \frac{L}{2} + x_{n-1}
\end{aligned}$$

Substituting $x_2 + x_3 + \cdots + x_n = (n-1)x_1 + \frac{L}{2}(n-1)$ from the second equation into the first equation gives

$$\begin{aligned}
\frac{1}{n}\Big\{\overbrace{x_1 + (n-1)x_1}^{nx_1} + \frac{L}{2}(n-1)\Big\} = \frac{L}{2} &\implies x_1 + \frac{L}{2}\Big(1 - \frac{1}{n}\Big) = \frac{L}{2}\Big(\frac{1}{2}\Big)\\
&\implies x_1 = \frac{L}{2}\Big(\frac{1}{n}\Big)
\end{aligned}$$

Substituting $x_3 + \cdots + x_n = (n-2)x_2 + \frac{L}{2}(n-2)$ from the third equation into the second equation gives

$$\begin{aligned}
\frac{1}{n-1}\Big\{\overbrace{x_2 + (n-2)x_2}^{(n-1)x_2} + \frac{L}{2}(\overbrace{n-2}^{(n-1)-1})\Big\} &= \frac{L}{2} + x_1 = \frac{L}{2}\Big(1+\frac{1}{n}\Big)\\
\implies x_2 + \frac{L}{2}\Big(1 - \frac{1}{n-1}\Big) &= \frac{L}{2}\Big(1+\frac{1}{n}\Big)\\
\implies x_2 &= \frac{L}{2}\Big(\frac{1}{n-1} + \frac{1}{n}\Big)
\end{aligned}$$

Just keep going. We end up with

$$\begin{aligned}
x_1 &= \frac{L}{2}\Big(\frac{1}{n}\Big)\\
x_2 &= \frac{L}{2}\Big(\frac{1}{n-1} + \frac{1}{n}\Big)\\
x_3 &= \frac{L}{2}\Big(\frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}\Big)\\
&\vdots\\
x_{n-2} &= \frac{L}{2}\Big(\frac{1}{3} + \cdots + \frac{1}{n}\Big)\\
x_{n-1} &= \frac{L}{2}\Big(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\Big)\\
x_n &= \frac{L}{2}\Big(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\Big)
\end{aligned}$$

Our overhang is $x_n = \frac{L}{2}\big(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\big)$. This is $\frac{L}{2}$ times the n^{th} partial sum of the harmonic series $\sum_{m=1}^{\infty} \frac{1}{m}$. As we saw in Example 3.3.6 (the p test), the harmonic series diverges. So, as n goes to infinity $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ also goes to infinity. We may make the overhang as large [9] as we like!

3.3.8 ▸▸ Optional — The Root Test

There is another test that is very similar in spirit to the ratio test. It also comes from a reexamination of the geometric series

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} ar^n$$

The ratio test was based on the observation that r, which largely determines whether or not the series converges, could be found by computing the ratio $r = a_{n+1}/a_n$. The root test is based on the observation that $|r|$ can also be determined by looking that the n^{th} root of the n^{th} term with n very large:

$$\lim_{n\to\infty} \sqrt[n]{\big|ar^n\big|} = |r| \lim_{n\to\infty} \sqrt[n]{\big|a\big|} = |r| \qquad \text{if } a \neq 0$$

9 At least if our table is strong enough.

Of course, in general, the n^{th} term is not exactly ar^n. However, if for very large n, the n^{th} term is approximately proportional to r^n, with $|r|$ given by the above limit, we would expect the series to converge when $|r| < 1$ and diverge when $|r| > 1$. That is indeed the case.

Theorem 3.3.24 Root Test.

Assume that

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

exists or is $+\infty$.

a If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

b If $L > 1$, or $L = +\infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Warning 3.3.25

Beware that the root test provides absolutely no conclusion about the convergence or divergence of the series $\sum_{n=1}^{\infty} a_n$ if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

Proof. (a) Pick any number R obeying $L < R < 1$. We are assuming that $\sqrt[n]{|a_n|}$ approaches L as $n \to \infty$. In particular there must be some natural number M so that $\sqrt[n]{|a_n|} \le R$ for all $n \ge M$. So $|a_n| \le R^n$ for all $n \ge M$ and the series $\sum_{n=1}^{\infty} a_n$ converges by comparison to the geometric series $\sum_{n=1}^{\infty} R^n$
(b) We are assuming that $\sqrt[n]{|a_n|}$ approaches $L > 1$ (or grows unboundedly) as $n \to \infty$. In particular there must be some natural number M so that $\sqrt[n]{|a_n|} \ge 1$ for all $n \ge M$. So $|a_n| \ge 1$ for all $n \ge M$ and the series diverges by the divergence test.

$\square$

Example 3.3.26 $\sum_{n=1}^{\infty} \frac{(-3)^n \sqrt{n+1}}{2n+3} x^n$.

We have already used the ratio test, in Example 3.3.23, to show that this series converges when $|x| < \frac{1}{3}$ and diverges when $|x| > \frac{1}{3}$. We'll now use the root test to draw the same conclusions.

- Write $a_n = \frac{(-3)^n \sqrt{n+1}}{2n+3} x^n$.

- We compute

$$\begin{aligned}\sqrt[n]{|a_n|} &= \sqrt[n]{\left|\frac{(-3)^n\sqrt{n+1}}{2n+3}x^n\right|}\\ &= 3|x|\big(n+1\big)^{\frac{1}{2n}}\big(2n+3\big)^{-\frac{1}{n}}\end{aligned}$$

- We'll now show that the limit of $\big(n+1\big)^{\frac{1}{2n}}$ as $n \to \infty$ is exactly 1. To do, so we first compute the limit of the logarithm.

$$\begin{aligned}\lim_{n\to\infty}\log\big(n+1\big)^{\frac{1}{2n}} &= \lim_{n\to\infty}\frac{\log\big(n+1\big)}{2n} && \text{now apply Theorem 3.1.6}\\ &= \lim_{t\to\infty}\frac{\log\big(t+1\big)}{2t}\\ &= \lim_{t\to\infty}\frac{\frac{1}{t+1}}{2} && \text{by l'Hôpital}\\ &= 0\end{aligned}$$

So

$$\lim_{n\to\infty}\big(n+1\big)^{\frac{1}{2n}} = \lim_{n\to\infty}\exp\Big\{\log\big(n+1\big)^{\frac{1}{2n}}\Big\} = e^0 = 1$$

An essentially identical computation also gives that $\lim_{n\to\infty}\big(2n+3\big)^{-\frac{1}{n}} = e^0 = 1$.

- So

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = 3|x|$$

and the root test also tells us that if $3|x| > 1$ the series diverges, while when $3|x| < 1$ the series converges.

Example 3.3.26

We have done the last example once, in Example 3.3.23, using the ratio test and once, in Example 3.3.26, using the root test. It was clearly much easier to use the ratio test. Here is an example that is most easily handled by the root test.

Example 3.3.27 $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$.

Write $a_n = \left(\frac{n}{n+1}\right)^{n^2}$. Then

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \left(\frac{n}{n+1}\right)^n = \left(1+\frac{1}{n}\right)^{-n}$$

Now we take the limit,

$$\begin{aligned} \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n} &= \lim_{X\to\infty}\left(1+\frac{1}{X}\right)^{-X} && \text{by Theorem 3.1.6} \\ &= \lim_{x\to 0}\left(1+x\right)^{-1/x} && \text{where } x=\frac{1}{X} \\ &= e^{-1} \end{aligned}$$

by Example 3.7.20 in the CLP-1 text with $a = -1$. As the limit is strictly smaller than 1, the series $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$ converges.
To draw the same conclusion using the ratio test, one would have to show that the limit of

$$\frac{a_{n+1}}{a_n} = \left(\frac{n+1}{n+2}\right)^{(n+1)^2}\left(\frac{n+1}{n}\right)^{n^2}$$

as $n \to \infty$ is strictly smaller than 1. It's clearly better to stick with the root test.

Example 3.3.27

3.3.9 ▸▸ Optional — Harmonic and Basel Series

3.3.9.1 ▸▸▸ The Harmonic Series

The series

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

that appeared in Warning 3.3.3, is called the Harmonic series [10], and its partial sums

$$H_N = \sum_{n=1}^{N}\frac{1}{n}$$

are called the Harmonic numbers. Though these numbers have been studied at least as far back as Pythagoras, the divergence of the series was first proved in around 1350 by Nicholas Oresme (1320-5 – 1382), though the proof was lost for many years and rediscovered by Mengoli (1626–1686) and the Bernoulli brothers (Johann 1667–1748 and Jacob 1655–1705).

Oresme's proof is beautiful and all the more remarkable that it was produced more than 300 years before calculus was developed by Newton and Leibnitz. It starts by

10 The interested reader should use their favourite search engine to read more on the link between this series and musical harmonics. You can also find interesting links between the Harmonic series and the so-called "jeep problem" and also the problem of stacking a tower of dominoes to create an overhang that does not topple over.

grouping the terms of the harmonic series carefully:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n} &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots \\
&= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{15}+\frac{1}{16}\right)+\cdots \\
&> 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\frac{1}{16}+\cdots+\frac{1}{16}+\frac{1}{16}\right)+\cdots \\
&= 1+\frac{1}{2}+\left(\frac{2}{4}\right)+\left(\frac{4}{8}\right)+\left(\frac{8}{16}\right)+\cdots
\end{aligned}$$

So one can see that this is $1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$ and so must diverge [11].

There are many variations on Oresme's proof — for example, using groups of two or three. A rather different proof relies on the inequality

$$e^x > 1+x \qquad \text{for } x>0$$

which follows immediately from the Taylor series for e^x given in Theorem 3.6.7. From this we can bound the exponential of the Harmonic numbers:

$$\begin{aligned}
e^{H_n} &= e^{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}} \\
&= e^1\cdot e^{1/2}\cdot e^{1/3}\cdot e^{1/4}\cdots e^{1/n} \\
&> (1+1)\cdot(1+1/2)\cdot(1+1/3)\cdot(1+1/4)\cdots(1+1/n) \\
&= \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots\frac{n+1}{n} \\
&= n+1
\end{aligned}$$

Since e^{H_n} grows unboundedly with n, the harmonic series diverges.

3.3.9.2 ⯮ The Basel Problem

The problem of determining the exact value of the sum of the series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

is called the Basel problem. The problem is named after the home town of Leonhard Euler, who solved it. One can use telescoping series to show that this series must converge. Notice that

$$\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1}-\frac{1}{n}$$

11 The grouping argument can be generalised further and the interested reader should look up Cauchy's condensation test.

Hence we can bound the partial sum:

$$\begin{aligned} S_k = \sum_{n=1}^{k} \frac{1}{n^2} &< 1 + \sum_{n=2}^{k} \frac{1}{n(n-1)} && \text{avoid dividing by 0} \\ &= 1 + \sum_{n=2}^{k} \left(\frac{1}{n-1} - \frac{1}{n} \right) && \text{which telescopes to} \\ &= 1 + 1 - \frac{1}{k} \end{aligned}$$

Thus, as k increases, the partial sum S_k increases (the series is a sum of positive terms), but is always smaller than 2. So the sequence of partial sums converges.

Mengoli posed the problem of evaluating the series exactly in 1644 and it was solved — not entirely rigorously — by Euler in 1734. A rigorous proof had to wait another 7 years. Euler used some extremely cunning observations and manipulations of the sine function to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

He used the Maclaurin series

$$\sin x = 1 - \frac{x^3}{6} + \frac{x^5}{24} - \cdots$$

and a product formula for sine

$$\begin{aligned} \sin x &= x \cdot \left(1 - \frac{x}{\pi}\right) \cdot \left(1 + \frac{x}{\pi}\right) \cdot \left(1 - \frac{x}{2\pi}\right) \cdot \left(1 + \frac{x}{2\pi}\right) \cdot \left(1 - \frac{x}{3\pi}\right) \cdot \left(1 + \frac{x}{3\pi}\right) \cdots \\ &= x \cdot \left(1 - \frac{x^2}{\pi}\right) \cdot \left(1 - \frac{x^2}{4\pi}\right) \cdot \left(1 - \frac{x^2}{9\pi}\right) \cdots \end{aligned} \qquad (\star)$$

Extracting the coefficient of x^3 from both expansions gives the desired result. The proof of the product formula is well beyond the scope of this course. But notice that at least the values of x which make the left hand side of $(\star)$ zero, namely $x = n\pi$ with n integer, are exactly the same as the values of x which make the right hand side of $(\star)$ zero [12].

This approach can also be used to compute $\sum_{n=1}^{\infty} n^{-2p}$ for $p = 1, 2, 3, \cdots$ and show that they are rational multiples [13] of π^{2p}. The corresponding series of odd powers are significantly nastier and getting closed form expressions for them remains a famous open problem.

3.3.10 ➤ Optional — Some Proofs

In this optional section we provide proofs of two convergence tests. We shall repeatedly use the fact that any sequence a_1, a_2, a_3, $\cdots$, of real numbers which is increasing (i.e.

12 Knowing that the left and right hand sides of $(\star)$ are zero for the same values of x is far from the end of the story. Two functions $f(x)$ and $g(x)$ having the same zeros, need not be equal. It is certainly possible that $f(x) = g(x) * A(x)$ where $A(x)$ is a function that is nowhere zero. The interested reader should look up the Weierstrass factorisation theorem.

13 Search-engine your way to “Riemann zeta function”.

$a_{n+1} \geq a_n$ for all n) and bounded (i.e. there is a constant M such that $a_n \leq M$ for all n) converges. We shall not prove this fact [14].

We start with the comparison test, and then move on to the alternating series test.

Theorem 3.3.28 The Comparison Test (stated again).

Let N_0 be a natural number and let $K > 0$.

a If $|a_n| \leq K c_n$ for all $n \geq N_0$ and $\sum\limits_{n=0}^{\infty} c_n$ converges, then $\sum\limits_{n=0}^{\infty} a_n$ converges.

b If $a_n \geq K d_n \geq 0$ for all $n \geq N_0$ and $\sum\limits_{n=0}^{\infty} d_n$ diverges, then $\sum\limits_{n=0}^{\infty} a_n$ diverges.

Proof. (a) By hypothesis $\sum_{n=0}^{\infty} c_n$ converges. So it suffices to prove that $\sum_{n=0}^{\infty}[Kc_n - a_n]$ converges, because then, by our Arithmetic of series Theorem 3.2.9,

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} Kc_n - \sum_{n=0}^{\infty}[Kc_n - a_n]$$

will converge too. But for all $n \geq N_0$, $Kc_n - a_n \geq 0$ so that, for all $N \geq N_0$, the partial sums

$$S_N = \sum_{n=0}^{N}[Kc_n - a_n]$$

increase with N, but never gets bigger than the finite number $\sum\limits_{n=0}^{N_0}[Kc_n - a_n] + K \sum\limits_{n=N_0+1}^{\infty} c_n$. So the partial sums S_N converge as $N \to \infty$.

(b) For all $N > N_0$, the partial sum

$$S_N = \sum_{n=0}^{N} a_n \geq \sum_{n=0}^{N_0} a_n + K \sum_{n=N_0+1}^{N} d_n$$

By hypothesis, $\sum_{n=N_0+1}^{N} d_n$, and hence S_N, grows without bound as $N \to \infty$. So $S_N \to \infty$ as $N \to \infty$.

$\square$

14 It is one way to state a property of the real number system called "completeness". The interested reader should use their favourite search engine to look up "completeness of the real numbers".

Theorem 3.3.29 Alternating Series Test (stated again).

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

i $a_n \geq 0$ for all $n \geq 1$ and

ii $a_{n+1} \leq a_n$ for all $n \geq 1$ (i.e. the sequence is monotone decreasing) and

iii $\lim_{n\to\infty} a_n = 0$.

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$$

converges and, for each natural number N, $S - S_N$ is between 0 and (the first dropped term) $(-1)^N a_{N+1}$. Here S_N is, as previously, the N^{th} partial sum $\sum_{n=1}^{N} (-1)^{n-1} a_n$.

Proof. Let $2n$ be an even natural number. Then the $2n^{\text{th}}$ partial sum obeys

$$\begin{aligned} S_{2n} &= \overbrace{(a_1 - a_2)}^{\geq 0} + \overbrace{(a_3 - a_4)}^{\geq 0} + \cdots + \overbrace{(a_{2n-1} - a_{2n})}^{\geq 0} \\ &\leq \overbrace{(a_1 - a_2)}^{\geq 0} + \overbrace{(a_3 - a_4)}^{\geq 0} + \cdots + \overbrace{(a_{2n-1} - a_{2n})}^{\geq 0} + \overbrace{(a_{2n+1} - a_{2n+2})}^{\geq 0} \\ &= S_{2(n+1)} \end{aligned}$$

and

$$\begin{aligned} S_{2n} &= a_1 - \overbrace{(a_2 - a_3)}^{\geq 0} - \overbrace{(a_4 - a_5)}^{\geq 0} - \cdots - \overbrace{(a_{2n-2} - a_{2n-1})}^{\geq 0} - \overbrace{a_{2n}}^{\geq 0} \\ &\leq a_1 \end{aligned}$$

So the sequence S_2, S_4, S_6, $\cdots$ of even partial sums is a bounded, increasing sequence and hence converges to some real number S. Since $S_{2n+1} = S_{2n} + a_{2n+1}$ and a_{2n+1} converges zero as $n \to \infty$, the odd partial sums S_{2n+1} also converge to S. That $S - S_N$ is between 0 and (the first dropped term) $(-1)^N a_{N+1}$ was already proved in §3.3.4.

□

3.3.11 ▸▸ Exercises

Exercises — Stage 1

1. Select the series below that diverge by the divergence test.

(A) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

(B) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n+1}$

(C) $\displaystyle\sum_{n=1}^{\infty} \sin n$

(D) $\displaystyle\sum_{n=1}^{\infty} \sin(\pi n)$

2. Select the series below whose terms satisfy the conditions to apply the integral test.

(A) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

(B) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n+1}$

(C) $\displaystyle\sum_{n=1}^{\infty} \sin n$

(D) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n + 1}{n^2}$

3. Suppose there is some threshold after which a person is considered old, and before which they are young.
Let Olaf be an old person, and let Yuan be a young person.

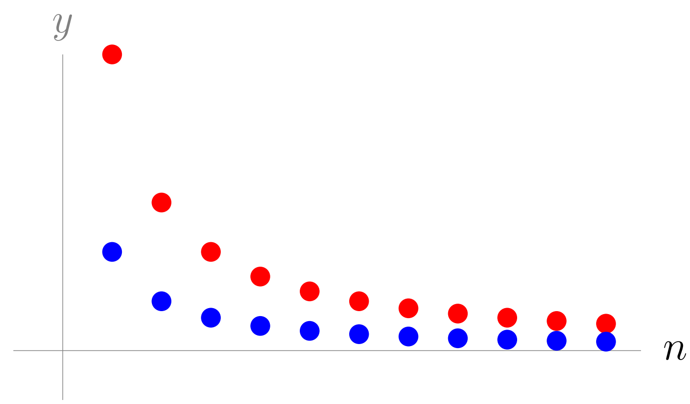
a Suppose I am older than Olaf. Am I old?

b Suppose I am younger than Olaf. Am I old?

c Suppose I am older than Yuan. Am I young?

d Suppose I am younger than Yuan. Am I young?

4. Below are graphs of two sequences with positive terms. Assume the sequences continue as shown. Fill in the table with conclusions that can be made from the direct comparison test, if any.



	if $\sum a_n$ converges	if $\sum a_n$ diverges
and if $\{a_n\}$ is the red series	then $\sum b_n$	then $\sum b_n$
and if $\{a_n\}$ is the blue series	then $\sum b_n$	then $\sum b_n$

5. For each pair of series below, decide whether the second series is a valid comparison series to determine the convergence of the first series, using the *direct* comparison test and/or the *limit* comparison test.

a $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n-1}$, compared to the divergent series $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n}$.

b $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2+1}$, compared to the convergent series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

c $\displaystyle\sum_{n=5}^{\infty} \frac{n^3+5n+1}{n^6-2}$, compared to the convergent series $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n^3}$.

d $\displaystyle\sum_{n=5}^{\infty} \frac{1}{\sqrt{n}}$, compared to the divergent series $\displaystyle\sum_{n=5}^{\infty} \frac{1}{\sqrt[4]{n}}$.

6. Suppose a_n is a sequence with $\displaystyle\lim_{n\to\infty} a_n = \frac{1}{2}$. Does $\displaystyle\sum_{n=7}^{\infty} a_n$ converge or diverge, or is it not possible to determine this from the information given? Why?

7. What flaw renders the following reasoning invalid?

Q: Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n}$ converges or diverges.

A: First, we will evaluate $\displaystyle\lim_{n\to\infty} \frac{\sin n}{n}$.

- Note $\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$ for $n \geq 1$.
- Note also that $\lim_{n\to\infty} \frac{-1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$.
- Therefore, by the Squeeze Theorem, $\lim_{n\to\infty} \frac{\sin n}{n} = 0$ as well.

So, by the divergence test, $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ converges.

8. What flaw renders the following reasoning invalid?

Q: Determine whether $\sum_{n=1}^{\infty} (\sin(\pi n) + 2)$ converges or diverges.

A: We use the integral test. Let $f(x) = \sin(\pi x) + 2$. Note $f(x)$ is always positive, since $\sin(x) + 2 \geq -1 + 2 = 1$. Also, $f(x)$ is continuous.

$$\begin{aligned}
\int_1^{\infty} [\sin(\pi x) + 2]\mathrm{d}x &= \lim_{b\to\infty} \int_1^b [\sin(\pi x) + 2]\mathrm{d}x \\
&= \lim_{b\to\infty} \left[-\frac{1}{\pi}\cos(\pi x) + 2x \Big|_1^b \right] \\
&= \lim_{b\to\infty} \left[-\frac{1}{\pi}\cos(\pi b) + 2b + \frac{1}{\pi}(-1) - 2 \right] \\
&= \infty
\end{aligned}$$

By the integral test, since the integral diverges, also $\sum_{n=1}^{\infty} (\sin(\pi n) + 2)$ diverges.

9. What flaw renders the following reasoning invalid?

Q: Determine whether the series $\sum_{n=1}^{\infty} \frac{2^{n+1}n^2}{e^n + 2n}$ converges or diverges.

A: We want to compare this series to the series $\sum_{n=1}^{\infty} \frac{2^{n+1}}{e^n}$. Note both this series and the series in the question have positive terms.

First, we find that $\frac{2^{n+1}n^2}{e^n + 2n} > \frac{2^{n+1}}{e^n}$ when n is sufficiently large. The justification for this claim is as follows:

- We note that $e^n(n^2 - 1) > n^2 - 1 > 2n$ for n sufficiently large.
- Therefore, $e^n \cdot n^2 > e^n + 2n$
- Therefore, $2^{n+1} \cdot e^n \cdot n^2 > 2^{n+1}(e^n + 2n)$

- Since $e^n + 2n$ and e^n are both expressions that work out to be positive for the values of n under consideration, we can divide both sides of the inequality by these terms without having to flip the inequality. So, $\dfrac{2^{n+1}n^2}{e^n + 2n} > \dfrac{2^{n+1}}{e^n}$.

Now, we claim $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{e^n}$ converges.

Note $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{e^n} = 2\sum_{n=1}^{\infty} \frac{2^n}{e^n} = 2\sum_{n=1}^{\infty} \left(\frac{2}{e}\right)^n$. This is a geometric series with $r = \frac{2}{e}$. Since $2/e < 1$, the series converges.

Now, by the Direct Comparison Test, we conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}n^2}{e^n + 2n}$ converges.

10. Which of the series below are alternating?

(A) $\displaystyle\sum_{n=1}^{\infty} \sin n$

(B) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^3}$

(C) $\displaystyle\sum_{n=1}^{\infty} \frac{7}{(-n)^{2n}}$

(D) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{3^{n+1}}$

11. Give an example of a convergent series for which the ratio test is inconclusive.

12. Imagine you're taking an exam, and you momentarily forget exactly how the inequality in the ratio test works. You remember there's a ratio, but you don't remember which term goes on top; you remember there's something about the limit being greater than or less than one, but you don't remember which way implies convergence.
Explain why

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$$

or, equivalently,

$$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| < 1$$

should mean that the sum $\sum_{n=1}^{\infty} a_n$ *diverges* (rather than converging).

13. Give an example of a series $\sum_{n=a}^{\infty} a_n$, with a function $f(x)$ such that $f(n) = a_n$ for all whole numbers n, such that:

- $\int_a^{\infty} f(x)\,\mathrm{d}x$ diverges, while
- $\sum_{n=a}^{\infty} a_n$ converges.

14. $*$. Suppose that you want to use the Limit Comparison Test on the series $\sum_{n=0}^{\infty} a_n$ where $a_n = \dfrac{2^n + n}{3^n + 1}$.
Write down a sequence $\{b_n\}$ such that $\lim_{n\to\infty} \dfrac{a_n}{b_n}$ exists and is nonzero. (You don't have to carry out the Limit Comparison Test)

15. $*$. Decide whether each of the following statements is true or false. If false, provide a counterexample. If true provide a brief justification.

a If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

b If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

c If $0 \le a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Exercises — Stage 2

16. $*$. Does the series $\sum_{n=2}^{\infty} \dfrac{n^2}{3n^2 + \sqrt{n}}$ converge?

17. $*$. Determine, with explanation, whether the series $\sum_{n=1}^{\infty} \dfrac{5^k}{4^k + 3^k}$ converges or diverges.

18. $*$. Determine whether the series $\sum_{n=0}^{\infty} \dfrac{1}{n + \frac{1}{2}}$ is convergent or divergent. If it is convergent, find its value.

19. Does the following series converge or diverge? $\sum_{k=1}^{\infty} \dfrac{1}{\sqrt{k}\sqrt{k+1}}$

20. Evaluate the following series, or show that it diverges: $\displaystyle\sum_{k=30}^{\infty} 3(1.001)^k$.

21. Evaluate the following series, or show that it diverges: $\displaystyle\sum_{n=3}^{\infty} \left(\frac{-1}{5}\right)^n$.

22. Does the following series converge or diverge? $\displaystyle\sum_{n=7}^{\infty} \sin(\pi n)$

23. Does the following series converge or diverge? $\displaystyle\sum_{n=7}^{\infty} \cos(\pi n)$

24. Does the following series converge or diverge? $\displaystyle\sum_{k=1}^{\infty} \frac{e^k}{k!}$.

25. Evaluate the following series, or show that it diverges: $\displaystyle\sum_{k=0}^{\infty} \frac{2^k}{3^{k+2}}$.

26. Does the following series converge or diverge? $\displaystyle\sum_{n=1}^{\infty} \frac{n!n!}{(2n)!}$.

27. Does the following series converge or diverge? $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{2n^4+n}$.

28. $*$. Show that the series $\displaystyle\sum_{n=3}^{\infty} \frac{5}{n(\log n)^{3/2}}$ converges.

29. $*$. Find the values of p for which the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges.

30. $*$. Does $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ converge or diverge?

31. $*$. Use the comparison test (not the limit comparison test) to show whether the series

$$\sum_{n=2}^{\infty} \frac{\sqrt{3n^2-7}}{n^3}$$

converges or diverges.

32. $*$. Determine whether the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k^4+1}}{\sqrt{k^5+9}}$ converges.

33. $*$. Does $\displaystyle\sum_{n=1}^{\infty} \frac{n^4 2^{n/3}}{(2n+7)^4}$ converge or diverge?

34. $*$. Determine, with explanation, whether each of the following series converge or diverge.

a $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

b $\displaystyle\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{2^n}$

35. $*$. Determine whether the series

$$\sum_{k=1}^{\infty} \frac{k^4-2k^3+2}{k^5+k^2+k}$$

converges or diverges.

36. $*$. Determine whether each of the following series converge or diverge.

a $\displaystyle\sum_{n=2}^{\infty} \frac{n^2+n+1}{n^5-n}$

b $\displaystyle\sum_{m=1}^{\infty} \frac{3m+\sin\sqrt{m}}{m^2}$

37. Evaluate the following series, or show that it diverges: $\displaystyle\sum_{n=5}^{\infty} \frac{1}{e^n}$.

38. $*$. Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{6}{7^n}$ is convergent or divergent. If it is convergent, find its value.

39. $*$. Determine, with explanation, whether each of the following series converge or diverge.

a $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots$.

b $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{2^{2n+1}}$

40. $*$. Determine, with explanation, whether each of the following series converges or diverges.

a $\displaystyle\sum_{k=2}^{\infty} \frac{\sqrt[3]{k}}{k^2 - k}$.

b $\displaystyle\sum_{k=1}^{\infty} \frac{k^{10} 10^k (k!)^2}{(2k)!}$.

c $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k(\log k)(\log\log k)}$.

41. $*$. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^3 - 4}{2n^5 - 6n}$ is convergent or divergent.

42. $*$. What is the smallest value of N such that the partial sum $\displaystyle\sum_{n=1}^{N} \frac{(-1)^n}{n \cdot 10^n}$ approximates $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 10^n}$ within an accuracy of 10^{-6}?

43. $*$. It is known that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$ (you don't have to show this). Find N so that S_N, the N^{th} partial sum of the series, satisfies $|\frac{\pi^2}{12} - S_N| \le 10^{-6}$. Be sure to say why your method can be applied to this particular series.

44. $*$. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)^2}$ converges to some number S (you don't have to prove this). According to the Alternating Series Estimation Theorem, what is the smallest value of N for which the N^{th} partial sum of the series is at most $\frac{1}{100}$ away from S? For this value of N, write out the N^{th} partial sum of the series.

Exercises — Stage 3 A number of phenomena roughly follow a distribution called Zipf's law. We discuss some of these in Questions 52 and 53.

45. $*$. Determine, with explanation, whether the following series converge or diverge.

a $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{9^n n!}$

b $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\log n}}$

46. $*$. (a) Prove that $\displaystyle\int_2^\infty \frac{x+\sin x}{1+x^2}\,\mathrm{d}x$ diverges.

(b) Explain why you *cannot* conclude that $\displaystyle\sum_{n=1}^\infty \frac{n+\sin n}{1+n^2}$ diverges from part (a) and the Integral Test.

(c) Determine, with explanation, whether $\displaystyle\sum_{n=1}^\infty \frac{n+\sin n}{1+n^2}$ converges or diverges.

47. $*$. Show that $\displaystyle\sum_{n=1}^\infty \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ converges and find an interval of length 0.05 or less that contains its exact value.

48. $*$. Suppose that the series $\displaystyle\sum_{n=1}^\infty a_n$ converges and that $1 > a_n \geq 0$ for all n. Prove that the series $\displaystyle\sum_{n=1}^\infty \frac{a_n}{1-a_n}$ also converges.

49. $*$. Suppose that the series $\sum_{n=0}^\infty (1-a_n)$ converges, where $a_n > 0$ for $n = 0, 1, 2, 3, \cdots$. Determine whether the series $\sum_{n=0}^\infty 2^n a_n$ converges or diverges.

50. $*$. Assume that the series $\displaystyle\sum_{n=1}^\infty \frac{na_n - 2n + 1}{n+1}$ converges, where $a_n > 0$ for $n = 1, 2, \cdots$. Is the following series

$$-\log a_1 + \sum_{n=1}^\infty \log\left(\frac{a_n}{a_{n+1}}\right)$$

convergent? If your answer is NO, justify your answer. If your answer is YES, evaluate the sum of the series $-\log a_1 + \sum_{n=1}^\infty \log\left(\frac{a_n}{a_{n+1}}\right)$.

51. $*$. Prove that if $a_n \geq 0$ for all n and if the series $\displaystyle\sum_{n=1}^\infty a_n$ converges, then the series $\displaystyle\sum_{n=1}^\infty a_n^2$ also converges.

52. Suppose the frequency of word use in a language has the following pattern:

> The n-th most frequently used word accounts for $\dfrac{\alpha}{n}$ percent of the total words used.

So, in a text of 100 words, we expect the most frequently used word to appear α times, while the second-most-frequently used word should appear about $\frac{\alpha}{2}$ times, and so on.
If books written in this language use $20,000$ distinct words, then the most commonly used word accounts for roughly what percentage of total words used?

53.

Suppose the sizes of cities in a country adhere to the following pattern: if the largest city has population α, then the n-th largest city has population $\frac{\alpha}{n}$.
If the largest city in this country has 2 million people and the smallest city has 1 person, then the population of the entire country is $\sum_{n=1}^{2\times10^6} \frac{2\times10^6}{n}$. (For many n's in this sum $\frac{2\times10^6}{n}$ is not an integer. Ignore that.) Evaluate this sum approximately, with an error of no more than 1 million people.

3.4▲ Absolute and Conditional Convergence

We have now seen examples of series that converge and of series that diverge. But we haven't really discussed how robust the convergence of series is — that is, can we tweak the coefficients in some way while leaving the convergence unchanged. A good example of this is the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

This is a simple geometric series and we know it converges. We have also seen, as examples 3.3.20 and 3.3.21 showed us, that we can multiply or divide the n^{th} term by n and it will still converge. We can even multiply the n^{th} term by $(-1)^n$ (making it an alternating series), and it will still converge. Pretty robust.

On the other hand, we have explored the Harmonic series and its relatives quite a lot and we know it is much more delicate. While

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges, we also know the following two series converge:

$$\sum_{n=1}^{\infty} \frac{1}{n^{1.00000001}} \qquad \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}.$$

This suggests that the divergence of the Harmonic series is much more delicate. In this section, we discuss one way to characterise this sort of delicate convergence — especially in the presence of changes of sign.

3.4.1 ▸▸ Definitions

Definition 3.4.1 Absolute and conditional convergence.

a A series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

b If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges we say that $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

If you consider these definitions for a moment, it should be clear that absolute convergence is a stronger condition than just simple convergence. All the terms in $\sum_n |a_n|$ are forced to be positive (by the absolute value signs), so that $\sum_n |a_n|$ must be bigger than $\sum_n a_n$— making it easier for $\sum_n |a_n|$ to diverge. This is formalised by the following theorem, which is an immediate consequence of the comparison test, Theorem 3.3.8.a, with $c_n = |a_n|$.

Theorem 3.4.2 Absolute convergence implies convergence.

If the series $\sum_{n=1}^{\infty} |a_n|$ converges then the series $\sum_{n=1}^{\infty} a_n$ also converges. That is, absolute convergence implies convergence.

Recall that some of our convergence tests (for example, the integral test) may only be applied to series with positive terms. Theorem 3.4.2 opens up the possibility of applying "positive only" convergence tests to series whose terms are not all positive, by checking for "absolute convergence" rather than for plain "convergence".

Example 3.4.3 $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$.

The alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ of Example 3.3.15 converges (by the alternating series test). But the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ of Example 3.3.6 diverges (by the integral test). So the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges conditionally.

Example 3.4.4 $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$.

Because the series $\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{1}{n^2}\right| = \sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ of Example 3.3.6 converges (by the integral test), the series $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ converges absolutely, and hence converges.

Example 3.4.5 Random signs.

Imagine flipping a coin infinitely many times. Set $\sigma_n = +1$ if the n^{th} flip comes up heads and $\sigma_n = -1$ if the n^{th} flip comes up tails. The series $\sum_{n=1}^{\infty}(-1)^{\sigma_n}\frac{1}{n^2}$ is not in general an alternating series. But we know that the series $\sum_{n=1}^{\infty}\left|(-1)^{\sigma_n}\frac{1}{n^2}\right| = \sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ converges. So $\sum_{n=1}^{\infty}(-1)^{\sigma_n}\frac{1}{n^2}$ converges absolutely, and hence converges.

3.4.2 ⏩ Optional — The delicacy of conditionally convergent series

Conditionally convergent series have to be treated with great care. For example, switching the order of the terms in a finite sum does not change its value.

$$1+2+3+4+5+6=6+3+5+2+4+1$$

The same is true for absolutely convergent series. But it is *not true* for conditionally convergent series. In fact by reordering *any* conditionally convergent series, you can make it add up to *any* number you like, including $+\infty$ and $-\infty$. This very strange result is known as Riemann's rearrangement theorem, named after Bernhard Riemann (1826–1866). The following example illustrates the phenomenon.

Example 3.4.6 The alternating Harmonic series.

The alternating Harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$

is a very good example of conditional convergence. We can show, quite explicitly, how we can rearrange the terms to make it add up to two different numbers. Later, in Example 3.5.20, we'll show that this series is equal to $\log 2$. However, by rearranging the terms we can make it sum to $\frac{1}{2}\log 2$. The usual order is

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$$

For the moment think of the terms being paired as follows:

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots$$

so the denominators go odd-even odd-even. Now rearrange the terms so the denominators are odd-even-even odd-even-even:

$$\left(1-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right)+\left(\frac{1}{5}-\frac{1}{10}-\frac{1}{12}\right)+\cdots$$

Now notice that the first term in each triple is exactly twice the second term. If we now combine those terms we get

$$\begin{aligned}&\left(\underbrace{1-\frac{1}{2}}_{=1/2}-\frac{1}{4}\right)+\left(\underbrace{\frac{1}{3}-\frac{1}{6}}_{=1/6}-\frac{1}{8}\right)+\left(\underbrace{\frac{1}{5}-\frac{1}{10}}_{=1/10}-\frac{1}{12}\right)+\cdots\\ &=\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{6}-\frac{1}{8}\right)+\left(\frac{1}{10}-\frac{1}{12}\right)+\cdots\end{aligned}$$

We can now extract a factor of $\frac{1}{2}$ from each term, so

$$\begin{aligned}&=\frac{1}{2}\left(\frac{1}{1}-\frac{1}{2}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{4}\right)+\frac{1}{2}\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots\\ &=\frac{1}{2}\left[\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots\right]\end{aligned}$$

So by rearranging the terms, the sum of the series is now exactly half the original sum!

Example 3.4.6

In fact, we can go even further, and show how we can rearrange the terms of the alternating harmonic series to add up to any given number [1]. For the purposes of the example we have chosen 1.234, but it could really be any number. The example below can actually be formalised to give a proof of the rearrangement theorem.

Example 3.4.7 Reorder summands to get 1.234.

We'll show how to reorder the conditionally convergent series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ so that it adds up to exactly 1.234 (but the reader should keep in mind that any fixed number will work).

1 This is reminiscent of the accounting trick of pushing all the company's debts off to next year so that this year's accounts look really good and you can collect your bonus.

- First create two lists of numbers — the first list consisting of the positive terms of the series, in order, and the second consisting of the negative numbers of the series, in order.

$$1, \ \frac{1}{3}, \ \frac{1}{5}, \ \frac{1}{7}, \ \cdots \qquad \text{and} \qquad -\frac{1}{2}, \ -\frac{1}{4}, \ -\frac{1}{6}, \ \cdots$$

- Notice that that if we add together the numbers in the second list,we get

$$-\frac{1}{2}\left[1+\frac{1}{2}+\frac{1}{3}+\cdots\right]$$

which is just $-\frac{1}{2}$ times the harmonic series. So the numbers in the second list add up to $-\infty$.

Also, if we add together the numbers in the first list, we get

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\cdots \quad \text{which is greater than} \quad \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots$$

That is, the sum of the first set of numbers must be bigger than the sum of the second set of numbers (which is just -1 times the second list). So the numbers in the first list add up to $+\infty$.

- Now we build up our reordered series. Start by moving just enough numbers from the beginning of the first list into the reordered series to get a sum bigger than 1.234.

$$1+\frac{1}{3}=1.3333$$

We know that we can do this, because the sum of the terms in the first list diverges to $+\infty$.

- Next move just enough numbers from the beginning of the second list into the reordered series to get a number less than 1.234.

$$1+\frac{1}{3}-\frac{1}{2}=0.8333$$

Again, we know that we can do this because the sum of the numbers in the second list diverges to $-\infty$.

- Next move just enough numbers from the beginning of the remaining part of the first list into the reordered series to get a number bigger than 1.234.

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}=1.2873$$

Again, this is possible because the sum of the numbers in the first list diverges. Even though we have already used the first few numbers, the sum of the rest of the list will still diverge.

- Next move just enough numbers from the beginning of the remaining part of the second list into the reordered series to get a number less than 1.234.

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}-\frac{1}{4}=1.0373$$

- At this point the idea is clear, just keep going like this. At the end of each step, the difference between the sum and 1.234 is smaller than the magnitude of the first unused number in the lists. Since the numbers in both lists tend to zero as you go farther and farther up the list, this procedure will generate a series whose sum is exactly 1.234. Since in each step we remove at least one number from a list and we alternate between the two lists, the reordered series will contain all of the terms from $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$, with each term appearing exactly once.

Example 3.4.7

3.4.3 ▸▸ Exercises

Exercises — Stage 1

1. $*$. Decide whether the following statement is true or false. If false, provide a counterexample. If true provide a brief justification.

- If $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ converges, then $\sum_{n=1}^{\infty}b_n$ also converges.

2. Describe the series $\sum_{n=1}^{\infty}a_n$ based on whether $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}|a_n|$ converge or diverge, using vocabulary from this section where possible.

	$\sum a_n$ converges	$\sum a_n$ diverges
$\sum \lvert a_n\rvert$ converges		
$\sum \lvert a_n\rvert$ diverges		

Exercises — Stage 2

3. $*$. Determine whether the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{9n+5}$ is absolutely convergent, conditionally convergent, or divergent; justify your answer.

4. $*$. Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{1+n}$ is absolutely convergent, conditionally convergent, or divergent.

5. $*$. The series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1+4^n}{3+2^{2n}}$ either:

- converges absolutely;
- converges conditionally;
- diverges;
- or none of the above.

Determine which is correct.

6. $*$. Does the series $\displaystyle\sum_{n=5}^{\infty} \frac{\sqrt{n}\cos n}{n^2-1}$ converge conditionally, converge absolutely, or diverge?

7. $*$. Determine (with justification!) whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-\sin n}{n^6+n^2}$ converges absolutely, converges but not absolutely, or diverges.

8. $*$. Determine (with justification!) whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{(n^2+1)(n!)^2}$ converges absolutely, converges but not absolutely, or diverges.

9. $*$. Determine (with justification!) whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^{101}}$ converges absolutely, converges but not absolutely, or diverges.

10. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges.

11. Show that the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\sin n}{4} - \frac{1}{8}\right)^n$ converges.

12. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n - \cos^2 n + \frac{1}{2}}{2^n}$ converges.

Exercises — Stage 3

13. $*$. Both parts of this question concern the series $S = \sum_{n=1}^{\infty}(-1)^{n-1}24n^2e^{-n^3}$.

a Show that the series S converges absolutely.

b Suppose that you approximate the series S by its fifth partial sum S_5. Give an upper bound for the error resulting from this approximation.

14. You may assume without proof the following:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} = \cos(1)$$

Using this fact, approximate $\cos 1$ as a rational number, accurate to within $\frac{1}{1000}$.
Check your answer against a calculator's approximation of $\cos(1)$: what was your actual error?

15. Let a_n be defined as

$$a_n = \begin{cases} -e^{n/2} & \text{if } n \text{ is prime} \\ n^2 & \text{if } n \text{ is not prime} \end{cases}$$

Show that the series $\sum_{n=1}^{\infty} \frac{a_n}{e^n}$ converges.

3.5▲ Power Series

Let's return to the simple geometric series

$$\sum_{n=0}^{\infty} x^n$$

where x is some real number. As we have seen (back in Example 3.2.4 and Lemma 3.2.5), for $|x| < 1$ this series converges to a limit, that varies with x, while for $|x| \geq 1$ the series diverges. Consequently we can consider this series to be a function of x

$$f(x) = \sum_{n=0}^{\infty} x^n \qquad \text{on the domain } |x| < 1.$$

Furthermore (also from Example 3.2.4 and Lemma 3.2.5) we know what the function is.

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Hence we can consider the series $\sum_{n=0}^{\infty} x^n$ as a new way of representing the function $\frac{1}{1-x}$ when $|x| < 1$. This series is an example of a power series.

Of course, representing a function as simple as $\frac{1}{1-x}$ by a series doesn't seem like it is going to make life easier. However the idea of representing a function by a series turns out to be extremely helpful. Power series turn out to be very robust mathematical objects and interact very nicely with not only standard arithmetic operations, but also with differentiation and integration (see Theorem 3.5.13). This means, for example, that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{1-x}\right\} &= \frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty} x^n && \text{provided } |x| < 1 \\ &= \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} x^n && \text{just differentiate term by term} \\ &= \sum_{n=0}^{\infty} n x^{n-1} \end{aligned}$$

and in a very similar way

$$\begin{aligned} \int \frac{1}{1-x}\mathrm{d}x &= \int \sum_{n=0}^{\infty} x^n \mathrm{d}x && \text{provided } |x| < 1 \\ &= \sum_{n=0}^{\infty} \int x^n \mathrm{d}x && \text{just integrate term by term} \\ &= C + \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} \end{aligned}$$

We are hiding some mathematics under the word "just" in the above, but you can see that once we have a power series representation of a function, differentiation and integration become very straightforward.

So we should set as our goal for this section, the development of machinery to define and understand power series. This will allow us to answer questions [1] like

$$\text{Is } \; e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \;?$$

Our starting point (now that we have equipped ourselves with basic ideas about series), is the definition of power series.

1 Recall that $n! = 1 \times 2 \times 3 \times \cdots \times n$ is called "n factorial". By convention $0! = 1$.

3.5.1 ➤ Definitions

Definition 3.5.1

A series of the form

$$A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 + \cdots = \sum_{n=0}^{\infty} A_n(x-c)^n$$

is called a *power series in* $(x-c)$ or a *power series centered on* c. The numbers A_n are called the coefficients of the power series.
One often considers power series centered on $c=0$ and then the series reduces to

$$A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots = \sum_{n=0}^{\infty} A_nx^n$$

For example $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is the power series with $c=0$ and $A_n = \frac{1}{n!}$. Typically, as in that case, the coefficients A_n are given fixed numbers, but the "x" is to be thought of as a variable. Thus each power series is really a whole family of series — a different series for each value of x.

One possible value of x is $x=c$ and then the series reduces [2] to

$$\begin{aligned}\sum_{n=0}^{\infty} A_n(x-c)^n\Big|_{x=c} &= \sum_{n=0}^{\infty} A_n(c-c)^n \\ &= \underbrace{A_0}_{n=0} + \underbrace{0}_{n=1} + \underbrace{0}_{n=2} + \underbrace{0}_{n=3} + \cdots\end{aligned}$$

and so simply converges to A_0.

We now know that a power series converges when $x=c$. We can now use our convergence tests to determine for what other values of x the series converges. Perhaps most straightforward is the ratio test. The n^{th} term in the series $\sum_{n=0}^{\infty} A_n(x-c)^n$ is $a_n = A_n(x-c)^n$. To apply the ratio test we need to compute the limit

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{A_{n+1}(x-c)^{n+1}}{A_n(x-c)^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| \cdot |x-c| \\ &= |x-c| \cdot \lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|.\end{aligned}$$

When we do so there are several possible outcomes.

- If the limit of ratios exists and is nonzero

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = A \neq 0,$$

2 *By convention*, when the term $(x-c)^0$ appears in a power series, it has value 1 for all values of x, even $x=c$.

then the ratio test says that the series $\sum_{n=0}^{\infty} A_n(x-c)^n$

- converges when $A \cdot |x-c| < 1$, i.e. when $|x-c| < \frac{1}{A}$, and
- diverges when $A \cdot |x-c| > 1$, i.e. when $|x-c| > \frac{1}{A}$.

Because of this, when the limit exists, the quantity

Equation 3.5.2 Radius of convergence.

$$R = \frac{1}{A} = \left[\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|\right]^{-1}$$

is called the *radius of convergence* of the series [3].

- If the limit of ratios exists and is zero

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = 0$$

then $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right||x-c| = 0$ for every x and the ratio test tells us that the series $\sum_{n=0}^{\infty} A_n(x-c)^n$ converges for every number x. In this case we say that the series has an infinite radius of convergence.

- If the limit of ratios diverges to $+\infty$

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = +\infty$$

then $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right||x-c| = +\infty$ for every $x \neq c$. The ratio test then tells us that the series $\sum_{n=0}^{\infty} A_n(x-c)^n$ diverges for every number $x \neq c$. As we have seen above, when $x = c$, the series reduces to $A_0 + 0 + 0 + 0 + 0 + \cdots$, which of course converges. In this case we say that the series has radius of convergence zero.

- If $\left|\frac{A_{n+1}}{A_n}\right|$ does not approach a limit as $n \to \infty$, then we learn nothing from the ratio test and we must use other tools to understand the convergence of the series.

All of these possibilities do happen. We give an example of each below. But first, the concept of "radius of convergence" is important enough to warrant a formal definition.

3 The use of the word "radius" might seem a little odd here, since we are really describing the interval in the real line where the series converges. However, when one starts to consider power series over complex numbers, the radius of convergence does describe a circle inside the complex plane and so "radius" is a more natural descriptor.

Definition 3.5.3

a Let $0 < R < \infty$. If $\sum_{n=0}^{\infty} A_n(x-c)^n$ converges for $|x-c| < R$, and diverges for $|x-c| > R$, then we say that the series has radius of convergence R.

b If $\sum_{n=0}^{\infty} A_n(x-c)^n$ converges for every number x, we say that the series has an infinite radius of convergence.

c If $\sum_{n=0}^{\infty} A_n(x-c)^n$ diverges for every $x \neq c$, we say that the series has radius of convergence zero.

Example 3.5.4 Finite nonzero radius of convergence.

We already know that, if $a \neq 0$, the geometric series $\sum\limits_{n=0}^{\infty} ax^n$ converges when $|x| < 1$ and diverges when $|x| \geq 1$. So, in the terminology of Definition 3.5.3, the geometric series has radius of convergence $R = 1$. As a consistency check, we can also compute R using 3.5.2. The series $\sum\limits_{n=0}^{\infty} ax^n$ has $A_n = a$. So

$$R = \left[\lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right|\right]^{-1} = \left[\lim_{n\to\infty} 1\right]^{-1} = 1$$

as expected.

Example 3.5.5 Radius of convergence $= +\infty$.

The series $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ has $A_n = \frac{1}{n!}$. So

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| &= \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \\
&= \lim_{n\to\infty} \frac{n!}{(n+1)!} \\
&= \lim_{n\to\infty} \frac{1 \times 2 \times 3 \times \cdots \times n}{1 \times 2 \times 3 \times \cdots \times n \times (n+1)} \\
&= \lim_{n\to\infty} \frac{1}{n+1} \\
&= 0
\end{aligned}$$

and $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ has radius of convergence ∞. It converges for every x.

Example 3.5.6 Radius of convergence = 0.

The series $\sum_{n=0}^{\infty} n!x^n$ has $A_n = n!$. So

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| &= \lim_{n\to\infty}\frac{(n+1)!}{n!} = \lim_{n\to\infty}\frac{1\times 2\times 3\times 4\times\cdots\times n\times(n+1)}{1\times 2\times 3\times 4\times\cdots\times n}\\ &= \lim_{n\to\infty}(n+1)\\ &= +\infty\end{aligned}$$

and $\sum_{n=0}^{\infty} n!x^n$ has radius of convergence zero [a]. It converges only for $x = 0$, where it takes the value $0! = 1$.

[a] Because of this, it might seem that such a series is fairly pointless. However there are all sorts of mathematical games that can be played with them without worrying about their convergence. Such "formal" power series can still impart useful information and the interested reader is invited to look up "generating functions" with their preferred search engine.

Example 3.5.7 An awkward series to test.

Comparing the series

$$1 + 2x + x^2 \quad + 2x^3 \quad + x^4 \quad + 2x^5 \quad + \cdots$$

to

$$\sum_{n=1}^{\infty} A_n x^n = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \cdots$$

we see that

$$A_0 = 1 \quad A_1 = 2 \qquad A_2 = 1 \qquad A_3 = 2 \qquad A_4 = 1 \qquad A_5 = 2 \qquad \cdots$$

so that

$$\frac{A_1}{A_0} = 2 \qquad \frac{A_2}{A_1} = \frac{1}{2} \qquad \frac{A_3}{A_2} = 2 \qquad \frac{A_4}{A_3} = \frac{1}{2} \qquad \frac{A_5}{A_4} = 2 \qquad \cdots$$

and $\frac{A_{n+1}}{A_n}$ does not converge as $n \to \infty$. Since the limit of the ratios does not exist, we cannot tell anything from the ratio test. Nonetheless, we can still figure out for which x's our power series converges.

- Because every coefficient A_n is either 1 or 2, the n^{th} term in our series obeys

$$\left|A_n x^n\right| \le 2|x|^n$$

and so is smaller than the n^{th} term in the geometric series $\sum_{n=0}^{\infty} 2|x|^n$. This geometric series converges if $|x| < 1$. So, by the comparison test, our series converges for $|x| < 1$ too.

- Since every A_n is at least one, the n^{th} term in our series obeys

$$\left|A_n x^n\right| \ge |x|^n$$

If $|x| \ge 1$, this $a_n = A_n x^n$ cannot converge to zero as $n \to \infty$, and our series diverges by the divergence test.

In conclusion, our series converges if and only if $|x| < 1$, and so has radius of convergence 1.

Example 3.5.7

Example 3.5.8 A series from π.

Lets construct a series from the digits of π. Now to avoid dividing by zero, let us set

$$A_n = 1 + \text{the } n^{\text{th}} \text{ digit of } \pi$$

Since $\pi = 3.141591\ldots$

$$A_0 = 4 \quad A_1 = 2 \quad A_2 = 5 \quad A_3 = 2 \quad A_4 = 6 \quad A_5 = 10 \quad A_6 = 2 \quad \cdots$$

Consequently every A_n is an integer between 1 and 10 and gives us the series

$$\sum_{n=0}^{\infty} A_n x^n = 4 + 2x + 5x^2 + 2x^3 + 6x^4 + 10x^5 + \cdots$$

The number π is irrational [a] and consequently the ratio $\frac{A_{n+1}}{A_n}$ cannot have a limit as $n \to \infty$. If you do not understand why this is the case then don't worry too much about it [b]. As in the last example, the limit of the ratios does not exist and we cannot tell anything from the ratio test. But we can still figure out for which x's it converges.

- Because every coefficient A_n is no bigger (in magnitude) than 10, the n^{th} term in our series obeys

$$\left|A_n x^n\right| \le 10|x|^n$$

and so is smaller than the n^{th} term in the geometric series $\sum_{n=0}^{\infty} 10|x|^n$. This geometric series converges if $|x| < 1$. So, by the comparison test, our series converges for $|x| < 1$ too.

- Since every A_n is at least one, the n^{th} term in our series obeys

$$\left|A_n x^n\right| \geq |x|^n$$

If $|x| \geq 1$, this $a_n = A_n x^n$ cannot converge to zero as $n \to \infty$, and our series diverges by the divergence test.

In conclusion, our series converges if and only if $|x| < 1$, and so has radius of convergence 1.

a We give a proof of this in the optional §3.7 at the end of this chapter.
b This is a little beyond the scope of the course. Roughly speaking, think about what would happen if the limit of the ratios did exist. If the limit were smaller than 1, then it would tell you that the terms of our series must be getting smaller and smaller and smaller — which is impossible because they are all integers between 1 and 10. Similarly if the limit existed and were bigger than 1 then the terms of the series would have to get bigger and bigger and bigger — also impossible. Hence if the ratio exists then it must be equal to 1 — but in that case because the terms are integers, they would have to be all equal when n became big enough. But that means that the expansion of π would be eventually periodic — something that only rational numbers do (a proof is given in the optional §3.7 at the end of this chapter).

Example 3.5.8

Though we won't prove it, it is true that every power series has a radius of convergence, whether or not the limit $\lim\limits_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right|$ exists.

Theorem 3.5.9

Let $\sum\limits_{n=0}^{\infty} A_n (x-c)^n$ be a power series. Then one of the following alternatives must hold.

a The power series converges for every number x. In this case we say that the radius of convergence is ∞.

b There is a number $0 < R < \infty$ such that the series converges for $|x-c| < R$ and diverges for $|x-c| > R$. Then R is called the radius of convergence.

c The series converges for $x = c$ and diverges for all $x \neq c$. In this case, we say that the radius of convergence is 0.

Definition 3.5.10

Consider the power series

$$\sum_{n=0}^{\infty} A_n(x-c)^n.$$

The set of real x-values for which it converges is called the interval of convergence of the series.

Suppose that the power series $\sum_{n=0}^{\infty} A_n(x-c)^n$ has radius of convergence R. Then from Theorem 3.5.9, we have that

- if $R=\infty$, then its interval of convergence is $-\infty < x < \infty$, which is also denoted $(-\infty, \infty)$, and
- if $R=0$, then its interval of convergence is just the point $x=c$, and
- if $0<R<\infty$, then we know that the series converges for any x which obeys

  $$\begin{aligned} |x-c|<R \quad &\text{or equivalently} \quad -R<x-c<R \\ &\text{or equivalently} \quad c-R<x<c+R \end{aligned}$$

  But we do not (yet) know whether or not the series converges at the two end points of that interval. We do know, however, that its interval of convergence must be one of

  - $c-R<x<c+R$, which is also denoted $(c-R\,,\,c+R)$, or
  - $c-R\le x<c+R$, which is also denoted $[c-R\,,\,c+R)$, or
  - $c-R<x\le c+R$, which is also denoted $(c-R\,,\,c+R]$, or
  - $c-R\le x\le c+R$, which is also denoted $[c-R\,,\,c+R]$.

To reiterate — while the radius convergence, R with $0<R<\infty$, tells us that the series converges for $|x-c|<R$ and diverges for $|x-c|>R$, it does not (by itself) tell us whether or not the series converges when $|x-c|=R$, i.e. when $x=c\pm R$. The following example shows that all four possibilities can occur.

Example 3.5.11 The series $\sum_{n=1}^{\infty} \frac{x^n}{n^p}$.

Let p be any real number[a] and consider the series $\sum_{n=1}^{\infty} \frac{x^n}{n^p}$. This series has $A_n=\frac{1}{n^p}$. Since

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\frac{n^p}{(n+1)^p} = \lim_{n\to\infty}\frac{1}{(1+\frac{1}{n})^p} = 1$$

the series has radius of convergence 1. So it certainly converges for $|x|<1$ and diverges for $|x|>1$. That just leaves $x=\pm 1$.

- When $x=1$, the series reduces to $\sum_{n=1}^{\infty}\frac{1}{n^p}$. We know, from Example 3.3.6, that this series converges if and only if $p > 1$.

- When $x = -1$, the series reduces to $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$. By the alternating series test, Theorem 3.3.14, this series converges whenever $p > 0$ (so that $\frac{1}{n^p}$ tends to zero as n tends to infinity). When $p \le 0$ (so that $\frac{1}{n^p}$ does *not* tend to zero as n tends to infinity), it diverges by the divergence test, Theorem 3.3.1.

So

- The power series $\sum_{n=1}^{\infty} x^n$ (i.e. $p = 0$) has interval of convergence $-1 < x < 1$.
- The power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ (i.e. $p = 1$) has interval of convergence $-1 \le x < 1$.
- The power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$ (i.e. $p = 1$) has interval of convergence $-1 < x \le 1$.
- The power series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ (i.e. $p = 2$) has interval of convergence $-1 \le x \le 1$.

[a] We avoid problems with 0^p by starting the series from $n = 1$.

Example 3.5.11

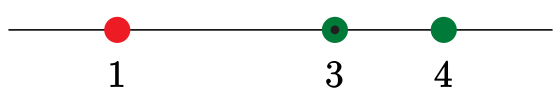
Example 3.5.12 Playing with intervals of convergence.

We are told that a certain power series with centre $c = 3$, converges at $x = 4$ and diverges at $x = 1$. What else can we say about the convergence or divergence of the series for other values of x?

We are told that the series is centred at 3, so its terms are all powers of $(x - 3)$ and it is of the form

$$\sum_{n \ge 0} A_n (x - 3)^n.$$

A good way to summarise the convergence data we are given is with a figure like the one below. Green dots mark the values of x where the series is known to converge. (Recall that every power series converges at its centre.) The red dot marks the value of x where the series is known to diverge. The bull's eye marks the centre.



Can we say more about the convergence and/or divergence of the series for other values of x? Yes!

Let us think about the radius of convergence, R, of the series. We know that it must exist and the information we have been given allows us to bound R. Recall that

- the series converges at x provided that $|x-3| < R$ and
- the series diverges at x if $|x-3| > R$.

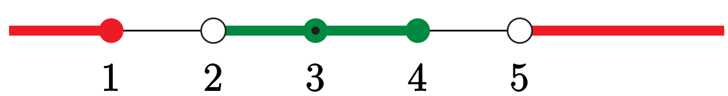
We have been told that

- the series converges when $x = 4$, which tells us that
  - $x = 4$ cannot obey $|x-3| > R$ so
  - $x = 4$ must obey $|x-3| \leq R$, i.e. $|4-3| \leq R$, i.e. $R \geq 1$
- the series diverges when $x = 1$ so we also know that
  - $x = 1$ cannot obey $|x-3| < R$ so
  - $x = 1$ must obey $|x-3| \geq R$, i.e. $|1-3| \geq R$, i.e. $R \leq 2$

We still don't know R exactly. But we do know that $1 \leq R \leq 2$. Consequently,

- since 1 is the smallest that R could be, the series certainly converges at x if $|x-3| < 1$, i.e. if $2 < x < 4$ and
- since 2 is the largest that R could be, the series certainly diverges at x if $|x-3| > 2$, i.e. if $x > 5$ or if $x < 1$.

The following figure provides a resume of all of this convergence data — there is convergence at green x's and divergence at red x's.



Notice that from the data given we cannot say anything about the convergence or divergence of the series on the intervals $(1, 2]$ and $(4, 5]$.
One lesson that we can derive from this example is that,

- if a series has centre c and converges at a,
- then it also converges at all points between c and a, as well as at all points of distance strictly less than $|a-c|$ from c on the other side of c from a.

Example 3.5.12

3.5.2 ⏩ Working With Power Series

Just as we have done previously with limits, differentiation and integration, we can construct power series representations of more complicated functions by using those of simpler functions. Here is a theorem that helps us to do so.

Theorem 3.5.13 Operations on Power Series.

Assume that the functions $f(x)$ and $g(x)$ are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n(x-c)^n \qquad g(x) = \sum_{n=0}^{\infty} B_n(x-c)^n$$

for all x obeying $|x-c| < R$. In particular, we are assuming that both power series have radius of convergence at least R. Also let K be a constant. Then

$$\begin{aligned}
f(x) + g(x) &= \sum_{n=0}^{\infty} [A_n + B_n]\,(x-c)^n \\
K f(x) &= \sum_{n=0}^{\infty} K\, A_n\,(x-c)^n \\
(x-c)^N f(x) &= \sum_{n=0}^{\infty} A_n\,(x-c)^{n+N} \quad \text{for any integer } N \geq 1 \\
&= \sum_{k=N}^{\infty} A_{k-N}\,(x-c)^k \quad \text{where } k = n+N \\
f'(x) &= \sum_{n=0}^{\infty} A_n\, n\,(x-c)^{n-1} = \sum_{n=1}^{\infty} A_n\, n\,(x-c)^{n-1} \\
\int_c^x f(t)\mathrm{d}t &= \sum_{n=0}^{\infty} A_n \frac{(x-c)^{n+1}}{n+1} \\
\int f(x)\ \mathrm{d}x &= \left[\sum_{n=0}^{\infty} A_n \frac{(x-c)^{n+1}}{n+1}\right] + C \quad \text{with } C \text{ an arbitrary constant}
\end{aligned}$$

for all x obeying $|x-c| < R$.
In particular the radius of convergence of each of the six power series on the right hand sides is at least R. In fact, if R is the radius of convergence of $\sum_{n=0}^{\infty} A_n(x-c)^n$, then R is also the radius of convergence of all of the above right hand sides, with the possible exceptions of $\sum_{n=0}^{\infty} [A_n + B_n]\,(x-c)^n$ and $\sum_{n=0}^{\infty} K A_n\,(x-c)^n$ when $K = 0$.

Example 3.5.14 More on the last part of Theorem 3.5.13.

The last statement of Theorem 3.5.13 might seem a little odd, but consider the following two power series centred at 0:

$$\sum_{n=0}^{\infty} 2^n x^n \text{ and } \sum_{n=0}^{\infty} (1-2^n)x^n.$$

The ratio test tells us that they both have radius of convergence $R = \frac{1}{2}$. However their sum is

$$\sum_{n=0}^{\infty} 2^n x^n + \sum_{n=0}^{\infty} (1-2^n)x^n = \sum_{n=0}^{\infty} x^n$$

which has the larger radius of convergence 1.
A more extreme example of the same phenomenon is supplied by the two series

$$\sum_{n=0}^{\infty} 2^n x^n \text{ and } \sum_{n=0}^{\infty} (-2^n)x^n.$$

They are both geometric series with radius of convergence $R = \frac{1}{2}$. But their sum is

$$\sum_{n=0}^{\infty} 2^n x^n + \sum_{n=0}^{\infty} (-2^n)x^n = \sum_{n=0}^{\infty} (0)x^n$$

which has radius of convergence $+\infty$.

We'll now use this theorem to build power series representations for a bunch of functions out of the one simple power series representation that we know — the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{for all } |x| < 1$$

Example 3.5.15 $\frac{1}{1-x^2}$.

Find a power series representation for $\frac{1}{1-x^2}$.
Solution: The secret to finding power series representations for a good many functions is to manipulate them into a form in which $\frac{1}{1-y}$ appears and use the geometric series representation $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$. We have deliberately renamed the variable to y here — it does not have to be x. We can use that strategy to find a power series expansion for

$\frac{1}{1-x^2}$ — we just have to recognize that $\frac{1}{1-x^2}$ is the same as $\frac{1}{1-y}$ if we set y to x^2.

$$\begin{aligned}\frac{1}{1-x^2} &= \left.\frac{1}{1-y}\right|_{y=x^2} = \left[\sum_{n=0}^{\infty} y^n\right]_{y=x^2} \qquad \text{if } |y|<1\text{, i.e. } |x|<1\\ &= \sum_{n=0}^{\infty}\left(x^2\right)^n = \sum_{n=0}^{\infty} x^{2n}\\ &= 1+x^2+x^4+x^6+\cdots\end{aligned}$$

This is a perfectly good power series. There is nothing wrong with the power of x being $2n$. (This just means that the coefficients of all odd powers of x are zero.) In fact, you should try to always write power series in forms that are as easy to understand as possible. The geometric series that we used at the end of the first line converges for

$$|y|<1 \iff \left|x^2\right|<1 \iff |x|<1$$

So our power series has radius of convergence 1 and interval of convergence $-1<x<1$.

Example 3.5.15

Example 3.5.16 $\frac{x}{2+x^2}$.

Find a power series representation for $\frac{x}{2+x^2}$.

Solution: This example is just a more algebraically involved variant of the last one. Again, the strategy is to manipulate $\frac{x}{2+x^2}$ into a form in which $\frac{1}{1-y}$ appears.

$$\begin{aligned}\frac{x}{2+x^2} &= \frac{x}{2}\,\frac{1}{1+\frac{x^2}{2}} = \frac{x}{2}\,\frac{1}{1-\left(-\frac{x^2}{2}\right)} \qquad \text{set } -\frac{x^2}{2}=y\\ &= \frac{x}{2}\left.\frac{1}{1-y}\right|_{y=-\frac{x^2}{2}} = \frac{x}{2}\left[\sum_{n=0}^{\infty} y^n\right]_{y=-\frac{x^2}{2}} \qquad \text{if } |y|<1\end{aligned}$$

Now use Theorem 3.5.13 twice

$$\begin{aligned}&= \frac{x}{2}\sum_{n=0}^{\infty}\left(-\frac{x^2}{2}\right)^n = \frac{x}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}x^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{2n+1}\\ &= \frac{x}{2}-\frac{x^3}{4}+\frac{x^5}{8}-\frac{x^7}{16}+\cdots\end{aligned}$$

The geometric series that we used in the second line converges when

$$\begin{aligned}|y|<1 &\iff \left|-\frac{x^2}{2}\right|<1\\ &\iff |x|^2<2 \iff |x|<\sqrt{2}\end{aligned}$$

So the given power series has radius of convergence $\sqrt{2}$ and interval of convergence $-\sqrt{2}<x<\sqrt{2}$.

Example 3.5.17 Nonzero centre.

Find a power series representation for $\frac{1}{5-x}$ with centre 3.

Solution: The new wrinkle in this example is the requirement that the centre be 3. That the centre is to be 3 means that we need a power series in powers of $x-c$, with $c=3$. So we are looking for a power series of the form $\sum_{n=0}^{\infty} A_n(x-3)^n$. The easy way to find such a series is to force an $x-3$ to appear by adding and subtracting a 3.

$$\frac{1}{5-x} = \frac{1}{5-(x-3)-3} = \frac{1}{2-(x-3)}$$

Now we continue, as in the last example, by manipulating $\frac{1}{2-(x-3)}$ into a form in which $\frac{1}{1-y}$ appears.

$$\begin{aligned}
\frac{1}{5-x} = \frac{1}{2-(x-3)} &= \frac{1}{2}\,\frac{1}{1-\frac{x-3}{2}} && \text{set } \frac{x-3}{2} = y \\
&= \frac{1}{2}\,\frac{1}{1-y}\bigg|_{y=\frac{x-3}{2}} = \frac{1}{2}\left[\sum_{n=0}^{\infty} y^n\right]_{y=\frac{x-3}{2}} && \text{if } |y|<1 \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x-3}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(x-3)^n}{2^{n+1}} \\
&= \frac{x-3}{2} + \frac{(x-3)^2}{4} + \frac{(x-3)^3}{8} + \cdots
\end{aligned}$$

The geometric series that we used in the second line converges when

$$\begin{aligned}
|y|<1 &\iff \left|\frac{x-3}{2}\right| < 1 \\
&\iff |x-3| < 2 \\
&\iff -2 < x-3 < 2 \\
&\iff 1 < x < 5
\end{aligned}$$

So the power series has radius of convergence 2 and interval of convergence $1<x<5$.

In the previous two examples, to construct a new series from an existing series, we replaced x by a simple function. The following theorem gives us some more (but certainly not all) commonly used substitutions.

Theorem 3.5.18 Substituting in a Power Series.

Assume that the function $f(x)$ is given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n x^n$$

for all x in the interval I. Also let K and k be real constants. Then

$$f\left(Kx^k\right) = \sum_{n=0}^{\infty} A_n K^n \, x^{kn}$$

whenever Kx^k is in I. In particular, if $\sum_{n=0}^{\infty} A_n x^n$ has radius of convergence R, K is nonzero and k is a natural number, then $\sum_{n=0}^{\infty} A_n K^n \, x^{kn}$ has radius of convergence $\sqrt[k]{R/|K|}$.

Example 3.5.19 $\frac{1}{(1-x)^2}$.

Find a power series representation for $\frac{1}{(1-x)^2}$.

Solution: Once again the trick is to express $\frac{1}{(1-x)^2}$ in terms of $\frac{1}{1-x}$. Notice that

$$\begin{aligned}
\frac{1}{(1-x)^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{1-x}\right\} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n=0}^{\infty} x^n\right\} \\
&= \sum_{n=1}^{\infty} nx^{n-1} \qquad \text{by Theorem 3.5.13}
\end{aligned}$$

Note that the $n = 0$ term has disappeared because, for $n = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = \frac{\mathrm{d}}{\mathrm{d}x}x^0 = \frac{\mathrm{d}}{\mathrm{d}x}1 = 0$$

Also note that the radius of convergence of this series is one. We can see this via Theorem 3.5.13. That theorem tells us that the radius of convergence of a power series is not changed by differentiation — and since $\sum_{n=0}^{\infty} x^n$ has radius of convergence one, so too does its derivative.

Without much more work we can determine the interval of convergence by testing at $x = \pm 1$. When $x = \pm 1$ the terms of the series do not go to zero as $n \to \infty$ and so, by the divergence test, the series does not converge there. Hence the interval of convergence for the series is $-1 < x < 1$.

Notice that, in this last example, we differentiated a known series to get to our answer. As per Theorem 3.5.13, the radius of convergence didn't change. In addition, in this particular example, the interval of convergence didn't change. This is not always the case. Differentiation of some series causes the interval of convergence to shrink. In particular the differentiated series may no longer be convergent at the end points of the interval [4]. Similarly, when we integrate a power series the radius of convergence is unchanged, but the interval of convergence may expand to include one or both ends, as illustrated by the next example.

Example 3.5.20 $\log(1+x)$.

Find a power series representation for $\log(1+x)$.

Solution: Recall that $\frac{\mathrm{d}}{\mathrm{d}x}\log(1+x) = \frac{1}{1+x}$ so that $\log(1+t)$ is an antiderivative of $\frac{1}{1+t}$ and

$$\begin{aligned}\log(1+x) &= \int_0^x \frac{\mathrm{d}t}{1+t} = \int_0^x \Big[\sum_{n=0}^\infty (-t)^n\Big]\mathrm{d}t \\ &= \sum_{n=0}^\infty \int_0^x (-t)^n \mathrm{d}t \qquad \text{by Theorem 3.5.13} \\ &= \sum_{n=0}^\infty (-1)^n \frac{x^{n+1}}{n+1} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\end{aligned}$$

Theorem 3.5.13 guarantees that the radius of convergence is exactly one (the radius of convergence of the geometric series $\sum_{n=0}^\infty (-t)^n$) and that

$$\log(1+x) = \sum_{n=0}^\infty (-1)^n \frac{x^{n+1}}{n+1} \qquad \text{for all} \quad -1 < x < 1$$

When $x = -1$ our series reduces to $\sum_{n=0}^\infty \frac{-1}{n+1}$, which is (minus) the harmonic series and so diverges. That's no surprise — $\log(1+(-1)) = \log 0 = -\infty$. When $x = 1$, the series converges by the alternating series test. It is possible to prove, by continuity, though we won't do so here, that the sum is $\log 2$. So the interval of convergence is $-1 < x \le 1$.

Example 3.5.21 $\arctan x$.

Find a power series representation for $\arctan x$.

Solution: Recall that $\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$ so that $\arctan t$ is an antiderivative of $\frac{1}{1+t^2}$ and

$$\arctan x = \int_0^x \frac{\mathrm{d}t}{1+t^2} = \int_0^x \Big[\sum_{n=0}^\infty \left(-t^2\right)^n\Big]\mathrm{d}t = \sum_{n=0}^\infty \int_0^x (-1)^n t^{2n} \mathrm{d}t$$

4 Consider the power series $\sum_{n=1}^\infty \frac{x^n}{n}$. We know that its interval of convergence is $-1 \le x < 1$. (Indeed see the next example.) When we differentiate the series we get the geometric series $\sum_{n=0}^\infty x^n$ which has interval of convergence $-1 < x < 1$.

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

Theorem 3.5.13 guarantees that the radius of convergence is exactly one (the radius of convergence of the geometric series $\sum_{n=0}^{\infty}(-t^2)^n$) and that

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad \text{for all } -1 < x < 1$$

When $x = \pm 1$, the series converges by the alternating series test. So the interval of convergence is $-1 \le x \le 1$. It is possible to prove, though once again we won't do so here, that when $x = \pm 1$, the series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ converges to the value of the left hand side, $\arctan x$, at $x = \pm 1$. That is, to $\arctan(\pm 1) = \pm\frac{\pi}{4}$.

Example 3.5.21

The operations on power series dealt with in Theorem 3.5.13 are fairly easy to apply. Unfortunately taking the product, ratio or composition of two power series is more involved and is beyond the scope of this course [5]. Unfortunately Theorem 3.5.13 alone will not get us power series representations of many of our standard functions (like e^x and $\sin x$). Fortunately we can find such representations by extending Taylor polynomials [6] to Taylor series.

3.5.3 ▸▸ Exercises

Exercises — Stage 1

1. Suppose $f(x) = \sum_{n=0}^{\infty} \left(\frac{3-x}{4}\right)^n$. What is $f(1)$?

2. Suppose $f(x) = \sum_{n=1}^{\infty} \frac{(x-5)^n}{n!+2}$. Give a power series representation of $f'(x)$.

5 As always, a quick visit to your favourite search engine will direct the interested reader to more information.

6 Now is a good time to review your notes from last term, though we'll give you a whirlwind review over the next page or two.

3. Let $f(x) = \sum_{n=a}^{\infty} A_n(x-c)^n$ for some positive constants a and c, and some sequence of constants $\{A_n\}$. For which values of x does $f(x)$ definitely converge?

4. Let $f(x)$ be a power series centred at $c = 5$. If $f(x)$ converges at $x = -1$, and diverges at $x = 11$, what is the radius of convergence of $f(x)$?

Exercises — Stage 2

5. $*$. (a) Find the radius of convergence of the series

$$\sum_{k=0}^{\infty} (-1)^k 2^{k+1} x^k$$

(b) You are given the formula for the sum of a geometric series, namely:

$$1 + r + r^2 + \cdots = \frac{1}{1-r}, \qquad |r| < 1$$

Use this fact to evaluate the series in part (a).

6. $*$. Find the radius of convergence for the power series $\sum_{k=0}^{\infty} \frac{x^k}{10^{k+1}(k+1)!}$

7. $*$. Find the radius of convergence for the power series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$.

8. $*$. Consider the power series $\sum_{n=1}^{\infty} \frac{(-1)^n(x+2)^n}{\sqrt{n}}$, where x is a real number. Find the interval of convergence of this series.

9. $*$. Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\frac{x+1}{3}\right)^n$$

10. $*$. Find the *interval of convergence* for the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^{4/5}(5^n-4)}.$$

11. $*$. Find all values x for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x+2)^n}{n^2}$ converges.

12. $*$. Find the interval of convergence for $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{n}(x-1)^n$.

13. $*$. Find, with explanation, the radius of convergence and the interval of convergence of the power series

$$\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{2^n(n+2)}$$

14. $*$. Find the *interval of convergence* for the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n n^2 (x-a)^{2n}$ where a is a constant.

15. $*$. Find the interval of convergence of the following series:

a $\displaystyle\sum_{k=1}^{\infty} \frac{(x+1)^k}{k^2 9^k}$.

b $\displaystyle\sum_{k=1}^{\infty} a_k (x-1)^k$, where $a_k > 0$ for $k = 1, 2, \cdots$ and $\displaystyle\sum_{k=1}^{\infty} \left(\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}}\right) = \frac{a_1}{a_2}$.

16. $*$. Find a power series representation for $\dfrac{x^3}{1-x}$.

17. Suppose $f'(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{n+2}$, and $\displaystyle\int_5^x f(t)\mathrm{d}t = 3x + \sum_{n=1}^{\infty} \frac{(x-1)^{n+1}}{n(n+1)^2}$.
Give a power series representation of $f(x)$.

Exercises — Stage 3

18. $*$. Determine the values of x for which the series

$$\sum_{n=2}^{\infty} \frac{x^n}{3^{2n} \log n}$$

converges absolutely, converges conditionally, or diverges.

19. $*$. (a) Find the power–series representation for $\displaystyle\int \frac{1}{1+x^3}\,\mathrm{d}x$ centred at 0 (i.e. in powers of x).

(b) The power series above is used to approximate $\displaystyle\int_0^{1/4} \frac{1}{1+x^3}\,\mathrm{d}x$. How many terms are required to guarantee that the resulting approximation is within 10^{-5} of the exact value? Justify your answer.

20. $*$. (a) Show that $\displaystyle\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $-1 < x < 1$.

(b) Express $\displaystyle\sum_{n=0}^{\infty} n^2x^n$ as a ratio of polynomials. For which x does this series converge?

21. $*$. Suppose that you have a sequence $\{b_n\}$ such that the series $\sum_{n=0}^{\infty}(1-b_n)$ converges. Using the tests we've learned in class, prove that the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} b_nx^n$ is equal to 1.

22. $*$. Assume $\{a_n\}$ is a sequence such that na_n decreases to C as $n \to \infty$ for some real number $C > 0$

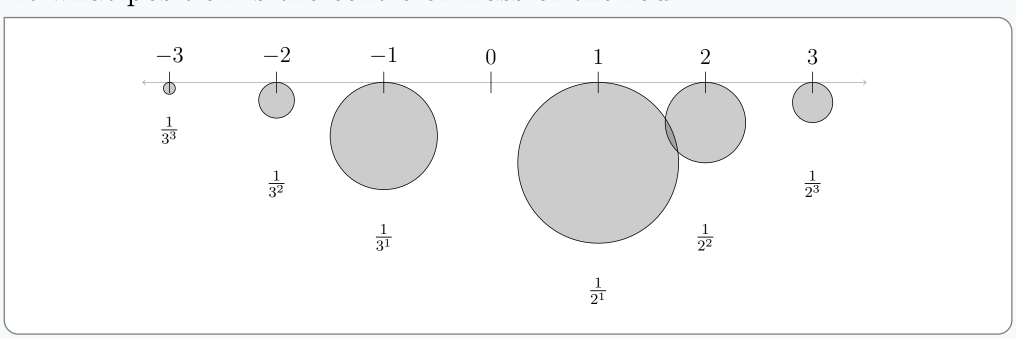
(a) Find the radius of convergence of $\displaystyle\sum_{n=1}^{\infty} a_nx^n$. Justify your answer carefully.

(b) Find the interval of convergence of the above power series, that is, find all x for which the power series in (a) converges. Justify your answer carefully.

23. An infinitely long, straight rod of negligible mass has the following weights:

- At every whole number n, a mass of weight $\frac{1}{2^n}$ at position n, and
- a mass of weight $\frac{1}{3^n}$ at position $-n$.

At what position is the centre of mass of the rod?



24. Let $f(x) = \sum_{n=0}^{\infty} A_n(x-c)^n$, for some constant c and a sequence of constants $\{A_n\}$. Further, let $f(x)$ have a positive radius of covergence.
If $A_1 = 0$, show that $y = f(x)$ has a critical point at $x = c$. What is the relationship between the behaviour of the graph at that point and the value of A_2?

25. Evaluate $\sum_{n=3}^{\infty} \frac{n}{5^{n-1}}$.

26. Find a polynomial that approximates $f(x) = \log(1+x)$ to within an error of 10^{-5} for all values of x in $\left(0, \frac{1}{10}\right)$.
Then, use your polynomial to approximate $\log(1.05)$ as a rational number.

27. Find a polynomial that approximates $f(x) = \arctan x$ to within an error of 10^{-5} for all values of x in $\left(-\frac{1}{4}, \frac{1}{4}\right)$.

3.6 ▲ Taylor Series

3.6.1 ▶▶ Extending Taylor Polynomials

Recall[1] that Taylor polynomials provide a hierarchy of approximations to a given function $f(x)$ near a given point a. Typically, the quality of these approximations improves as we move up the hierarchy.

- The crudest approximation is the constant approximation $f(x) \approx f(a)$.
- Then comes the linear, or tangent line, approximation $f(x) \approx f(a) + f'(a)\,(x-a)$.
- Then comes the quadratic approximation

$$f(x) \approx f(a) + f'(a)\,(x-a) + \frac{1}{2}f''(a)\,(x-a)^2$$

- In general, the Taylor polynomial of degree n, for the function $f(x)$, about the expansion point a, is the polynomial, $T_n(x)$, determined by the requirements that $f^{(k)}(a) = T_n^{(k)}(a)$ for all $0 \le k \le n$. That is, f and T_n have the same derivatives at a, up to order n. Explicitly,

$$f(x) \approx T_n(x)$$

1 Please review your notes from last term if this material is feeling a little unfamiliar.

$$= f(a) + f'(a)\,(x-a) + \frac{1}{2}f''(a)\,(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)\,(x-a)^n$$
$$= \sum_{k=0}^{n} \frac{1}{k!}f^{(k)}(a)\,(x-a)^k$$

These are, of course, approximations — often very good approximations near $x = a$ — but still just approximations. One might hope that if we let the degree, n, of the approximation go to infinity then the error in the approximation might go to zero. If that is the case then the "infinite" Taylor polynomial would be an exact representation of the function. Let's see how this might work.

Fix a real number a and suppose that all derivatives of the function $f(x)$ exist. Then, we saw in (3.4.33) of the CLP-1 text that, for any natural number n,

Equation 3.6.1

$$f(x) = T_n(x) + E_n(x)$$

where $T_n(x)$ is the Taylor polynomial of degree n for the function $f(x)$ expanded about a, and $E_n(x) = f(x) - T_n(x)$ is the error in the approximation $f(x) \approx T_n(x)$. The Taylor polynomial[2] is given by the formula

Equation 3.6.2

$$T_n(x) = f(a) + f'(a)\,(x-a) + \cdots + \tfrac{1}{n!}f^{(n)}(a)\,(x-a)^n$$

while the error satisfies[3]

Equation 3.6.3

$$E_n(x) = \tfrac{1}{(n+1)!}f^{(n+1)}(c)\,(x-a)^{n+1}$$

for some c strictly between a and x.

2 Did you take a quick look at your notes?
3 This is probably the most commonly used formula for the error. But there is another fairly commonly used formula. It, and some less commonly used formulae, are given in the next (optional) subsection "More about the Taylor Remainder".

Note that we typically do not know the value of c in the formula for the error. Instead we use the bounds on c to find bounds on $f^{(n+1)}(c)$ and so bound the error[4].

In order for our Taylor polynomial to be an exact representation of the function $f(x)$ we need the error $E_n(x)$ to be zero. This will not happen when n is finite unless $f(x)$ is a polynomial. However it can happen in the limit as $n \to \infty$, and in that case we can write $f(x)$ as the limit

$$f(x) = \lim_{n\to\infty} T_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \tfrac{1}{k!} f^{(k)}(a)\,(x-a)^k$$

This is really a limit of partial sums, and so we can write

$$f(x) = \sum_{k=0}^{\infty} \tfrac{1}{k!} f^{(k)}(a)\,(x-a)^k$$

which is a power series representation of the function. Let us formalise this in a definition.

Definition 3.6.4 Taylor series.

The Taylor series for the function $f(x)$ expanded around a is the power series

$$\sum_{n=0}^{\infty} \tfrac{1}{n!} f^{(n)}(a)\,(x-a)^n$$

When $a = 0$ it is also called the Maclaurin series of $f(x)$. If $\lim_{n\to\infty} E_n(x) = 0$, then

$$f(x) = \sum_{n=0}^{\infty} \tfrac{1}{n!} f^{(n)}(a)\,(x-a)^n$$

Demonstrating that, for a given function, $\lim_{n\to\infty} E_n(x) = 0$ can be difficult, but for many of the standard functions you are used to dealing with, it turns out to be pretty easy. Let's compute a few Taylor series and see how we do it.

Example 3.6.5 Exponential Series.

Find the Maclaurin series for $f(x) = e^x$.

Solution: Just as was the case for computing Taylor polynomials, we need to compute the derivatives of the function at the particular choice of a. Since we are asked for a Maclaurin series, $a = 0$. So now we just need to find $f^{(k)}(0)$ for all integers $k \geq 0$.

We know that $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$ and so

$$\begin{aligned} e^x = f(x) = f'(x) = f''(x) = \cdots = f^{(k)}(x) = \cdots & \qquad \text{which gives} \\ 1 = f(0) = f'(0) = f''(0) = \cdots = f^{(k)}(0) = \cdots . & \end{aligned}$$

4 The discussion here is only supposed to jog your memory. If it is feeling insufficiently jogged, then please look at your notes from last term.

Equations 3.6.1 and 3.6.2 then give us

$$e^x = f(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + E_n(x)$$

We shall see, in the optional Example 3.6.8 below, that, for any fixed x, $\lim_{n\to\infty} E_n(x) = 0$. Consequently, for all x,

$$e^x = \lim_{n\to\infty} \left[1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n\right] = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

Example 3.6.5

We have now seen power series representations for the functions

$$\frac{1}{1-x} \qquad \frac{1}{(1-x)^2} \qquad \log(1+x) \qquad \arctan(x) \qquad e^x.$$

We do not think that you, the reader, will be terribly surprised to see that we develop series for sine and cosine next.

Example 3.6.6 Sine and Cosine Series.

The trigonometric functions $\sin x$ and $\cos x$ also have widely used Maclaurin series expansions (i.e. Taylor series expansions about $a = 0$). To find them, we first compute all derivatives at general x.

$$\begin{array}{llll} f(x) = \sin x & f'(x) = \cos x & f''(x) = -\sin x & f^{(3)}(x) = -\cos x \\ & f^{(4)}(x) = \sin x & \cdots & \\ g(x) = \cos x & g'(x) = -\sin x & g''(x) = -\cos x & g^{(3)}(x) = \sin x \\ & g^{(4)}(x) = \cos x & \cdots & \end{array}$$

Now set $x = a = 0$.

$$\begin{array}{lllll} f(x) = \sin x & f(0) = 0 & f'(0) = 1 & f''(0) = 0 & f^{(3)}(0) = -1 \\ & f^{(4)}(0) = 0 & \cdots & & \\ g(x) = \cos x & g(0) = 1 & g'(0) = 0 & g''(0) = -1 & g^{(3)}(0) = 0 \\ & g^{(4)}(0) = 1 & \cdots & & \end{array}$$

For $\sin x$, all even numbered derivatives (at $x = 0$) are zero, while the odd numbered derivatives alternate between 1 and -1. Very similarly, for $\cos x$, all odd numbered derivatives (at $x = 0$) are zero, while the even numbered derivatives alternate between 1 and -1. So, the Taylor polynomials that best approximate $\sin x$ and $\cos x$ near $x = a = 0$ are

$$\sin x \approx x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots$$

$$\cos x \approx 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \cdots$$

We shall see, in the optional Example 3.6.10 below, that, for both $\sin x$ and $\cos x$, we have $\lim\limits_{n\to\infty} E_n(x) = 0$ so that

$$\begin{aligned} f(x) &= \lim_{n\to\infty} \left[f(0) + f'(0)\,x + \cdots + \tfrac{1}{n!}f^{(n)}(0)\,x^n\right] \\ g(x) &= \lim_{n\to\infty} \left[g(0) + g'(0)\,x + \cdots + \tfrac{1}{n!}g^{(n)}(0)\,x^n\right] \end{aligned}$$

Reviewing the patterns we found in the derivatives, we conclude that, for all x,

$$\begin{aligned} \sin x &= x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots = \sum_{n=0}^{\infty}(-1)^n \tfrac{1}{(2n+1)!}x^{2n+1} \\ \cos x &= 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \cdots = \sum_{n=0}^{\infty}(-1)^n \tfrac{1}{(2n)!}x^{2n} \end{aligned}$$

and, in particular, both of the series on the right hand sides converge for all x.
We could also test for convergence of the series using the ratio test. Computing the ratios of successive terms in these two series gives us

$$\begin{aligned} \left|\frac{A_{n+1}}{A_n}\right| &= \frac{|x|^{2n+3}/(2n+3)!}{|x|^{2n+1}/(2n+1)!} = \frac{|x|^2}{(2n+3)(2n+2)} \\ \left|\frac{A_{n+1}}{A_n}\right| &= \frac{|x|^{2n+2}/(2n+2)!}{|x|^{2n}/(2n)!} = \frac{|x|^2}{(2n+2)(2n+1)} \end{aligned}$$

for sine and cosine respectively. Hence as $n \to \infty$ these ratios go to zero and consequently both series are convergent for all x. (This is very similar to what was observed in Example 3.5.5.)

Example 3.6.6

We have developed power series representations for a number of important functions[5]. Here is a theorem that summarizes them.

5 The reader might ask whether or not we will give the series for other trigonometric functions or their inverses. While the tangent function has a perfectly well defined series, its coefficients are not as simple as those of the series we have seen — they form a sequence of numbers known (perhaps unsurprisingly) as the "tangent numbers". They, and the related Bernoulli numbers, have many interesting properties, links to which the interested reader can find with their favourite search engine. The Maclaurin series for inverse sine is $\arcsin(x) = \sum_{n=0}^{\infty} \frac{4^{-n}}{2n+1}\frac{(2n)!}{(n!)^2}x^{2n+1}$ which is quite tidy, but proving it is beyond the scope of the course.

Theorem 3.6.7

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} && = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots && \text{for all } -\infty < x < \infty \\
\sin(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} && = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots && \text{for all } -\infty < x < \infty \\
\cos(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} && = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots && \text{for all } -\infty < x < \infty \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n && = 1 + x + x^2 + x^3 + \cdots && \text{for all } -1 < x < 1 \\
\log(1+x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} && = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots && \text{for all } -1 < x \le 1 \\
\arctan x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} && = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots && \text{for all } -1 \le x \le 1
\end{aligned}$$

Notice that the series for sine and cosine sum to something that looks very similar to the series for e^x:

$$\begin{aligned}
\sin(x) + \cos(x) &= \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right) + \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right) \\
&= 1 + x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 - \cdots \\
e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots
\end{aligned}$$

So both series have coefficients with the same absolute value (namely $\frac{1}{n!}$), but there are differences in sign[6]. This is not a coincidence and we direct the interested reader to the optional Section 3.6.3 where will show how these series are linked through $\sqrt{-1}$.

Example 3.6.8 Optional — Why $\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ is e^x.

We have already seen, in Example 3.6.5, that

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + E_n(x)$$

By (3.6.3)

$$E_n(x) = \frac{1}{(n+1)!}e^c x^{n+1}$$

6 Warning: antique sign–sine pun. No doubt the reader first saw it many years syne.

for some (unknown) c between 0 and x. Fix any real number x. We'll now show that $E_n(x)$ converges to zero as $n \to \infty$.
To do this we need get bound the size of e^c, and to do this, consider what happens if x is positive or negative.

- If $x < 0$ then $x \leq c \leq 0$ and hence $e^x \leq e^c \leq e^0 = 1$.
- On the other hand, if $x \geq 0$ then $0 \leq c \leq x$ and so $1 = e^0 \leq e^c \leq e^x$.

In either case we have that $0 \leq e^c \leq 1 + e^x$. Because of this the error term

$$|E_n(x)| = \left|\frac{e^c}{(n+1)!}x^{n+1}\right| \leq [e^x + 1]\frac{|x|^{n+1}}{(n+1)!}$$

We claim that this upper bound, and hence the error $E_n(x)$, quickly shrinks to zero as $n \to \infty$.
Call the upper bound (except for the factor $e^x + 1$, which is independent of n) $e_n(x) = \frac{|x|^{n+1}}{(n+1)!}$. To show that this shrinks to zero as $n \to \infty$, let's write it as follows.

$$e_n(x) = \frac{|x|^{n+1}}{(n+1)!} = \overbrace{\frac{|x|}{1} \cdot \frac{|x|}{2} \cdot \frac{|x|}{3} \cdots \frac{|x|}{n} \cdot \frac{|x|}{|n+1|}}^{n+1 \text{ factors}}$$

Now let k be an integer bigger than $|x|$. We can split the product

$$\begin{aligned} e_n(x) &= \overbrace{\left(\frac{|x|}{1} \cdot \frac{|x|}{2} \cdot \frac{|x|}{3} \cdots \frac{|x|}{k}\right)}^{k \text{ factors}} \cdot \left(\frac{|x|}{k+1} \cdots \frac{|x|}{|n+1|}\right) \\ &\leq \underbrace{\left(\frac{|x|}{1} \cdot \frac{|x|}{2} \cdot \frac{|x|}{3} \cdots \frac{|x|}{k}\right)}_{=Q(x)} \cdot \left(\frac{|x|}{k+1}\right)^{n+1-k} \\ &= Q(x) \cdot \left(\frac{|x|}{k+1}\right)^{n+1-k} \end{aligned}$$

Since k does not depend not n (though it does depend on x), the function $Q(x)$ does not change as we increase n. Additionally, we know that $|x| < k+1$ and so $\frac{|x|}{k+1} < 1$. Hence as we let $n \to \infty$ the above bound must go to zero.
Alternatively, compare $e_n(x)$ and $e_{n+1}(x)$.

$$\frac{e_{n+1}(x)}{e_n(x)} = \frac{\frac{|x|^{n+2}}{(n+2)!}}{\frac{|x|^{n+1}}{(n+1)!}} = \frac{|x|}{n+2}$$

When n is bigger than, for example $2|x|$, we have $\frac{e_{n+1}(x)}{e_n(x)} < \frac{1}{2}$. That is, increasing the index on $e_n(x)$ by one decreases the size of $e_n(x)$ by a factor of at least two. As a result $e_n(x)$ must tend to zero as $n \to \infty$.
Consequently, for all x, $\lim_{n\to\infty} E_n(x) = 0$, as claimed, and we really have

$$e^x = \lim_{n\to\infty}\left[1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n\right] = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$$

Example 3.6.8

There is another way to prove that the series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges to the function e^x. Rather than looking at how the error term $E_n(x)$ behaves as $n \to \infty$, we can show that the series satisfies the same simple differential equation[7] and the same initial condition as the function.

Example 3.6.9 Optional — Another approach to showing that $\sum_{n=0}^{\infty}\frac{1}{n!}x^n$ is e^x.

We already know from Example 3.5.5, that the series $\sum_{n=0}^{\infty}\frac{1}{n!}x^n$ converges to some function $f(x)$ for all values of x . All that remains to do is to show that $f(x)$ is really e^x. We will do this by showing that $f(x)$ and e^x satisfy the same differential equation with the same initial conditions[a]. We know that $y = e^x$ satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \qquad \text{and} \qquad y(0) = 1$$

and by Theorem 2.4.4 (with $a = 1$, $b = 0$ and $y(0) = 1$), this is the only solution. So it suffices to show that $f(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ satisfies

$$\frac{\mathrm{d}f}{\mathrm{d}x} = f(x) \qquad \text{and} \qquad f(0) = 1.$$

- By Theorem 3.5.13,

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n=0}^{\infty}\frac{1}{n!}x^n\right\} = \sum_{n=1}^{\infty}\frac{n}{n!}x^{n-1} = \sum_{n=1}^{\infty}\frac{1}{(n-1)!}x^{n-1}\\ &= \overbrace{1}^{n=1} + \overbrace{x}^{n=2} + \overbrace{\frac{x^2}{2!}}^{n=3} + \overbrace{\frac{x^3}{3!}}^{n=4} + \cdots\\ &= f(x)\end{aligned}$$

7 Recall, you studied that differential equation in the section on separable differential equations (Theorem 2.4.4 in Section 2.4) as well as wayyyy back in the section on exponential growth and decay in differential calculus.

- When we substitute $x = 0$ into the series we get (see the discussion after Definition 3.5.1)

$$f(0) = 1 + \frac{0}{1!} + \frac{0}{2!} + \cdots = 1.$$

Hence $f(x)$ solves the same initial value problem and we must have $f(x) = e^x$.

a Recall that when we solve of a separable differential equation our general solution will have an arbitrary constant in it. That constant cannot be determined from the differential equation alone and we need some extra data to find it. This extra information is often information about the system at its beginning (for example when position or time is zero) — hence "initial conditions". Of course the reader is already familiar with this because it was covered back in Section 2.4.

Example 3.6.9

We can show that the error terms in Maclaurin polynomials for sine and cosine go to zero as $n \to \infty$ using very much the same approach as in Example 3.6.8.

Example 3.6.10 Optional — Why $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sin x$ and $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \cos x$.

Let $f(x)$ be either $\sin x$ or $\cos x$. We know that every derivative of $f(x)$ will be one of $\pm \sin(x)$ or $\pm \cos(x)$. Consequently, when we compute the error term using equation 3.6.3 we always have $\left|f^{(n+1)}(c)\right| \le 1$ and hence

$$|E_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}.$$

In Example 3.6.5, we showed that $\frac{|x|^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$ — so all the hard work is already done. Since the error term shrinks to zero for both $f(x) = \sin x$ and $f(x) = \cos x$, and

$$f(x) = \lim_{n\to\infty} \left[f(0) + f'(0)\, x + \cdots + \tfrac{1}{n!} f^{(n)}(0)\, x^n \right]$$

as required.

3.6.1.1 ⯮ Optional — More about the Taylor Remainder

In this section, we fix a real number a and a natural number n, suppose that all derivatives of the function $f(x)$ exist, and we study the error

$$E_n(a,x) = f(x) - T_n(a,x)$$
$$\text{where } T_n(a,x) = f(a) + f'(a)\,(x-a) + \cdots + \tfrac{1}{n!} f^{(n)}(a)\,(x-a)^n$$

made when we approximate $f(x)$ by the Taylor polynomial $T_n(a,x)$ of degree n for the function $f(x)$, expanded about a. We have already seen, in (3.6.3), one formula, probably the most commonly used formula, for $E_n(a,x)$. In the next theorem, we repeat that formula and give a second, commonly used, formula. After an example, we give a second theorem that contains some less commonly used formulae.

Theorem 3.6.11 Commonly used formulae for the Taylor remainder.

The Taylor remainder $E_n(a,x)$ is given by

a (integral form)

$$E_n(a,x) = \int_a^x \frac{1}{n!} f^{(n+1)}(t)\,(x-t)^n \,\mathrm{d}t$$

b (Lagrange form)

$$E_n(a,x) = \frac{1}{(n+1)!} f^{(n+1)}(c)\,(x-a)^{n+1}$$

for some c strictly between a and x.

Notice that the integral form of the error is explicit - we could, in principle, compute it exactly. (Of course if we could do that, we probably wouldn't need to use a Taylor expansion to approximate f.) This contrasts with the Lagrange form which is an 'existential' statement - it tells us that 'c' exists, but not how to compute it.

Proof.

a We will give two proofs. The first is shorter and simpler, but uses some trickery. The second is longer, but is more straightforward. It uses a technique called mathematical induction.

Proof 1: We are going to use a little trickery to get a simple proof. We simply view x as being fixed and study the dependence of $E_n(a,x)$ on a. To emphasise that that is what we are doing, we define

$$\begin{aligned} S(t) = f(x) - f(t) - f'(t)\,(x-t) - \tfrac{1}{2} f''(t)\,(x-t)^2 \\ - \cdots - \tfrac{1}{n!} f^{(n)}(t)\,(x-t)^n \qquad (*) \end{aligned}$$

and observe that $E_n(a,x) = S(a)$.

So, by the fundamental theorem of calculus (Theorem 1.3.1), the function $S(t)$ is determined by its derivative, $S'(t)$, and its value at a single point. Finding a value of $S(t)$ for one value of t is easy. Substitute $t = x$ into $(*)$ to yield $S(x) = 0$. To find $S'(t)$, apply $\frac{\mathrm{d}}{\mathrm{d}t}$ to both sides of $(*)$. Recalling that x is just a constant parameter,

$$S'(t) = 0 - f'(t) - \big[f''(t)(x-t) - f'(t)\big]$$

$$\begin{aligned}
&- \left[\tfrac{1}{2} f^{(3)}(t)(x-t)^2 - f''(t)(x-t)\right] \\
&- \cdots - \left[\tfrac{1}{n!} f^{(n+1)}(t)\,(x-t)^n - \tfrac{1}{(n-1)!} f^{(n)}(t)\,(x-t)^{n-1}\right] \\
= &-\tfrac{1}{n!} f^{(n+1)}(t)\,(x-t)^n
\end{aligned}$$

So, by the fundamental theorem of calculus, $S(x) = S(a) + \int_a^x S'(t)\,\mathrm{d}t$ and

$$\begin{aligned}
E_n(a,x) &= -\big[S(x) - S(a)\big] = -\int_a^x S'(t)\,\mathrm{d}t \\
&= \int_a^x \frac{1}{n!} f^{(n+1)}(t)\,(x-t)^n\,\mathrm{d}t
\end{aligned}$$

Proof 2: The proof that we have just given was short, but also very tricky — almost noone could create that proof without big hints. Here is another much less tricky, but also commonly used, proof.

- First consider the case $n = 0$. When $n = 0$,

  $$E_0(a,x) = f(x) - T_0(a,x) = f(x) - f(a)$$

  The fundamental theorem of calculus gives

  $$f(x) - f(a) = \int_a^x f'(t)\,\mathrm{d}t$$

  so that

  $$E_0(a,x) = \int_a^x f'(t)\,\mathrm{d}t$$

  That is exactly the $n = 0$ case of part (a).
- Next fix any integer $n \geq 0$ and suppose that we already know that

  $$E_n(a,x) = \int_a^x \frac{1}{n!} f^{(n+1)}(t)\,(x-t)^n\,\mathrm{d}t$$

  Apply integration by parts (Theorem 1.7.2) to this integral with

  $$\begin{aligned}
  u(t) &= f^{(n+1)}(t) \\
  \mathrm{d}v &= \frac{1}{n!}(x-t)^n\,\mathrm{d}t, \qquad v(t) = -\frac{1}{(n+1)!}(x-t)^{n+1}
  \end{aligned}$$

  Since $v(x) = 0$, integration by parts gives

  $$\begin{aligned}
  E_n(a,x) &= u(x)v(x) - u(a)v(a) - \int_a^x v(t)u'(t)\,\mathrm{d}t \\
  &= \frac{1}{(n+1)!} f^{(n+1)}(a)\,(x-a)^{n+1} \\
  &\quad + \int_a^x \frac{1}{(n+1)!} f^{(n+2)}(t)\,(x-t)^{n+1}\,\mathrm{d}t \qquad (**)
  \end{aligned}$$

Now, we defined

$$E_n(a,x) = f(x) - f(a) - f'(a)\,(x-a) - \tfrac{1}{2}f''(a)\,(x-a)^2 \\ - \cdots - \tfrac{1}{n!}f^{(n)}(a)\,(x-a)^n$$

so

$$E_{n+1}(a,x) = E_n(a,x) - \tfrac{1}{(n+1)!}f^{(n+1)}(a)\,(x-a)^{n+1}$$

This formula expresses $E_{n+1}(a,x)$ in terms of $E_n(a,x)$. That's called a reduction formula. Combining the reduction formula with (∗∗) gives

$$E_{n+1}(a,x) = \int_a^x \frac{1}{(n+1)!} f^{(n+2)}(t)\,(x-t)^{n+1}\,\mathrm{d}t$$

- Let's pause to summarise what we have learned in the last two bullets. Use the notation $P(n)$ to stand for the statement "$E_n(a,x) = \int_a^x \frac{1}{n!} f^{(n+1)}(t)\,(x-t)^n\,\mathrm{d}t$". To prove part (a) of the theorem, we need to prove that the statement $P(n)$ is true for all integers $n \geq 0$. In the first bullet, we showed that the statement $P(0)$ is true. In the second bullet, we showed that if, for some integer $n \geq 0$, the statement $P(n)$ is true, then the statement $P(n+1)$ is also true. Consequently,
  - $P(0)$ is true by the first bullet and then
  - $P(1)$ is true by the second bullet with $n = 0$ and then
  - $P(2)$ is true by the second bullet with $n = 1$ and then
  - $P(3)$ is true by the second bullet with $n = 2$
  - and so on, for ever and ever.

  That tells us that $P(n)$ is true for all integers $n \geq 0$, which is exactly part (a) of the theorem. This proof technique is called mathematical induction[a].

b We have already seen one proof in the optional Section 3.4.9 of the CLP-1 text. We will see two more proofs here.

Proof 1: We apply the generalised mean value theorem, which is Theorem 3.4.38 in the CLP-1 text. It says that

$$\frac{F(b)-F(a)}{G(b)-G(a)} = \frac{F'(c)}{G'(c)} \tag{GMVT}$$

for some c strictly between[b] a and b. We apply (GMVT) with $b = x$, $F(t) = S(t)$ and $G(t) = (x-t)^{n+1}$. This gives

$$\begin{aligned} E_n(a,x) &= -\big[S(x) - S(a)\big] = -\frac{S'(c)}{G'(c)}\big[G(x) - G(a)\big] \\ &= -\frac{-\frac{1}{n!}f^{(n+1)}(c)\,(x-c)^n}{-(n+1)(x-c)^n}\ \big[0 - (x-a)^{n+1}\big] \end{aligned}$$

$$= \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Don't forget, when computing $G'(c)$, that G is a function of t with x just a fixed parameter.

Proof 2: We apply Theorem 2.2.10 (the mean value theorem for weighted integrals). If $a < x$, we use the weight function $w(t) = \frac{1}{n!}(x-t)^n$, which is strictly positive for all $a < t < x$. By part (a) this gives

$$\begin{aligned} E_n(a,x) &= \int_a^x \frac{1}{n!} f^{(n+1)}(t)\,(x-t)^n\,\mathrm{d}t \\ &= f^{(n+1)}(c) \int_a^x \frac{1}{n!}(x-t)^n\,\mathrm{d}t \qquad \text{for some } a < c < x \\ &= f^{(n+1)}(c) \left[-\frac{1}{n!}\frac{(x-t)^{n+1}}{n+1} \right]_a^x \\ &= \frac{1}{(n+1)!} f^{(n+1)}(c)\,(x-a)^{n+1} \end{aligned}$$

If $x < a$, we instead use the weight function $w(t) = \frac{1}{n!}(t-x)^n$, which is strictly positive for all $x < t < a$. This gives

$$\begin{aligned} E_n(a,x) &= \int_a^x \frac{1}{n!} f^{(n+1)}(t)\,(x-t)^n\,\mathrm{d}t \\ &= -(-1)^n \int_x^a \frac{1}{n!} f^{(n+1)}(t)\,(t-x)^n\,\mathrm{d}t \\ &= (-1)^{n+1} f^{(n+1)}(c) \int_x^a \frac{1}{n!}(t-x)^n\,\mathrm{d}t \qquad \text{for some } x < c < a \\ &= (-1)^{n+1} f^{(n+1)}(c) \left[\frac{1}{n!}\frac{(t-x)^{n+1}}{n+1} \right]_x^a \\ &= \frac{1}{(n+1)!} f^{(n+1)}(c)\,(-1)^{n+1}(a-x)^{n+1} \\ &= \frac{1}{(n+1)!} f^{(n+1)}(c)\,(x-a)^{n+1} \end{aligned}$$

□

a While the use of the ideas of induction goes back over 2000 years, the first recorded rigorous use of induction appeared in the work of Levi ben Gershon (1288–1344, better known as Gersonides). The first explicit formulation of mathematical induction was given by the French mathematician Blaise Pascal in 1665.

b In Theorem 3.4.38 in the CLP-1 text, we assumed, for simplicity, that $a < b$. To get (GVMT) when $b < a$ simply exchange a and b in Theorem 3.4.38.

Theorem 3.6.11 has provided us with two formulae for the Taylor remainder $E_n(a,x)$.

The formula of part (b), $E_n(a,x) = \frac{1}{(n+1)!} f^{(n+1)}(c)\,(x-a)^{n+1}$, is probably the easiest to use, and the most commonly used, formula for $E_n(a,x)$. The formula of part (a), $E_n(a,x) = \int_a^x \frac{1}{n!} f^{(n+1)}(t)\,(x-t)^n\,\mathrm{d}t$, while a bit harder to apply, gives a bit better bound than that of part (b) (in the proof of Theorem 3.6.11 we showed that part (b) follows from part (a)). Here is an example in which we use both parts.

Example 3.6.12

In Theorem 3.6.7 we stated that

$$\log(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{S1}$$

for all $-1 < x \le 1$. But, so far, we have not justified this statement. We do so now, using (both parts of) Theorem 3.6.11. We start by setting $f(x) = \log(1+x)$ and finding the Taylor polynomials $T_n(0,x)$, and the corresponding errors $E_n(0,x)$, for $f(x)$.

$$\begin{aligned}
f(x) &= \log(1+x) & f(0) &= \log 1 = 0\\
f'(x) &= \frac{1}{1+x} & f'(0) &= 1\\
f''(x) &= \frac{-1}{(1+x)^2} & f''(0) &= -1\\
f'''(x) &= \frac{2}{(1+x)^3} & f'''(1) &= 2\\
f^{(4)}(x) &= \frac{-2\times 3}{(1+x)^4} & f^{(4)}(0) &= -3!\\
f^{(5)}(x) &= \frac{2\times 3\times 4}{(1+x)^5} & f^{(5)}(0) &= 4!\\
&\vdots & &\vdots\\
f^{(n)}(x) &= \frac{(-1)^{n+1}(n-1)!}{(1+x)^n} & f^{(n)}(0) &= (-1)^{n+1}(n-1)!
\end{aligned}$$

So the Taylor polynomial of degree n for the function $f(x) = \log(1+x)$, expanded about $a = 0$, is

$$\begin{aligned}
T_n(0,x) &= f(0) + f'(0)\,x + \cdots + \tfrac{1}{n!} f^{(n)}(0)\,x^n\\
&= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \cdots + \frac{(-1)^{n+1}}{n} x^n
\end{aligned}$$

Theorem 3.6.11 gives us two formulae for the error $E_n(0,x) = f(x) - T_n(0,x)$ made when we approximate $f(x)$ by $T_n(0,x)$. Part (a) of the theorem gives

$$E_n(0,x) = \int_0^x \frac{1}{n!} f^{(n+1)}(t)\,(x-t)^n\,\mathrm{d}t = (-1)^n \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}}\,\mathrm{d}t \tag{Ea}$$

and part (b) gives

$$E_n(0,x) = \frac{1}{(n+1)!}\, f^{(n+1)}(c)\, x^{n+1} = (-1)^n\, \frac{1}{n+1}\, \frac{x^{n+1}}{(1+c)^{n+1}} \tag{Eb}$$

for some (unknown) c between 0 and x. The statement (S1), that we wish to prove, is equivalent to the statement

$$\lim_{n\to\infty} E_n(0,x) = 0 \qquad \text{for all } -1 < x \le 1 \tag{S2}$$

and we will now show that (S2) is true.

The case $x = 0$: This case is trivial, since, when $x = 0$, $E_n(0,x) = 0$ for all n.

The case $0 < x \le 1$: This case is relatively easy to deal with using (Eb). In this case $0 < x \le 1$, so that the c of (Eb) must be positive and

$$\begin{aligned} |E_n(0,x)| &= \frac{1}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}} \\ &\le \frac{1}{n+1}\frac{1^{n+1}}{(1+0)^{n+1}} \\ &= \frac{1}{n+1} \end{aligned}$$

converges to zero as $n \to \infty$.

The case $-1 < x < 0$: When $-1 < x < 0$ is close to -1, (Eb) is not sufficient to show that (S2) is true. To see this, let's consider the example $x = -0.8$. All we know about the c of (Eb) is that it has to be between 0 and -0.8. For example, (Eb) certainly allows c to be -0.6 and then

$$\begin{aligned} &\left|(-1)^n\frac{1}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}}\right|_{\substack{x=-0.8 \\ c=-0.6}} \\ &\quad= \frac{1}{n+1}\frac{0.8^{n+1}}{(1-0.6)^{n+1}} \\ &\quad= \frac{1}{n+1}2^{n+1} \end{aligned}$$

goes to $+\infty$ as $n \to \infty$.

Note that, while this does tell us that (Eb) is not sufficient to prove (S2), when x is close to -1, it does not also tell us that $\lim_{n\to\infty} |E_n(0,-0.8)| = +\infty$ (which would imply that (S2) is false) — c could equally well be -0.2 and then

$$\left|(-1)^n\frac{1}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}}\right|_{\substack{x=-0.8 \\ c=-0.2}}$$

$$= \frac{1}{n+1}\frac{0.8^{n+1}}{(1-0.2)^{n+1}}$$
$$= \frac{1}{n+1}$$

goes to 0 as $n \to \infty$.

We'll now use (Ea) (which has the advantage of not containing any unknown free parameter c) to verify (S2) when $-1 < x < 0$. Rewrite the right hand side of (Ea)

$$\begin{aligned}(-1)^n \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}}\,\mathrm{d}t &= -\int_x^0 \frac{(t-x)^n}{(1+t)^{n+1}}\,\mathrm{d}t \\ &= -\int_0^{-x} \frac{s^n}{(1+x+s)^{n+1}}\,\mathrm{d}s \quad s = t-x,\ \mathrm{d}s = \mathrm{d}t\end{aligned}$$

The exact evaluation of this integral is very messy and not very illuminating. Instead, we bound it. Note that, for $1+x>0$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{s}{1+x+s}\right) &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1+x+s-(1+x)}{1+x+s}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\left(1-\frac{1+x}{1+x+s}\right) \\ &= \frac{1+x}{(1+x+s)^2} > 0\end{aligned}$$

so that $\frac{s}{1+x+s}$ increases as s increases. Consequently, the biggest value that $\frac{s}{1+x+s}$ takes on the domain of integration $0 \le s \le -x = |x|$ is

$$\left.\frac{s}{1+x+s}\right|_{s=-x} = -x = |x|$$

and the integrand

$$\begin{aligned}0 \le \frac{s^n}{[1+x+s]^{n+1}} &= \left(\frac{s}{1+x+s}\right)^n \frac{1}{1+x+s} \\ &\le \frac{|x|^n}{1+x+s}\end{aligned}$$

Consequently,

$$\begin{aligned}|E_n(0,x)| &= \left|(-1)^n \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}}\,\mathrm{d}t\right| \\ &= \int_0^{-x} \frac{s^n}{[1+x+s]^{n+1}}\,\mathrm{d}s \\ &\le |x|^n \int_0^{-x} \frac{1}{1+x+s}\,\mathrm{d}s\end{aligned}$$

$$= |x|^n \Big[\log(1+x+s) \Big]_{s=0}^{s=-x}$$
$$= |x|^n [-\log(1+x)]$$

converges to zero as $n \to \infty$ for each fixed $-1 < x < 0$.

So we have verified (S2), as desired.

Example 3.6.12

As we said above, Theorem 3.6.11 gave the two most commonly used formulae for the Taylor remainder. Here are some less commonly used, but occasionally useful, formulae.

Theorem 3.6.13 More formulae for the Taylor remainder.

a If $G(t)$ is differentiable[a] and $G'(c)$ is nonzero for all c strictly between a and x, then the Taylor remainder

$$E_n(a,x) = \frac{1}{n!} f^{(n+1)}(c) \, \frac{G(x)-G(a)}{G'(c)} \, (x-c)^n$$

for some c strictly between a and x.

b (Cauchy form)

$$E_n(a,x) = \frac{1}{n!} f^{(n+1)}(c) \, (x-c)^n (x-a)$$

for some c strictly between a and x.

[a] Note that the function G need not be related to f. It just has to be differentiable with a nonzero derivative.

Proof. As in the proof of Theorem 3.6.11, we define

$$S(t) = f(x) - f(t) - f'(t)\,(x-t) - \tfrac{1}{2} f''(t)\,(x-t)^2 - \cdots - \tfrac{1}{n!} f^{(n)}(t)\,(x-t)^n$$

and observe that $E_n(a,x) = S(a)$ and $S(x) = 0$ and $S'(t) = -\frac{1}{n!} f^{(n+1)}(t)\,(x-t)^n$.

a Recall that the generalised mean-value theorem, which is Theorem 3.4.38 in the CLP-1 text, says that

$$\frac{F(b)-F(a)}{G(b)-G(a)} = \frac{F'(c)}{G'(c)} \tag{GMVT}$$

for some c strictly between a and b. We apply this theorem with $b = x$ and $F(t) = S(t)$. This gives

$$\begin{aligned} E_n(a,x) &= -\big[S(x) - S(a)\big] = -\frac{S'(c)}{G'(c)}\big[G(x) - G(a)\big] \\ &= -\frac{-\frac{1}{n!} f^{(n+1)}(c)\,(x-c)^n}{G'(c)}\,\big[G(x) - G(a)\big] \\ &= \frac{1}{n!} f^{(n+1)}(c)\,\frac{G(x)-G(a)}{G'(c)}\,(x-c)^n \end{aligned}$$

b Apply part (a) with $G(x) = x$. This gives

$$\begin{aligned} E_n(a,x) &= \frac{1}{n!} f^{(n+1)}(c)\,\frac{x-a}{1}\,(x-c)^n \\ &= \frac{1}{n!} f^{(n+1)}(c)\,(x-c)^n(x-a) \end{aligned}$$

for some c strictly between a and b.

□

Example 3.6.14 Example 3.6.12, continued.

In Example 3.6.12 we verified that

$$\log(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{S1}$$

for all $-1 < x \le 1$. There we used the Lagrange form,

$$E_n(a,x) = \frac{1}{(n+1)!}\, f^{(n+1)}(c)\,(x-a)^{n+1}$$

for the Taylor remainder to verify (S1) when $0 \le x \le 1$, but we also saw that it is not possible to use the Lagrange form to verify (S1) when x is close to -1. We instead used the integral form

$$E_n(a,x) = \int_a^x \frac{1}{n!} f^{(n+1)}(t)\,(x-t)^n\,\mathrm{d}t$$

We will now use the Cauchy form (part (b) of Theorem 3.6.13)

$$E_n(a,x) = \frac{1}{n!} f^{(n+1)}(c)\,(x-c)^n(x-a)$$

to verify

$$\lim_{n\to\infty} E_n(0,x) = 0 \tag{S2}$$

when $-1 < x < 0$. We have already noted that (S2) is equivalent to (S1). Write $f(x) = \log(1+x)$. We saw in Example 3.6.12 that

$$f^{(n+1)}(x) = \frac{(-1)^n n!}{(1+x)^{n+1}}$$

So, in this example, the Cauchy form is

$$E_n(0,x) = (-1)^n \frac{(x-c)^n x}{(1+c)^{n+1}}$$

for some $x < c < 0$. When $-1 < x < c < 0$,

- c and x are negative and $1+x$, $1+c$ and $c-x$ are (strictly) positive so that

$$\begin{aligned} c(1+x) < 0 \implies c < -cx \implies c - x < -x - xc = |x|(1+c) \\ \implies \left|\frac{x-c}{1+c}\right| = \frac{c-x}{1+c} < |x| \end{aligned}$$

  so that $\left|\frac{x-c}{1+c}\right|^n < |x|^n$ and

- the distance from -1 to c, namely $c-(-1) = 1+c$ is greater than the distance from -1 to x, namely $x-(-1) = 1+x$, so that $\frac{1}{1+c} < \frac{1}{1+x}$.

So, for $-1 < x < c < 0$,

$$|E_n(0,x)| = \left|\frac{x-c}{1+c}\right|^n \frac{|x|}{1+c} < \frac{|x|^{n+1}}{1+c} < \frac{|x|^{n+1}}{1+x}$$

goes to zero as $n \to \infty$.

Example 3.6.14

3.6.2 ►► Computing with Taylor Series

Taylor series have a great many applications. (Hence their place in this course.) One of the most immediate of these is that they give us an alternate way of computing many functions. For example, the first definition we see for the sine and cosine functions is in terms of triangles. Those definitions, however, do not lend themselves to computing sine and cosine except at very special angles. Armed with power series representations, however, we can compute them to very high precision at any angle. To illustrate this, consider the computation of π — a problem that dates back to the Babylonians.

Example 3.6.15 Computing the number π.

There are numerous methods for computing π to any desired degree of accuracy[a]. Many of them use the Maclaurin expansion

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

of Theorem 3.6.7. Since $\arctan(1) = \frac{\pi}{4}$, the series gives us a very pretty formula for π:

$$\begin{aligned}\frac{\pi}{4} = \arctan 1 &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\\ \pi &= 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)\end{aligned}$$

Unfortunately, this series is not very useful for computing π because it converges so slowly. If we approximate the series by its N^{th} partial sum, then the alternating series test (Theorem 3.3.14) tells us that the error is bounded by the first term we drop. To guarantee that we have 2 decimal digits of π correct, we need to sum about the first 200 terms!

A much better way to compute π using this series is to take advantage of the fact that $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$:

$$\begin{aligned}\pi = 6\arctan\left(\frac{1}{\sqrt{3}}\right) &= 6\sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}\;\frac{1}{(\sqrt{3})^{2n+1}}\\ &= 2\sqrt{3}\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}\;\frac{1}{3^n}\\ &= 2\sqrt{3}\Big(1 - \frac{1}{3\times 3} + \frac{1}{5\times 9} - \frac{1}{7\times 27} + \frac{1}{9\times 81} - \frac{1}{11\times 243} + \cdots\Big)\end{aligned}$$

Again, this is an alternating series and so (via Theorem 3.3.14) the error we introduce by truncating it is bounded by the first term dropped. For example, if we keep ten terms, stopping at $n = 9$, we get $\pi = 3.141591$ (to 6 decimal places) with an error between zero and

$$\frac{2\sqrt{3}}{21\times 3^{10}} < 3\times 10^{-6}$$

In 1699, the English astronomer/mathematician Abraham Sharp (1653–1742) used 150 terms of this series to compute 72 digits of π — by hand!

This is just one of very many ways to compute π. Another one, which still uses the Maclaurin expansion of $\arctan x$, but is much more efficient, is

$$\pi = 16\arctan\frac{1}{5} - 4\arctan\frac{1}{239}$$

This formula was used by John Machin in 1706 to compute π to 100 decimal digits — again, by hand.

a The computation of π has a very, very long history and your favourite search engine will turn up many sites that explore the topic. For a more comprehensive history one can turn to books such as "A history of Pi" by Petr Beckmann and "The joy of π" by David Blatner.

Example 3.6.15

Power series also give us access to new functions which might not be easily expressed in terms of the functions we have been introduced to so far. The following is a good example of this.

Example 3.6.16 Error function.

The *error function*

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \mathrm{d}t$$

is used in computing "bell curve" probabilities. The indefinite integral of the integrand e^{-t^2} cannot be expressed in terms of standard functions. But we can still evaluate the integral to within any desired degree of accuracy by using the Taylor expansion of the exponential. Start with the Maclaurin series for e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

and then substitute $x = -t^2$ into this:

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}$$

We can then apply Theorem 3.5.13 to integrate term-by-term:

$$\begin{aligned}\mathrm{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x \left[\sum_{n=0}^{\infty} \frac{\left(-t^2\right)^n}{n!}\right] \mathrm{d}t \\ &= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!}\end{aligned}$$

For example, for the bell curve, the probability of being within one standard deviation of the mean[a], is

$$\begin{aligned}\mathrm{erf}\left(\frac{1}{\sqrt{2}}\right) &= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{\sqrt{2}}\right)^{2n+1}}{(2n+1)n!} = \frac{2}{\sqrt{2\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)2^n n!} \\ &= \sqrt{\frac{2}{\pi}}\left(1 - \frac{1}{3 \times 2} + \frac{1}{5 \times 2^2 \times 2} - \frac{1}{7 \times 2^3 \times 3!} + \frac{1}{9 \times 2^4 \times 4!} - \cdots\right)\end{aligned}$$

This is yet another alternating series. If we keep five terms, stopping at $n = 4$, we get 0.68271 (to 5 decimal places) with, by Theorem 3.3.14 again, an error between zero and the first dropped term, which is minus

$$\sqrt{\frac{2}{\pi}}\,\frac{1}{11 \times 2^5 \times 5!} < 2 \times 10^{-5}$$

a If you don't know what this means (forgive the pun) don't worry, because it is not part of the course. Standard deviation is a way of quantifying variation within a population.

Example 3.6.16

Example 3.6.17 Two nice series.

Evaluate

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n3^n} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n3^n}$$

Solution. There are not very many series that can be easily evaluated exactly. But occasionally one encounters a series that can be evaluated simply by realizing that it is exactly one of the series in Theorem 3.6.7, just with a specific value of x. The left hand given series is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\,\frac{1}{3^n} = \frac{1}{3} - \frac{1}{2}\,\frac{1}{3^2} + \frac{1}{3}\,\frac{1}{3^3} - \frac{1}{4}\,\frac{1}{3^4} + \cdots$$

The series in Theorem 3.6.7 that this most closely resembles is

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

Indeed

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\,\frac{1}{3^n} &= \frac{1}{3} - \frac{1}{2}\,\frac{1}{3^2} + \frac{1}{3}\,\frac{1}{3^3} - \frac{1}{4}\,\frac{1}{3^4} + \cdots \\
&= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right]_{x=\frac{1}{3}} \\
&= \Big[\log(1+x)\Big]_{x=\frac{1}{3}} \\
&= \log\frac{4}{3}
\end{aligned}$$

The right hand series above differs from the left hand series above only that the signs of the left hand series alternate while those of the right hand series do not. We can flip every second sign in a power series just by using a negative x.

$$\Big[\log(1+x)\Big]_{x=-\frac{1}{3}} = \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right]_{x=-\frac{1}{3}}$$

$$= -\frac{1}{3} - \frac{1}{2}\frac{1}{3^2} - \frac{1}{3}\frac{1}{3^3} - \frac{1}{4}\frac{1}{3^4} + \cdots$$

which is exactly minus the desired right hand series. So

$$\sum_{n=1}^{\infty} \frac{1}{n3^n} = -\Big[\log(1+x)\Big]_{x=-\frac{1}{3}} = -\log\frac{2}{3} = \log\frac{3}{2}$$

Example 3.6.17

Example 3.6.18 Finding a derivative from a series.

Let $f(x) = \sin(2x^3)$. Find $f^{(15)}(0)$, the fifteenth derivative of f at $x = 0$.
Solution: This is a bit of a trick question. We could of course use the product and chain rules to directly apply fifteen derivatives and then set $x = 0$, but that would be extremely tedious[a]. There is a much more efficient approach that exploits two pieces of knowledge that we have.

- From equation 3.6.2, we see that the coefficient of $(x-a)^n$ in the Taylor series of $f(x)$ with expansion point a is exactly $\frac{1}{n!}f^{(n)}(a)$. So $f^{(n)}(a)$ is exactly $n!$ times the coefficient of $(x-a)^n$ in the Taylor series of $f(x)$ with expansion point a.
- We know, or at least can easily find, the Taylor series for $\sin(2x^3)$.

Let's apply that strategy.

- First, we know that, for all y,

$$\sin y = y - \frac{1}{3!}y^3 + \frac{1}{5!}y^5 - \cdots$$

- Just substituting $y = 2x^3$, we have

$$\begin{aligned}\sin(2x^3) &= 2x^3 - \frac{1}{3!}(2x^3)^3 + \frac{1}{5!}(2x^3)^5 - \cdots \\ &= 2x^3 - \frac{8}{3!}x^9 + \frac{2^5}{5!}x^{15} - \cdots\end{aligned}$$

- So the coefficient of x^{15} in the Taylor series of $f(x) = \sin(2x^3)$ with expansion point $a = 0$ is $\frac{2^5}{5!}$

and we have

$$f^{(15)}(0) = 15! \times \frac{2^5}{5!} = 348{,}713{,}164{,}800$$

a We could get a computer algebra system to do it for us without much difficulty — but we wouldn't learn much in the process. The point of this example is to illustrate that one can do more than just represent a function with Taylor series. More on this in the next section.

Example 3.6.19 Optional — Computing the number e.

Back in Example 3.6.8, we saw that

$$e^x = 1 + x + \tfrac{x^2}{2!} + \cdots + \tfrac{x^n}{n!} + \tfrac{1}{(n+1)!} e^c x^{n+1}$$

for some (unknown) c between 0 and x. This can be used to approximate the number e, with any desired degree of accuracy. Setting $x = 1$ in this equation gives

$$e = 1 + 1 + \tfrac{1}{2!} + \cdots + \tfrac{1}{n!} + \tfrac{1}{(n+1)!} e^c$$

for some c between 0 and 1. Even though we don't know c exactly, we can bound that term quite readily. We do know that e^c in an increasing function[a] of c, and so $1 = e^0 \le e^c \le e^1 = e$. Thus we know that

$$\frac{1}{(n+1)!} \le e - \big(1 + 1 + \tfrac{1}{2!} + \cdots + \tfrac{1}{n!}\big) \le \frac{e}{(n+1)!}$$

So we have a lower bound on the error, but our upper bound involves the e — precisely the quantity we are trying to get a handle on.

But all is not lost. Let's look a little more closely at the right-hand inequality when $n = 1$:

$$\begin{aligned} e - (1+1) &\le \frac{e}{2} && \text{move the } e\text{'s to one side} \\ \frac{e}{2} &\le 2 && \text{and clean it up} \\ e &\le 4. \end{aligned}$$

Now this is a pretty crude bound[b] but it isn't hard to improve. Try this again with $n = 2$:

$$\begin{aligned} e - (1 + 1 + \frac{1}{2}) &\le \frac{e}{6} && \text{move } e\text{'s to one side} \\ \frac{5e}{6} &\le \frac{5}{2} \\ e &\le 3. \end{aligned}$$

Better. Now we can rewrite our bound:

$$\frac{1}{(n+1)!} \le e - \big(1 + 1 + \tfrac{1}{2!} + \cdots + \tfrac{1}{n!}\big) \le \frac{e}{(n+1)!} \le \frac{3}{(n+1)!}$$

If we set $n = 4$ in this we get

$$\frac{1}{120} = \frac{1}{5!} \le e - \left(1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24}\right) \le \frac{3}{120}$$

So the error is between $\frac{1}{120}$ and $\frac{3}{120} = \frac{1}{40}$ — this approximation isn't guaranteed to give us the first 2 decimal places. If we ramp n up to 9 however, we get

$$\frac{1}{10!} \le e - \left(1 + 1 + \frac{1}{2} + \cdots + \frac{1}{9!}\right) \le \frac{3}{10!}$$

Since $10! = 3628800$, the upper bound on the error is $\frac{3}{3628800} < \frac{3}{3000000} = 10^{-6}$, and we can approximate e by

$$\begin{aligned}
&1 + 1 + \tfrac{1}{2!} + \tfrac{1}{3!} + \tfrac{1}{4!} + \tfrac{1}{5!} + \tfrac{1}{6!} + \tfrac{1}{7!} \\
&\qquad + \tfrac{1}{8!} + \tfrac{1}{9!} \\
=&1 + 1 + 0.5 + 0.1\dot{6} + 0.041\dot{6} + 0.008\dot{3} + 0.0013\dot{8} + 0.0001984 \\
&\qquad + 0.0000248 + 0.0000028 \\
=&2.718282
\end{aligned}$$

and it is correct to six decimal places.

a Check the derivative!

b The authors hope that by now we all "know" that e is between 2 and 3, but maybe we don't know how to prove it.

Example 3.6.19

3.6.3 ▸▸ Optional — Linking e^x with trigonometric functions

Let us return to the observation that we made earlier about the Maclaurin series for sine, cosine and the exponential functions:

$$\begin{aligned}
\cos x + \sin x &= 1 + x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 - \cdots \\
e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots
\end{aligned}$$

We see that these series are identical except for the differences in the signs of the coefficients. Let us try to make them look even more alike by introducing extra constants A, B and q into the equations. Consider

$$\begin{aligned}
A\cos x + B\sin x &= A + Bx - \frac{A}{2!}x^2 - \frac{B}{3!}x^3 + \frac{A}{4!}x^4 + \frac{B}{5!}x^5 - \cdots \\
e^{qx} &= 1 + qx + \frac{q^2}{2!}x^2 + \frac{q^3}{3!}x^3 + \frac{q^4}{4!}x^4 + \frac{q^5}{5!}x^5 + \cdots
\end{aligned}$$

Let's try to choose A, B and q so that these to expressions are equal. To do so we must make sure that the coefficients of the various powers of x agree. Looking just at the coefficients of x^0 and x^1, we see that we need

$$A = 1 \qquad \text{and} \qquad B = q$$

Substituting this into our expansions gives

$$\cos x + q\sin x = 1 + qx - \frac{1}{2!}x^2 - \frac{q}{3!}x^3 + \frac{1}{4!}x^4 + \frac{q}{5!}x^5 - \cdots$$

$$e^{qx} = 1 + qx + \frac{q^2}{2!}x^2 + \frac{q^3}{3!}x^3 + \frac{q^4}{4!}x^4 + \frac{q^5}{5!}x^5 + \cdots$$

Now the coefficients of x^0 and x^1 agree, but the coefficient of x^2 tells us that we need q to be a number so that $q^2 = -1$, or

$$q = \sqrt{-1}$$

We know that no such *real* number q exists. But for the moment let us see what happens if we just assume[8] that we can find q so that $q^2 = -1$. Then we will have that

$$q^3 = -q \qquad q^4 = 1 \qquad q^5 = q \qquad \cdots$$

so that the series for $\cos x + q\sin x$ and e^{qx} are identical. That is

$$e^{qx} = \cos x + q\sin x$$

If we now write this with the more usual notation $q = \sqrt{-1} = i$ we arrive at what is now known as Euler's formula

Equation 3.6.20

$$e^{ix} = \cos x + i\sin x$$

Euler's proof of this formula (in 1740) was based on Maclaurin expansions (much like our explanation above). Euler's formula[9] is widely regarded as one of the most important and beautiful in all of mathematics.

Of course having established Euler's formula one can find slicker demonstrations. For example, let

$$f(x) = e^{-ix}(\cos x + i\sin x)$$

Differentiating (with product and chain rules and the fact that $i^2 = -1$) gives us

$$\begin{aligned} f'(x) &= -ie^{-ix}(\cos x + i\sin x) + e^{-ix}(-\sin x + i\cos x) \\ &= 0 \end{aligned}$$

8 We do not wish to give a primer on imaginary and complex numbers here. The interested reader can start by looking at Appendix B.

9 It is worth mentioning here that history of this topic is perhaps a little rough on Roger Cotes (1682–1716) who was one of the strongest mathematicians of his time and a collaborator of Newton. Cotes published a paper on logarithms in 1714 in which he states $ix = \log(\cos x + i\sin x)$. (after translating his results into more modern notation). He proved this result by computing in two different ways the surface area of an ellipse rotated about one axis and equating the results. Unfortunately Cotes died only 2 years later at the age of 33. Upon hearing of his death Newton is supposed to have said "If he had lived, we might have known something." The reader might think this a rather weak statement, however coming from Newton it was high praise.

Since the derivative is zero, the function $f(x)$ must be a constant. Setting $x = 0$ tells us that

$$f(0) = e^0 (\cos 0 + i \sin 0) = 1.$$

Hence $f(x) = 1$ for all x. Rearranging then arrives at

$$e^{ix} = \cos x + i \sin x$$

as required.

Substituting $x = \pi$ into Euler's formula we get Euler's identity

$$e^{i\pi} = -1$$

which is more often stated

Equation 3.6.21 Euler's identity.

$$e^{i\pi} + 1 = 0$$

which links the 5 most important constants in mathematics, $1, 0, \pi, e$ and $\sqrt{-1}$.

3.6.4 ►► Evaluating Limits using Taylor Expansions

Taylor polynomials provide a good way to understand the behaviour of a function near a specified point and so are useful for evaluating complicated limits. Here are some examples.

Example 3.6.22 A simple limit from a Taylor expansion.

In this example, we'll start with a relatively simple limit, namely

$$\lim_{x \to 0} \frac{\sin x}{x}$$

The first thing to notice about this limit is that, as x tends to zero, both the numerator, $\sin x$, and the denominator, x, tend to 0. So we may not evaluate the limit of the ratio by simply dividing the limits of the numerator and denominator. To find the limit, or show that it does not exist, we are going to have to exhibit a cancellation between the numerator and the denominator. Let's start by taking a closer look at the numerator. By Example 3.6.6,

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

Consequently[a]

$$\frac{\sin x}{x} = 1 - \frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \cdots$$

Every term in this series, except for the very first term, is proportional to a strictly positive power of x. Consequently, as x tends to zero, all terms in this series, except for the very first term, tend to zero. In fact the sum of all terms, starting with the second term, also tends to zero. That is,

$$\lim_{x\to 0}\Big[-\frac{1}{3!}x^2+\frac{1}{5!}x^4-\cdots\Big]=0$$

We won't justify that statement here, but it will be justified in the following (optional) subsection. So

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x}{x}&=\lim_{x\to 0}\Big[1-\frac{1}{3!}x^2+\frac{1}{5!}x^4-\cdots\Big]\\&=1+\lim_{x\to 0}\Big[-\frac{1}{3!}x^2+\frac{1}{5!}x^4-\cdots\Big]\\&=1\end{aligned}$$

a We are hiding some mathematics behind this "consequently". What we are really using our knowledge of Taylor polynomials to write $f(x)=\sin(x)=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+E_5(x)$ where $E_5(x)=\frac{f^{(6)}(c)}{6!}x^6$ and c is between 0 and x. We are effectively hiding "$E_5(x)$" inside the "$\cdots$". Now we can divide both sides by x (assuming $x\neq 0$): $\frac{\sin(x)}{x}=1-\frac{1}{3!}x^2+\frac{1}{5!}x^4+\frac{E_5(x)}{x}$. and everything is fine provided the term $\frac{E_5(x)}{x}$ stays well behaved.

Example 3.6.22

The limit in the previous example can also be evaluated relatively easily using l'Hôpital's rule[10]. While the following limit can also, in principal, be evaluated using l'Hôpital's rule, it is much more efficient to use Taylor series[11].

Example 3.6.23 A not so easy limit made easier.

In this example we evaluate

$$\lim_{x\to 0}\frac{\arctan x-x}{\sin x-x}$$

Once again, the first thing to notice about this limit is that, as x tends to zero, the numerator tends to $\arctan 0-0$, which is 0, and the denominator tends to $\sin 0-0$, which is also 0. So we may not evaluate the limit of the ratio by simply dividing the limits of the numerator and denominator. Again, to find the limit, or show that it does not exist, we are going to have to exhibit a cancellation between the numerator and the denominator. To get a more detailed understanding of the behaviour of the numerator

10 Many of you learned about l'Hôptial's rule in school and all of you should have seen it last term in your differential calculus course.

11 It takes 3 applications of l'Hôpital's rule and some careful cleaning up of the intermediate expressions. Oof!

and denominator near $x = 0$, we find their Taylor expansions. By Example 3.5.21,

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

so the numerator

$$\arctan x - x = -\frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

By Example 3.6.6,

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

so the denominator

$$\sin x - x = -\frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

and the ratio

$$\frac{\arctan x - x}{\sin x - x} = \frac{-\frac{x^3}{3} + \frac{x^5}{5} - \cdots}{-\frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots}$$

Notice that every term in both the numerator and the denominator contains a common factor of x^3, which we can cancel out.

$$\frac{\arctan x - x}{\sin x - x} = \frac{-\frac{1}{3} + \frac{x^2}{5} - \cdots}{-\frac{1}{3!} + \frac{1}{5!}x^2 - \cdots}$$

As x tends to zero,

- the numerator tends to $-\frac{1}{3}$, which is not 0, and
- the denominator tends to $-\frac{1}{3!} = -\frac{1}{6}$, which is also not 0.

so we may now legitimately evaluate the limit of the ratio by simply dividing the limits of the numerator and denominator.

$$\begin{aligned}
\lim_{x\to 0} \frac{\arctan x - x}{\sin x - x} &= \lim_{x\to 0} \frac{-\frac{1}{3} + \frac{x^2}{5} - \cdots}{-\frac{1}{3!} + \frac{1}{5!}x^2 - \cdots} \\
&= \frac{\lim_{x\to 0}\big[-\frac{1}{3} + \frac{x^2}{5} - \cdots\big]}{\lim_{x\to 0}\big[-\frac{1}{3!} + \frac{1}{5!}x^2 - \cdots\big]} \\
&= \frac{-\frac{1}{3}}{-\frac{1}{3!}} \\
&= 2
\end{aligned}$$

Example 3.6.23

3.6.5 ▸▸ Optional — The Big O Notation

In Example 3.6.22 we used, without justification[12], that, as x tends to zero, not only does every term in

$$\frac{\sin x}{x} - 1 = -\frac{1}{3!}x^2 + \frac{1}{5!}x^4 - \cdots = \sum_{n=1}^{\infty}(-1)^n \frac{1}{(2n+1)!}x^{2n}$$

converge to zero, but in fact the sum of all infinitely many terms also converges to zero. We did something similar twice in Example 3.6.23; once in computing the limit of the numerator and once in computing the limit of the denominator.

We'll now develop some machinery that provides the justification. We start by recalling, from equation 3.6.1, that if, for some natural number n, the function $f(x)$ has $n+1$ derivatives near the point a, then

$$f(x) = T_n(x) + E_n(x)$$

where

$$T_n(x) = f(a) + f'(a)\,(x-a) + \cdots + \tfrac{1}{n!}f^{(n)}(a)\,(x-a)^n$$

is the Taylor polynomial of degree n for the function $f(x)$ and expansion point a and

$$E_n(x) = f(x) - T_n(x) = \tfrac{1}{(n+1)!}f^{(n+1)}(c)\,(x-a)^{n+1}$$

is the error introduced when we approximate $f(x)$ by the polynomial $T_n(x)$. Here c is some unknown number between a and x. As c is not known, we do not know exactly what the error $E_n(x)$ is. But that is usually not a problem.

In the present context[13] we are interested in taking the limit as $x \to a$. So we are only interested in x-values that are very close to a, and because c lies between x and a, c is also very close to a. Now, as long as $f^{(n+1)}(x)$ is continuous at a, as $x \to a$, $f^{(n+1)}(c)$ must approach $f^{(n+1)}(a)$ which is some finite value. This, in turn, means that there must be constants $M, D > 0$ such that $\big|f^{(n+1)}(c)\big| \le M$ for all c's within a distance D of a. If so, there is another constant C (namely $\frac{M}{(n+1)!}$) such that

$$\big|E_n(x)\big| \le C|x-a|^{n+1} \qquad \text{whenever } |x-a| \le D$$

There is some notation for this behaviour.

Definition 3.6.24 Big O.

Let a and m be real numbers. We say that the function "$g(x)$ is of order $|x-a|^m$ near a" and we write $g(x) = O\big(|x-a|^m\big)$ if there exist constants[a] $C, D > 0$ such

12 Though there were a few comments in a footnote.

13 It is worth pointing out that our Taylor series must be expanded about the point to which we are limiting — i.e. a. To work out a limit as $x \to a$ we need Taylor series expanded about a and not some other point.

that

$$\big|g(x)\big| \le C|x-a|^m \qquad \text{whenever } |x-a| \le D \tag{$\star$}$$

Whenever $O\big(|x-a|^m\big)$ appears in an algebraic expression, it just stands for some (unknown) function $g(x)$ that obeys ($\star$). This is called "big O" notation.

a To be precise, C and D do not depend on x, though they may, and usually do, depend on m.

How should we parse the big O notation when we see it? Consider the following

$$g(x) = O(|x-3|^2)$$

First of all, we know from the definition that the notation only tells us something about $g(x)$ for x near the point a. The equation above contains "$O(|x-3|^2)$" which tells us something about what the function looks like when x is close to 3. Further, because it is "$|x-3|$" squared, it says that the graph of the function lies below a parabola $y = C(x-3)^2$ and above a parabola $y = -C(x-3)^2$ near $x = 3$. The notation doesn't tell us anything more than this — we don't know, for example, that the graph of $g(x)$ is concave up or concave down. It also tells us that Taylor expansion of $g(x)$ around $x = 3$ does not contain any constant or linear term — the first nonzero term in the expansion is of degree at least two. For example, all of the following functions are $O(|x-3|^2)$.

$$5(x-3)^2 + 6(x-3)^3, \qquad -7(x-3)^2 - 8(x-3)^4, \qquad (x-3)^3, \qquad (x-3)^{\frac{5}{2}}$$

In the next few examples we will rewrite a few of the Taylor polynomials that we know using this big O notation.

Example 3.6.25 Sine and the big O.

Let $f(x) = \sin x$ and $a = 0$. Then

$$\begin{array}{rrrr} f(x) = \sin x & f'(x) = \cos x & f''(x) = -\sin x & f^{(3)}(x) = -\cos x \\ f(0) = 0 & f'(0) = 1 & f''(0) = 0 & f^{(3)}(0) = -1 \\ f^{(4)}(x) = \sin x & \cdots & & \\ f^{(4)}(0) = 0 & \cdots & & \end{array}$$

and the pattern repeats. So every derivative is plus or minus either sine or cosine and, as we saw in previous examples, this makes analysing the error term for the sine and cosine series quite straightforward. In particular, $\big|f^{(n+1)}(c)\big| \le 1$ for all real numbers c and all natural numbers n. So the Taylor polynomial of, for example, degree 3 and its error term are

$$\sin x = x - \tfrac{1}{3!}x^3 + \tfrac{\cos c}{5!}x^5$$

$$= x - \tfrac{1}{3!}x^3 + O(|x|^5)$$

under Definition 3.6.24, with $C = \frac{1}{5!}$ and any $D > 0$. Similarly, for any natural number n,

Equation 3.6.26

$$\sin x = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots + (-1)^n \tfrac{1}{(2n+1)!}x^{2n+1} + O\big(|x|^{2n+3}\big)$$
$$\cos x = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \cdots + (-1)^n \tfrac{1}{(2n)!}x^{2n} + O\big(|x|^{2n+2}\big)$$

Example 3.6.26

When we studied the error in the expansion of the exponential function (way back in optional Example 3.6.8), we had to go to some length to understand the behaviour of the error term well enough to prove convergence for all numbers x. However, in the big O notation, we are free to assume that x is close to 0. Furthermore we do not need to derive an explicit bound on the size of the coefficient C. This makes it quite a bit easier to verify that the big O notation is correct.

Example 3.6.27 Exponential and the big O.

Let n be any natural number. Since $\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$, we know that $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left\{e^x\right\} = e^x$ for every integer $k \geq 0$. Thus

$$e^x = 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \cdots + \tfrac{x^n}{n!} + \tfrac{e^c}{(n+1)!}x^{n+1}$$

for some c between 0 and x. If, for example, $|x| \leq 1$, then $|e^c| \leq e$, so that the error term

$$\Big|\tfrac{e^c}{(n+1)!}x^{n+1}\Big| \leq C|x|^{n+1} \qquad \text{with } C = \tfrac{e}{(n+1)!} \qquad \text{whenever } |x| \leq 1$$

So, under Definition 3.6.24, with $C = \frac{e}{(n+1)!}$ and $D = 1$,

Equation 3.6.28

$$e^x = 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \cdots + \tfrac{x^n}{n!} + O\big(|x|^{n+1}\big)$$

You can see that, because we only have to consider x's that are close to the expansion point (in this example, 0) it is relatively easy to derive the bounds that are required to justify the use of the big O notation.

Example 3.6.29 Logarithms and the big O.

Let $f(x) = \log(1+x)$ and $a = 0$. Then

$$\begin{array}{lll} f'(x) = \frac{1}{1+x} & f''(x) = -\frac{1}{(1+x)^2} & f^{(3)}(x) = \frac{2}{(1+x)^3} \\ f'(0) = 1 & f''(0) = -1 & f^{(3)}(0) = 2 \\ f^{(4)}(x) = -\frac{2\times 3}{(1+x)^4} & f^{(5)}(x) = \frac{2\times 3\times 4}{(1+x)^5} & \\ f^{(4)}(0) = -3! & f^{(5)}(0) = 4! & \end{array}$$

We can see a pattern for $f^{(n)}(x)$ forming here — $f^{(n)}(x)$ is a sign times a ratio with

- the sign being $+$ when n is odd and being $-$ when n is even. So the sign is $(-1)^{n-1}$.
- The denominator is $(1+x)^n$.
- The numerator[a] is the product $2 \times 3 \times 4 \times \cdots \times (n-1) = (n-1)!$.

Thus[b], for any natural number n,

$$f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n} \qquad \text{which means that}$$

$$\frac{1}{n!}f^{(n)}(0)\,x^n = (-1)^{n-1}\frac{(n-1)!}{n!}x^n = (-1)^{n-1}\frac{x^n}{n}$$

so

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + E_n(x)$$

with

$$E_n(x) = \frac{1}{(n+1)!}f^{(n+1)}(c)\,(x-a)^{n+1} = \frac{1}{n+1}\cdot\frac{(-1)^n}{(1+c)^{n+1}}\cdot x^{n+1}$$

If we choose, for example $D = \frac{1}{2}$, then[c] for any x obeying $|x| \le D = \frac{1}{2}$, we have $|c| \le \frac{1}{2}$ and $|1+c| \ge \frac{1}{2}$ so that

$$|E_n(x)| \le \frac{1}{(n+1)(1/2)^{n+1}}|x|^{n+1} = O\big(|x|^{n+1}\big)$$

under Definition 3.6.24, with $C = \frac{2^{n+1}}{n+1}$ and $D = \frac{1}{2}$. Thus we may write

Equation 3.6.30

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + O\big(|x|^{n+1}\big)$$

a Remember that $n! = 1 \times 2 \times 3 \times \cdots \times n$, and that we use the convention $0! = 1$.

b It is not too hard to make this rigorous using the principle of mathematical induction. The interested reader should do a little search-engine-ing. Induction is a very standard technique for proving statements of the form "For every natural number $n, \ldots$". For example For every natural number n, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, or For every natural number n, $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\{\log(1+x)\} = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$. It was also used by Polya (1887–1985) to give a very convincing (but subtly (and deliberately) flawed) proof that all horses have the same colour.

c Since $|c| \le \frac{1}{2}$, $-\frac{1}{2} \le c \le \frac{1}{2}$. If we now add 1 to every term we get $\frac{1}{2} \le 1+c \le \frac{3}{2}$ and so $|1+c| \ge \frac{1}{2}$. You can also do this with the triangle inequality which tells us that for any x, y we know that $|x+y| \le |x| + |y|$. Actually, you want the reverse triangle inequality (which is a simple corollary of the triangle inequality) which says that for any x, y we have $|x+y| \ge \big||x| - |y|\big|$.

Example 3.6.30

Remark 3.6.31 The big O notation has a few properties that are useful in computations and taking limits. All follow immediately from Definition 3.6.24.

a If $p > 0$, then
$$\lim_{x\to 0} O(|x|^p) = 0$$

b For any real numbers p and q,
$$O(|x|^p)\; O(|x|^q) = O(|x|^{p+q})$$
(This is just because $C|x|^p \times C'|x|^q = (CC')|x|^{p+q}$.) In particular,
$$ax^m\, O(|x|^p) = O(|x|^{p+m})$$
for any constant a and any integer m.

c For any real numbers p and q,
$$O(|x|^p) + O(|x|^q) = O(|x|^{\min\{p,q\}})$$
(For example, if $p = 2$ and $q = 5$, then $C|x|^2 + C'|x|^5 = \big(C + C'|x|^3\big)|x|^2 \le (C + C')|x|^2$ whenever $|x| \le 1$.)

d For any real numbers p and q with $p > q$, any function which is $O(|x|^p)$ is also $O(|x|^q)$ because $C|x|^p = C|x|^{p-q}|x|^q \le C|x|^q$ whenever $|x| \le 1$.

e All of the above observations also hold for more general expressions with $|x|$ replaced by $|x - a|$, i.e. for $O(|x-a|^p)$. The only difference being in (a) where we must take the limit as $x \to a$ instead of $x \to 0$.

3.6.6 ►► Optional — Evaluating Limits Using Taylor Expansions — More Examples

Example 3.6.32 Example 3.6.22 revisited.

In this example, we'll return to the limit

$$\lim_{x\to 0}\frac{\sin x}{x}$$

of Example 3.6.22 and treat it more carefully. By Example 3.6.25,

$$\sin x = x - \frac{1}{3!}x^3 + O(|x|^5)$$

That is, for small x, $\sin x$ is the same as $x - \frac{1}{3!}x^3$, up to an error that is bounded by some constant times $|x|^5$. So, dividing by x, $\frac{\sin x}{x}$ is the same as $1 - \frac{1}{3!}x^2$, up to an error that is bounded by some constant times x^4 — see Remark 3.6.31(b). That is

$$\frac{\sin x}{x} = 1 - \frac{1}{3!}x^2 + O(x^4)$$

But any function that is bounded by some constant times x^4 (for all x smaller than some constant $D > 0$) necessarily tends to 0 as $x \to 0$ — see Remark 3.6.31(a). . Thus

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\left[1 - \frac{1}{3!}x^2 + O(x^4)\right] = \lim_{x\to 0}\left[1 - \frac{1}{3!}x^2\right] = 1$$

Reviewing the above computation, we see that we did a little more work than we had to. It wasn't necessary to keep track of the $-\frac{1}{3!}x^3$ contribution to $\sin x$ so carefully. We could have just said that

$$\sin x = x + O(|x|^3)$$

so that

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{x + O(|x|^3)}{x} = \lim_{x\to 0}\left[1 + O(x^2)\right] = 1$$

We'll spend a little time in the later, more complicated, examples learning how to choose the number of terms we keep in our Taylor expansions so as to make our computations as efficient as possible.

Example 3.6.33 Practicing using Taylor polynomials for limits.

In this example, we'll use the Taylor polynomial of Example 3.6.29 to evaluate $\lim\limits_{x\to 0}\frac{\log(1+x)}{x}$ and $\lim\limits_{x\to 0}(1+x)^{a/x}$. The Taylor expansion of equation 3.6.30 with $n = 1$

tells us that

$$\log(1+x) = x + O(|x|^2)$$

That is, for small x, $\log(1+x)$ is the same as x, up to an error that is bounded by some constant times x^2. So, dividing by x, $\frac{1}{x}\log(1+x)$ is the same as 1, up to an error that is bounded by some constant times $|x|$. That is

$$\frac{1}{x}\log(1+x) = 1 + O(|x|)$$

But any function that is bounded by some constant times $|x|$, for all x smaller than some constant $D > 0$, necessarily tends to 0 as $x \to 0$. Thus

$$\lim_{x\to 0}\frac{\log(1+x)}{x} = \lim_{x\to 0}\frac{x + O(|x|^2)}{x} = \lim_{x\to 0}\big[1 + O(|x|)\big] = 1$$

We can now use this limit to evaluate

$$\lim_{x\to 0}(1+x)^{a/x}.$$

Now, we could either evaluate the limit of the logarithm of this expression, or we can carefully rewrite the expression as $e^{(\text{something})}$. Let us do the latter.

$$\begin{aligned}\lim_{x\to 0}(1+x)^{a/x} &= \lim_{x\to 0} e^{\frac{a}{x}\log(1+x)} \\ &= \lim_{x\to 0} e^{\frac{a}{x}[x+O(|x|^2)]} \\ &= \lim_{x\to 0} e^{a+O(|x|)} = e^a\end{aligned}$$

Here we have used that if $F(x) = O(|x|^2)$ then $\frac{a}{x}F(x) = O(x)$ — see Remark 3.6.31(b). We have also used that the exponential is continuous — as x tends to zero, the exponent of $e^{a+O(|x|)}$ tends to a so that $e^{a+O(|x|)}$ tends to e^a — see Remark 3.6.31(a).

Example 3.6.33

Example 3.6.34 A difficult limit.

In this example, we'll evaluate[a] the harder limit

$$\lim_{x\to 0}\frac{\cos x - 1 + \frac{1}{2}x\sin x}{[\log(1+x)]^4}$$

The first thing to notice about this limit is that, as x tends to zero, the numerator

$$\cos x - 1 + \frac{1}{2}x\sin x \to \cos 0 - 1 + \frac{1}{2}\cdot 0\cdot\sin 0 = 0$$

and the denominator

$$[\log(1+x)]^4 \to [\log(1+0)]^4 = 0$$

too. So both the numerator and denominator tend to zero and we may not simply evaluate the limit of the ratio by taking the limits of the numerator and denominator and dividing.

To find the limit, or show that it does not exist, we are going to have to exhibit a cancellation between the numerator and the denominator. To develop a strategy for evaluating this limit, let's do a "little scratch work", starting by taking a closer look at the denominator. By Example 3.6.29,

$$\log(1+x) = x + O(x^2)$$

This tells us that $\log(1+x)$ looks a lot like x for very small x. So the denominator $[x+O(x^2)]^4$ looks a lot like x^4 for very small x. Now, what about the numerator?

- If the numerator looks like some constant times x^p with $p > 4$, for very small x, then the ratio will look like the constant times $\frac{x^p}{x^4} = x^{p-4}$ and, as $p-4 > 0$, will tend to 0 as x tends to zero.
- If the numerator looks like some constant times x^p with $p < 4$, for very small x, then the ratio will look like the constant times $\frac{x^p}{x^4} = x^{p-4}$ and will, as $p-4 < 0$, tend to infinity, and in particular diverge, as x tends to zero.
- If the numerator looks like Cx^4, for very small x, then the ratio will look like $\frac{Cx^4}{x^4} = C$ and will tend to C as x tends to zero.

The moral of the above "scratch work" is that we need to know the behaviour of the numerator, for small x, up to order x^4. Any contributions of order x^p with $p > 4$ may be put into error terms $O(|x|^p)$.

Now we are ready to evaluate the limit. Because the expressions are a little involved, we will simplify the numerator and denominator separately and then put things together. Using the expansions we developed in Example 3.6.25, the numerator,

$$\begin{aligned}\cos x - 1 + \frac{1}{2}x\sin x &= \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + O(|x|^6)\right)\\ &\qquad - 1 + \frac{x}{2}\left(x - \frac{1}{3!}x^3 + O(|x|^5)\right) && \text{expand}\\ &= \left(\frac{1}{24} - \frac{1}{12}\right)x^4 + O(|x|^6) + \frac{x}{2}O(|x|^5)\end{aligned}$$

Then by Remark 3.6.31(b)

$$= -\frac{1}{24}x^4 + O(|x|^6) + O(|x|^6)$$

and now by Remark3.6.31(c)

$$= -\frac{1}{24}x^4 + O(|x|^6)$$

Similarly, using the expansion that we developed in Example 3.6.29,

$$\begin{aligned} [\log(1+x)]^4 &= \big[x + O(|x|^2)\big]^4 \\ &= \big[x + xO(|x|)\big]^4 && \text{by Remark 3.6.31(b)} \\ &= x^4[1 + O(|x|)]^4 \end{aligned}$$

Now put these together and take the limit as $x \to 0$:

$$\begin{aligned} \lim_{x\to 0} \frac{\cos x - 1 + \frac{1}{2}x\sin x}{[\log(1+x)]^4} &= \lim_{x\to 0} \frac{-\frac{1}{24}x^4 + O(|x|^6)}{x^4[1+O(|x|)]^4} \\ &= \lim_{x\to 0} \frac{-\frac{1}{24}x^4 + x^4O(|x|^2)}{x^4[1+O(|x|)]^4} && \text{by Remark 3.6.31(b)} \\ &= \lim_{x\to 0} \frac{-\frac{1}{24} + O(|x|^2)}{[1+O(|x|)]^4} \\ &= -\frac{1}{24} && \text{by Remark 3.6.31(a).} \end{aligned}$$

a Use of l'Hôpital's rule here could be characterised as a "courageous decision". The interested reader should search-engine their way to Sir Humphrey Appleby and "Yes Minister" to better understand this reference (and the workings of government in the Westminster system). Discretion being the better part of valour, we'll stop and think a little before limiting (ha) our choices.

Example 3.6.34

The next two limits have much the same flavour as those above — expand the numerator and denominator to high enough order, do some cancellations and then take the limit. We have increased the difficulty a little by introducing "expansions of expansions".

Example 3.6.35 Another difficult limit.

In this example we'll evaluate another harder limit, namely

$$\lim_{x\to 0} \frac{\log\left(\frac{\sin x}{x}\right)}{x^2}$$

The first thing to notice about this limit is that, as x tends to zero, the denominator x^2 tends to 0. So, yet again, to find the limit, we are going to have to show that the numerator also tends to 0 and we are going to have to exhibit a cancellation between the numerator and the denominator.

Because the denominator is x^2 any terms in the numerator, $\log\left(\frac{\sin x}{x}\right)$ that are of order x^3 or higher will contribute terms in the ratio $\frac{\log(\frac{\sin x}{x})}{x^2}$ that are of order x or higher. Those terms in the ratio will converge to zero as $x \to 0$. The moral of this discussion is that we need to compute $\log \frac{\sin x}{x}$ to order x^2 with errors of order x^3. Now we saw, in

Example 3.6.32, that

$$\frac{\sin x}{x} = 1 - \frac{1}{3!}x^2 + O(x^4)$$

We also saw, in equation 3.6.30 with $n = 1$, that

$$\log(1 + X) = X + O(X^2)$$

Substituting[a] $X = -\frac{1}{3!}x^2 + O(x^4)$, and using that $X^2 = O(x^4)$ (by Remark 3.6.31(b,c)), we have that the numerator

$$\log\left(\frac{\sin x}{x}\right) = \log(1 + X) = X + O(X^2) = -\frac{1}{3!}x^2 + O(x^4)$$

and the limit

$$\begin{aligned} \lim_{x\to 0} \frac{\log\left(\frac{\sin x}{x}\right)}{x^2} &= \lim_{x\to 0} \frac{-\frac{1}{3!}x^2 + O(x^4)}{x^2} = \lim_{x\to 0} \Big[-\frac{1}{3!} + O(x^2)\Big] = -\frac{1}{3!} \\ &= -\frac{1}{6} \end{aligned}$$

a In our derivation of $\log(1+X) = X+O(X^2)$ in Example 3.6.29, we required only that $|X| \le \frac{1}{2}$. So we are free to substitute $X = -\frac{1}{3!}x^2+O(x^4)$ for any x that is small enough that $\left|-\frac{1}{3!}x^2+O(x^4)\right| < \frac{1}{2}$.

Example 3.6.35

Example 3.6.36 Yet another difficult limit.

Evaluate

$$\lim_{x\to 0} \frac{e^{x^2} - \cos x}{\log(1 + x) - \sin x}$$

Solution: *Step 1*: Find the limit of the denominator.

$$\lim_{x\to 0} \big[\log(1 + x) - \sin x\big] = \log(1 + 0) - \sin 0 = 0$$

This tells us that we can't evaluate the limit just by finding the limits of the numerator and denominator separately and then dividing.

Step 2: Determine the leading order behaviour of the denominator near $x = 0$. By equations 3.6.30 and 3.6.26,

$$\begin{aligned} \log(1 + x) &= x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \cdots \\ \sin x &= x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \cdots \end{aligned}$$

Taking the difference of these expansions gives

$$\log(1 + x) - \sin x = -\tfrac{1}{2}x^2 + \left(\tfrac{1}{3} + \tfrac{1}{3!}\right)x^3 + \cdots$$

This tells us that, for x near zero, the denominator is $-\frac{x^2}{2}$ (that's the leading order term) plus contributions that are of order x^3 and smaller. That is

$$\log(1+x) - \sin x = -\tfrac{x^2}{2} + O(|x|^3)$$

Step 3: Determine the behaviour of the numerator near $x = 0$ to order x^2 with errors of order x^3 and smaller (just like the denominator). By equation 3.6.28

$$e^X = 1 + X + O\big(X^2\big)$$

Substituting $X = x^2$

$$\begin{aligned} e^{x^2} &= 1 + x^2 + O\big(x^4\big) \\ \cos x &= 1 - \tfrac{1}{2}x^2 + O\big(x^4\big) \end{aligned}$$

by equation 3.6.26. Subtracting, the numerator

$$e^{x^2} - \cos x = \tfrac{3}{2}x^2 + O\big(x^4\big)$$

Step 4: Evaluate the limit.

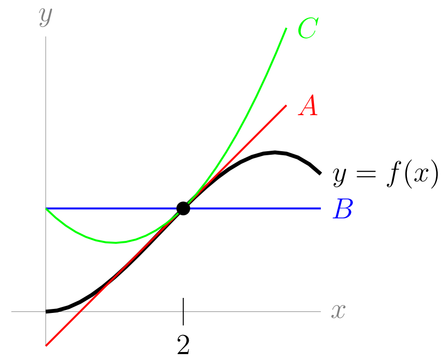
$$\begin{aligned} \lim_{x\to 0} \frac{e^{x^2} - \cos x}{\log(1+x) - \sin x} &= \lim_{x\to 0} \frac{\frac{3}{2}x^2 + O(x^4)}{-\frac{x^2}{2} + O(|x|^3)} \\ &= \lim_{x\to 0} \frac{\frac{3}{2} + O(x^2)}{-\frac{1}{2} + O(|x|)} \\ &= \frac{\frac{3}{2}}{-\frac{1}{2}} = -3 \end{aligned}$$

Example 3.6.36

3.6.7 ▸▸ Exercises

Exercises — Stage 1

1. Below is a graph of $y = f(x)$, along with the constant approximation, linear approximation, and quadratic approximation centred at $a = 2$. Which is which?



2. Suppose $T(x)$ is the Taylor series for $f(x) = \arctan^3\left(e^x + 7\right)$ centred at $a = 5$. What is $T(5)$?

3. Below are a list of common functions, and their Taylor series representations. Match the function to the Taylor series.

function	series
A. $\dfrac{1}{1-x}$	I. $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$
B. $\log(1+x)$	II. $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$
C. $\arctan x$	III. $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$
D. e^x	IV. $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$
E. $\sin x$	V. $\displaystyle\sum_{n=0}^{\infty} x^n$
F. $\cos x$	VI. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

4.

a Suppose $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{n^2}{(n!+1)}(x-3)^n$ for all real x. What is $f^{(20)}(3)$ (the twentieth derivative of $f(x)$ at $x = 3$)?

b Suppose $g(x) = \displaystyle\sum_{n=0}^{\infty} \frac{n^2}{(n!+1)}(x-3)^{2n}$ for all real x. What is $g^{(20)}(3)$?

c If $h(x) = \dfrac{\arctan(5x^2)}{x^4}$, what is $h^{(20)}(0)$? What is $h^{(22)}(0)$?

Exercises — Stage 2 In Questions 5 through 8, you will create Taylor series from scratch. In practice, it is often preferable to modify an existing series, rather than creating a new one, but you should understand both ways.

5. Using the definition of a Taylor series, find the Taylor series for $f(x) = \log(x)$ centred at $x = 1$.

6. Find the Taylor series for $f(x) = \sin x$ centred at $a = \pi$.

7. Using the definition of a Taylor series, find the Taylor series for $g(x) = \dfrac{1}{x}$ centred at $x = 10$. What is the interval of convergence of the resulting series?

8. Using the definition of a Taylor series, find the Taylor series for $h(x) = e^{3x}$ centred at $x = a$, where a is some constant. What is the radius of convergence of the resulting series?

In Questions 9 through 16, practice creating new Taylor series by modifying known Taylor series, rather than creating your series from scratch.

9. $*$. Find the Maclaurin series for $f(x) = \dfrac{1}{2x-1}$.

10. $*$. Let $\displaystyle\sum_{n=0}^{\infty} b_n x^n$ be the Maclaurin series for $f(x) = \dfrac{3}{x+1} - \dfrac{1}{2x-1}$,
i.e. $\displaystyle\sum_{n=0}^{\infty} b_n x^n = \frac{3}{x+1} - \frac{1}{2x-1}$.
Find b_n.

11. $*$. Find the coefficient c_5 of the fifth degree term in the Maclaurin series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ for e^{3x}.

12. $*$. Express the Taylor series of the function

$$f(x) = \log(1 + 2x)$$

about $x = 0$ in summation notation.

13. $*$. The first two terms in the Maclaurin series for $x^2 \sin(x^3)$ are $ax^5 + bx^{11}$, where a and b are constants. Find the values of a and b.

14. $*$. Give the first two nonzero terms in the Maclaurin series for $\displaystyle\int \frac{e^{-x^2}-1}{x}\,dx$.

15. $*$. Find the Maclaurin series for $\displaystyle\int x^4 \arctan(2x)\,dx$.

16. $*$. Suppose that $\dfrac{df}{dx} = \dfrac{x}{1+3x^3}$ and $f(0) = 1$. Find the Maclaurin series for $f(x)$.

In past chapters, we were only able to exactly evaluate very specific types of series: geometric and telescoping. In Questions 17 through 25, we expand our range by relating given series to Taylor series.

17. $*$. The Maclaurin series for $\arctan x$ is given by

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

which has radius of convergence equal to 1. Use this fact to compute the exact value of the series below:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$$

18. $*$. Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$.

19. $*$. Evaluate $\displaystyle\sum_{k=0}^{\infty} \frac{1}{e^k k!}$.

20. $*$. Evaluate the sum of the convergent series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\pi^k k!}$.

21. $*$. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,2^n}$.

22. $*$. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n!} e^n$.

23. Evaluate $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n}$, or show that it diverges.

24. Evaluate

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{4}\right)^{2n+1}\left(1+2^{2n+1}\right)$$

or show that it diverges.

25. *. (a) Show that the power series $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$ converges absolutely for all real numbers x.

(b) Evaluate $\displaystyle\sum_{n=0}^{\infty}\frac{1}{(2n)!}$.

26.

a Using the fact that $\arctan(1) = \frac{\pi}{4}$, how many terms of the Taylor series for arctangent would you have to add up to approximate π with an error of at most 4×10^{-5}?

b Example 3.6.15 mentions the formula

$$\pi = 16\arctan\frac{1}{5} - 4\arctan\frac{1}{239}$$

Using the Taylor series for arctangent, how many terms would you have to add up to approximate π with an error of at most 4×10^{-5}?

c Assume without proof the following:

$$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\left(\frac{3+2}{2\cdot 3 - 1}\right)$$

Using the Taylor series for arctangent, how many terms would you have to add up to approximate π with an error of at most 4×10^{-5}?

27. Suppose you wanted to approximate the number $\log(1.5)$ as a rational number using the Taylor expansion of $\log(1 + x)$. How many terms would you need to add to get 10 decimal places of accuracy? (That is, an absolute error less than 5×10^{-11}.)

28. Suppose you wanted to approximate the number e as a rational number using the Maclaurin expansion of e^x. How many terms would you need to add to get 10 decimal places of accuracy? (That is, an absolute error less than 5×10^{-11}.) You may assume without proof that $2 < e < 3$.

29. Suppose you wanted to approximate the number $\log(0.9)$ as a rational number using the Taylor expansion of $\log(1-x)$. Which partial sum should you use to get 10 decimal places of accuracy? (That is, an absolute error less than 5×10^{-11}.)

30. Define the hyperbolic sine function as

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Suppose you wanted to approximate the number $\sinh(b)$ using the Maclaurin series of $\sinh x$, where b is some number in $(-2, 1)$. Which partial sum should you use to guarantee 10 decimal places of accuracy? (That is, an absolute error less than 5×10^{-11}.)
You may assume without proof that $2 < e < 3$.

31. Let $f(x)$ be a function with

$$f^{(n)}(x) = \frac{(n-1)!}{2}\left[(1-x)^{-n} + (-1)^{n-1}(1+x)^{-n}\right]$$

for all $n \geq 1$.
Give reasonable bounds (both upper and lower) on the error involved in approximating $f\left(-\frac{1}{3}\right)$ using the partial sum S_6 of the Taylor series for $f(x)$ centred at $a = \frac{1}{2}$.
Remark: One function with this quality is the inverse hyperbolic tangent function[a].

a Of course it is! Actually, hyperbolic tangent is $\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, and inverse hyperbolic tangent is its functional inverse.

Exercises — Stage 3

32. $*$. Use series to evaluate $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{1+x-e^x}$.

33. $*$. Evaluate $\displaystyle\lim_{x\to 0} \frac{\sin x - x + \frac{x^3}{6}}{x^5}$.

34. Evaluate $\displaystyle\lim_{x\to 0}\left(1+x+x^2\right)^{2/x}$ using a Taylor series for the natural logarithm.

35. Use series to evaluate

$$\lim_{x\to\infty}\left(1+\frac{1}{2x}\right)^x$$

36. Evaluate the series $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{7^n}$ or show that it diverges.

37. Write the series $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+4}}{(2n+1)(2n+2)}$ as a combination of familiar functions.

38.

a Find the Maclaurin series for $f(x) = (1-x)^{-1/2}$. What is its radius of convergence?

b Manipulate the series you just found to find the Maclaurin series for $g(x) = \arcsin x$. What is its radius of convergence?

39. $*$. Find the Taylor series for $f(x) = \log(x)$ centred at $a = 2$. Find the interval of convergence for this series.

40. $*$. Let $I(x) = \displaystyle\int_0^x \frac{1}{1+t^4}\,\mathrm{d}t$.

a Find the Maclaurin series for $I(x)$.

b Approximate $I(1/2)$ to within ± 0.0001.

c Is your approximation in (b) larger or smaller than the true value of $I(1/2)$? Explain.

41. $*$. Using a Maclaurin series, the number $a = 1/5 - 1/7 + 1/18$ is found to be an approximation for $I = \displaystyle\int_0^1 x^4 e^{-x^2}\,\mathrm{d}x$. Give the best upper bound you can for $|I - a|$.

42. $*$. Find an interval of length 0.0002 or less that contains the number

$$I = \int_0^{\frac{1}{2}} x^2 e^{-x^2}\,\mathrm{d}x$$

43. $*$. Let $I(x) = \displaystyle\int_0^x \frac{e^{-t}-1}{t}\,\mathrm{d}t$.

a Find the Maclaurin series for $I(x)$.

b Approximate $I(1)$ to within ± 0.01.

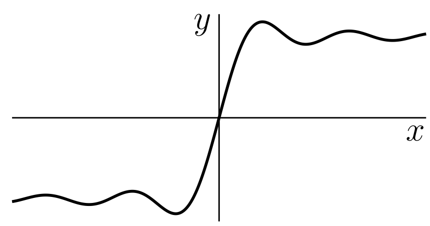
c Explain why your answer to part (b) has the desired accuracy.

44. $*$. The function $\Sigma(x)$ is defined by $\Sigma(x) = \displaystyle\int_0^x \frac{\sin t}{t}\,\mathrm{d}t$.

a Find the Maclaurin series for $\Sigma(x)$.

b It can be shown that $\Sigma(x)$ has an absolute maximum which occurs at its smallest positive critical point (see the graph of $\Sigma(x)$ below). Find this critical point.

c Use the previous information to find the maximum value of $\Sigma(x)$ to within ± 0.01.



45. $*$. Let $I(x) = \displaystyle\int_0^x \frac{\cos t - 1}{t^2}\,\mathrm{d}t$.

a Find the Maclaurin series for $I(x)$.

b Use this series to approximate $I(1)$ to within ± 0.01

c Is your estimate in (b) greater than $I(1)$? Explain.

46. $*$. Let $I(x) = \displaystyle\int_0^x \frac{\cos t + t\sin t - 1}{t^2}\,\mathrm{d}t$

a Find the Maclaurin series for $I(x)$.

b Use this series to approximate $I(1)$ to within ± 0.001

c Is your estimate in (b) greater than or less than $I(1)$?

47. $*$. Define $f(x) = \displaystyle\int_0^x \frac{1-e^{-t}}{t}\,\mathrm{d}t$.

a Show that the Maclaurin series for $f(x)$ is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot n!} x^n$.

b Use the ratio test to determine the values of x for which the Maclaurin series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot n!} x^n$ converges.

48. $*$. Show that $\displaystyle\int_0^1 \frac{x^3}{e^x - 1}\,\mathrm{d}x \le \frac{1}{3}$.

49. $*$. Let $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$.

a Find the power series expansion of $\cosh(x)$ about $x_0 = 0$ and determine its interval of convergence.

b Show that $3\frac{2}{3} \le \cosh(2) \le 3\frac{2}{3} + 0.1$.

c Show that $\cosh(t) \le e^{\frac{1}{2}t^2}$ for all t.

50. The law of the instrument says "If you have a hammer then everything looks like a nail" — it is really a description of the "tendency of jobs to be adapted to tools rather than adapting tools to jobs"[a]. Anyway, this is a long way of saying that just because we know how to compute things using Taylor series doesn't mean we should neglect other techniques.

a Using Newton's method, approximate the constant $\sqrt[3]{2}$ as a root of the function $g(x) = x^3 - 2$. Using a calculator, make your estimation accurate to within 0.01.

b You may assume without proof that

$$\sqrt[3]{x} = 1 + \frac{1}{6}(x-1) + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{(2)(5)(8)\cdots(3n-4)}{3^n\, n!}(x-1)^n.$$

for all real numbers x. Using the fact that this is an alternating series, how many terms would you have to add for the partial sum to estimate $\sqrt[3]{2}$ with an error less than 0.01?

a Quote from Silvan Tomkins's *Computer Simulation of Personality: Frontier of Psychological Theory*. See also Birmingham screwdrivers.

51. Let $f(x) = \arctan(x^3)$. Write $f^{(10)}\left(\frac{1}{5}\right)$ as a sum of rational numbers with an error less than 10^{-6} using the Maclaurin series for arctangent.

52. Consider the following function:

$$f(x) = \begin{cases} e^{-1/x^2} & x \ne 0 \\ 0 & x = 0 \end{cases}$$

a Sketch $y = f(x)$.

b Assume (without proof) that $f^{(n)}(0) = 0$ for all whole numbers n. Find the Maclaurin series for $f(x)$.

c Where does the Maclaurin series for $f(x)$ converge?

d For which values of x is $f(x)$ equal to its Maclaurin series?

53. Suppose $f(x)$ is an odd function, and $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$. Simplify $\sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n}$.

3.7▲ Optional — Rational and irrational numbers

In this optional section we shall use series techniques to look a little at rationality and irrationality of real numbers. We shall see the following results.

- A real number is rational (i.e. a ratio of two integers) if and only if its decimal expansion is eventually periodic. "Eventually periodic" means that, if we denote the n^{th} decimal place by d_n, then there are two positive integers k and p such that $d_{n+p} = d_n$ whenever $n > k$. So the part of the decimal expansion after the decimal point looks like
$$.\underbrace{a_1 a_2 a_3 \cdots a_k}\underbrace{b_1 b_2 \cdots b_p}\underbrace{b_1 b_2 \cdots b_p}\underbrace{b_1 b_2 \cdots b_p}\cdots$$
It is possible that a finite number of decimal places right after the decimal point do not participate in the periodicity. It is also possible that $p = 1$ and $b_1 = 0$, so that the decimal expansion ends with an infinite string of zeros.
- e is irrational.
- π is irrational.

3.7.1 ▶▶ Decimal expansions of rational numbers

We start by showing that a real number is rational if and only if its decimal expansion is eventually periodic. We need only consider the expansions of numbers $0 < x < 1$. If a number is negative then we can just multiply it by -1 and not change the expansion. Similarly if the number is larger than 1 then we can just subtract off the integer part of the number and leave the expansion unchanged.

3.7.2 ▶▶ Eventually periodic implies rational

Let us assume that a number $0 < x < 1$ has a decimal expansion that is eventually periodic. Hence we can write

$$x = 0.\underbrace{a_1 a_2 a_3 \cdots a_k}\underbrace{b_1 b_2 \cdots b_p}\underbrace{b_1 b_2 \cdots b_p}\underbrace{b_1 b_2 \cdots b_p}\cdots$$

Let $\alpha = a_1 a_2 a_3 \cdots a_k$ and $\beta = b_1 b_2 \cdots b_p$. In particular, α has at most k digits and β has at most p digits. Then we can (carefully) write

$$\begin{aligned} x &= \frac{\alpha}{10^k} + \frac{\beta}{10^{k+p}} + \frac{\beta}{10^{k+2p}} + \frac{\beta}{10^{k+3p}} + \cdots \\ &= \frac{\alpha}{10^k} + \frac{\beta}{10^{k+p}} \sum_{j=0}^{\infty} 10^{-p} \end{aligned}$$

This sum is just a geometric series (see Lemma 3.2.5) and we can evaluate it

$$\begin{aligned} &= \frac{\alpha}{10^k} + \frac{\beta}{10^{k+p}} \cdot \frac{1}{1-10^{-p}} = \frac{\alpha}{10^k} + \frac{\beta}{10^k} \cdot \frac{1}{10^p - 1} \\ &= \frac{1}{10^k}\left(\alpha + \frac{\beta}{10^p - 1}\right) = \frac{\alpha(10^p-1)+\beta}{10^k(10^p-1)} \end{aligned}$$

This is a ratio of integers, so x is a rational number.

3.7.3 ➤ Rational implies eventually periodic

Let $0 < x < 1$ be rational with $x = \frac{a}{b}$, where a and b are positive integers. We wish to show that x's decimal expansion is eventually periodic. Start by looking at the last formula we derived in the "eventually periodic implies rational" subsection. If we can express the denominator b in the form $\frac{10^k(10^p-1)}{q}$ with k, p and q integers, we will be in business because $\frac{a}{b} = \frac{aq}{10^k(10^p-1)}$. From this we can generate the desired decimal expansion by running the argument of the last subsection backwards. So we want to find integers k, p, q such that $10^{k+p} - 10^k = b \cdot q$. To do so consider the powers of 10 up to 10^b:

$$1, 10^1, 10^2, 10^3, \cdots, 10^b$$

For each $j = 0, 1, 2, \cdots, b$, find integers c_j and $0 \le r_j < b$ so that

$$10^j = b \cdot c_j + r_j$$

To do so, start with 10^j and repeatedly subtract b from it until the remainder drops strictly below b. The r_j's can take at most b different values, namely 0, 1, 2, $\cdots$, $b-1$, and we now have $b+1$ r_j's, namely r_0, r_1, $\cdots$, r_b. So we must be able to find two powers of 10 which give the same remainder [1]. That is there must be $0 \le k < l \le b$ so that $r_k = r_l$. Hence

$$\begin{aligned} 10^l - 10^k &= (bc_l + r_l) - (bc_k + r_k) \\ &= b(c_l - c_k) && \text{since } r_k = r_l. \end{aligned}$$

1 This is an application of the pigeon hole principle — the very simple but surprisingly useful idea that if you have n items which you have to put in m boxes, and if $n > m$, then at least one box must contain more than one item.

and we have

$$b = \frac{10^k(10^p - 1)}{q}$$

where $p = l - k$ and $q = c_l - c_k$ are both strictly positive integers, since $l > k$ so that $10^l - 10^k > 0$. Thus we can write

$$\frac{a}{b} = \frac{aq}{10^k(10^p - 1)}$$

Next divide the numerator aq by $10^p - 1$ and compute the remainder. That is, write $aq = \alpha(10^p - 1) + \beta$ with $0 \leq \beta < 10^p - 1$. Notice that $0 \leq \alpha < 10^k$, as otherwise $x = \frac{a}{b} \geq 1$. That is, α has at most k digits and β has at most p digits. This, finally, gives us

$$\begin{aligned}
x = \frac{a}{b} &= \frac{\alpha(10^p - 1) + \beta}{10^k(10^p - 1)} \\
&= \frac{\alpha}{10^k} + \frac{\beta}{10^k(10^p - 1)} \\
&= \frac{\alpha}{10^k} + \frac{\beta}{10^{k+p}(1 - 10^{-p})} \\
&= \frac{\alpha}{10^k} + \frac{\beta}{10^{k+p}} \sum_{j=0}^{\infty} 10^{-pj}
\end{aligned}$$

which gives the required eventually periodic expansion.

3.7.4 ⏩ Irrationality of e

We will give 2 proofs that the number e is irrational, the first due to Fourier (1768–1830) and the second due to Pennisi (1918–2010). Both are proofs by contradiction [2] — we first assume that e is rational and then show that this implies a contradiction. In both cases we reach the contradiction by showing that a given quantity (related to the series expression for e) must be both a positive integer and also strictly less than 1.

3.7.4.1 ⏩⏩ Proof 1

This proof is due to Fourier. Let us assume that the number e is rational so we can write it as

$$e = \frac{a}{b}$$

2 Proof by contradiction is a standard and very powerful method of proof in mathematics. It relies on the law of the excluded middle which states that any given mathematical statement P is either true or false. Because of this, if we can show that the statement P being false implies something contradictory — like $1 = 0$ or $a > a$ — then we can conclude that P must be true. The interested reader can certainly find many examples (and a far more detailed explanation) using their favourite search engine.

where a, b are positive integers. Using the Maclaurin series for e^x we have

$$\frac{a}{b} = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Now multiply both sides by $b!$ to get

$$a\frac{b!}{b} = \sum_{n=0}^{\infty} \frac{b!}{n!}$$

The left-hand side of this expression is an integer. We complete the proof by showing that the right-hand side cannot be an integer (and hence that we have a contradiction).

First split the series on the right-hand side into two piece as follows

$$\sum_{n=0}^{\infty} \frac{b!}{n!} = \underbrace{\sum_{n=0}^{b} \frac{b!}{n!}}_{=A} + \underbrace{\sum_{n=b+1}^{\infty} \frac{b!}{n!}}_{=B}$$

The first sum, A, is finite sum of integers:

$$A = \sum_{n=0}^{b} \frac{b!}{n!} = \sum_{n=0}^{b} (n+1)(n+2)\cdots(b-1)b.$$

Consequently A must be an integer. Notice that we simplified the ratio of factorials using the fact that when $b \geq n$ we have

$$\frac{b!}{n!} = \frac{1 \cdot 2 \cdots n(n+1)(n+2)\cdots(b-1)b}{1 \cdot 2 \cdots n} = (n+1)(n+2)\cdots(b-1)b.$$

Now we turn to the second sum. Since it is a sum of strictly positive terms we must have

$$B > 0$$

We complete the proof by showing that $B < 1$. To do this we bound each term from above:

$$\begin{aligned}
\frac{b!}{n!} &= \frac{1}{\underbrace{(b+1)(b+2)\cdots(n-1)n}_{n-b \text{ factors}}} \\
&\leq \frac{1}{\underbrace{(b+1)(b+1)\cdots(b+1)(b+1)}_{n-b \text{ factors}}} = \frac{1}{(b+1)^{n-b}}
\end{aligned}$$

Indeed the inequality is strict except when $n = b+1$. Hence we have that

$$B < \sum_{n=b+1}^{\infty} \frac{1}{(b+1)^{n-b}}$$

$$= \frac{1}{(b+1)} + \frac{1}{(b+1)^2} + \frac{1}{(b+1)^3} + \cdots$$

This is just a geometric series (see Lemma 3.2.5) and equals

$$\begin{aligned} &= \frac{1}{(b+1)} \frac{1}{1 - \frac{1}{b+1}} \\ &= \frac{1}{b+1-1} = \frac{1}{b} \end{aligned}$$

And since b is a positive integer, we have shown that

$$0 < B < 1$$

and thus B cannot be an integer.

Thus we have that

$$\underbrace{a\frac{b!}{b}}_{\text{integer}} = \underbrace{A}_{\text{integer}} + \underbrace{B}_{\text{not integer}}$$

which gives a contradiction. Thus e cannot be rational.

3.7.4.2 ⏵⏵⏵ Proof 2

This proof is due to Pennisi (1953). Let us (again) assume that the number e is rational. Hence it can be written as

$$e = \frac{a}{b},$$

where a, b are positive integers. This means that we can write

$$e^{-1} = \frac{b}{a}.$$

Using the Maclaurin series for e^x we have

$$\frac{b}{a} = e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

Before we do anything else, we multiply both sides by $(-1)^{a+1}a!$ — this might seem a little strange at this point, but the reason will become clear as we proceed through the proof. The expression is now

$$(-1)^{a+1} b \frac{a!}{a} = \sum_{n=0}^{\infty} \frac{(-1)^{n+a+1} a!}{n!}$$

The left-hand side of the expression is an integer. We again complete the proof by showing that the right-hand side cannot be an integer.

We split the series on the right-hand side into two pieces:

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+a+1}a!}{n!} = \underbrace{\sum_{n=0}^{a} \frac{(-1)^{n+a+1}a!}{n!}}_{=A} + \underbrace{\sum_{n=a+1}^{\infty} \frac{(-1)^{n+a+1}a!}{n!}}_{=B}$$

We will show that A is an integer while $0 < B < 1$; this gives the required contradiction.

Every term in the sum A is an integer. To see this we simplify the ratio of factorials as we did in the previous proof:

$$A = \sum_{n=0}^{a} \frac{(-1)^{n+a+1}a!}{n!} = \sum_{n=0}^{a} (-1)^{n+a+1}(n+1)(n+2)\cdots(a-1)a$$

Let us now examine the series B. Again clean up the ratio of factorials:

$$\begin{aligned} B &= \sum_{n=a+1}^{\infty} \frac{(-1)^{n+a+1}a!}{n!} = \sum_{n=a+1}^{\infty} \frac{(-1)^{n+a+1}}{(a+1)\cdot(a+2)\cdots(n-1)\cdot n} \\ &= \frac{(-1)^{2a+2}}{a+1} + \frac{(-1)^{2a+3}}{(a+1)(a+2)} + \frac{(-1)^{2a+4}}{(a+1)(a+2)(a+3)} + \cdots \\ &= \frac{1}{a+1} - \frac{1}{(a+1)(a+2)} + \frac{1}{(a+1)(a+2)(a+3)} - \cdots \end{aligned}$$

Hence B is an alternating series of decreasing terms and by the alternating series test (Theorem 3.3.14) it converges. Further, it must converge to a number between its first and second partial sums (see the discussion before Theorem 3.3.14). Hence the right-hand side lies between

$$\frac{1}{a+1} \qquad \text{and} \qquad \frac{1}{a+1} - \frac{1}{(a+1)(a+2)} = \frac{1}{a+2}$$

Since a is a positive integer the above tells us that B converges to a real number strictly greater than 0 and strictly less than 1. Hence it cannot be an integer.

This gives us a contradiction and hence e cannot be rational.

3.7.5 ▸▸ Irrationality of π

This proof is due to Niven (1946) and doesn't require any mathematics beyond the level of this course. Much like the proofs above we will start by assuming that π is rational and then reach a contradiction. Again this contradiction will be that a given quantity must be an integer but at the same time must lie strictly between 0 and 1.

Assume that π is a rational number and so can be written as $\pi = \frac{a}{b}$ with a, b positive integers. Now let n be a positive integer and define the polynomial

$$f(x) = \frac{x^n(a-bx)^n}{n!}.$$

It is certainly not immediately obvious why and how Niven chose this polynomial, but you will see that it has been very carefully crafted to make the proof work. In particular we will show — under our assumption that π is rational — that, if n is really big, then

$$I_n = \int_0^\pi f(x)\sin(x)\mathrm{d}x$$

is an integer and it also lies strictly between 0 and 1, giving the required contradiction.

3.7.5.1 ►►► Bounding the integral

Consider again the polynomial

$$f(x) = \frac{x^n(a-bx)^n}{n!}.$$

Notice that

$$\begin{aligned} f(0) &= 0 \\ f(\pi) &= f(a/b) = 0. \end{aligned}$$

Furthermore, for $0 \le x \le \pi = a/b$, we have $x \le \frac{a}{b}$ and $a - bx \le a$ so that

$$0 \le x(a-bx) \le a^2/b.$$

We could work out a more precise [3] upper bound, but this one is sufficient for the analysis that follows. Hence

$$0 \le f(x) \le \left(\frac{a^2}{b}\right)^n \frac{1}{n!}$$

We also know that for $0 \le x \le \pi = a/b$, $0 \le \sin(x) \le 1$. Thus

$$0 \le f(x)\sin(x) \le \left(\frac{a^2}{b}\right)^n \frac{1}{n!}$$

for all $0 \le x \le 1$. Using this inequality we bound

$$0 < I_n = \int_0^\pi f(x)\sin(x)\mathrm{d}x < \left(\frac{a^2}{b}\right)^n \frac{1}{n!}.$$

We will later show that, if n is really big, then $\left(\frac{a^2}{b}\right)^n \frac{1}{n!} < 1$. We'll first show, starting now, that I_n is an integer.

3 You got lots of practice finding the maximum and minimum values of continuous functions on closed intervals when you took calculus last term.

3.7.5.2 ▸▸▸ Integration by parts

In order to show that the value of this integral is an integer we will use integration by parts. You have already practiced using integration by parts to integrate quantities like

$$\int x^2 \sin(x)\, \mathrm{d}x$$

and this integral isn't much different. For the moment let us just use the fact that $f(x)$ is a polynomial of degree $2n$. Using integration by parts with $u = f(x)$, $\mathrm{d}v = \sin(x)$ and $v = -\cos(x)$ gives us

$$\int f(x) \sin(x)\, \mathrm{d}x = -f(x)\cos(x) + \int f'(x)\cos(x)\, \mathrm{d}x$$

Use integration by parts again with $u = f'(x)$, $\mathrm{d}v = \cos(x)$ and $v = \sin(x)$.

$$= -f(x)\cos(x) + f'(x)\sin(x) - \int f''(x)\sin(x)\, \mathrm{d}x$$

Use integration by parts yet again, with $u = f''(x)$, $\mathrm{d}v = \sin(x)$ and $v = -\cos(x)$.

$$= -f(x)\cos(x) + f'(x)\sin(x) + f''(x)\cos(x) - \int f'''(x)\cos(x)\, \mathrm{d}x$$

And now we can see the pattern; we get alternating signs, and then derivatives multiplied by sines and cosines:

$$\begin{aligned}\int f(x)\sin(x)\mathrm{d}x &= \cos(x)\left(-f(x) + f''(x) - f^{(4)}(x) + f^{(6)}(x) - \cdots\right)\\ &\quad + \sin(x)\left(f'(x) - f'''(x) + f^{(5)}(x) - f^{(7)}(x) + \cdots\right)\end{aligned}$$

This terminates at the $2n^{\text{th}}$ derivative since $f(x)$ is a polynomial of degree $2n$. We can check this computation by differentiating the terms on the right-hand side:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos(x)\left(-f(x) + f''(x) - f^{(4)}(x) + f^{(6)}(x) - \cdots\right)\right)\\ &= -\sin(x)\left(-f(x) + f''(x) - f^{(4)}(x) + f^{(6)}(x) - \cdots\right)\\ &\quad + \cos(x)\left(-f'(x) + f'''(x) - f^{(5)}(x) + f^{(7)}(x) - \cdots\right)\end{aligned}$$

and similarly

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin(x)\left(f'(x) - f'''(x) + f^{(5)}(x) - f^{(7)}(x) + \cdots\right)\right)\\ &= \cos(x)\left(f'(x) - f'''(x) + f^{(5)}(x) - f^{(7)}(x) + \cdots\right)\\ &\quad + \sin(x)\left(f''(x) - f^{(4)}(x) + f^{(6)}(x) - \cdots\right)\end{aligned}$$

When we add these two expressions together all the terms cancel except $f(x)\sin(x)$, as required.

Now when we take the definite integral from 0 to π , all the sine terms give 0 because $\sin(0) = \sin(\pi) = 0$. Since $\cos(\pi) = -1$ and $\cos(0) = +1$, we are just left with:

$$\begin{aligned}\int_0^\pi f(x)\sin(x)\mathrm{d}x &= \left(f(0) - f''(0) + f^{(4)}(0) - f^{(6)}(0) + \cdots + (-1)^n f^{(2n)}(0)\right) \\ &\quad + \left(f(\pi) - f''(\pi) + f^{(4)}(\pi) - f^{(6)}(\pi) + \cdots + (-1)^n f^{(2n)}(\pi)\right)\end{aligned}$$

So to show that I_n is an integer, it now suffices to show that $f^{(j)}(0)$ and $f^{(j)}(\pi)$ are integers.

3.7.5.3 ▶▶▶ The derivatives are integers

Recall that

$$f(x) = \frac{x^n(a-bx)^n}{n!}$$

and expand it:

$$f(x) = \frac{c_0}{n!}x^0 + \frac{c_1}{n!}x^1 + \cdots + \frac{c_n}{n!}x^n + \cdots + \frac{c_{2n}}{n!}x^{2n}$$

All the c_j are integers, and clearly $c_j = 0$ for all $j = 0, 1, \cdots, n-1$, because of the factor x^n in $f(x)$.

Now take the k^{th} derivative and set $x = 0$. Note that, if $j < k$, then $\frac{\mathrm{d}^k}{\mathrm{d}x^k}x^j = 0$ for all x and, if $j > k$, then $\frac{\mathrm{d}^k}{\mathrm{d}x^k}x^j$ is some number times x^{j-k} which evaluates to zero when we set $x = 0$. So

$$f^{(k)}(0) = \frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\frac{c_k}{k!}x^k\right) = \frac{k!c_k}{n!}$$

If $k < n$, then this is zero since $c_k = 0$. If $k > n$, this is an integer because c_k is an integer and $k!/n! = (n+1)(n+2)\cdots(k-1)k$ is an integer. If $k = n$, then $f^{(k)}(0) = c_n$ is again an integer. Thus all the derivatives of $f(x)$ evaluated at $x = 0$ are integers.

But what about the derivatives at $\pi = a/b$? To see this, we can make use of a handy symmetry. Notice that

$$f(x) = f(\pi - x) = f(a/b - x)$$

You can confirm this by just grinding through the algebra:

$$\begin{aligned} f(x) &= \frac{x^n(a-bx)^n}{n!} && \text{now replace } x \text{ with } a/b - x \\ f(a/b - x) &= \frac{(a/b - x)^n(a - b(a/b - x))^n}{n!} && \text{start cleaning this up:} \\ &= \frac{\left(\frac{a-bx}{b}\right)^n (a - a + bx)^n}{n!} \end{aligned}$$

$$= \frac{\left(\frac{a-bx}{b}\right)^n (bx)^n}{n!}$$
$$= \frac{(a-bx)^n x^n}{n!} = f(x)$$

Using this symmetry (and the chain rule) we see that

$$f'(x) = -f'(\pi - x)$$

and if we keep differentiating

$$f^{(k)}(x) = (-1)^k f^{(k)}(\pi - x)$$

Setting $x = 0$ in this tells us that

$$f^{(k)}(0) = (-1)^k f^{(k)}(\pi)$$

So because all the derivatives at $x = 0$ are integers, we know that all the derivatives at $x = \pi$ are also integers.

Hence the integral we are interested in

$$\int_0^\pi f(x)\sin(x)\mathrm{d}x$$

must be an integer.

3.7.5.4 ▶▶▶ Putting it together

Based on our assumption that $\pi = a/b$ is rational, we have shown that the integral

$$I_n = \int_0^\pi \frac{x^n(a-bx)}{n!}\sin(x)\mathrm{d}x$$

satisfies

$$0 < I_n < \left(\frac{a^2}{b}\right)^n \frac{1}{n!}$$

and also that I_n is an integer.

We are, however, free to choose n to be any positive integer we want. If we take n to be very large — in particular much much larger than a — then $n!$ will be much much larger than a^{2n} (we showed this in Example 3.6.8), and consequently

$$0 < I_n < \left(\frac{a^2}{b}\right)^n \frac{1}{n!} < 1$$

Which means that the integral cannot be an integer. This gives the required contradiction, showing that π is irrational.

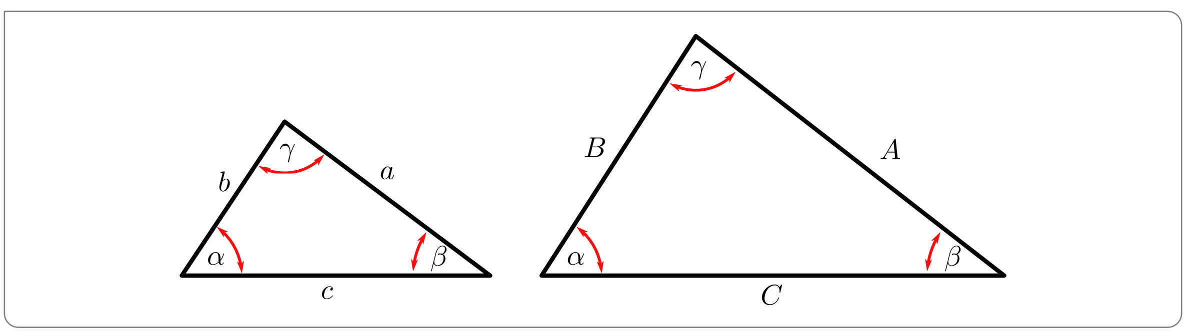
Appendix A

High School Material

This chapter is really split into three parts.

- Sections A.1 to A.11 contains results that we expect you to understand and know.
- Then Section A.14 contains results that we don't expect you to memorise, but that we think you should be able to quickly derive from other results you know.
- The remaining sections contain some material (that may be new to you) that is related to topics covered in the main body of these notes.

A.1 ▲ Similar Triangles



Two triangles T_1, T_2 are similar when

- (AAA — angle angle angle) The angles of T_1 are the same as the angles of T_2.
- (SSS — side side side) The ratios of the side lengths are the same. That is

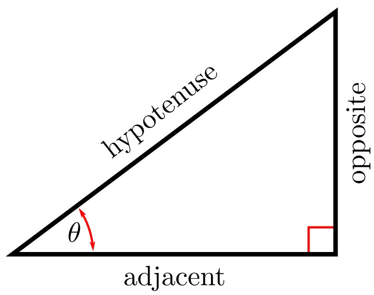
$$\frac{A}{a} = \frac{B}{b} = \frac{C}{c}$$

- (SAS — side angle side) Two sides have lengths in the same ratio and the angle between them is the same. For example

$$\frac{A}{a} = \frac{C}{c} \text{ and angle } \beta \text{ is same}$$

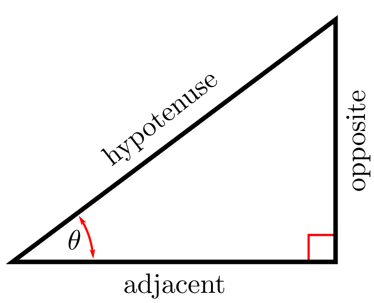
A.2 ▲ Pythagoras

For a right-angled triangle the length of the hypotenuse is related to the lengths of the other two sides by



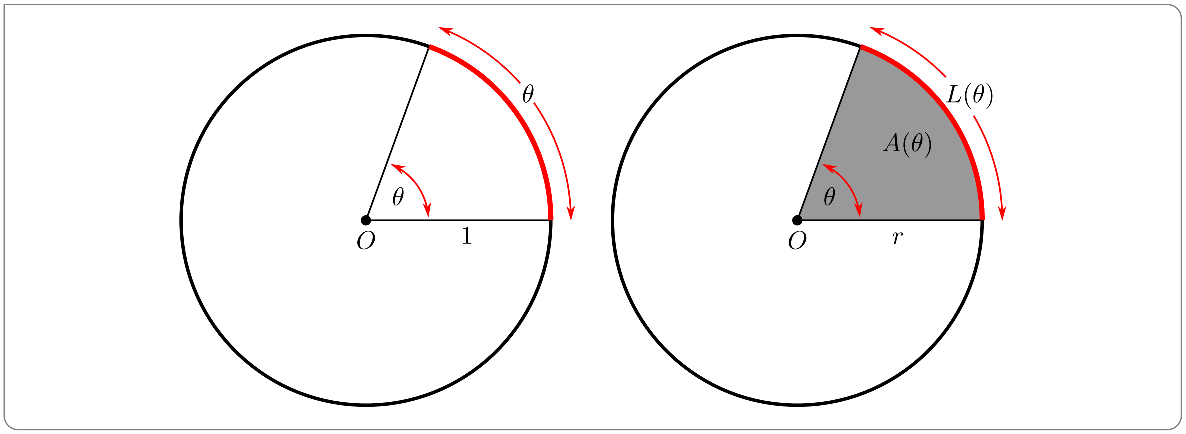
$$(\text{adjacent})^2 + (\text{opposite})^2 = (\text{hypotenuse})^2$$

A.3 ▲ Trigonometry — Definitions



$$\begin{aligned}
\sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}} & \csc\theta &= \frac{1}{\sin\theta} \\
\cos\theta &= \frac{\text{adjacent}}{\text{hypotenuse}} & \sec\theta &= \frac{1}{\cos\theta} \\
\tan\theta &= \frac{\text{opposite}}{\text{adjacent}} & \cot\theta &= \frac{1}{\tan\theta}
\end{aligned}$$

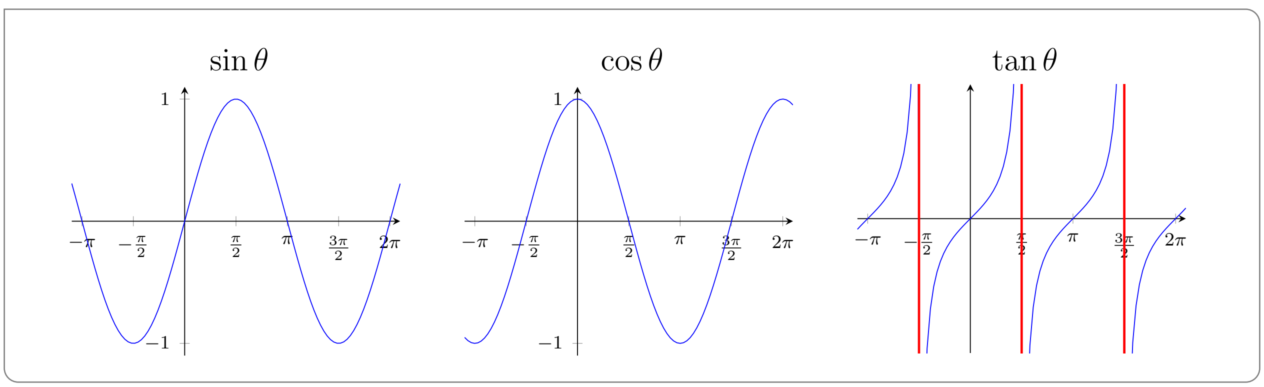
A.4 ▴ Radians, Arcs and Sectors



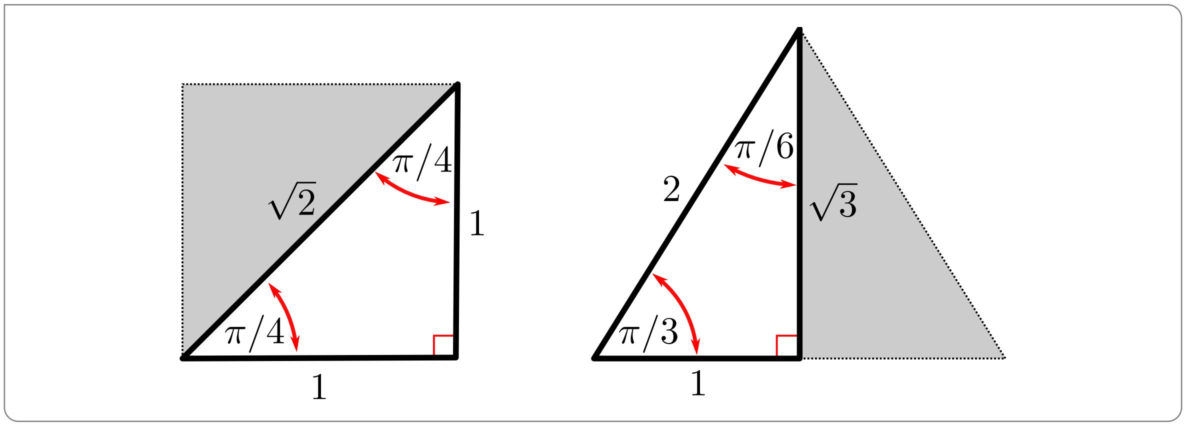
For a circle of radius r and angle of θ radians:

- Arc length $L(\theta) = r\theta$.
- Area of sector $A(\theta) = \frac{\theta}{2}r^2$.

A.5 ▴ Trigonometry — Graphs



A.6 ▲ Trigonometry — Special Triangles



From the above pair of special triangles we have

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

A.7 ▲ Trigonometry — Simple Identities

- Periodicity

$$\sin(\theta + 2\pi) = \sin(\theta) \qquad \cos(\theta + 2\pi) = \cos(\theta)$$

- Reflection

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta)$$

- Reflection around $\pi/4$

$$\sin\left(\tfrac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\tfrac{\pi}{2} - \theta\right) = \sin\theta$$

- Reflection around $\pi/2$

$$\sin(\pi - \theta) = \sin\theta \qquad \cos(\pi - \theta) = -\cos\theta$$

- Rotation by π

$$\sin(\theta + \pi) = -\sin\theta \qquad\qquad \cos(\theta + \pi) = -\cos\theta$$

- Pythagoras

$$\sin^2\theta + \cos^2\theta = 1$$

A.8▲ Trigonometry — Add and Subtract Angles

- Sine

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$

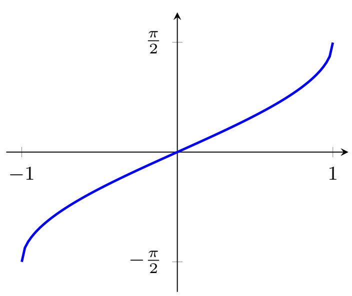
- Cosine

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

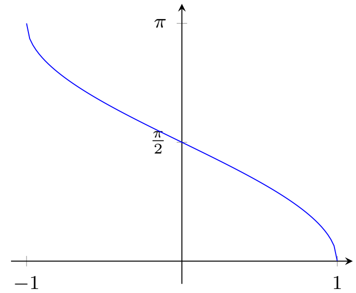
A.9▲ Inverse Trigonometric Functions

Some of you may not have studied inverse trigonometric functions in highschool, however we still expect you to know them by the end of the course.

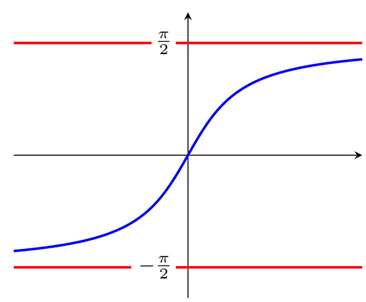
$\arcsin x$
Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}$



$\arccos x$
Domain: $-1 \le x \le 1$
Range: $0 \le \arccos x \le \pi$



$\arctan x$
Domain: all real numbers
Range: $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$



Since these functions are inverses of each other we have

$$\arcsin(\sin\theta) = \theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\arccos(\cos\theta) = \theta \qquad 0 \le \theta \le \pi$$

$$\arctan(\tan\theta) = \theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

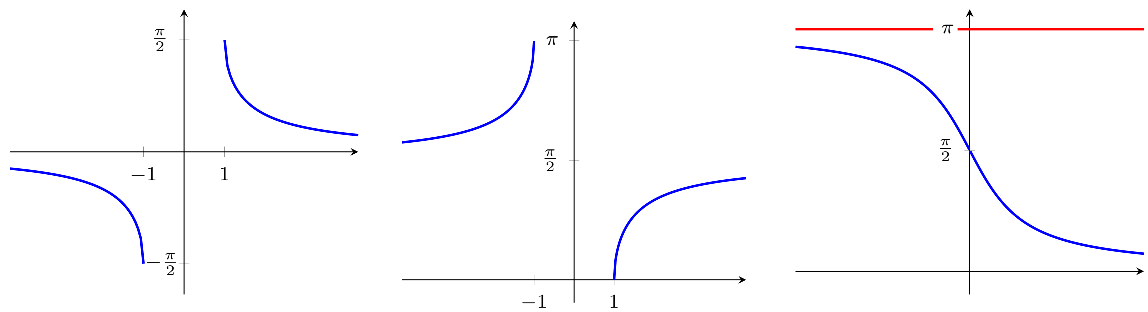
and also

$$\sin(\arcsin x) = x \qquad -1 \le x \le 1$$

$$\cos(\arccos x) = x \qquad -1 \le x \le 1$$

$$\tan(\arctan x) = x \qquad \text{any real } x$$

$\operatorname{arccsc} x$	$\operatorname{arcsec} x$	$\operatorname{arccot} x$
Domain: $\lvert x\rvert \geq 1$	Domain: $\lvert x\rvert \geq 1$	Domain: all real numbers
Range: $-\frac{\pi}{2} \leq \operatorname{arccsc} x \leq \frac{\pi}{2}$	Range: $0 \leq \operatorname{arcsec} x \leq \pi$	Range: $0 < \operatorname{arccot} x < \pi$
$\operatorname{arccsc} x \neq 0$	$\operatorname{arcsec} x \neq \dfrac{\pi}{2}$	



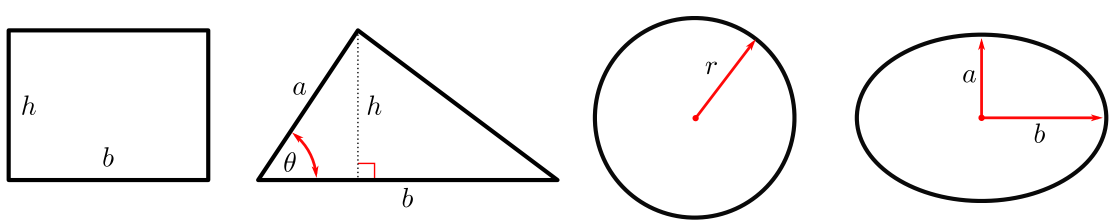
Again

$$\begin{aligned} \operatorname{arccsc}(\csc\theta) &= \theta & -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},\ \theta \neq 0 \\ \operatorname{arcsec}(\sec\theta) &= \theta & 0 \leq \theta \leq \pi,\ \theta \neq \frac{\pi}{2} \\ \operatorname{arccot}(\cot\theta) &= \theta & 0 < \theta < \pi \end{aligned}$$

and

$$\begin{aligned} \csc(\operatorname{arccsc} x) &= x & |x| \geq 1 \\ \sec(\operatorname{arcsec} x) &= x & |x| \geq 1 \\ \cot(\operatorname{arccot} x) &= x & \text{any real } x \end{aligned}$$

A.10 ▲ Areas



- Area of a rectangle

$$A = bh$$

- Area of a triangle

$$A = \frac{1}{2}bh = \frac{1}{2}ab\sin\theta$$

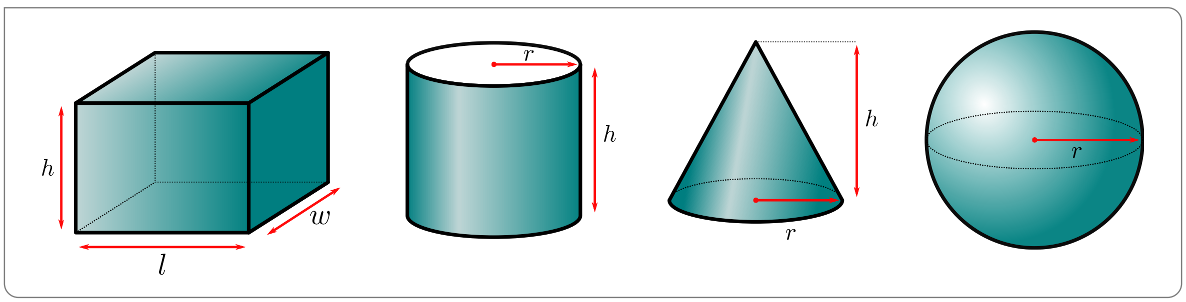
- Area of a circle

$$A = \pi r^2$$

- Area of an ellipse

$$A = \pi ab$$

A.11 ▲ Volumes



- Volume of a rectangular prism

$$V = lwh$$

- Volume of a cylinder

$$V = \pi r^2 h$$

- Volume of a cone

$$V = \frac{1}{3}\pi r^2 h$$

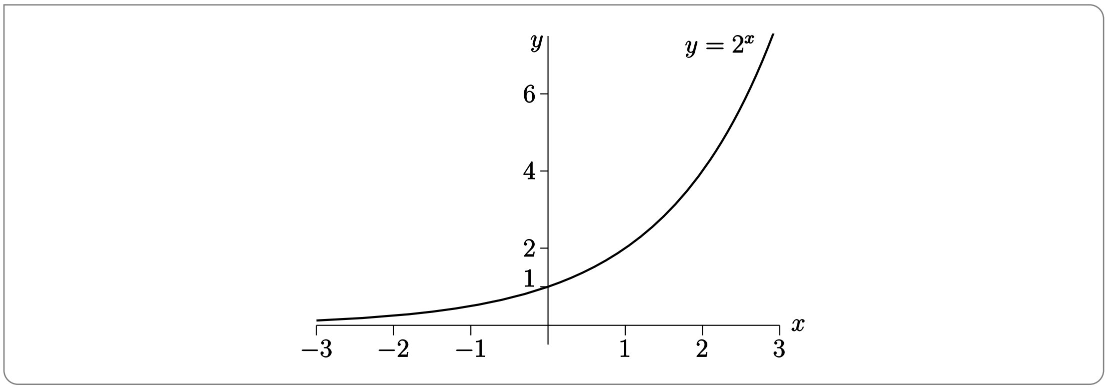
- Volume of a sphere

$$V = \frac{4}{3}\pi r^3$$

A.12 ▲ Powers

In the following, x and y are arbitrary real numbers, and q is an arbitrary constant that is strictly bigger than zero.

- $q^0 = 1$
- $q^{x+y} = q^x q^y$, $q^{x-y} = \frac{q^x}{q^y}$
- $q^{-x} = \frac{1}{q^x}$
- $\left(q^x\right)^y = q^{xy}$
- $\lim\limits_{x\to\infty} q^x = \infty$, $\lim\limits_{x\to-\infty} q^x = 0$ if $q > 1$
- $\lim\limits_{x\to\infty} q^x = 0$, $\lim\limits_{x\to-\infty} q^x = \infty$ if $0 < q < 1$
- The graph of 2^x is given below. The graph of q^x, for any $q > 1$, is similar.

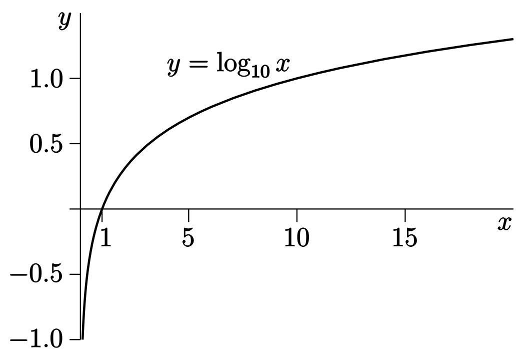


A.13 ▲ Logarithms

In the following, x and y are arbitrary real numbers that are strictly bigger than 0, and p and q are arbitrary constants that are strictly bigger than one.

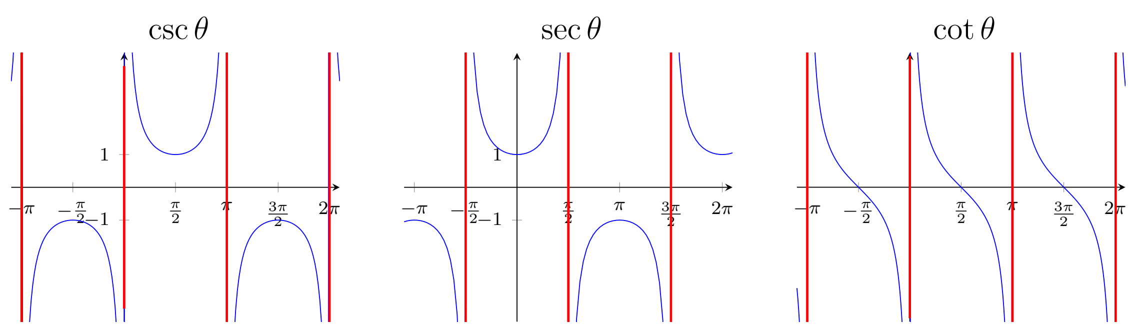
- $q^{\log_q x} = x$, $\quad \log_q\left(q^x\right) = x$
- $\log_q x = \frac{\log_p x}{\log_p q}$
- $\log_q 1 = 0$, $\quad \log_q q = 1$
- $\log_q(xy) = \log_q x + \log_q y$
- $\log_q\left(\frac{x}{y}\right) = \log_q x - \log_q y$

- $\log_q\left(\frac{1}{y}\right) = -\log_q y$,
- $\log_q(x^y) = y\log_q x$
- $\lim_{x\to\infty} \log_q x = \infty, \qquad \lim_{x\to 0+} \log_q x = -\infty$
- The graph of $\log_{10} x$ is given below. The graph of $\log_q x$, for any $q > 1$, is similar.



A.14▲ Highschool Material You Should be Able to Derive

- Graphs of $\csc\theta, \sec\theta$ and $\cot\theta$:



- More Pythagoras

$$\sin^2\theta + \cos^2\theta = 1 \quad \xrightarrow{\text{divide by } \cos^2\theta} \quad \tan^2\theta + 1 = \sec^2\theta$$

$$\sin^2\theta + \cos^2\theta = 1 \quad \xrightarrow{\text{divide by } \sin^2\theta} \quad 1 + \cot^2\theta = \csc^2\theta$$

- Sine — double angle (set $\beta = \alpha$ in sine angle addition formula)

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

- Cosine — double angle (set $\beta = \alpha$ in cosine angle addition formula)

$$\begin{aligned}\cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha) \\ &= 2\cos^2(\alpha) - 1 && (\text{use } \sin^2(\alpha) = 1 - \cos^2(\alpha)) \\ &= 1 - 2\sin^2(\alpha) && (\text{use } \cos^2(\alpha) = 1 - \sin^2(\alpha))\end{aligned}$$

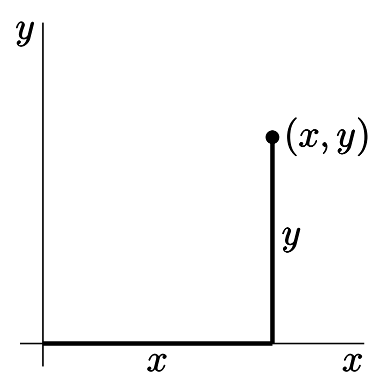
- Composition of trigonometric and inverse trigonometric functions:

$$\cos(\arcsin x) = \sqrt{1-x^2} \qquad \sec(\arctan x) = \sqrt{1+x^2}$$

  and similar expressions.

A.15 ▲ Cartesian Coordinates

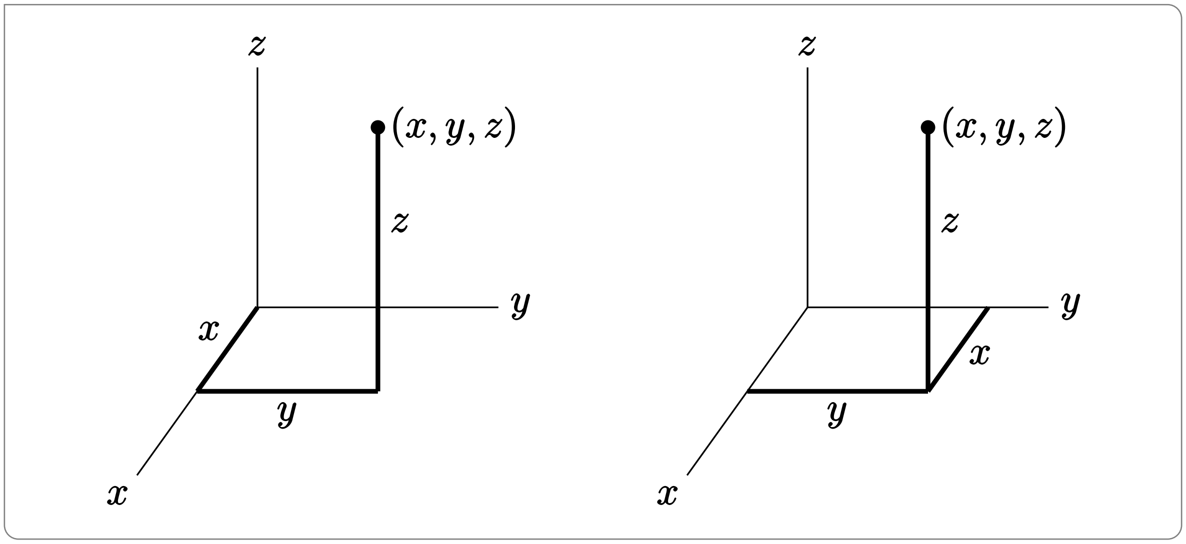
Each point in two dimensions may be labeled by two coordinates (x, y) which specify the position of the point in some units with respect to some axes as in the figure below.



The set of all points in two dimensions is denoted $\mathbb{R}^2$. Observe that

- the distance from the point (x, y) to the x-axis is $|y|$
- the distance from the point (x, y) to the y-axis is $|x|$
- the distance from the point (x, y) to the origin $(0, 0)$ is $\sqrt{x^2 + y^2}$

Similarly, each point in three dimensions may be labeled by three coordinates (x, y, z), as in the two figures below.

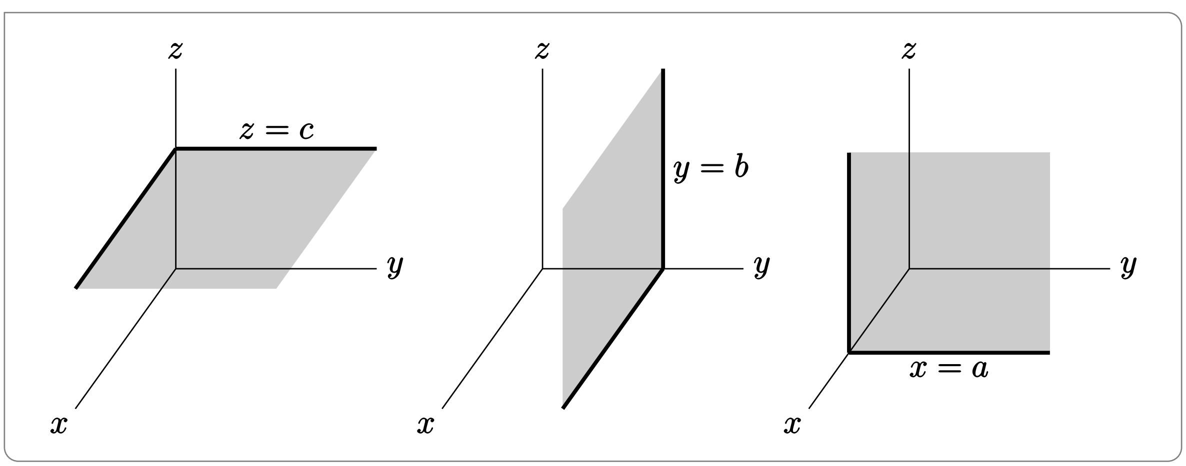


The set of all points in three dimensions is denoted $\mathbb{R}^3$. The plane that contains, for example, the x- and y-axes is called the xy-plane.

- The xy-plane is the set of all points (x, y, z) that obey $z = 0$.
- The xz-plane is the set of all points (x, y, z) that obey $y = 0$.
- The yz-plane is the set of all points (x, y, z) that obey $x = 0$.

More generally,

- The set of all points (x, y, z) that obey $z = c$ is a plane that is parallel to the xy-plane and is a distance $|c|$ from it. If $c > 0$, the plane $z = c$ is above the xy-plane. If $c < 0$, the plane $z = c$ is below the xy-plane. We say that the plane $z = c$ is a signed distance c from the xy-plane.
- The set of all points (x, y, z) that obey $y = b$ is a plane that is parallel to the xz-plane and is a signed distance b from it.
- The set of all points (x, y, z) that obey $x = a$ is a plane that is parallel to the yz-plane and is a signed distance a from it.



Observe that

- the distance from the point (x, y, z) to the xy-plane is $|z|$
- the distance from the point (x, y, z) to the xz-plane is $|y|$
- the distance from the point (x, y, z) to the yz-plane is $|x|$
- the distance from the point (x, y, z) to the origin $(0, 0, 0)$ is $\sqrt{x^2 + y^2 + z^2}$

The distance from the point (x, y, z) to the point (x', y', z') is

$$\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

so that the equation of the sphere centered on $(1, 2, 3)$ with radius 4, that is, the set of all points (x, y, z) whose distance from $(1, 2, 3)$ is 4, is

$$(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 16$$

A.16▲ Roots of Polynomials

Being able to factor polynomials is a very important part of many of the computations in this course. Related to this is the process of finding roots (or zeros) of polynomials. That is, given a polynomial $P(x)$, find all numbers r so that $P(r) = 0$.

In the case of a quadratic $P(x) = ax^2 + bx + c$, we can use the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The corresponding formulas for cubics and quartics [1] are extremely cumbersome, and no such formula exists for polynomials of degree 5 and higher [2].

Despite this there are many tricks [3] for finding roots of polynomials that work well in some situations but not all. Here we describe approaches that will help you find integer and rational roots of polynomials that will work well on exams, quizzes and homework assignments.

Consider the quadratic equation $x^2 - 5x + 6 = 0$. We could [4] solve this using the quadratic formula

$$x = \frac{5 \pm \sqrt{25 - 4 \times 1 \times 6}}{2} = \frac{5 \pm 1}{2} = 2, 3.$$

1 The method for cubics was developed in the 15th century by del Ferro, Cardano and Ferrari (Cardano's student). Ferrari then went on to discover a formula for the roots of a quartic. His formula requires the solution of an associated cubic polynomial.

2 This is the famous Abel-Ruffini theorem.

3 There is actually a large body of mathematics devoted to developing methods for factoring polynomials. Polynomial factorisation is a fundamental problem for most computer algebra systems. The interested reader should make use of their favourite search engine to find out more.

4 We probably shouldn't do it this way for such a simple polynomial, but for pedagogical purposes we do here.

Hence $x^2 - 5x + 6$ has roots $x = 2, 3$ and so it factors as $(x - 3)(x - 2)$. Notice [5] that the numbers 2 and 3 divide the constant term of the polynomial, 6. This happens in general and forms the basis of our first trick.

Lemma A.16.1 A very useful trick.

If r or $-r$ is an integer root of a polynomial $P(x) = a_n x^n + \cdots + a_1 x + a_0$ with integer coefficients, then r is a factor of the constant term a_0.

Proof. If r is a root of the polynomial we know that $P(r) = 0$. Hence

$$a_n \cdot r^n + \cdots + a_1 \cdot r + a_0 = 0$$

If we isolate a_0 in this expression we get

$$a_0 = -\big[a_n r^n + \cdots + a_1 r\big]$$

We can see that r divides every term on the right-hand side. This means that the right-hand side is an integer times r. Thus the left-hand side, being a_0, is an integer times r, as required. The argument for when $-r$ is a root is almost identical.

□

Let us put this observation to work.

Example A.16.2 Integer roots of $x^3 - x^2 + 2$.

Find the integer roots of $P(x) = x^3 - x^2 + 2$.
Solution:

- The constant term in this polynomial is 2.
- The only divisors of 2 are $1, 2$. So the only candidates for integer roots are $\pm 1, \pm 2$.
- Trying each in turn

$$P(1) = 2 \qquad P(-1) = 0$$
$$P(2) = 6 \qquad P(-2) = -10$$

- Thus the only integer root is -1.

5 Many of you may have been taught this approach in highschool.

Example A.16.3 Integer roots of $3x^3 + 8x^2 - 5x - 6$.

Find the integer roots of $P(x) = 3x^3 + 8x^2 - 5x - 6$.
Solution:

- The constant term is -6.
- The divisors of 6 are $1, 2, 3, 6$. So the only candidates for integer roots are $\pm 1, \pm 2, \pm 3, \pm 6$.
- We try each in turn (it is tedious but not difficult):

$$\begin{aligned} P(1) &= 0 & P(-1) &= 4 \\ P(2) &= 40 & P(-2) &= 12 \\ P(3) &= 132 & P(-3) &= 0 \\ P(6) &= 900 & P(-6) &= -336 \end{aligned}$$

- Thus the only integer roots are 1 and -3.

We can generalise this approach in order to find rational roots. Consider the polynomial $6x^2 - x - 2$. We can find its zeros using the quadratic formula:

$$x = \frac{1 \pm \sqrt{1+48}}{12} = \frac{1 \pm 7}{12} = -\frac{1}{2}, \frac{2}{3}.$$

Notice now that the numerators, 1 and 2, both divide the constant term of the polynomial (being 2). Similarly, the denominators, 2 and 3, both divide the coefficient of the highest power of x (being 6). This is quite general.

Lemma A.16.4 Another nice trick.

If b/d or $-b/d$ is a rational root in lowest terms (i.e. b and d are integers with no common factors) of a polynomial $Q(x) = a_n x^n + \cdots + a_1 x + a_0$ with integer coefficients, then the numerator b is a factor of the constant term a_0 and the denominator d is a factor of a_n.

Proof. Since $\frac{b}{d}$ is a root of $P(x)$ we know that

$$a_n (b/d)^n + \cdots + a_1 (b/d) + a_0 = 0$$

Multiply this equation through by d^n to get

$$a_n b^n + \cdots + a_1 b d^{n-1} + a_0 d^n = 0$$

Move terms around to isolate $a_0 d^n$:

$$a_0 d^n = -\left[a_n b^n + \cdots + a_1 b d^{n-1}\right]$$

Now every term on the right-hand side is some integer times b. Thus the left-hand side must also be an integer times b. We know that d does not contain any factors of b, hence a_0 must be some integer times b (as required).
Similarly we can isolate the term $a_n b^n$:

$$a_n b^n = -\left[a_{n-1}b^{n-1}d + \cdots + a_1 b d^{n-1} + a_0 d^n\right]$$

Now every term on the right-hand side is some integer times d. Thus the left-hand side must also be an integer times d. We know that b does not contain any factors of d, hence a_n must be some integer times d (as required).
The argument when $-\frac{b}{d}$ is a root is nearly identical.

□

We should put this to work:

Example A.16.5 Rational roots of $2x^2 - x - 3$.

$P(x) = 2x^2 - x - 3$.
Solution:

- The constant term in this polynomial is $3 = 1\times 3$ and the coefficient of the highest power of x is $2 = 1 \times 2$.
- Thus the only candidates for integer roots are $\pm 1,\ \pm 3$.
- By our newest trick, the only candidates for fractional roots are $\pm\frac{1}{2},\ \pm\frac{3}{2}$.
- We try each in turn [a]

$$\begin{aligned} P(1) &= -2 & \qquad P(-1) &= 0\\ P(3) &= 12 & P(-3) &= 18\\ P\left(\tfrac{1}{2}\right) &= -3 & P\left(-\tfrac{1}{2}\right) &= -2\\ P\left(\tfrac{3}{2}\right) &= 0 & P\left(-\tfrac{3}{2}\right) &= 3 \end{aligned}$$

so the roots are -1 and $\frac{3}{2}$.

a Again, this is a little tedious, but not difficult. Its actually pretty easy to code up for a computer to do. Modern polynomial factoring algorithms do more sophisticated things, but these are a pretty good way to start.

The tricks above help us to find integer and rational roots of polynomials. With a little extra work we can extend those methods to help us factor polynomials. Say we have a polynomial $P(x)$ of degree p and have established that r is one of its roots. That is, we know $P(r) = 0$. Then we can factor $(x - r)$ out from $P(x)$ — it is always possible to find a polynomial $Q(x)$ of degree $p-1$ so that

$$P(x) = (x-r)Q(x)$$

In sufficiently simple cases, you can probably do this factoring by inspection. For example, $P(x) = x^2 - 4$ has $r = 2$ as a root because $P(2) = 2^2 - 4 = 0$. In this case, $P(x) = (x-2)(x+2)$ so that $Q(x) = (x+2)$. As another example, $P(x) = x^2 - 2x - 3$ has $r = -1$ as a root because $P(-1) = (-1)^2 - 2(-1) - 3 = 1 + 2 - 3 = 0$. In this case, $P(x) = (x+1)(x-3)$ so that $Q(x) = (x-3)$.

For higher degree polynomials we need to use something more systematic — long divison.

Lemma A.16.6 Long Division.

Once you have found a root r of a polynomial, even if you cannot factor $(x-r)$ out of the polynomial by inspection, you can find $Q(x)$ by dividing $P(x)$ by $x-r$, using the long division algorithm you learned [a] in school, but with 10 replaced by x.

[a] This is a standard part of most highschool mathematics curricula, but perhaps not all. You should revise this carefully.

Example A.16.7 Roots of $x^3 - x^2 + 2$.

Factor $P(x) = x^3 - x^2 + 2$.

Solution:

- We can go hunting for integer roots of the polynomial by looking at the divisors of the constant term. This tells us to try $x = \pm 1, \pm 2$.
- A quick computation shows that $P(-1) = 0$ while $P(1), P(-2), P(2) \neq 0$. Hence $x = -1$ is a root of the polynomial and so $x+1$ must be a factor.
- So we divide $\frac{x^3 - x^2 + 2}{x+1}$. The first term, x^2, in the quotient is chosen so that when you multiply it by the denominator, $x^2(x+1) = x^3 + x^2$, the leading term, x^3, matches the leading term in the numerator, $x^3 - x^2 + 2$, exactly.

$$\begin{array}{r|l}
 & x^2 \\
\hline
x+1 & x^3 - x^2 + \qquad 2 \\
 & x^3 + x^2 \qquad\qquad \longleftarrow x^2(x+1)
\end{array}$$

- When you subtract $x^2(x+1) = x^3 + x^2$ from the numerator $x^3 - x^2 + 2$ you get the remainder $-2x^2 + 2$. Just like in public school, the 2 is not normally "brought down" until it is actually needed.

$$
\begin{array}{r|l l}
 & x^2 & \\
\cline{2-2}
x+1 & x^3 - \ x^2 + \qquad 2 & \\
 & x^3 + \ x^2 & \longleftarrow x^2(x+1) \\
\cline{2-2}
 & \quad -2x^2 &
\end{array}
$$

- The next term, $-2x$, in the quotient is chosen so that when you multiply it by the denominator, $-2x(x+1) = -2x^2 - 2x$, the leading term $-2x^2$ matches the leading term in the remainder exactly.

$$
\begin{array}{r|l l}
 & x^2 - 2x & \\
\cline{2-2}
x+1 & x^3 - \ x^2 + \qquad 2 & \\
 & x^3 + \ x^2 & \longleftarrow x^2(x+1) \\
\cline{2-2}
 & \quad -2x^2 & \\
 & \quad -2x^2 - 2x & \longleftarrow -2x(x+1)
\end{array}
$$

And so on.

$$
\begin{array}{r|l l}
 & x^2 - 2x + 2 & \\
\cline{2-2}
x+1 & x^3 - \ x^2 + \qquad 2 & \\
 & x^3 + \ x^2 & \longleftarrow x^2(x+1) \\
\cline{2-2}
 & \quad -2x^2 & \\
 & \quad -2x^2 - 2x & \longleftarrow -2x(x+1) \\
\cline{2-2}
 & \qquad\qquad 2x + 2 & \\
 & \qquad\qquad 2x + 2 & \longleftarrow 2(x+1) \\
\cline{2-2}
 & \qquad\qquad\quad 0 &
\end{array}
$$

- Note that we finally end up with a remainder 0. A nonzero remainder would have signalled a computational error, since we know that the denominator $x - (-1)$ must divide the numerator $x^3 - x^2 + 2$ exactly.

- We conclude that

$$(x+1)(x^2 - 2x + 2) = x^3 - x^2 + 2$$

To check this, just multiply out the left hand side explicitly.

- Applying the high school quadratic root formula $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ to $x^2 - 2x + 2$ tells us that it has no real roots and that we cannot factor it further [a].

a Because we are not permitted to use complex numbers.

Example A.16.7

We finish by describing an alternative to long division. The approach is roughly equivalent, but is perhaps more straightforward at the expense of requiring more algebra.

Example A.16.8 Roots of $x^3 - x^2 + 2$ again.

Factor $P(x) = x^3 - x^2 + 2$, again.

Solution: Let us do this again but avoid long division.

- From the previous example, we know that $\frac{x^3-x^2+2}{x+1}$ must be a polynomial (since -1 is a root of the numerator) of degree 2. So write

  $$\frac{x^3 - x^2 + 2}{x+1} = ax^2 + bx + c$$

  for some, as yet unknown, coefficients a, b and c.

- Cross multiplying and simplifying gives us

  $$\begin{aligned} x^3 - x^2 + 2 &= (ax^2 + bx + c)(x+1) \\ &= ax^3 + (a+b)x^2 + (b+c)x + c \end{aligned}$$

- Now matching coefficients of the various powers of x on the left and right hand sides

  $$\begin{aligned} &\text{coefficient of } x^3: & a &= 1 \\ &\text{coefficient of } x^2: & a + b &= -1 \\ &\text{coefficient of } x^1: & b + c &= 0 \\ &\text{coefficient of } x^0: & c &= 2 \end{aligned}$$

- This gives us a system of equations that we can solve quite directly. Indeed it tells us immediately that that $a = 1$ and $c = 2$. Subbing $a = 1$ into $a + b = -1$ tells us that $1 + b = -1$ and hence $b = -2$.

- Thus

  $$x^3 - x^2 + 2 = (x+1)(x^2 - 2x + 2).$$

Appendix B

Complex Numbers and Exponentials

B.1 ▲ Definition and Basic Operations

We'll start with the definition of a complex number and its addition and multiplication rules. You may find the multiplication rule quite mysterious. Don't worry. We'll soon gets lots of practice using it and seeing how useful it is.

Definition B.1.1

(a) The complex plane is simply the xy-plane equipped with an addition operation and a multiplication operation. A complex number is nothing more than a point in that xy-plane. It is conventional to use the notation $x+iy$[a] to stand for the complex number (x, y). In other words, it is conventional to write x in place of $(x, 0)$ and i in place of $(0, 1)$.

(b) The first component, x, of the complex number $x+iy$ is called its real part and the second component, y, is called its imaginary part, even though there is nothing imaginary[b] about it.

(c) The sum of the complex numbers $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$ is defined by

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

That is, you just separately add the real parts and the imaginary parts.

(d) The product of the complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is defined by

$$z_1z_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

Do not memorise this multiplication rule. We'll see a simple, effective memory aid for it shortly. The heart of that memory aid is the observation that the complex number i has the special property that

$$i^2 = (0+1i)(0+1i) = (0 \times 0 - 1 \times 1) + i(0 \times 1 + 1 \times 0) = -1$$

a In electrical engineering it is cooonventional to use $x + jy$ instead of $x + iy$.

b Don't attempt to attribute any special significance to the word "complex" in "complex number", or to the word "real" in "real number" and "real part", or to the word "imaginary" in "imaginary part". All are *just names*. The name "imaginary" was introduced by René Descartes in 1637. René Descartes (1596–1650) was a French scientist and philosopher, who lived in the Dutch Republic for roughly twenty years after serving in the (mercenary) Dutch States Army. Originally, "imaginary" was a derogatory term and imaginary numbers were thought to be useless. But they turned out to be incredibly useful!

Addition and multiplication of complex numbers obey the familiar addition rules

$$\begin{aligned} z_1 + z_2 &= z_2 + z_1 \\ z_1 + (z_2 + z_3) &= (z_1 + z_2) + z_3 \\ 0 + z_1 &= z_1 \end{aligned}$$

and multiplication rules

$$\begin{aligned} z_1 z_2 &= z_2 z_1 \\ z_1(z_2 z_3) &= (z_1 z_2) z_3 \\ 1 z_1 &= z_1 \end{aligned}$$

and distributive laws

$$\begin{aligned} z_1(z_2 + z_3) &= z_1 z_2 + z_1 z_3 \\ (z_1 + z_2) z_3 &= z_1 z_3 + z_2 z_3 \end{aligned}$$

To remember how to multiply complex numbers, you just have to supplement the familiar rules of the real number system with $i^2 = -1$. The previous sentence is the memory aid referred to in Definition B.1.1(d).

Example B.1.2

If $z = 1 + 2i$ and $w = 3 + 4i$, then

$$\begin{aligned} z + w &= (1 + 2i) + (3 + 4i) = 4 + 6i \\ zw &= (1 + 2i)(3 + 4i) \qquad = 3 + 4i + 6i + 8i^2 = 3 + 4i + 6i - 8 \\ &= -5 + 10i \end{aligned}$$

Definition B.1.3

(a) The negative of any complex number $z = x + iy$ is defined by

$$-z = -x + (-y)i$$

and obviously obeys $z + (-z) = 0$.

(b) The reciprocal[a], z^{-1} or $\frac{1}{z}$, of any complex number $z = x + iy$, other than 0, is defined by

$$\frac{1}{z}z = 1$$

We shall see below that it is given by the formula

$$\frac{1}{z} = \frac{x}{x^2+y^2} + \frac{-y}{x^2+y^2}i$$

a The reciprocal z^{-1} is also called the multiplicative inverse of z.

Example B.1.4

It is possible to derive the formula for $\frac{1}{z}$ by observing that

$$(a+ib)(x+iy) = [ax-by] + i[ay+bx]$$

equals $1 = 1 + i0$ if and only if

$$\begin{aligned} ax - by &= 1 \\ ay + bx &= 0 \end{aligned}$$

and solving these equations for a and b. We will see a much shorter derivation in Remark B.1.6 below. For now, we'll content ourselves with just verifying that $\frac{x}{x^2+y^2} + \frac{-y}{x^2+y^2}i$ is the inverse of $x + iy$ by multiplying out

$$\begin{aligned} &\left(\frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i\right)(x+iy) \\ &\qquad = \frac{x^2}{x^2+y^2} - \frac{xy}{x^2+y^2}i + \frac{xy}{x^2+y^2}i - \frac{y^2}{x^2+y^2}i^2 \\ &\qquad = \frac{x^2 - i^2y^2}{x^2+y^2} = \frac{x^2+y^2}{x^2+y^2} = 1 \end{aligned}$$

Definition B.1.5

(a) The complex conjugate of $z = x + iy$ is denoted $\bar{z}$ and is defined to be

$$\bar{z} = \overline{x + iy} = x - iy$$

That is, to take the complex conjugate, one replaces every i by $-i$ and vice versa.

(b) The distance from $z = x + iy$ (recall that this is the point (x, y) in the xy-plane) to 0 is denoted $|z|$ and is called the absolute value, or modulus, of z. It is given by

$$|z| = \sqrt{x^2 + y^2}$$

Note that

$$z\bar{z} = (x + iy)(x - iy) = x^2 - ixy + ixy + y^2 = x^2 + y^2$$

is always a nonnegative real number and that

$$|z| = \sqrt{z\,\bar{z}}$$

Remark B.1.6 Let $z = x + iy$ with x and y real. Since $|z|^2 = z\,\bar{z}$, we have

$$\frac{1}{z} = \frac{1}{z}\frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} + \frac{-y}{x^2 + y^2}i$$

which is the formula for $\frac{1}{z}$ given in Definition B.1.3(b).

Example B.1.7

It is easy to divide a complex number by a real number. For example

$$\frac{11 + 2i}{25} = \frac{11}{25} + \frac{2}{25}i$$

In general, the complex conjugate provides us with a trick for rewriting any ratio of complex numbers as a ratio with a real denominator. For example, suppose that we want to find $\frac{1+2i}{3+4i}$. The trick is to multiply by $1 = \frac{3-4i}{3-4i}$. The number $3-4i$ is the complex conjugate of the denominator $3 + 4i$. Since $(3 + 4i)(3 - 4i) = 9 - 12i + 12i - 16i^2 = 9 + 16 = 25$

$$\frac{1 + 2i}{3 + 4i} = \frac{1 + 2i}{3 + 4i}\,\frac{3 - 4i}{3 - 4i} = \frac{(1 + 2i)(3 - 4i)}{25} = \frac{11 + 2i}{25} = \frac{11}{25} + \frac{2}{25}i$$

Definition B.1.8

The notations[a] $\Re z$ and $\Im z$ stand for the real and imaginary parts of the complex number z, respectively. If $z = x + iy$ (with x and y real) they are defined by

$$\Re z = x \qquad \Im z = y$$

Note that both $\Re z$ and $\Im z$ are real numbers. Just subbing in $\bar{z} = x - iy$, you can verify that

$$\Re z = \frac{1}{2}(z + \bar{z}) \qquad \Im z = \frac{1}{2i}(z - \bar{z})$$

a The symbols $\Re$ and $\Im$ are the letters R and I in the Fraktur font, which was created in the early 1500's and became common in the German-speaking world. A standard alternative notation is $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$.

Lemma B.1.9

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$|z_1 z_2| = |z_1|\,|z_2|$$

Proof. Since $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$,

$$\begin{aligned}
|z_1 z_2| &= \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2} \\
&= \sqrt{x_1^2x_2^2 - 2x_1x_2y_1y_2 + y_1^2y_2^2 + x_1^2y_2^2 + 2x_1y_2x_2y_1 + x_2^2y_1^2} \\
&= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2} = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \\
&= |z_1|\,|z_2|
\end{aligned}$$

□

B.2 ▲ The Complex Exponential

B.2.1 ⏩ Definition and Basic Properties

There are two equivalent standard definitions of the exponential, e^z, of the complex number $z = x + iy$. For the more intuitive definition, one simply replaces the real number x in the Taylor series expansion[1] $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ with the complex number z,

1 See Theorem 3.6.7.

giving

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{EZ}$$

We instead highlight the more computationally useful definition.

Definition B.2.1

For any complex number $z = x + iy$, with x and y real, the exponential e^z, is defined by

$$e^{x+iy} = e^x \cos y + ie^x \sin y$$

In particular[a], $e^{iy} = \cos y + i \sin y$.

a The equation $e^{iy} = \cos y + i \sin y$ is known as Euler's formula. Leonhard Euler (1707–1783) was a Swiss mathematician and physicist who spent most of his adult life in Saint Petersberg and Berlin. He gave the name π to the ratio of a circle's circumference to its diameter. He also developed the constant e. His collected works fill 92 volumes.

We will not fully prove that the intuitive definition (EZ) and the computational Definition B.2.1 are equivalent. But we will do so in the special case that $z = iy$, with y real. Under (EZ),

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \frac{(iy)^6}{6!} + \cdots$$

The even terms in this expansion are

$$1 + \frac{(iy)^2}{2!} + \frac{(iy)^4}{4!} + \frac{(iy)^6}{6!} + \cdots = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots = \cos y$$

and the odd terms in this expansion are

$$iy + \frac{(iy)^3}{3!} + \frac{(iy)^5}{5!} + \cdots = i\Big(y - \frac{y^3}{3!} + \frac{y^5}{5!} + \cdots\Big) = i \sin y$$

Adding the even and odd terms together gives us that, under (EZ), e^{iy} is indeed equal to $\cos y + i \sin y$.[2] As a consequence, we have

$$e^{i\pi} = -1$$

which gives an amazing linking between calculus (e), geometry (π), algebra (i) and the basic number -1.

In the next lemma we verify that the complex exponential obeys a couple of familiar computational properties.

2 It is obvious that, in the special case that $z = x$ with x real, the definitions (EZ) and B.2.1 are equivalent. So to complete the proof of equivalence in the general case $z = x + iy$, it suffices to prove that $e^{x+iy} = e^x e^{iy}$ under both (EZ) and Definition B.2.1. For Definition B.2.1, this follows from Lemma B.2.2, below.

Lemma B.2.2

(a) For any complex numbers z_1 and z_2,

$$e^{z_1+z_2} = e^{z_1}e^{z_2}$$

(b) For any complex number c,

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{ct} = ce^{ct}$$

Proof.

(a) For any two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, with x_1, y_1, x_2, y_2 real,

$$\begin{aligned}
e^{z_1}e^{z_2} &= e^{x_1}(\cos y_1 + i\sin y_1)e^{x_2}(\cos y_2 + i\sin y_2) \\
&= e^{x_1+x_2}(\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2) \\
&= e^{x_1+x_2}\left\{(\cos y_1\cos y_2 - \sin y_1\sin y_2) + i(\cos y_1\sin y_2 + \cos y_2\sin y_1)\right\} \\
&= e^{x_1+x_2}\left\{\cos(y_1+y_2) + i\sin(y_1+y_2)\right\}
\end{aligned}$$

by the trig identities of Appendix A.8. This says exactly that

$$e^{z_1}e^{z_2} = e^{(x_1+x_2)+i(y_1+y_2)} = e^{z_1+z_2}$$

and that the familiar multiplication formula also applies to complex exponentials.

(b) For any real number t and any complex number $c = \alpha + i\beta$, with α, β real,

$$e^{ct} = e^{\alpha t + i\beta t} = e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)]$$

so that the derivative with respect to t

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}e^{ct} &= \alpha e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)] + e^{\alpha t}[-\beta\sin(\beta t) + i\beta\cos(\beta t)] \\
&= (\alpha + i\beta)e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)] \\
&= ce^{ct}
\end{aligned}$$

is also the familiar one.

□

B.2.2 ▸▸ Relationship with sin and cos

Equation B.2.3

When θ is a real number

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta = \overline{e^{i\theta}} \end{aligned}$$

are complex numbers of modulus one.

Solving for $\cos\theta$ and $\sin\theta$ (by adding and subtracting the two equations) gives

Equation B.2.4

$$\begin{aligned} \cos\theta &= \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \ = \Re e^{i\theta} \\ \sin\theta &= \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \Im e^{i\theta} \end{aligned}$$

Example B.2.5

These formulae make it easy derive trig identities. For example,

$$\begin{aligned} \cos\theta\cos\phi &= \frac{1}{4}\big(e^{i\theta} + e^{-i\theta}\big)\big(e^{i\phi} + e^{-i\phi}\big) \\ &= \frac{1}{4}\big(e^{i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)} + e^{-i(\theta+\phi)}\big) \\ &= \frac{1}{4}\big(e^{i(\theta+\phi)} + e^{-i(\theta+\phi)} + e^{i(\theta-\phi)} + e^{i(-\theta+\phi)}\big) \\ &= \frac{1}{2}\big(\cos(\theta+\phi) + \cos(\theta-\phi)\big) \end{aligned}$$

and, using $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$,

$$\begin{aligned} \sin^3\theta &= -\frac{1}{8i}\big(e^{i\theta} - e^{-i\theta}\big)^3 \\ &= -\frac{1}{8i}\big(e^{i3\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-i3\theta}\big) \\ &= \frac{3}{4}\frac{1}{2i}\big(e^{i\theta} - e^{-i\theta}\big) - \frac{1}{4}\frac{1}{2i}\big(e^{i3\theta} - e^{-i3\theta}\big) \end{aligned}$$

$$= \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta)$$

and

$$\begin{aligned}\cos(2\theta) &= \Re\big(e^{2\theta i}\big) = \Re\big(e^{i\theta}\big)^2\\ &= \Re\big(\cos\theta + i\sin\theta\big)^2\\ &= \Re\big(\cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta\big)\\ &= \cos^2\theta - \sin^2\theta\end{aligned}$$

Example B.2.5

B.2.3 ▸▸ Polar Coordinates

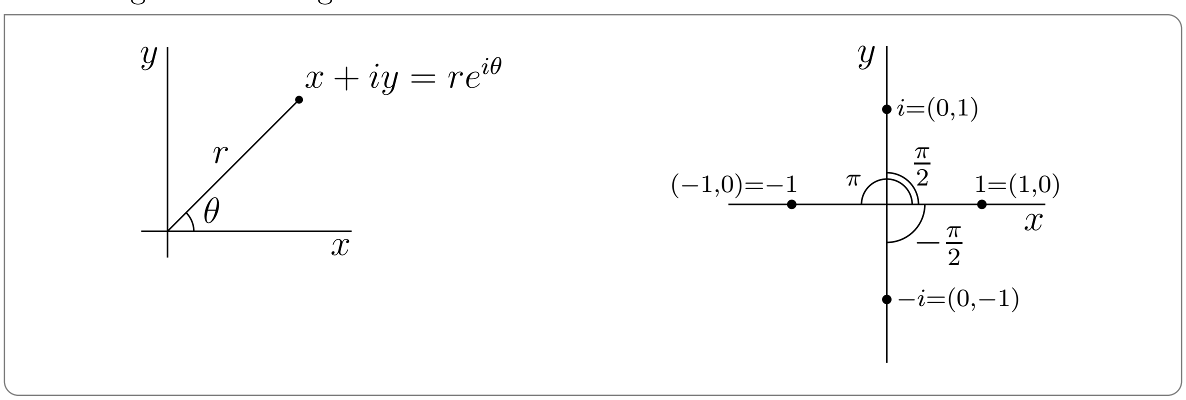
Let $z = x + iy$ be any complex number. Writing x and y in polar coordinates in the usual way, i.e. $x = r\cos(\theta)$, $y = r\sin(\theta)$, gives

$$x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}$$

See the figure on the left below. In particular

$$\begin{aligned} 1 &= e^{i0} = e^{2\pi i} = e^{2k\pi i} && \text{for } k = 0, \pm1, \pm2, \cdots\\ -1 &= e^{i\pi} = e^{3\pi i} = e^{(1+2k)\pi i} && \text{for } k = 0, \pm1, \pm2, \cdots\\ i &= e^{i\pi/2} = e^{\frac{5}{2}\pi i} = e^{(\frac{1}{2}+2k)\pi i} && \text{for } k = 0, \pm1, \pm2, \cdots\\ -i &= e^{-i\pi/2} = e^{\frac{3}{2}\pi i} = e^{(-\frac{1}{2}+2k)\pi i} && \text{for } k = 0, \pm1, \pm2, \cdots \end{aligned}$$

See the figure on the right below.



The polar coordinate $\theta = \arctan\frac{y}{x}$ associated with the complex number $z = x + iy$, i.e. the point (x, y) in the xy-plane, is also called the argument of z.

The polar coordinate representation makes it easy to find square roots, third roots and so on. Fix any positive integer n. The n^{th} roots of unity are, by definition, all solutions z of

$$z^n = 1$$

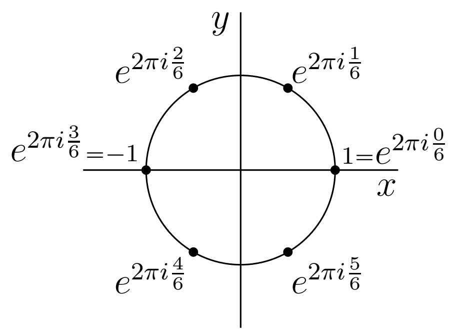
Writing $z = re^{i\theta}$

$$r^n e^{n\theta i} = 1e^{0i}$$

The polar coordinates (r, θ) and (r', θ') represent the same point in the xy-plane if and only if $r = r'$ and $\theta = \theta' + 2k\pi$ for some integer k. So $z^n = 1$ if and only if $r^n = 1$, i.e. $r = 1$, and $n\theta = 2k\pi$ for some integer k. The n^{th} roots of unity are all the complex numbers $e^{2\pi i \frac{k}{n}}$ with k integer. There are precisely n distinct n^{th} roots of unity because $e^{2\pi i \frac{k}{n}} = e^{2\pi i \frac{k'}{n}}$ if and only if $2\pi\frac{k}{n} - 2\pi\frac{k'}{n} = 2\pi\frac{k-k'}{n}$ is an integer multiple of 2π. That is, if and only if $k - k'$ is an integer multiple of n. The n distinct n^{th} roots of unity are

$$1\ ,\ e^{2\pi i \frac{1}{n}}\ ,\ e^{2\pi i \frac{2}{n}}\ ,\ e^{2\pi i \frac{3}{n}}\ ,\ \cdots\ ,\ e^{2\pi i \frac{n-1}{n}}$$

For example, the 6^{th} roots of unity are depicted below.



B.2.4 ▸▸ Exploiting Complex Exponentials in Calculus Computations

You have learned how to evaluate integrals involving trigonometric functions by using integration by parts, various trigonometric identities and various substitutions. It is often much easier to just use (B.2.3) and (B.2.4). Part of the utility of complex numbers comes from how well they interact with calculus through the exponential function. Here are two examples.

Example B.2.6

$$\begin{aligned}\int e^x \cos x \,\mathrm{d}x &= \frac{1}{2}\int e^x\big[e^{ix} + e^{-ix}\big]\,\mathrm{d}x = \frac{1}{2}\int \big[e^{(1+i)x} + e^{(1-i)x}\big]\,\mathrm{d}x \\ &= \frac{1}{2}\left[\frac{1}{1+i}e^{(1+i)x} + \frac{1}{1-i}e^{(1-i)x}\right] + C\end{aligned}$$

This form of the indefinite integral looks a little weird because of the i's. While it looks complex because of the i's, it is actually purely real (and correct), because $\frac{1}{1-i}e^{(1-i)x}$ is the complex conjugate of $\frac{1}{1+i}e^{(1+i)x}$. We can convert the indefinite integral into a more familiar form just by subbing back in $e^{\pm ix} = \cos x \pm i \sin x$, $\frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2}$ and

$\frac{1}{1-i} = \overline{\frac{1}{1+i}} = \frac{1+i}{2}$.

$$\begin{aligned}\int e^x \cos x \ \mathrm{d}x &= \frac{1}{2}e^x\left[\frac{1}{1+i}e^{ix} + \frac{1}{1-i}e^{-ix}\right] + C \\ &= \frac{1}{2}e^x\left[\frac{1-i}{2}(\cos x + i\sin x) + \frac{1+i}{2}(\cos x - i\sin x)\right] + C \\ &= \frac{1}{2}e^x\cos x + \frac{1}{2}e^x \sin x + C\end{aligned}$$

You can quickly verify this by differentiating (or by comparing with Example 1.7.11).

Example B.2.6

Example B.2.7

Evaluating the integral $\int \cos^n x \ \mathrm{d}x$ using the methods of Section 1.8 can be a real pain. It is much easier if we convert to complex exponentials. Using $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$,

$$\begin{aligned}\int \cos^4 x \ \mathrm{d}x &= \frac{1}{2^4}\int \left[e^{ix} + e^{-ix}\right]^4 \mathrm{d}x \\ &= \frac{1}{2^4}\int \left[e^{4ix} + 4e^{2ix} + 6 + 4e^{-2ix} + e^{-4ix}\right] \mathrm{d}x \\ &= \frac{1}{2^4}\left[\frac{1}{4i}e^{4ix} + \frac{4}{2i}e^{2ix} + 6x + \frac{4}{-2i}e^{-2ix} + \frac{1}{-4i}e^{-4ix}\right] + C \\ &= \frac{1}{2^4}\left[\frac{1}{2}\frac{1}{2i}(e^{4ix} - e^{-4ix}) + \frac{4}{2i}(e^{2ix} - e^{-2ix}) + 6x\right] + C \\ &= \frac{1}{2^4}\left[\frac{1}{2}\sin 4x + 4\sin 2x + 6x\right] + C \\ &= \frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3}{8}x + C\end{aligned}$$

B.2.5 ➠ Exploiting Complex Exponentials in Differential Equation Computations

Complex exponentials are also widely used to simplify the process of guessing solutions to ordinary differential equations. We'll start with (possibly a review of) some basic definitions and facts about differential equations.

Definition B.2.8

a A **differential equation** is an equation for an unknown function that contains the derivatives of that unknown function. For example $y''(t) + y(t) = 0$ is a differential equation for the unknown function $y(t)$.

b In the differential calculus text CLP-1, we treated only derivatives of functions of one variable. Such derivatives are called ordinary derivatives. A differential equation is called an **ordinary differential equation** (often shortened to "ODE") if only ordinary derivatives appear. That is, if the unknown function has only a single independent variable.

In CLP-3 we will treat derivatives of functions of more than one variable. For example, let $f(x, y)$ be a function of two variables. If you treat y as a constant and take the derivative of the resulting function of the single variable x, the result is called the partial derivative of f with respect to x. A differential equation is called a **partial differential equation** (often shortened to "PDE") if partial derivatives appear. That is, if the unknown function has more than one independent variable. For example $y''(t)+y(t) = 0$ is an ODE while $\frac{\partial^2 u}{\partial t^2}(x, t) = c^2 \frac{\partial^2 u}{\partial x^2}(x, t)$ is a PDE.

c The **order** of a differential equation is the order of the highest derivative that appears. For example $y''(t) + y(t) = 0$ is a second order ODE.

d An ordinary differential equation that is of the form

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_n(t)y(t) = F(t) \tag{B.2.1}$$

with given coefficient functions $a_0(t)$, $\cdots$, $a_n(t)$ and $F(t)$ is said to be **linear**. Otherwise, the ODE is said to be **nonlinear**. For example, $y'(t)^2+y(t) = 0$, $y'(t)y''(t) + y(t) = 0$ and $y'(t) = e^{y(t)}$ are all nonlinear.

e The ODE (B.2.1) is said to have **constant coefficients** if the coefficients $a_0(t)$, $a_1(t)$, $\cdots$, $a_n(t)$ are all constants. Otherwise, it is said to have **variable coefficients**. For example, the ODE $y''(t) + 7y(t) = \sin t$ is constant coefficient, while $y''(t) + ty(t) = \sin t$ is variable coefficient.

f The ODE (B.2.1) is said to be **homogeneous** if $F(t)$ is identically zero. Otherwise, it is said to be **inhomogeneous** or **nonhomogeneous**. For example, the ODE $y''(t) + 7y(t) = 0$ is homogeneous, while $y''(t) + 7y(t) = \sin t$ is inhomogeneous. A homogeneous ODE always has the trivial solution $y(t) = 0$.

g An **initial value problem** is a problem in which one is to find an unknown function $y(t)$ that satisfies both a given ODE and given initial conditions, like $y(t_0) = 1$, $y'(t_0) = 0$. Note that all of the conditions involve the function $y(t)$ (or its derivatives) evaluated at a single time $t = t_0$.

h A **boundary value problem** is a problem in which one is to find an unknown function $y(t)$ that satisfies both a given ODE and given boundary conditions, like $y(t_0) = 0$, $y(t_1) = 0$. Note that the conditions involve the function $y(t)$ (or its derivatives) evaluated at two different times.

The following theorem gives the form of solutions to the linear[3] ODE (B.2.1).

Theorem B.2.9

Assume that the coefficients $a_0(t)$, $a_1(t)$, $\cdots$, $a_{n-1}(t)$, $a_n(t)$ and $F(t)$ are continuous functions and that $a_0(t)$ is not zero.

a The general solution to the linear ODE (B.2.1) is of the form

$$y(t) = y_p(t) + C_1y_1(t) + C_2y_2(t) + \cdots + C_ny_n(t) \tag{B.2.2}$$

where

- n is the order of (B.2.1)
- $y_p(t)$ is ***any*** solution to (B.2.1)
- C_1, C_2, $\cdots$, C_n are arbitrary constants
- y_1, y_2, $\cdots$, y_n are n independent solutions to the homogenous equation

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \cdots + a_{n-1}(t)y'(t) + a_n(t)y(t) = 0$$

associated to (B.2.1). "Independent" just means that no y_i can be written as a linear combination of the other y_j's. For example, $y_1(t)$ cannot be expressed in the form $b_2y_2(t) + \cdots + b_ny_n(t)$.

In (B.2.2), y_p is called the "particular solution" and $C_1y_1(t)+C_2y_2(t)+\cdots+C_ny_n(t)$ is called the "complementary solution".

b Given any constants b_0, $\cdots$, b_{n-1} there is exactly one function $y(t)$ that obeys the ODE (B.2.1) and the initial conditions

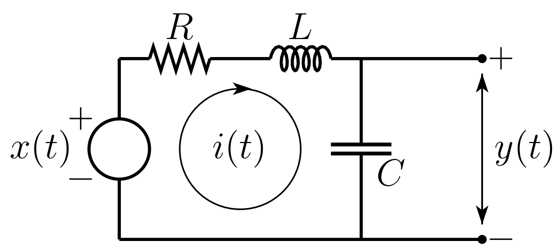
$$y(0) = b_0 \qquad y'(0) = b_1 \qquad \cdots \qquad y^{(n-1)}(0) = b_{n-1}$$

In the following example we'll derive one widely used linear constant coefficient ODE.

3 There are a some special classes of nonlinear ODE's, like the separable differential equations of §2.4, that are relatively easy to solve. But generally, nonlinear ODE's are much harder to solve than linear ODE's.

Example B.2.10 RLC circuit.

As an example of how ODE's arise, we consider the RLC circuit, which is the electrical circuit consisting of a resistor of resistance R, a coil (or solenoid) of inductance L, a capacitor of capacitance C and a voltage source arranged in series, as shown below. Here R, L and C are all nonnegative constants.



We're going to think of the voltage $x(t)$ as an input signal, and the voltage $y(t)$ as an output signal. The goal is to determine the output signal produced by a given input signal. If $i(t)$ is the current flowing at time t in the loop as shown and $q(t)$ is the charge on the capacitor, then the voltages across R, L and C, respectively, at time t are $Ri(t)$, $L\frac{\mathrm{d}i}{\mathrm{d}t}(t)$ and $y(t) = \frac{q(t)}{C}$. By the Kirchhoff's law[a] that says that the voltage between any two points has to be independent of the path used to travel between the two points, these three voltages must add up to $x(t)$ so that

$$Ri(t) + L\frac{\mathrm{d}i}{\mathrm{d}t}(t) + \frac{q(t)}{C} = x(t) \tag{B.2.3}$$

Assuming that R, L, C and $x(t)$ are known, this is still one differential equation in two unknowns, the current $i(t)$ and the charge $q(t)$. Fortunately, there is a relationship between the two. Because the current entering the capacitor is the rate of change of the charge on the capacitor

$$i(t) = \frac{\mathrm{d}q}{\mathrm{d}t}(t) = Cy'(t) \tag{B.2.4}$$

This just says that the capacitor cannot create or destroy charge on its own; all charging of the capacitor must come from the current. Substituting (B.2.4) into (B.2.3) gives

$$LCy''(t) + RCy'(t) + y(t) = x(t)$$

which is a second order linear constant coefficient ODE. As a concrete example, we'll take an ac voltage source and choose the origin of time so that $x(0) = 0$, $x(t) = E_0 \sin(\omega t)$. Then the differential equation becomes

$$LCy''(t) + RCy'(t) + y(t) = E_0 \sin(\omega t) \tag{B.2.5}$$

a Gustav Robert Kirchhoff (1824–1887) was a German physicist. There are several sets of Kirchhoff's laws that are named after him — Kirchhoff's circuit laws, that we are using in this example, Krichhoff's spectroscopy laws and Kirchhoff's law of thermochemistry. Kirchhoff and his collaborator Robert Bunsen, of Bunsen burner fame, invented the spectroscope.

Finally, here are two examples in which we use complex exponentials to solve an ODE.

Example B.2.11

By Theorem B.2.9(a), the general solution to the ordinary differential equation

$$y''(t) + 4y'(t) + 5y(t) = 0 \tag{ODE}$$

is of the form $C_1u_1(t)+C_2u_2(t)$ with $u_1(t)$ and $u_2(t)$ being two (independent) solutions to (ODE) and with C_1 and C_2 being arbitrary constants. The easiest way to find $u_1(t)$ and $u_2(t)$ is to guess them. And the easiest way to guess them is to try[a] $y(t) = e^{rt}$, with r being a constant to be determined. Substituting $y(t) = e^{rt}$ into (ODE) gives

$$r^2e^{rt} + 4re^{rt} + 5e^{rt} = 0 \iff \big(r^2 + 4r + 5\big)e^{rt} = 0 \iff r^2 + 4r + 5 = 0$$

This quadratic equation for r can be solved either by using the high school formula or by completing the square.

$$\begin{aligned} r^2 + 4r + 5 = 0 &\iff (r + 2)^2 + 1 = 0 \\ &\iff (r + 2)^2 = -1 \iff r + 2 = \pm i \\ &\iff r = -2 \pm i \end{aligned}$$

So the general solution to (ODE) is

$$y(t) = C_1e^{(-2+i)t} + C_2e^{(-2-i)t}$$

This is one way to write the general solution, but there are many others. In particular there are quite a few people in the world who are (foolishly) afraid[b] of complex exponentials. We can hide them by using (B.2.3) and (B.2.4).

$$\begin{aligned} y(t) &= C_1e^{(-2+i)t} + C_2e^{(-2-i)t} = C_1e^{-2t}e^{it} + C_2e^{-2t}e^{-it} \\ &= C_1e^{-2t}\big(\cos t + i\sin t\big) + C_2e^{-2t}\big(\cos t - i\sin t\big) \\ &= (C_1 + C_2)e^{-2t}\cos t + (iC_1 - iC_2)e^{-2t}\sin t \\ &= D_1e^{-2t}\cos t + D_2e^{-2t}\sin t \end{aligned}$$

with $D_1 = C_1 + C_2$ and $D_2 = iC_1 - iC_2$ being two other arbitrary constants. Don't make the mistake of thinking that D_2 must be complex because i appears in the formula $D_2 = iC_1-iC_2$ relating D_2 and C_1, C_2. No one said that C_1 and C_2 are real numbers. In fact, in typical applications, the arbitrary constants are determined by initial conditions and often D_1 and D_2 turn out to be real and C_1 and C_2 turn out to be complex. For example, the initial conditions $y(0) = 0$, $y'(0) = 2$ force

$$\begin{aligned} 0 &= y(0) = C_1 + C_2 \\ 2 &= y'(0) = (-2 + i)C_1 + (-2 - i)C_2 \end{aligned}$$

The first equation gives $C_2 = -C_1$ and then the second equation gives

$$(-2 + i)C_1 - (-2 - i)C_1 = 2 \iff 2iC_1 = 2 \iff iC_1 = 1$$

$$\iff C_1 = -i,\ C_2 = i$$

and

$$D_1 = C_1 + C_2 = 0 \qquad D_2 = iC_1 - iC_2 = 2$$

a The reason that $y(t) = e^{rt}$ is a good guess is that, with this guess, all of $y(t)$, $y'(t)$ and $y''(t)$ are constants times e^{rt}. So the left hand side of the differential equation is also a constant, that depends on r, times e^{rt}. So we just have to choose r so that the constant is zero.

b Embracing the complexity leads to simplicity.

Example B.2.11

Example B.2.12

We shall now guess one solution (i.e. a particular solution) to the differential equation

$$y''(t) + 2y'(t) + 3y(t) = \cos t \tag{ODE1}$$

Equations like this arise, for example, in the study of the RLC circuit. We shall simplify the computation by exploiting that $\cos t = \Re e^{it}$. First, we shall guess a function $Y(t)$ obeying

$$Y'' + 2Y' + 3Y = e^{it} \tag{ODE2}$$

Then, taking complex conjugates gives

$$\bar{Y}'' + 2\bar{Y}' + 3\bar{Y} = e^{-it}$$

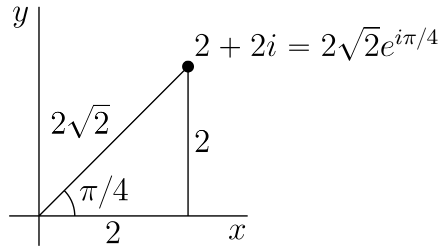
which we shall call $(\overline{\text{ODE2}})$. Then, adding $\frac{1}{2}$(ODE2) and $\frac{1}{2}(\overline{\text{ODE2}})$ together will give

$$(\Re Y)'' + 2(\Re Y)' + 3(\Re Y) = \Re e^{it} = \cos t$$

which shows that $\Re Y(t)$ is a solution to (ODE1). Let's try $Y(t) = Ae^{it}$, with A a constant to be determined. This is a solution of (ODE2) if and only if

$$\begin{aligned} && \frac{\mathrm{d}^2}{\mathrm{d}t^2}\big(Ae^{it}\big) + 2\frac{\mathrm{d}}{\mathrm{d}t}\big(Ae^{it}\big) + 3Ae^{it} = e^{it} \\ \iff && (i^2 + 2i + 3)Ae^{it} = e^{it} \\ \iff && A = \frac{1}{2+2i} \end{aligned}$$

So $\frac{e^{it}}{2+2i}$ is a solution to (ODE2) and $\Re\frac{e^{it}}{2+2i}$ is a solution to (ODE1). To simplify this, write $2+2i$ in polar coordinates. From the sketch below we have $2+2i = 2\sqrt{2}e^{i\frac{\pi}{4}}$.



So

$$\begin{aligned}
\frac{e^{it}}{2+2i} &= \frac{e^{it}}{2\sqrt{2}e^{i\frac{\pi}{4}}} = \frac{1}{2\sqrt{2}}e^{i(t-\frac{\pi}{4})} \\
\implies \Re\frac{e^{it}}{2+2i} &= \frac{1}{2\sqrt{2}}\Re e^{i(t-\frac{\pi}{4})} = \frac{1}{2\sqrt{2}}\cos\left(t-\frac{\pi}{4}\right)
\end{aligned}$$

Example B.2.12

Appendix C

More About Numerical Integration

C.1 ▲ Richardson Extrapolation

There are many approximation procedures in which one first picks a step size h and then generates an approximation $A(h)$ to some desired quantity $\mathcal{A}$. For example, $\mathcal{A}$ might be the value of some integral $\int_a^b f(x)\,\mathrm{d}x$. For the trapezoidal rule with n steps, $\Delta x = \frac{b-a}{n}$ plays the role of the step size. Often the order of the error generated by the procedure is known. This means

$$\mathcal{A} = A(h) + Kh^k + K_1 h^{k+1} + K_2 h^{k+2} + \cdots \tag{E1}$$

with k being some known constant, called the order of the error, and $K,\ K_1,\ K_2,\ \cdots$ being some other (usually unknown) constants. If $A(h)$ is the approximation to $\mathcal{A} = \int_a^b f(x)\,\mathrm{d}x$ produced by the trapezoidal rule with $\Delta x = h$, then $k = 2$. If Simpson's rule is used, $k = 4$.

Let's first suppose that h is small enough that the terms $K'h^{k+1} + K''h^{k+2} + \cdots$ in (E1) are small enough[1] that dropping them has essentially no impact. This would give

$$\mathcal{A} = A(h) + Kh^k \tag{E2}$$

Imagine that we know k, but that we do not know A or K, and think of (E2) as an equation that the unknowns $\mathcal{A}$ and K have to solve. It may look like we have one equation in the two unknowns K, $\mathcal{A}$, but that is *not* the case. The reason is that (E2)

1 Typically, we don't have access to, and don't care about, the exact error. We only care about its order of magnitude. So if h is small enough that $K_1 h^{k+1} + K_2 h^{k+2} + \cdots$ is a factor of at least, for example, one hundred smaller than Kh^k, then dropping $K_1 h^{k+1} + K_2 h^{k+2} + \cdots$ would not bother us at all.

is (essentially) true for all (sufficiently small) choices of h. If we pick some h, say h_1, and use the algorithm to determine $A(h_1)$ then (E2), with h replaced by h_1, gives one equation in the two unknowns $\mathcal{A}$ and K, and if we then pick some different h, say h_2, and use the algorithm a second time to determine $A(h_2)$ then (E2), with h replaced by h_2, gives a second equation in the two unknowns $\mathcal{A}$ and K. The two equations will then determine both $\mathcal{A}$ and K.

To be more concrete, suppose that we have picked some specific value of h, and have chosen $h_1 = h$ and $h_2 = \frac{h}{2}$, and that we have evaluated $A(h)$ and $A(h/2)$. Then the two equations are

$$\mathcal{A} = A(h) + Kh^k \tag{E3a}$$

$$\mathcal{A} = A(h/2) + K\left(\tfrac{h}{2}\right)^k \tag{E3b}$$

It is now easy to solve for both K and $\mathcal{A}$. To get K, just subtract (E3b) from (E3a).

$$\begin{aligned} \text{(E3a)} - \text{(E3b)}: \quad & 0 = A(h) - A(h/2) + \left(1 - \tfrac{1}{2^k}\right)Kh^k \\ \implies \quad & K = \frac{A(h/2) - A(h)}{[1 - 2^{-k}]h^k} \end{aligned} \tag{E4a}$$

To get $\mathcal{A}$ multiply (E3b) by 2^k and then subtract (E3a).

$$\begin{aligned} 2^k\text{(E3b)} - \text{(E3a)}: \quad & [2^k - 1]\mathcal{A} = 2^kA(h/2) - A(h) \\ \implies \quad & \mathcal{A} = \frac{2^kA(h/2) - A(h)}{2^k - 1} \end{aligned} \tag{E4b}$$

The generation of a "new improved" approximation for $\mathcal{A}$ from two $A(h)$'s with different values of h is called Richardson[2] Extrapolation. Here is a summary

Equation C.1.1 Richardson extrapolation.

Let $A(h)$ be a step size h approximation to $\mathcal{A}$. If

$$\mathcal{A} = A(h) + Kh^k$$

then

$$K = \frac{A(h/2) - A(h)}{[1 - 2^{-k}]h^k} \qquad \mathcal{A} = \frac{2^kA(h/2) - A(h)}{2^k - 1}$$

This works very well since, by computing $A(h)$ for two different h's, we can remove the biggest error term in (E1), and so get a much more precise approximation to $\mathcal{A}$ for little additional work.

2 Richardson extrapolation was introduced by the Englishman Lewis Fry Richardson (1881–1953) in 1911.

Example C.1.2 Richardson extrapolation with the trapezoidal rule.

Applying the trapezoidal rule (1.11.6) to the integral $\mathcal{A} = \int_0^1 \frac{4}{1+x^2}\mathrm{d}x$ with step sizes $\frac{1}{8}$ and $\frac{1}{16}$ (i.e. with $n = 8$ and $n = 16$) gives, with $h = \frac{1}{8}$,

$$A(h) = 3.1389884945 \qquad A(h/2) = 3.1409416120$$

So (E4b), with $k = 2$, gives us the "new improved" approximation

$$\frac{2^2 \times 3.1409416120 - 3.1389884945}{2^2 - 1} = 3.1415926512$$

We saw in Example 1.11.3 that $\int_0^1 \frac{4}{1+x^2}\mathrm{d}x = \pi$, so this new approximation really is "improved":

- $A(1/8)$ agrees with π to two decimal places,
- $A(1/16)$ agrees with π to three decimal places and
- the new approximation agrees with π to eight decimal places.

Beware that (E3b), namely $\mathcal{A} = A(h/2) + K\big(\frac{h}{2}\big)^k$, is saying that $K\big(\frac{h}{2}\big)^k$ is (approximately) the error in $A(h/2)$, *not* the error in $\mathcal{A}$. You cannot get an "even more improved" approximation by using (E4a) to compute K and then adding $K\big(\frac{h}{2}\big)^k$ to the "new improved" $\mathcal{A}$ of (E4b) — doing so just gives $\mathcal{A} + K\big(\frac{h}{2}\big)^k$, not a more accurate $\mathcal{A}$.

Example C.1.3 Example 1.11.16 revisited.

Suppose again that we wish to use Simpson's rule (1.11.9) to evaluate $\int_0^1 e^{-x^2}\,\mathrm{d}x$ to within an accuracy of 10^{-6}, but that we do not need the degree of certainty provided by Example 1.11.16. Observe that we need (approximately) that $|K|h^4 < 10^{-6}$, so if we can estimate K (using our Richardson trick) then we can estimate the required h. A commonly used strategy, based on this observation, is to

- first apply Simpson's rule twice with some relatively small number of steps and
- then use (E4a), with $k = 4$, to estimate K and
- then use the condition $|K|h^k \le 10^{-6}$ to determine, approximately, the number of steps required
- and finally apply Simpson's rule with the number of steps just determined.

Let's implement this strategy. First we estimate K by applying Simpson's rule with step sizes $\frac{1}{4}$ and $\frac{1}{8}$. Writing $\frac{1}{4} = h'$, we get

$$A(h') = 0.74685538 \qquad A(h'/2) = 0.74682612$$

so that (E4a), with $k = 4$ and h replaced by h', yields

$$K = \frac{0.74682612 - 0.74685538}{[1-2^{-4}](1/4)^4} = -7.990 \times 10^{-3}$$

We want to use a step size h obeying

$$|K|h^4 \le 10^{-6} \iff 7.990 \times 10^{-3} h^4 \le 10^{-6} \iff h \le \sqrt[4]{\frac{1}{7990}} = \frac{1}{9.45}$$

like, for example, $h = \frac{1}{10}$. Applying Simpson's rule with $h = \frac{1}{10}$ gives

$$A(1/10) = 0.74682495$$

The exact answer, to eight decimal places, is 0.74682413 so the error in $A(1/10)$ is indeed just under 10^{-6}.
Suppose now that we change our minds. We want an accuracy of 10^{-12}, rather than 10^{-6}. We have already estimated K. So now we want to use a step size h obeying

$$\begin{aligned} |K|h^4 \le 10^{-12} &\iff 7.99 \times 10^{-3} h^4 \le 10^{-12} \\ &\iff h \le \sqrt[4]{\frac{1}{7.99 \times 10^9}} = \frac{1}{299.0} \end{aligned}$$

like, for example, $h = \frac{1}{300}$. Applying Simpson's rule with $h = \frac{1}{300}$ gives, to fourteen decimal places,

$$A(1/300) = 0.74682413281344$$

The exact answer, to fourteen decimal places, is 0.74682413281243 so the error in $A(1/300)$ is indeed just over 10^{-12}.

Example C.1.3

C.2 ▲ Romberg Integration

The formulae (E4a,b) for K and $\mathcal{A}$ are, of course, only[1] approximate since they are based on (E2), which is an approximation to (E1). Let's repeat the derivation that leads to (E4), but using the full (E1),

$$\mathcal{A} = A(h) + Kh^k + K_1 h^{k+1} + K_2 h^{k+2} + \cdots$$

Once again, suppose that we have chosen some h and that we have evaluated $A(h)$ and $A(h/2)$. They obey

$$\mathcal{A} = A(h) + Kh^k + K_1 h^{k+1} + K_2 h^{k+2} + \cdots \tag{E5a}$$

1 "Only" is a bit strong. Don't underestimate the power of a good approximation (pun intended).

$$\mathcal{A} = A(h/2) + K\left(\tfrac{h}{2}\right)^k + K_1\left(\tfrac{h}{2}\right)^{k+1} + K_2\left(\tfrac{h}{2}\right)^{k+2} + \cdots \tag{E5b}$$

Now, as we did in the derivation of (E4b), multiply (E5b) by 2^k and then subtract (E5a). This gives

$$\left(2^k - 1\right)\mathcal{A} = 2^k A(h/2) - A(h) - \frac{1}{2}K_1 h^{k+1} - \tfrac{3}{4}K_2 h^{k+1} + \cdots$$

and then, dividing across by $\left(2^k - 1\right)$,

$$\mathcal{A} = \frac{2^k A(h/2) - A(h)}{2^k - 1} - \frac{1/2}{2^k - 1}K_1 h^{k+1} - \frac{3/4}{2^k - 1}K_2 h^{k+2} + \cdots$$

Hence if we define our "new improved approximation"

$$B(h) = \frac{2^k A(h/2) - A(h)}{2^k - 1} \text{ and } \tilde{K} = -\frac{1/2}{2^k - 1}K_1 \text{ and } \tilde{K}_1 = -\frac{3/4}{2^k - 1}K_2 \tag{E6}$$

we have

$$\mathcal{A} = B(h) + \tilde{K}h^{k+1} + \tilde{K}_1 h^{k+2} + \cdots$$

which says that $B(h)$ is an approximation to $\mathcal{A}$ whose error is of order $k+1$, one better[2] than $A(h)$'s.

If $A(h)$ has been computed for three values of h, we can generate $B(h)$ for two values of h and repeat the above procedure with a new value of k. And so on. One widely used numerical integration algorithm, called Romberg integration[3], applies this procedure repeatedly to the trapezoidal rule. It is known that the trapezoidal rule approximation $T(h)$ to an integral I has error behaviour (assuming that the integrand $f(x)$ is smooth)

$$I = T(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 + \cdots$$

Only even powers of h appear. Hence

$$T(h) \qquad \text{has error of order 2}$$

so that, using (E6) with $k = 2$,

$$T_1(h) = \tfrac{4T(h/2) - T(h)}{3} \qquad \text{has error of order 4}$$

so that, using (E6) with $k = 4$,

$$T_2(h) = \tfrac{16T_1(h/2) - T_1(h)}{15} \qquad \text{has error of order 6}$$

so that, using (E6) with $k = 6$,

$$T_3(h) = \tfrac{64T_2(h/2) - T_2(h)}{63} \qquad \text{has error of order 8}$$

and so on. We know another method which produces an error of order 4 — Simpson's rule. In fact, $T_1(h)$ is exactly Simpson's rule (for step size $\frac{h}{2}$).

2 That is, the error decays as h^{k+1} as opposed to h^k — so, as h decreases, it gets smaller faster.
3 Romberg Integration was introduced by the German Werner Romberg (1909–2003) in 1955.

Equation C.2.1 Romberg integration.

Let $T(h)$ be the trapezoidal rule approximation, with step size h, to an integral I. The Romberg integration algorithm is

$$\begin{aligned} T_1(h) &= \tfrac{4T(h/2)-T(h)}{3} \\ T_2(h) &= \tfrac{16T_1(h/2)-T_1(h)}{15} \\ T_3(h) &= \tfrac{64T_2(h/2)-T_2(h)}{63} \\ &\vdots \\ T_k(h) &= \tfrac{2^{2k}T_{k-1}(h/2)-T_{k-1}(h)}{2^{2k}-1} \\ &\vdots \end{aligned}$$

Example C.2.2 Finding π by Romberg integration.

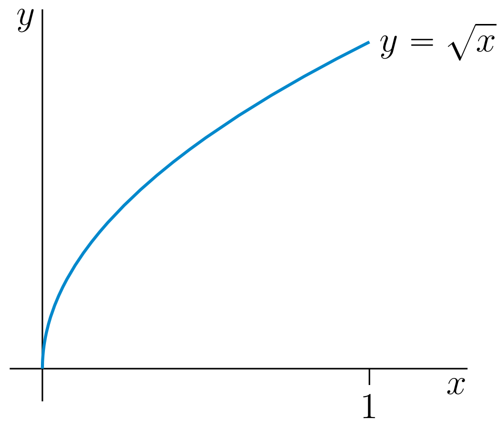
The following table[a] illustrates Romberg integration by applying it to the area A of the integral $A = \int_0^1 \frac{4}{1+x^2}\mathrm{d}x$. The exact value of this integral is π which is 3.14159265358979, to fourteen decimal places.

h	$T(h)$	$T_1(h)$	$T_2(h)$	$T_3(h)$
1/4	3.13118	3.14159250246	3.14159266114	3.14159265359003
1/8	3.13899	3.141592651225	3.141592653708	
1/16	3.14094	3.141592653553		
1/32	3.14143			

This computation required the evaluation of $f(x) = \frac{4}{1+x^2}$ only for $x = \frac{n}{32}$ with $0 \le n \le 32$ — that is, a total of 33 evaluations of f. Those 33 evaluations gave us 12 correct decimal places. By way of comparison, $T\left(\frac{1}{32}\right)$ used the same 33 evaluations of f, but only gave us 3 correct decimal places.

a The second column, for example, of the table only reports 5 decimal places for $T(h)$. But many more decimal places of $T(h)$ were used in the computations of $T_1(h)$ etc.

As we have seen, Richardson extrapolation can be used to choose the step size so as to achieve some desired degree of accuracy. We are next going to consider a family of algorithms that extend this idea to use small step sizes in the part of the domain of integration where it is hard to get good accuracy and large step sizes in the part of the domain of integration where it is easy to get good accuracy. We will illustrate the ideas by applying them to the integral $\int_0^1 \sqrt{x}\ \mathrm{d}x$. The integrand $\sqrt{x}$ changes very quickly when x is small and changes slowly when x is large. So we will make the step size small near $x = 0$ and make the step size large near $x = 1$.



C.3▲ Adaptive Quadrature

Richardson extrapolation is also used to choose the step size so as to achieve some desired degree of accuracy. "Adaptive quadrature" refers to a family of algorithms that use small step sizes in the part of the domain of integration where it is hard to get good accuracy and large step sizes in the part of the domain of integration where it is easy to get good accuracy.

We'll illustrate the idea using Simpson's rule applied to the integral $\int_a^b f(x)\,\mathrm{d}x$, and assuming that we want the error to be no more than (approximately) some fixed constant ε. For example, ε could be 10^{-6}. Denote by $S(a', b'\,;\, h')$, the answer given when Simpson's rule is applied to the integral $\int_{a'}^{b'} f(x)\,\mathrm{d}x$ with step size h'.

- *Step 1.* We start by applying Simpson's rule, combined with Richardson extrapolation so as to get an error estimate, with the largest possible step size h. Namely, set $h = \frac{b-a}{2}$ and compute

$$f(a) \quad f\big(a+\tfrac{h}{2}\big) \quad f(a+h) = f\big(\tfrac{a+b}{2}\big) \quad f\big(a+\tfrac{3h}{2}\big) \quad f(a+2h) = f(b)$$

Then

$$S\big(a, b\,;h\big) = \tfrac{h}{3}\big\{f(a) + 4f\big(a+h\big) + f(b)\big\}$$

and

$$\begin{aligned} S\big(a, b\,;\tfrac{h}{2}\big) &= \tfrac{h}{6}\big\{f(a) + 4f\big(a+\tfrac{h}{2}\big) + 2f\big(\tfrac{a+b}{2}\big) + 4f\big(a+\tfrac{3h}{2}\big) + f(b)\big\} \\ &= S\big(a, \tfrac{a+b}{2}\,;\tfrac{h}{2}\big) + S\big(\tfrac{a+b}{2}, b\,;\tfrac{h}{2}\big) \end{aligned}$$

with

$$\begin{aligned} S\big(a, \tfrac{a+b}{2}\,;\,\tfrac{h}{2}\big) &= \tfrac{h}{6}\big\{f(a) + 4f\big(a+\tfrac{h}{2}\big) + f\big(\tfrac{a+b}{2}\big)\big\} \\ S\big(\tfrac{a+b}{2}, b;\,\tfrac{h}{2}\big) &= \tfrac{h}{6}\big\{f\big(\tfrac{a+b}{2}\big) + 4f\big(a+\tfrac{3h}{2}\big) + f(b)\big\} \end{aligned}$$

Using the Richardson extrapolation formula (E4a) with $k = 4$ gives that the error in $S\big(a, b\,;\, \frac{h}{2}\big)$ is (approximately)

$$\big|K\big(\tfrac{h}{2}\big)^4\big| = \tfrac{1}{15}\Big|S\big(a,b\,;\,\tfrac{h}{2}\big) - S\big(a,b\,;\,h\big)\Big| \tag{E7}$$

If this is smaller than ε, we have (approximately) the desired accuracy and stop[1].

- *Step 2.* If (E7) is larger than ε, we divide the original integral $I = \int_a^b f(x)\,\mathrm{d}x$ into two "half-sized" integrals, $I_1 = \int_a^{\frac{a+b}{2}} f(x)\,\mathrm{d}x$ and $I_2 = \int_{\frac{a+b}{2}}^b f(x)\,\mathrm{d}x$ and repeat the procedure of Step 1 on each of them, but with h replaced by $\frac{h}{2}$ and ε replaced by $\frac{\varepsilon}{2}$ — if we can find an approximation, $\tilde{I}_1$, to I_1 with an error less than $\frac{\varepsilon}{2}$ and an approximation, $\tilde{I}_2$, to I_2 with an error less than $\frac{\varepsilon}{2}$, then $\tilde{I}_1 + \tilde{I}_2$ approximates I with an error less than ε. Here is more detail.
    - If the error in the approximation $\tilde{I}_1$ to I_1 and the error in the approximation $\tilde{I}_2$ to I_2 are both acceptable, then we use $\tilde{I}_1$ as our final approximation to I_1 and we use $\tilde{I}_2$ as our final approximation to I_2.
    - If the error in the approximation $\tilde{I}_1$ to I_1 is acceptable but the error in the approximation $\tilde{I}_2$ to I_2 is not acceptable, then we use $\tilde{I}_1$ as our final approximation to I_1 but we subdivide the integral I_2.
    - If the error in the approximation $\tilde{I}_1$ to I_1 is not acceptable but the error in the approximation $\tilde{I}_2$ to I_2 is acceptable, then we use $\tilde{I}_2$ as our final approximation to I_2 but we subdivide the integral I_1.
    - If the error in the approximation $\tilde{I}_1$ to I_1 and the error in the approximation $\tilde{I}_2$ to I_2 are both not acceptable, then we subdivide both of the integrals I_1 and I_2.

  So we *adapt* the step size as we go.

- *Steps 3, 4, 5,* $\cdots$ Repeat as required.

Example C.3.1 Adaptive quadrature.

Let's apply adaptive quadrature using Simpson's rule as above with the goal of computing $\int_0^1 \sqrt{x}\,\mathrm{d}x$ with an error of at most $\varepsilon = 0.0005 = 5\times 10^{-4}$. Observe that $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x} = \frac{1}{2\sqrt{x}}$ blows up as x tends to zero. The integrand changes very quickly when x is small. So we will probably need to make the step size small near the limit of integration $x = 0$.

- *Step 1 — the interval* $[0, 1]$. (The notation $[0, 1]$ stands for the interval $0 \le x \le 1$.)

$$S(0, 1\,;\,\tfrac{1}{2}) = 0.63807119$$
$$S(0, \tfrac{1}{2}\,;\,\tfrac{1}{4}) = 0.22559223$$

1 It is very common to build in a bit of a safety margin and require that, for example, $\big|K\big(\frac{h}{2}\big)^4\big|$ be smaller than $\frac{\varepsilon}{2}$ rather than ε.

$$\begin{aligned} S(\tfrac{1}{2},1\,;\tfrac{1}{4}) &= 0.43093403 \\ \text{error} &= \tfrac{1}{15}\left|S(0,\tfrac{1}{2}\,;\tfrac{1}{4}) + S(\tfrac{1}{2},1\,;\tfrac{1}{4}) - S(0,1\,;\tfrac{1}{2})\right| = 0.0012 \\ &> \varepsilon = 0.0005 \end{aligned}$$

This is unacceptably large, so we subdivide the interval $[0, 1]$ into the two halves $\big[0, \frac{1}{2}\big]$ and $\big[\frac{1}{2}, 1\big]$ and apply the procedure separately to each half.

- *Step 2a — the interval* $[0, \frac{1}{2}]$.

$$\begin{aligned} S(0,\tfrac{1}{2}\,;\tfrac{1}{4}) &= 0.22559223 \\ S(0,\tfrac{1}{4}\,;\tfrac{1}{8}) &= 0.07975890 \\ S(\tfrac{1}{4},\tfrac{1}{2}\,;\tfrac{1}{8}) &= 0.15235819 \\ \text{error} &= \tfrac{1}{15}\left|S(0,\tfrac{1}{4}\,;\tfrac{1}{8}) + S(\tfrac{1}{4},\tfrac{1}{2}\,;\tfrac{1}{8}) - S(0,\tfrac{1}{2}\,;\tfrac{1}{4})\right| = 0.00043 \\ &> \tfrac{\varepsilon}{2} = 0.00025 \end{aligned}$$

This error is unacceptably large.

- *Step 2b — the interval* $[\frac{1}{2}, 1]$.

$$\begin{aligned} S(\tfrac{1}{2},1\,;\tfrac{1}{4}) &= 0.43093403 \\ S(\tfrac{1}{2},\tfrac{3}{4}\,;\tfrac{1}{8}) &= 0.19730874 \\ S(\tfrac{3}{4},1\,;\tfrac{1}{8}) &= 0.23365345 \\ \text{error} &= \tfrac{1}{15}\left|S(\tfrac{1}{2},\tfrac{3}{4}\,;\tfrac{1}{8}) + S(\tfrac{3}{4},1\,;\tfrac{1}{8}) - S(\tfrac{1}{2},1\,;\tfrac{1}{4})\right| = 0.0000019 \\ &< \tfrac{\varepsilon}{2} = 0.00025 \end{aligned}$$

This error is acceptable.

- *Step 2 resumé.* The error for the interval $[\frac{1}{2}, 1]$ is small enough, so we accept

$$S(\tfrac{1}{2},1\,;\tfrac{1}{8}) = S(\tfrac{1}{2},\tfrac{3}{4}\,;\tfrac{1}{8}) + S(\tfrac{3}{4},1\,;\tfrac{1}{8}) = 0.43096219$$

as the approximate value of $\int_{1/2}^{1} \sqrt{x}\,\mathrm{d}x$.

The error for the interval $[0, \frac{1}{2}]$ is unacceptably large, so we subdivide the interval $[0, \frac{1}{2}]$ into the two halves $[0, \frac{1}{4}]$ and $[\frac{1}{4}, \frac{1}{2}]$ and apply the procedure separately to each half.

- *Step 3a — the interval* $[0, \frac{1}{4}]$.

$$\begin{aligned} S(0,\tfrac{1}{4}\,;\tfrac{1}{8}) &= 0.07975890 \\ S(0,\tfrac{1}{8}\,;\tfrac{1}{16}) &= 0.02819903 \\ S(\tfrac{1}{8},\tfrac{1}{4}\,;\tfrac{1}{16}) &= 0.05386675 \\ \text{error} &= \tfrac{1}{15}\left|S(0,\tfrac{1}{8}\,;\tfrac{1}{16}) + S(\tfrac{1}{8},\tfrac{1}{4}\,;\tfrac{1}{16}) - S(0,\tfrac{1}{4}\,;\tfrac{1}{8})\right| \\ &= 0.000153792 > \tfrac{\varepsilon}{4} = 0.000125 \end{aligned}$$

This error is unacceptably large.

- *Step 3b — the interval* $[\frac{1}{4}, \frac{1}{2}]$.

$$\begin{aligned}
S(\tfrac{1}{4}, \tfrac{1}{2}\,;\tfrac{1}{8}) &= 0.15235819\\
S(\tfrac{1}{4}, \tfrac{3}{8}\,;\tfrac{1}{16}) &= 0.06975918\\
S(\tfrac{3}{8}, \tfrac{1}{2}\,;\tfrac{1}{16}) &= 0.08260897\\
\text{error} &= \tfrac{1}{15}\left|S(\tfrac{1}{4}, \tfrac{3}{8}\,;\tfrac{1}{16}) + S(\tfrac{3}{8}, \tfrac{1}{2}\,;\tfrac{1}{16}) - S(\tfrac{1}{4}, \tfrac{1}{2}\,;\tfrac{1}{8})\right|\\
&= 0.00000066 < \tfrac{\varepsilon}{4} = 0.000125
\end{aligned}$$

This error is acceptable.

- *Step 3 resumé.* The error for the interval $[\frac{1}{4}, \frac{1}{2}]$ is small enough, so we accept

$$S(\tfrac{1}{4}, \tfrac{1}{2}\,;\tfrac{1}{16}) = S(\tfrac{1}{4}, \tfrac{3}{8}\,;\tfrac{1}{16}) + S(\tfrac{3}{8}, \tfrac{1}{2}\,;\tfrac{1}{16}) = 0.15236814$$

as the approximate value of $\int_{1/4}^{1/2} \sqrt{x}\,\mathrm{d}x$.

The error for the interval $[0, \frac{1}{4}]$ is unacceptably large, so we subdivide the interval $[0, \frac{1}{4}]$ into the two halves $[0, \frac{1}{8}]$ and $[\frac{1}{8}, \frac{1}{4}]$ and apply the procedure separately to each half.

- *Step 4a — the interval* $[0, \frac{1}{8}]$.

$$\begin{aligned}
S(0, \tfrac{1}{8}\,;\tfrac{1}{16}) &= 0.02819903\\
S(0, \tfrac{1}{16}\,;\tfrac{1}{32}) &= 0.00996986\\
S(\tfrac{1}{16}, \tfrac{1}{8}\,;\tfrac{1}{32}) &= 0.01904477\\
\text{error} &= \tfrac{1}{15}\left|S(0, \tfrac{1}{16}\,;\tfrac{1}{32}) + S(\tfrac{1}{16}, \tfrac{1}{8}\,;\tfrac{1}{32}) - S(0, \tfrac{1}{8}\,;\tfrac{1}{16})\right|\\
&= 0.000054 < \tfrac{\varepsilon}{8} = 0.0000625
\end{aligned}$$

This error is acceptable.

- *Step 4b — the interval* $[\frac{1}{8}, \frac{1}{4}]$.

$$\begin{aligned}
S(\tfrac{1}{8}, \tfrac{1}{4}\,;\tfrac{1}{16}) &= 0.05386675\\
S(\tfrac{1}{8}, \tfrac{3}{16}\,;\tfrac{1}{32}) &= 0.02466359\\
S(\tfrac{3}{16}, \tfrac{1}{4}\,;\tfrac{1}{32}) &= 0.02920668\\
\text{error} &= \tfrac{1}{15}\left|S(\tfrac{1}{8}, \tfrac{3}{16}\,;\tfrac{1}{32}) + S(\tfrac{3}{6}, \tfrac{1}{4}\,;\tfrac{1}{32}) - S(\tfrac{1}{8}, \tfrac{1}{4}\,;\tfrac{1}{16})\right|\\
&= 0.00000024 < \tfrac{\varepsilon}{8} = 0.0000625
\end{aligned}$$

This error is acceptable.

- *Step 4 resumé.* The error for the interval $[0, \frac{1}{8}]$ is small enough, so we accept

$$S(0, \tfrac{1}{8}\,;\tfrac{1}{32}) = S(0, \tfrac{1}{16}\,;\tfrac{1}{32}) + S(\tfrac{1}{16}, \tfrac{1}{8}\,;\tfrac{1}{32}) = 0.02901464$$

as the approximate value of $\int_0^{1/8} \sqrt{x}\,\mathrm{d}x$.

The error for the interval $[\frac{1}{8}, \frac{1}{4}]$ is small enough, so we accept

$$S(\tfrac{1}{8}, \tfrac{1}{4}\,; \tfrac{1}{32}) = S(\tfrac{1}{8}, \tfrac{3}{16}\,; \tfrac{1}{32}) + S(\tfrac{3}{16}, \tfrac{1}{4}\,; \tfrac{1}{32}) = 0.05387027$$

as the approximate value of $\int_{1/8}^{1/4} \sqrt{x}\,\mathrm{d}x$.

- *Conclusion.* The approximate value for $\int_0^1 \sqrt{x}\ \mathrm{d}x$ is

$$\begin{aligned} S(0, \tfrac{1}{8}\,; \tfrac{1}{32}) + S(\tfrac{1}{8}, \tfrac{1}{4}\,; \tfrac{1}{32}) + S(\tfrac{1}{4}, \tfrac{1}{2}\,; \tfrac{1}{16}) + S(\tfrac{1}{2}, 1\,; \tfrac{1}{8}) \\ = 0.66621525 \end{aligned} \tag{E8}$$

Of course the exact value of $\int_0^1 \sqrt{x}\ \mathrm{d}x = \frac{2}{3}$, so the actual error in our approximation is

$$\tfrac{2}{3} - 0.66621525 = 0.00045 < \varepsilon = 0.0005$$

Here is what Simpson's rule gives us when applied with some fixed step sizes.

$$\begin{aligned} S(0, 1\,; \tfrac{1}{8}) &= 0.66307928 \\ S(0, 1\,; \tfrac{1}{16}) &= 0.66539819 \\ S(0, 1\,; \tfrac{1}{32}) &= 0.66621818 \\ S(0, 1\,; \tfrac{1}{64}) &= 0.66650810 \end{aligned}$$

So to get an error comparable to that in (E8) from Simpson's rule with a fixed step size, we need to use $h = \frac{1}{32}$. In (E8) the step size $h = \frac{1}{32}$ was just used on the subinterval $\big[0, \frac{1}{4}\big]$.

Example C.3.1

Appendix D

Numerical Solution of ODE's

In Section 2.4 we solved a number of inital value problems of the form

$$\begin{aligned} y'(t) &= f\big(t, y(t)\big) \\ y(t_0) &= y_0 \end{aligned}$$

Here $f(t, y)$ is a given function, t_0 is a given initial time and y_0 is a given initial value for y. The unknown in the problem is the function $y(t)$. There are a number of other techniques for analytically solving some problems of this type. However it is often simply not possible to find an explicit solution. This appendix introduces some simple algorithms for generating approximate numerical solutions to such problems.

D.1 ▲ Simple ODE Solvers — Derivation

The first order of business is to derive three simple algorithms for generating approximate numerical solutions to the initial value problem

$$\begin{aligned} y'(t) &= f\big(t, y(t)\big) \\ y(t_0) &= y_0 \end{aligned}$$

The first is called Euler's method because it was developed by (surprise!) Euler[1].

1 Leonhard Euler (1707–1783) was a Swiss mathematician and physicist who spent most of his adult life in Saint Petersberg and Berlin. He gave the name π to the ratio of a circle's circumference to its diameter. He also developed the constant e.

D.1.1 ➤ Euler's Method

Our goal is to approximate (numerically) the unknown function

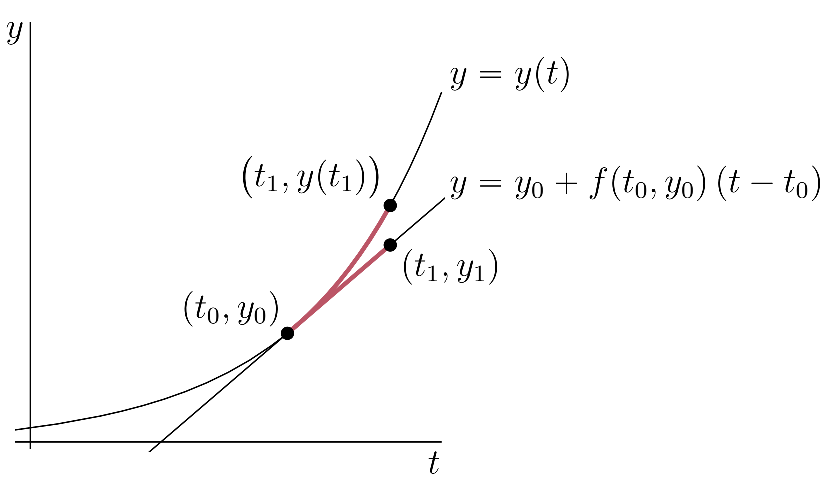
$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} y'(\tau)\,\mathrm{d}\tau \\ &= y(t_0) + \int_{t_0}^{t} f\big(\tau, y(\tau)\big)\,\mathrm{d}\tau \end{aligned}$$

for $t \geq t_0$. We are told explicitly the value of $y(t_0)$, namely y_0. So we know $f\big(\tau, y(\tau)\big)\big|_{\tau=t_0} = f\big(t_0, y_0\big)$. But we do not know the integrand $f\big(\tau, y(\tau)\big)$ for $\tau > t_0$. On the other hand, if τ is close t_0, then $f\big(\tau, y(\tau)\big)$ will remain close[2] to $f\big(t_0, y_0\big)$. So pick a small number h and define

$$\begin{aligned} t_1 &= t_0 + h \\ y_1 &= y(t_0) + \int_{t_0}^{t_1} f(t_0, y_0)\,\mathrm{d}\tau = y_0 + f\big(t_0, y_0\big)(t_1 - t_0) \\ &= y_0 + f\big(t_0, y_0\big)h \end{aligned}$$

By the above argument

$$y(t_1) \approx y_1$$



Now we start over from the new point (t_1, y_1). We now know an approximate value for y at time t_1. If $y(t_1)$ were exactly y_1, then the instantaneous rate of change of y at time t_1, namely $y'(t_1) = f\big(t_1, y(t_1)\big)$, would be exactly $f(t_1, y_1)$ and $f\big(t, y(t)\big)$ would remain close to $f(t_1, y_1)$ for t close to t_1. Defining

$$\begin{aligned} t_2 &= t_1 + h = t_0 + 2h \\ y_2 &= y_1 + \int_{t_1}^{t_2} f(t_1, y_1)\,\mathrm{d}t = y_1 + f\big(t_1, y_1\big)(t_2 - t_1) \end{aligned}$$

2 This will be the case as long as $f(t, y)$ is continuous.

$$= y_1 + f\big(t_1, y_1\big)h$$

we have

$$y(t_2) \approx y_2$$

We just repeat this argument ad infinitum. Define, for $n = 0, 1, 2, 3, \cdots$

$$t_n = t_0 + nh$$

Suppose that, for some value of n, we have already computed an approximate value y_n for $y(t_n)$. Then the rate of change of $y(t)$ for t close to t_n is $f\big(t, y(t)\big) \approx f\big(t_n, y(t_n)\big) \approx f\big(t_n, y_n\big)$ and

Equation D.1.1 Euler's Method.

$$y(t_{n+1}) \approx y_{n+1} = y_n + f\big(t_n, y_n\big)h$$

This algorithm is called *Euler's Method*. The parameter h is called the *step size*.

Here is a table applying a few steps of Euler's method to the initial value problem

$$\begin{aligned} y' &= -2t + y \\ y(0) &= 3 \end{aligned}$$

with step size $h = 0.1$. For this initial value problem

$$\begin{aligned} f(t, y) &= -2t + y \\ t_0 &= 0 \\ y_0 &= 3 \end{aligned}$$

Of course this initial value problem has been chosen for illustrative purposes only. The exact solution is[3] $y(t) = 2 + 2t + e^t$.

n	t_n	y_n	$f(t_n, y_n) = -2t_n + y_n$	$y_{n+1} = y_n + f(t_n, y_n) * h$
0	0.0	3.000	$-2 * 0.0 + 3.000 = 3.000$	$3.000 + 3.000 * 0.1 = 3.300$
1	0.1	3.300	$-2 * 0.1 + 3.300 = 3.100$	$3.300 + 3.100 * 0.1 = 3.610$
2	0.2	3.610	$-2 * 0.2 + 3.610 = 3.210$	$3.610 + 3.210 * 0.1 = 3.931$
3	0.3	3.931	$-2 * 0.3 + 3.931 = 3.331$	$3.931 + 3.331 * 0.1 = 4.264$
4	0.4	4.264	$-2 * 0.4 + 4.264 = 3.464$	$4.264 + 3.464 * 0.1 = 4.611$
5	0.5	4.611		

The exact solution at $t = 0.5$ is 4.6487, to four decimal places. We expect that Euler's method will become more accurate as the step size becomes smaller. But, of course, the amount of effort goes up as well. If we recompute using $h = 0.01$, we get (after much more work) 4.6446.

3 Even if you haven't learned how to solve initial value problems like this one, you can check that $y(t) = 2 + 2t + e^t$ obeys both $y'(t) = -2t + y(t)$ and $y(0) = 3$.

D.1.2 ▸▸ The Improved Euler's Method

Euler's method is one algorithm which generates approximate solutions to the initial value problem

$$\begin{aligned} y'(t) &= f\big(t, y(t)\big) \\ y(t_0) &= y_0 \end{aligned}$$

In applications, $f(t, y)$ is a given function and t_0 and y_0 are given numbers. The function $y(t)$ is unknown. Denote by $\varphi(t)$ the exact solution[4] for this initial value problem. In other words $\varphi(t)$ is the function that obeys

$$\begin{aligned} \varphi'(t) &= f\big(t, \varphi(t)\big) \\ \varphi(t_0) &= y_0 \end{aligned}$$

exactly.

Fix a step size h and define $t_n = t_0 + nh$. By turning the problem into one of approximating integrals, we now derive another algorithm that generates approximate values for φ at the sequence of equally spaced time values $t_0,\ t_1,\ t_2,\ \cdots$. We shall denote the approximate values y_n with

$$y_n \approx \varphi(t_n)$$

By the fundamental theorem of calculus and the differential equation, the exact solution obeys

$$\begin{aligned} \varphi(t_{n+1}) &= \varphi(t_n) + \int_{t_n}^{t_{n+1}} \varphi'(t)\ \mathrm{d}t \\ &= \varphi(t_n) + \int_{t_n}^{t_{n+1}} f\big(t, \varphi(t)\big)\ \mathrm{d}t \end{aligned}$$

Fix any n and suppose that we have already found $y_0,\ y_1,\ \cdots,\ y_n$. Our algorithm for computing y_{n+1} will be of the form

$$y_{n+1} = y_n + \text{ approximate value of } \int_{t_n}^{t_{n+1}} f\big(t, \varphi(t)\big)\ \mathrm{d}t$$

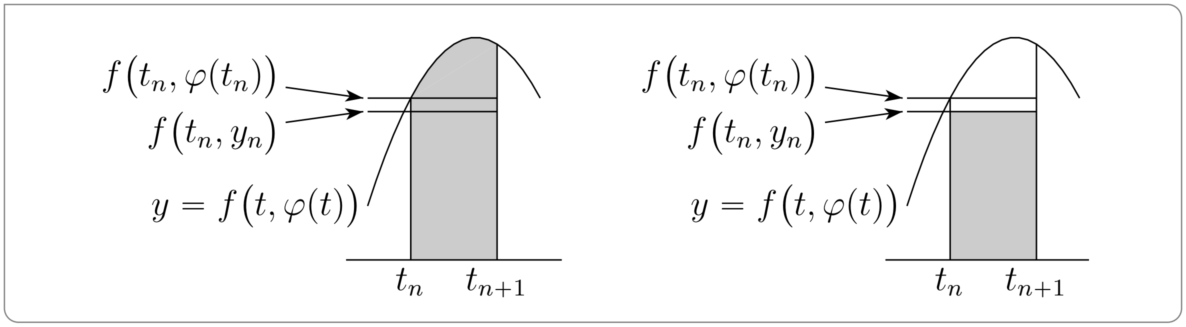
In Euler's method, we approximated $f\big(t, \varphi(t)\big)$ for $t_n \le t \le t_{n+1}$ by the constant $f\big(t_n, y_n\big)$. Thus

$$\text{Euler's approximate value for } \int_{t_n}^{t_{n+1}} f\big(t, \varphi(t)\big)\ \mathrm{d}t \text{ is}$$

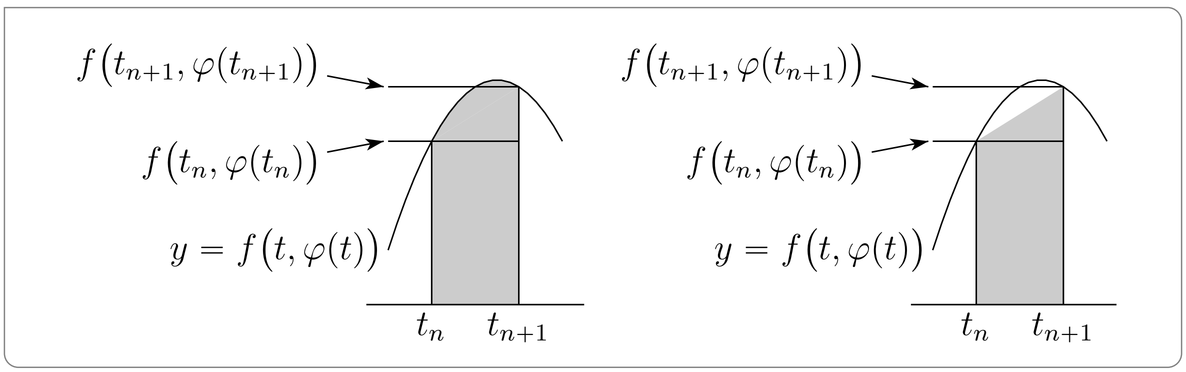
$$\int_{t_n}^{t_{n+1}} f\big(t_n, y_n\big)\ \mathrm{d}t = f\big(t_n, y_n\big)h$$

4 Under reasonable hypotheses on f, there is exactly one such solution. The interested reader should search engine their way to the Picard-Lindelöf theorem.

So Euler approximates the area of the complicated region $0 \le y \le f\big(t, \varphi(t)\big)$, $t_n \le t \le t_{n+1}$ (represented by the shaded region under the parabola in the left half of the figure below) by the area of the rectangle $0 \le y \le f\big(t_n, y_n\big)$, $t_n \le t \le t_{n+1}$ (the shaded rectangle in the right half of the figure below).



Our second algorithm, the improved Euler's method, gets a better approximation by using the trapezoidal rule. That is, we approximate the integral by the area of the trapezoid on the right below, rather than the rectangle on the right above.



The exact area of this trapezoid is the length h of the base multiplied by the average of the heights of the two sides, which is $\frac{1}{2}\big[f\big(t_n, \varphi(t_n)\big) + f\big(t_{n+1}, \varphi(t_{n+1})\big)\big]$. Of course we do not know $\varphi(t_n)$ or $\varphi(t_{n+1})$ exactly.

Recall that we have already found $y_0, \cdots, y_n$ and are in the process of finding y_{n+1}. So we already have an approximation for $\varphi(t_n)$, namely y_n. But we still need to approximate $\varphi(t_{n+1})$. We can do so by using one step of the original Euler method! That is

$$\varphi(t_{n+1}) \approx \varphi(t_n) + \varphi'(t_n)h \approx y_n + f(t_n, y_n)h$$

So our approximation of $\frac{1}{2}\big[f\big(t_n, \varphi(t_n)\big) + f\big(t_{n+1}, \varphi(t_{n+1})\big)\big]$ is

$$\frac{1}{2}\Big[f\big(t_n, y_n\big) + f\Big(t_{n+1}, y_n + f(t_n, y_n)h\Big)\Big]$$

and

$$\text{Improved Euler's approximate value for } \int_{t_n}^{t_{n+1}} f\big(t, \varphi(t)\big)\ \mathrm{d}t \text{ is}$$

$$\frac{1}{2}\Big[f\big(t_n, y_n\big) + f\Big(t_{n+1}, y_n + f(t_n, y_n)h\Big)\Big]h$$

Putting everything together[5], the improved Euler's method algorithm is

5 Notice that we have made a first approximation for $\varphi(t_{n+1})$ by using Euler's method. Then

Equation D.1.2 Improved Euler.

$$y(t_{n+1}) \approx y_{n+1} = y_n + \frac{1}{2}\Big[f\big(t_n, y_n\big) + f\Big(t_{n+1}, y_n + f(t_n, y_n)h\Big)\Big]h$$

Here are the first two steps of the improved Euler's method applied to

$$y' = -2t + y \qquad y(0) = 3$$

with $h = 0.1$. In each step we compute $f(t_n, y_n)$, followed by $y_n + f(t_n, y_n)h$, which we denote $\tilde{y}_{n+1}$, followed by $f(t_{n+1}, \tilde{y}_{n+1})$, followed by $y_{n+1} = y_n + \frac{1}{2}\big[f\big(t_n, y_n\big) + f\big(t_{n+1}, \tilde{y}_{n+1}\big)\big]h$.

$$\begin{aligned}
t_0 = 0 \qquad y_0 = 3 \quad &\Longrightarrow f(t_0, y_0) = -2 * 0 + 3 = 3\\
&\Longrightarrow \qquad \tilde{y}_1 = 3 + 3 * 0.1 = 3.3\\
&\Longrightarrow f(t_1, \tilde{y}_1) = -2 * 0.1 + 3.3 = 3.1\\
&\Longrightarrow \qquad y_1 = 3 + \frac{1}{2}[3 + 3.1] * 0.1 = 3.305\\
t_1 = 0.1 \quad y_1 = 3.305 \quad &\Longrightarrow f(t_1, y_1) = -2 * 0.1 + 3.305 = 3.105\\
&\Longrightarrow \qquad \tilde{y}_2 = 3.305 + 3.105 * 0.1 = 3.6155\\
&\Longrightarrow f(t_2, \tilde{y}_2) = -2 * 0.2 + 3.6155 = 3.2155\\
&\Longrightarrow \qquad y_2 = 3.305 + \frac{1}{2}[3.105 + 3.2155] * 0.1\\
&\Longrightarrow \qquad = 3.621025
\end{aligned}$$

Here is a table which gives the first five steps.

n	t_n	y_n	$f(t_n, y_n)$	$\tilde{y}_{n+1}$	$f(t_{n+1}, \tilde{y}_{n+1})$	y_{n+1}
0	0.0	3.000	3.000	3.300	3.100	3.305
1	0.1	3.305	3.105	3.616	3.216	3.621
2	0.2	3.621	3.221	3.943	3.343	3.949
3	0.3	3.949	3.349	4.284	3.484	4.291
4	0.4	4.291	3.491	4.640	3.640	4.647
5	0.5	4.647				

As we saw at the end of Section D.1.1, the exact $y(0.5)$ is 4.6487, to four decimal places, and Euler's method gave 4.611.

improved Euler uses the first approximation to build a better approximation for $\varphi(t_{n+1})$. Building an approximation on top of another approximation does not always work, but it works very well here.

D.1.3 ➤ The Runge-Kutta Method

The Runge-Kutta[6] algorithm is similar to the Euler and improved Euler methods in that it also uses, in the notation of the last subsection,

$$y_{n+1} = y_n + \text{ approximate value for } \int_{t_n}^{t_{n+1}} f\big(t, \varphi(t)\big)\ \mathrm{d}t$$

But rather than approximating $\int_{t_n}^{t_{n+1}} f\big(t, \varphi(t)\big)\ \mathrm{d}t$ by the area of a rectangle, as does Euler, or by the area of a trapezoid, as does improved Euler, it approximates by the area under a parabola. That is, it uses Simpson's rule. According to Simpson's rule (which is derived in Section 1.11.3)

$$\begin{aligned}\int_{t_n}^{t_n+h} & f\big(t, \varphi(t)\big)\ \mathrm{d}t \\ & \approx \tfrac{h}{6}\Big[f\big(t_n, \varphi(t_n)\big) + 4f\big(t_n + \tfrac{h}{2}, \varphi(t_n + \tfrac{h}{2})\big) + f\big(t_n + h, \varphi(t_n + h)\big)\Big]\end{aligned}$$

Analogously to what happened in our development of the improved Euler method, we don't know $\varphi(t_n)$, $\varphi(t_n + \frac{h}{2})$ or $\varphi(t_n + h)$. So we have to approximate them as well. The Runge-Kutta algorithm, incorporating all these approximations, is[7]

Equation D.1.3 Runge-Kutta.

$$\begin{aligned} k_{1,n} &= f(t_n, y_n) \\ k_{2,n} &= f(t_n + \tfrac{1}{2}h, y_n + \tfrac{h}{2}k_{1,n}) \\ k_{3,n} &= f(t_n + \tfrac{1}{2}h, y_n + \tfrac{h}{2}k_{2,n}) \\ k_{4,n} &= f(t_n + h, y_n + hk_{3,n}) \\ y_{n+1} &= y_n + \tfrac{h}{6}\left[k_{1,n} + 2k_{2,n} + 2k_{3,n} + k_{4,n}\right] \end{aligned}$$

That is, Runge-Kutta uses

- $k_{1,n}$ to approximate $f\big(t_n, \varphi(t_n)\big) = \varphi'(t_n)$,
- both $k_{2,n}$ and $k_{3,n}$ to approximate $f\big(t_n + \frac{h}{2}, \varphi(t_n + \frac{h}{2})\big) = \varphi'(t_n + \frac{h}{2})$, and
- $k_{4,n}$ to approximate $f\big(t_n + h, \varphi(t_n + h)\big)$.

Here are the first two steps of the Runge-Kutta algorithm applied to

$$y' = -2t + y \qquad y(0) = 3$$

6 Carl David Tolmé Runge (1856–1927) and Martin Wilhelm Kutta (1867–1944) were German mathematicians.

7 It is well beyond our scope to derive this algorithm, though the derivation is similar in flavour to that of the improved Euler method. You can find more in, for example, Wikipedia.

with $h = 0.1$.

$$
\begin{aligned}
&t_0 = 0 \quad y_0 = 3 \\
&\quad \Longrightarrow k_{1,0} = f(0,3) = -2*0+3 = 3 \\
&\quad \Longrightarrow \quad y_0 + \tfrac{h}{2}k_{1,0} = 3 + 0.05*3 = 3.15 \\
&\quad \Longrightarrow k_{2,0} = f(0.05, 3.15) = -2*0.05+3.15 = 3.05 \\
&\quad \Longrightarrow \quad y_0 + \tfrac{h}{2}k_{2,0} = 3 + 0.05*3.05 = 3.1525 \\
&\quad \Longrightarrow k_{3,0} = f(0.05, 3.1525) = -2*0.05+3.1525 = 3.0525 \\
&\quad \Longrightarrow \quad y_0 + hk_{3,0} = 3 + 0.1*3.0525 = 3.30525 \\
&\quad \Longrightarrow k_{4,0} = f(0.1, 3.30525) = -2*0.1+3.30525 = 3.10525 \\
&\quad \Longrightarrow \quad y_1 = 3 + \tfrac{0.1}{6}[3 + 2*3.05 + 2*3.0525 + 3.10525] = 3.3051708 \\
&t_1 = 0.1 \;\; y_1 = 3.3051708 \\
&\quad \Longrightarrow k_{1,1} = f(0.1, 3.3051708) = -2*0.1+3.3051708 = 3.1051708 \\
&\quad \Longrightarrow \quad y_1 + \tfrac{h}{2}k_{1,1} = 3.3051708 + 0.05*3.1051708 = 3.4604293 \\
&\quad \Longrightarrow k_{2,1} = f(0.15, 3.4604293) = -2*0.15+3.4604293 = 3.1604293 \\
&\quad \Longrightarrow \quad y_1 + \tfrac{h}{2}k_{2,1} = 3.3051708 + 0.05*3.1604293 = 3.4631923 \\
&\quad \Longrightarrow k_{3,1} = f(0.15, 3.4631923) = -2*0.15+3.4631923 = 3.1631923 \\
&\quad \Longrightarrow \quad y_1 + hk_{3,1} = 3.3051708 + 0.1*3.4631923 = 3.62149 \\
&\quad \Longrightarrow k_{4,1} = f(0.2, 3.62149) = -2*0.2+3.62149 = 3.22149 \\
&\quad \Longrightarrow \quad y_2 = 3.3051708 + \tfrac{0.1}{6}[3.1051708 + 2*3.1604293+ \\
&\qquad\qquad\qquad + 2*3.1631923 + 3.22149] = 3.6214025 \\
&t_2 = 0.2 \;\; y_2 = 3.6214025
\end{aligned}
$$

Now, while this might look intimidating written out in full like this, one should keep in mind that it is quite easy to write a program to do this. Here is a table giving the first five steps. The data is only given to three decimal places even though the computation has been done to many more.

n	t_n	y_n	$k_{1,n}$	$y_{n,1}$	$k_{2,n}$	$y_{n,2}$	$k_{3,n}$	$y_{n,3}$	$k_{4,n}$	y_{n+1}
0	0.0	3.000	3.000	3.150	3.050	3.153	3.053	3.305	3.105	3.305
1	0.1	3.305	3.105	3.460	3.160	3.463	3.163	3.621	3.221	3.621
2	0.2	3.621	3.221	3.782	3.282	3.786	3.286	3.950	3.350	3.949
3	0.3	3.950	3.350	4.117	3.417	4.121	3.421	4.292	3.492	4.291
4	0.4	4.292	3.492	4.466	3.566	4.470	3.570	4.649	3.649	4.648
5	0.5	4.6487206								

As we saw at the end of Section D.1.2, the exact $y(0.5)$ is 4.6487213, to seven decimal places, Euler's method gave 4.611, and improved Euler gave 4.647.

So far we have, hopefully, motivated the Euler, improved Euler and Runge-Kutta algorithms. We have not attempted to see how efficient and how accurate the algorithms are. A first look at those questions is provided in the next section.

D.2▲ Simple ODE Solvers — Error Behaviour

We now provide an introduction to the error behaviour of Euler's Method, the improved Euler's method and the Runge-Kutta algorithm for generating approximate solutions to the initial value problem

$$\begin{aligned} y'(t) &= f\big(t, y(t)\big) \\ y(t_0) &= y_0 \end{aligned}$$

Here $f(t, y)$ is a given known function, t_0 is a given initial time and y_0 is a given initial value for y. The unknown in the problem is the function $y(t)$.

Two obvious considerations in deciding whether or not a given algorithm is of any practical value are

(a) the amount of computational effort required to execute the algorithm and

(b) the accuracy that this computational effort yields.

For algorithms like our simple ODE solvers, the bulk of the computational effort usually goes into evaluating the function[1] $f(t, y)$. Euler's method uses one evaluation of $f(t, y)$ for each step; the improved Euler's method uses two evaluations of f per step; the Runge-Kutta algorithm uses four evaluations of f per step. So Runge-Kutta costs four times as much work per step as does Euler. But this fact is extremely deceptive because, as we shall see, you typically get the same accuracy with a few steps of Runge-Kutta as you do with hundreds of steps of Euler.

To get a first impression of the error behaviour of these methods, we apply them to a problem that we know the answer to. The solution to the first order constant coefficient linear initial value problem

$$\begin{aligned} y'(t) &= y - 2t \\ y(0) &= 3 \end{aligned}$$

is

$$y(t) = 2 + 2t + e^t$$

In particular, the exact value of $y(1)$, to ten decimal places, is $4 + e = 6.7182818285$. The following table lists the error in the approximate value for this number generated by our three methods applied with three different step sizes. It also lists the number of evaluations of f required to compute the approximation.

1 Typically, evaluating a complicated function will take a great many arithmetic operations, while the actual ODE solver method (as per, for example, (D.1.3)) takes only an additional handful of operations. So the great bulk of computational time goes into evaulating f and we want to do it as few times as possible.

	Euler		Improved Euler		Runge Kutta	
steps	error	#evals	error	#evals	error	#evals
5	2.3×10^{-1}	5	1.6×10^{-2}	10	3.1×10^{-5}	20
50	2.7×10^{-2}	50	1.8×10^{-4}	100	3.6×10^{-9}	200
500	2.7×10^{-3}	500	1.8×10^{-6}	1000	3.6×10^{-13}	2000

Observe

- Using 20 evaluations of f worth of Runge-Kutta gives an error 90 times smaller than 500 evaluations of f worth of Euler.
- With Euler's method, decreasing the step size by a factor of ten appears to reduce the error by about a factor of ten.
- With improved Euler, decreasing the step size by a factor of ten appears to reduce the error by about a factor of one hundred.
- With Runge-Kutta, decreasing the step size by a factor of ten appears to reduce the error by about a factor of about 10^4.

Use $A_E(h)$, $A_{IE}(h)$ and $A_{RK}(h)$ to denote the approximate value of $y(1)$ given by Euler, improved Euler and Runge-Kutta, respectively, with step size h. It looks like

Equation D.2.1

$$\begin{aligned} A_E(h) &\approx y(1) + K_E h \\ A_{IE}(h) &\approx y(1) + K_{IE} h^2 \\ A_{RK}(h) &\approx y(1) + K_{RK} h^4 \end{aligned}$$

with some constants K_E, K_{IE} and K_{RK}.

To test these conjectures further, we apply our three methods with about ten different step sizes of the form $\frac{1}{n} = \frac{1}{2^m}$ with m integer. Below are three graphs, one for each method. Each contains a plot of $Y = \log_2 e_n$, the (base 2) logarithm of the error for step size $\frac{1}{n}$, against the logarithm (of base 2) of n. The logarithm of base 2 is used because $\log_2 n = \log_2 2^m = m$ — nice and simple.

Here is why it is a good reason to plot $Y = \log_2 e_n$ against $x = \log_2 n$. If, for some algorithm, there are (unknown) constants K and k such that

$$\text{approx value of } y(1) \text{ with step size } h = y(1) + Kh^k$$

then the error with step size $\frac{1}{n}$ is $e_n = K\frac{1}{n^k}$ and obeys

$$\log_2 e_n = \log_2 K - k \log_2 n \tag{E1}$$

The graph of $Y = \log_2 e_n$ against $x = \log_2 n$ is the straight line $Y = -kx + \log_2 K$ of slope $-k$ and y intercept $\log_2 K$.

Remark D.2.2 This procedure can still be used even if we do not know the exact value of $y(1)$. Suppose, more generally, that we have some algorithm that generates approximate values for some (unknown) exact value A. Call A_h the approximate value with step size h. Suppose that

$$A_h = A + Kh^k$$

with K and k constant (but also unknown). Then plotting

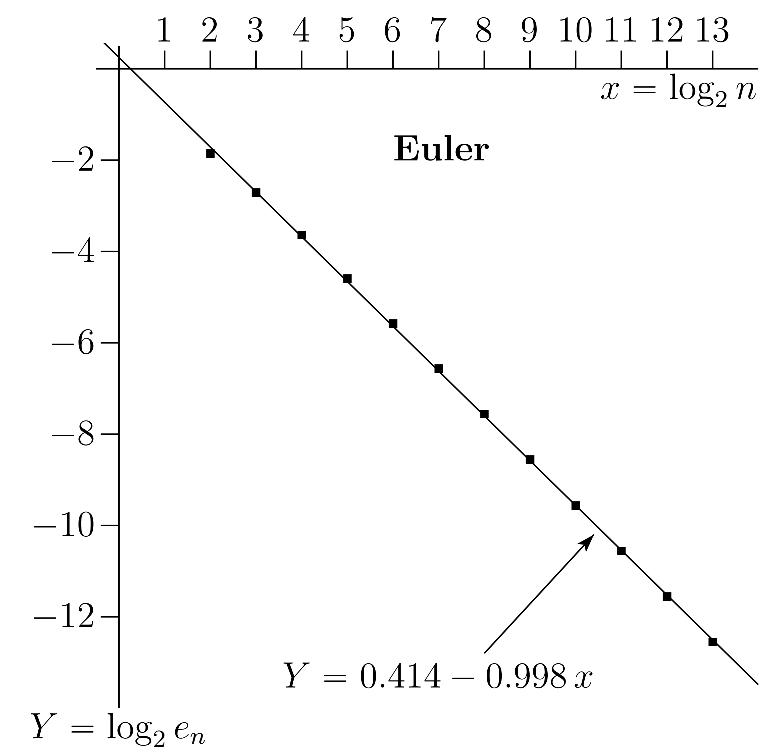
$$y = \log(A_h - A_{h/2}) = \log\left(Kh^k - K\left(\tfrac{h}{2}\right)^k\right) = \log\left(K - \tfrac{K}{2^k}\right) + k\log h$$

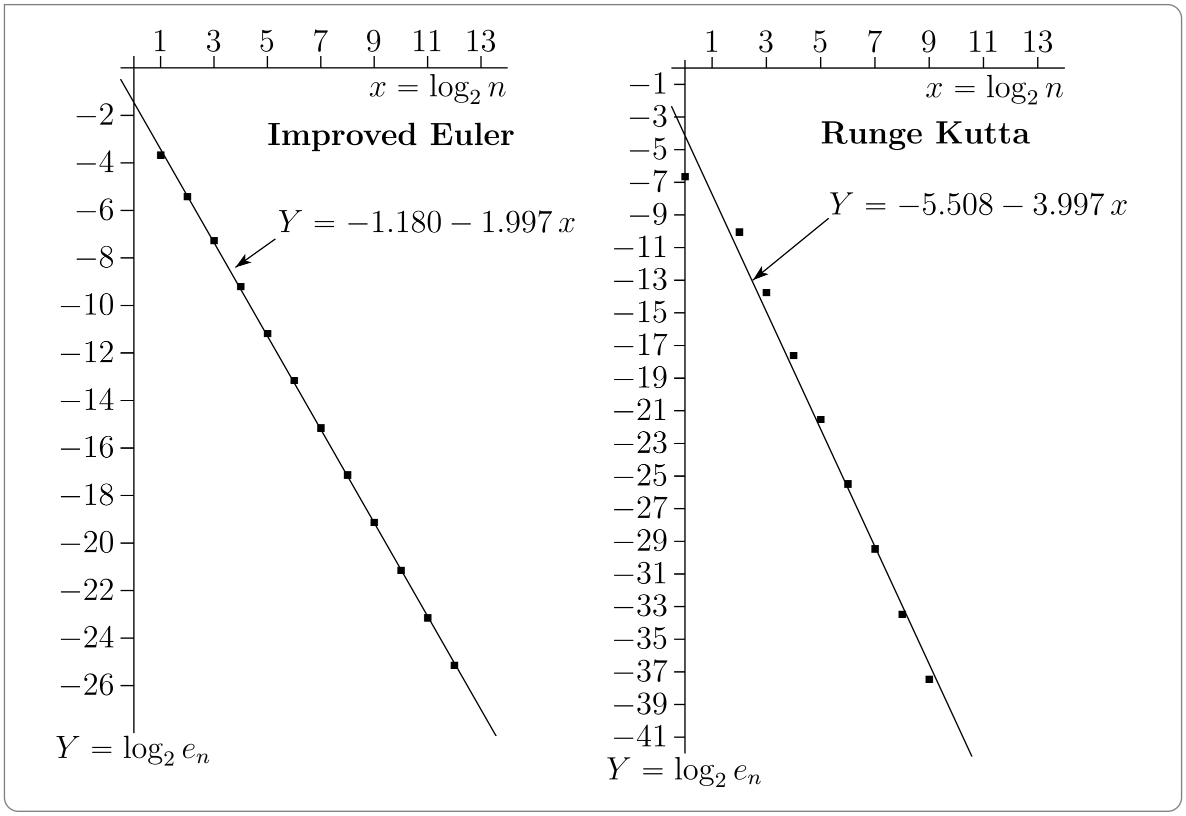
against $x = \log h$ gives the straight line $y = mx + b$ with slope $m = k$ and y intercept $b = \log\left(K - \frac{K}{2^k}\right)$. So we can

- read off k from the slope of the line and then
- compute $K = e^b\left(1 - \frac{1}{2^k}\right)^{-1}$ from the y intercept b and then
- compute[a] $A = A_h - Kh^k$.

a This is the type of strategy used by the Richardson extrapolation of Section C.1.

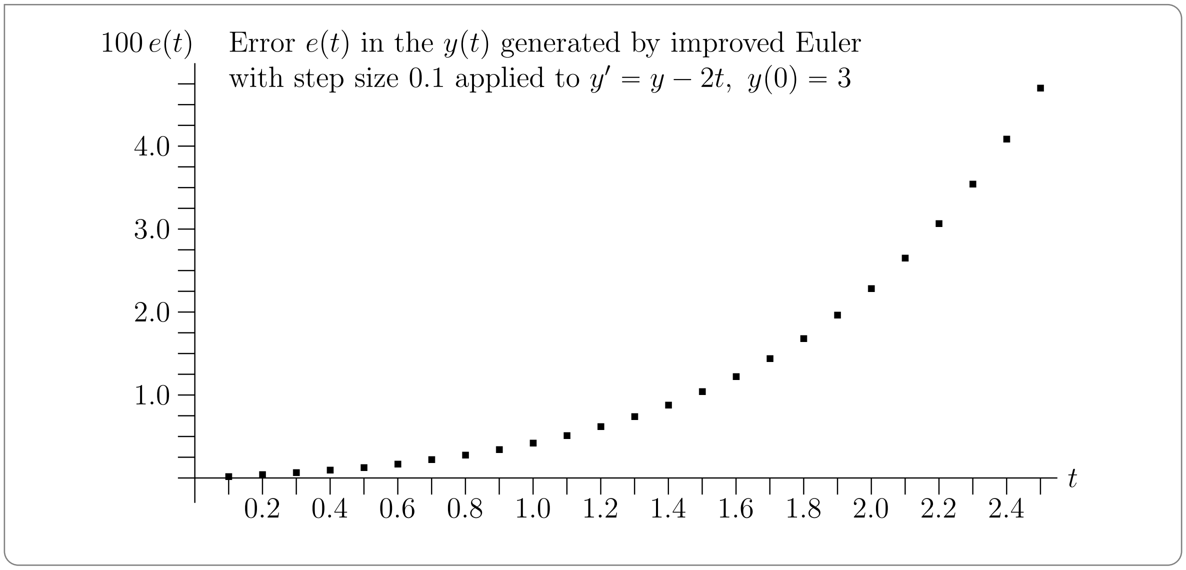
Here are the three graphs — one each for the Euler method, the improved Euler method and the Runge-Kutta method.



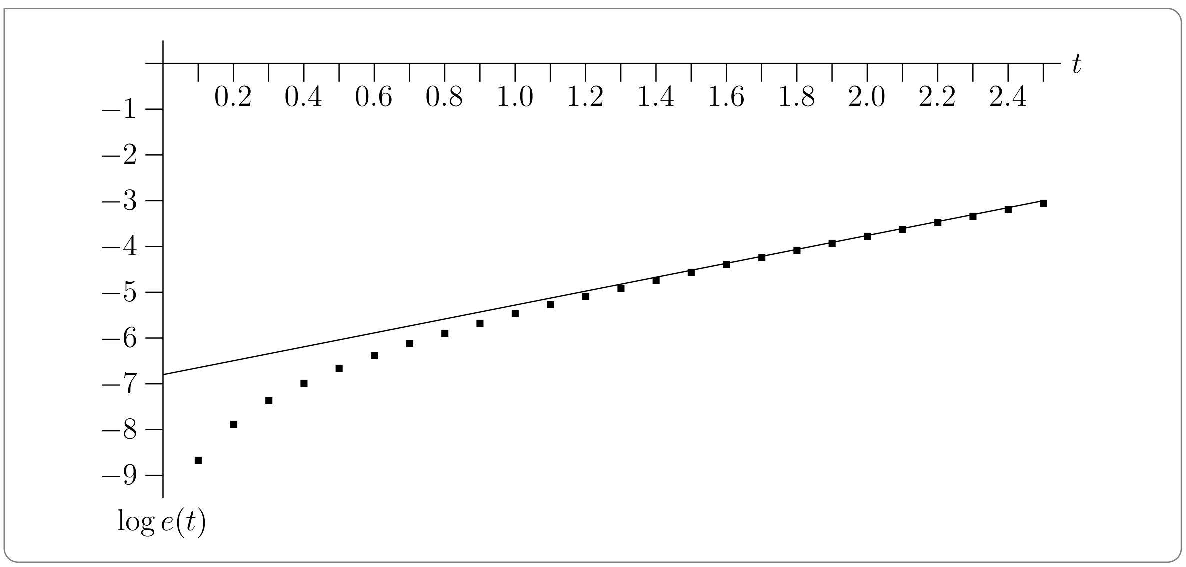


Each graph contains about a dozen data points, $(x, Y) = (\log_2 n, \log_2 e_n)$, and also contains a straight line, chosen by linear regression, to best fit the data. The method of linear regression for finding the straight line which best fits a given set of data points is covered in Example 2.9.11 of the CLP-3 text. The three straight lines have slopes -0.998 for Euler, -1.997 for improved Euler and -3.997 for Runge Kutta. Reviewing (E1), it sure looks like $k = 1$ for Euler, $k = 2$ for improved Euler and $k = 4$ for Runge-Kutta (at least if k is integer).

So far we have only looked at the error in the approximate value of $y(t_f)$ as a function of the step size h with t_f held fixed. The graph below illustrates how the error behaves as a function of t, with h held fixed. That is, we hold the step size fixed and look at the error as a function of the distance, t, from the initial point.



From the graph, it appears that the error grows exponentially with t. But it is not so easy to visually distinguish exponential curves from other upward curving curves. On the other hand, it is pretty easy to visually distinguish straight lines from other curves, and taking a logarithm converts the exponential curve $y = e^{kx}$ into the straight line $Y = \log y = k\,x$. Here is a graph of the logarithm, $\log e(t)$, of the error at time t, $e(t)$, against t. We have added a straight line as an aide to your eye.



It looks like the log of the error grows very quickly initially, but then settles into a straight line. Hence it really does look like, at least in this example, except at the very beginning, the error $e(t)$ grows exponentially with t.

The above numerical experiments have given a little intuition about the error behaviour of the Euler, improved Euler and Runge-Kutta methods. It's time to try and understand what is going on more rigorously.

D.2.1 ▸▸ Local Truncation Error for Euler's Method

We now try to develop some understanding as to why we got the above experimental results. We start with the error generated by a single step of Euler's method.

Definition D.2.3 Local truncation error.

The (signed) error generated by a single step of Euler's method, under the assumptions that we start the step with the exact solution and that there is no roundoff error, is called the *local truncation error* for Euler's method. That is, if $\phi(t)$ obeys $\phi'(t) = f\big(t, \phi(t)\big)$ and $\phi(t_n) = y_n$, and if $y_{n+1} = y_n + hf(t_n, y_n)$, then the local truncation error for Euler's method is

$$\phi\big(t_{n+1}\big) - y_{n+1}$$

That is, it is difference between the exact value, $\phi\big(t_{n+1}\big)$, and the approximate value generated by a single Euler method step, y_{n+1}, ignoring any numerical issues caused by storing numbers in a computer.

Denote by $\phi(t)$ the exact solution to the initial value problem

$$y'(t) = f(t, y) \qquad y(t_n) = y_n$$

That is, $\phi(t)$ obeys

$$\phi'(t) = f\big(t, \phi(t)\big) \qquad \phi(t_n) = y_n$$

for all t. Now execute one more step of Euler's method with step size h:

$$y_{n+1} = y_n + hf\big(t_n, y_n\big)$$

Because we are assuming that $y_n = \phi(t_n)$

$$y_{n+1} = \phi(t_n) + hf\big(t_n, \phi(t_n)\big)$$

Because $\phi(t)$ is the exact solution, $\phi'(t_n) = f\big(t_n, \phi(t_n)\big) = f(t_n, y_n)$ and

$$y_{n+1} = \phi(t_n) + h\phi'(t_n)$$

The local truncation error in y_{n+1} is, by definition, $\phi(t_{n+1}) - y_{n+1}$.

Taylor expanding (see (3.4.10) in the CLP-1 text) $\phi(t_{n+1}) = \phi(t_n + h)$ about t_n

$$\phi(t_n + h) = \phi(t_n) + \phi'(t_n)h + \tfrac{1}{2}\phi''(t_n)h^2 + \tfrac{1}{3!}\phi'''(t_n)h^3 + \cdots$$

so that

$$\begin{aligned}
&\phi(t_{n+1}) - y_{n+1} \\
&= \big[\phi(t_n) + \phi'(t_n)h + \tfrac{1}{2}\phi''(t_n)h^2 + \tfrac{1}{3!}\phi'''(t_n)h^3 + \cdots\big] - \big[\phi(t_n) + h\phi'(t_n)\big] \\
&= \tfrac{1}{2}\phi''(t_n)h^2 + \tfrac{1}{3!}\phi'''(t_n)h^3 + \cdots
\end{aligned} \tag{E2}$$

Notice that the constant and h^1 terms have cancelled out. So the first term that appears is proportional to h^2. Since h is typically a very small number, the h^3, h^4, $\cdots$ terms will usually be much smaller than the h^2 term.

We conclude that the local truncation error for Euler's method is h^2 times some unknown constant (we usually don't know the value of $\frac{1}{2}\phi''(t_n)$ because we don't usually know the solution $\phi(t)$ of the differential equation) plus smaller terms that are proportional to h^r with $r \geq 3$. This conclusion is typically written

Equation D.2.4

$$\text{Local truncation error for Euler's method } = Kh^2 + O(h^3)$$

The symbol $O(h^3)$ is used to designate any function that, for small h, is bounded by a constant times h^3. So, if h is very small, $O(h^3)$ will be a lot smaller than h^2.

To get from an initial time $t = t_0$ to a final time $t = t_f$ using steps of size h requires $(t_f - t_0)/h$ steps. If each step were to introduce an error[2] $Kh^2 + O(h^3)$, then the final error in the approximate value of $y(t_f)$ would be

$$\frac{t_f - t_0}{h}\,\Big[Kh^2 + O(h^3)\Big] = K(t_f - t_0)\,h + O(h^2)$$

This very rough estimate is consistent with the experimental data for the dependence of error on step size with t_f held fixed, shown on the first graph after Remark D.2.2. But it is not consistent with the experimental time dependence data above, which shows the error growing exponentially, rather than linearly, in $t_f - t_0$.

We can get some rough understanding of this exponential growth as follows. The general solution to $y' = y - 2t$ is $y = 2 + 2t + c_0e^t$. The arbitrary constant, c_0, is to be determined by initial conditions. When $y(0) = 3$, $c_0 = 1$. At the end of step 1, we have computed an approximation y_1 to $y(h)$. This y_1 is not exactly $y(h) = 2 + 2h + e^h$. Instead, it is a number that differs from $2 + 2h + e^h$ by $O(h^2)$. We choose to write the number $y_1 = 2 + 2h + e^h + O(h^2)$ as $2 + 2h + (1 + \epsilon)e^h$ with $\epsilon = e^{-h}O(h^2)$ of order of magnitude h^2. That is, we choose to write

$$y_1 = 2 + 2t + c_0e^t\Big|_{t=h} \qquad \text{with } c_0 = 1 + \epsilon$$

If we were to make no further errors we would end up with the solution to

$$y' = y - 2t, \qquad y(h) = 2 + 2h + (1 + \epsilon)e^h$$

which is[3]

$$y(t) = 2 + 2t + (1 + \epsilon)e^t = 2 + 2t + e^t + \epsilon e^t$$

2 For simplicity, we are assuming that K takes the same value in every step. If, instead, there is a different K in each of the $n = (t_f - t_0)/h$ steps, the final error would be $K_1h^2+K_2h^2+\cdots+K_nh^2+nO(h^3) = \bar{K}nh^2+nO(h^3) = \bar{K}(t_f - t_0)\,h+O(h^2)$, where $\bar{K}$ is the average of K_1, K_2, $\cdots$, K_n.

3 Note that this $y(t)$ obeys both the differential equation $y' = y - 2t$ and the initial condition $y(h) = 2 + 2h + (1 + \epsilon)e^h$.

$$= \phi(t) + \epsilon e^t$$

So, once as error has been introduced, the natural time evolution of the solutions to this differential equation cause the error to grow exponentially. Other differential equations with other time evolution characteristics will exhibit different t_f dependence of errors[4]. In the next section, we show that, for many differential equations, errors grow at worst exponentially with t_f.

D.2.2 ➤ Global Truncation Error for Euler's Method

Suppose once again that we are applying Euler's method with step size h to the initial value problem

$$y'(t) = f(t, y)$$
$$y(0) = y_0$$

Denote by $\phi(t)$ the exact solution to the initial value problem and by y_n the approximation to $\phi(t_n)$, $t_n = t_0 + nh$, given by n steps of Euler's method (applied without roundoff error).

Definition D.2.5 Global truncation error.

The (signed) error in y_n is $\phi(t_n) - y_n$ and is called the *global truncation error* at time t_n.

The word "truncation" is supposed to signify that this error is due solely to Euler's method and does not include any effects of roundoff error that might be introduced by our not writing down an infinite number of decimal digits for each number that we compute along the way. We now derive a bound on the global truncation error.

Define

$$\varepsilon_n = \phi(t_n) - y_n$$

The first half of the derivation is to find a bound on ε_{n+1} in terms of ε_n.

$$\begin{aligned}
\varepsilon_{n+1} &= \phi(t_{n+1}) - y_{n+1} \\
&= \phi(t_{n+1}) - y_n - hf(t_n, y_n) \\
&= [\phi(t_n) - y_n] + h[f(t_n, \phi(t_n)) - f(t_n, y_n)] \\
&\qquad\qquad + [\phi(t_{n+1}) - \phi(t_n) - hf(t_n, \phi(t_n))]
\end{aligned} \tag{E3}$$

where we have massaged the expression into three manageable pieces.

- The first $[\cdots]$ is exactly ε_n.

4 For example, if the solution is polynomial, then we might expect (by a similar argument) that the error also grows polynomially in t_f

- The third $[\cdots]$ is exactly the local truncation error. Assuming that $|\phi''(t)| \le A$ for all t of interest[5], we can bound the third $[\cdots]$ by

$$\big|\phi(t_{n+1}) - \phi(t_n) - hf\big(t_n, \phi(t_n)\big)\big| \le \frac{1}{2}Ah^2$$

This bound follows quickly from the Taylor expansion with remainder ((3.4.32) in the CLP-1 text),

$$\begin{aligned} \phi(t_{n+1}) &= \phi(t_n) + \phi'(t_n)h + \frac{1}{2}\phi''(\tilde{t})h^2 \\ &= \phi(t_n) + h\, f\big(t_n, \phi(t_n)\big) + \frac{1}{2}\phi''(\tilde{t})h^2 \end{aligned}$$

for some $t_n < \tilde{t} < t_{n+1}$.

- Finally, by the mean value theorem, the magnitude of the second $[\cdots]$ is h times

$$\begin{aligned} \big|f\big(t_n, \phi(t_n)\big) - f(t_n, y_n)\big| &= F_{t_n}\big(\phi(t_n)\big) - F_{t_n}(y_n) \quad \text{where} \quad F_{t_n}(y) = f\big(t_n, y\big) \\ &= \big|F'_{t_n}(\tilde{y})\big|\, |\phi(t_n) - y_n| \\ &\qquad\qquad\qquad \text{for some } \tilde{y} \text{ between } y_n \text{ and } \phi(t_n) \\ &= \big|F'_{t_n}(\tilde{y})\big|\, |\varepsilon_n| \\ &\le B|\varepsilon_n| \end{aligned}$$

assuming that $\big|F'_t(y)\big| \le B$ for all t and y of interest[6].

Substituting into (E3) gives

$$|\varepsilon_{n+1}| \le |\varepsilon_n| + Bh|\varepsilon_n| + \frac{1}{2}Ah^2 = (1 + Bh)|\varepsilon_n| + \frac{1}{2}Ah^2 \tag{E4-n}$$

Hence the (bound on the) magnitude of the total error, $|\varepsilon_{n+1}|$, consists of two parts. One part is the magnitude of the local truncation error, which is no more than $\frac{1}{2}Ah^2$ and which is present even if we start the step with no error at all, i.e. with $\varepsilon_n = 0$. The other part is due to the combined error from all previous steps. This is the ε_n term. At the beginning of step number $n+1$, the combined error has magnitude $|\varepsilon_n|$. During the step, this error gets magnified by no more than a factor of $1 + Bh$.

The second half of the derivation is to repeatedly apply (E4-n) with $n = 0, 1, 2, \cdots$. By definition $\phi(t_0) = y_0$ so that $\varepsilon_0 = 0$, so

$$\begin{aligned} &(\text{E4-0}) \implies |\varepsilon_1| \le (1 + Bh)|\varepsilon_0| + \tfrac{A}{2}h^2 = \tfrac{A}{2}h^2 \\ &(\text{E4-1}) \implies |\varepsilon_2| \le (1 + Bh)|\varepsilon_1| + \tfrac{A}{2}h^2 = (1 + Bh)\tfrac{A}{2}h^2 + \tfrac{A}{2}h^2 \\ &(\text{E4-2}) \implies |\varepsilon_3| \le (1 + Bh)|\varepsilon_2| + \tfrac{A}{2}h^2 = (1 + Bh)^2\tfrac{A}{2}h^2 + (1 + Bh)\tfrac{A}{2}h^2 + \tfrac{A}{2}h^2 \end{aligned}$$

5 We are assuming that the derivative $\phi'(t)$ doesn't change too rapidly. This will be the case if $f(t, y)$ is a reasonably smooth function.

6 Again, this will be the case if $f(t, y)$ is a reasonably smooth function.

Continuing in this way

$$|\varepsilon_n| \le (1+Bh)^{n-1}\tfrac{A}{2}h^2 + \cdots + (1+Bh)\tfrac{A}{2}h^2 + \tfrac{A}{2}h^2 = \sum_{m=0}^{n-1}(1+Bh)^m \tfrac{A}{2}h^2$$

This is the beginning of a geometric series, and we can sum it up by using $\sum\limits_{m=0}^{n-1} ar^m = \frac{r^n-1}{r-1}a$ (which is Theorem 1.1.6(a)) with $a = \frac{A}{2}h^2$ and $r = 1+Bh$ gives

$$|\varepsilon_n| \le \frac{(1+Bh)^n - 1}{(1+Bh)-1}\frac{A}{2}h^2 = \frac{A}{2B}\big[(1+Bh)^n - 1\big]h$$

We are interested in how this behaves as $t_n - t_0$ increases and/or h decreases. Now $n = \frac{t_n - t_0}{h}$ so that $(1+Bh)^n = (1+Bh)^{(t_n-t_0)/h}$. When h is small, the behaviour of $(1+Bh)^{(t_n-t_0)/h}$ is not so obvious. So we'll use a little trickery to make it easier to understand. Setting $x = Bh$ in

$$x \ge 0 \implies 1 + x \le 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots = e^x$$

(the exponential series $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$ was derived in Example 3.6.5. gives[7] $1 + Bh \le e^{Bh}$. Hence $(1+Bh)^n \le e^{Bhn} = e^{B(t_n-t_0)}$, since $t_n = t_0 + nh$, and we arrive at the conclusion

Equation D.2.6

$$|\varepsilon_n| \le \frac{A}{2B}\left[e^{B(t_n-t_0)} - 1\right]h$$

This is of the form $K(t_f)h^k$ with $k = 1$ and the coefficient $K(t_f)$ growing exponentially with $t_f - t_0$. If we keep h fixed and increase t_n we see exponential growth, but if we fix t_n and decrease h we see the error decrease linearly. This is just what our experimental data suggested.

D.3▲ Variable Step Size Methods

We now introduce a family of procedures that decide by themselves what step size to use. In all of these procedures the user specifies an acceptable error rate and the procedure attempts to adjust the step size so that each step introduces error at no more than that rate. That way the procedure uses a small step size when it is hard

7 When $x = Bh$ is large, it is not wise to bound the linear $1+x$ by the much larger exponential e^x. However when x is small, $1+x$ and e^x are almost the same.

to get an accurate approximation, and a large step size when it is easy to get a good approximation.

Suppose that we wish to generate an approximation to the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0$$

for some range of t's and we want the error introduced per unit increase[1] of t to be no more than about some small fixed number ε. This means that if $y_n \approx y(t_0 + nh)$ and $y_{n+1} \approx y(t + (n+1)h)$, then we want the local truncation error in the step from y_n to y_{n+1} to be no more than about εh. Suppose further that we have already produced the approximate solution as far as t_n. The rough strategy is as follows. We do the step from t_n to $t_n + h$ twice using two different algorithms, giving two different approximations to $y(t_{n+1})$, that we call $A_{1,n+1}$ and $A_{2,n+1}$. The two algorithms are chosen so that

(1) we can use $A_{1,n+1} - A_{2,n+1}$ to compute an approximate local truncation error and

(2) for efficiency, the two algorithms use almost the same evaluations of f. Remember that evaluating the function f is typically the most time-consuming part of our computation.

In the event that the local truncation error, divided by h, (i.e. the error per unit increase of t) is smaller than ε, we set $t_{n+1} = t_n + h$, accept $A_{2,n+1}$ as the approximate value[2] for $y(t_{n+1})$, and move on to the next step. Otherwise we pick, using what we have learned from $A_{1,n+1} - A_{2,n+1}$, a new trial step size h and start over again at t_n.

Now for the details. We start with a very simple procedure. We will later soup it up to get a much more efficient procedure.

D.3.1 ▸▸ Euler and Euler-2step (preliminary version)

Denote by $\phi(t)$ the exact solution to $y' = f(t, y)$ that satisfies the initial condition $\phi(t_n) = y_n$. If we apply one step of Euler with step size h, giving

$$A_{1,n+1} = y_n + hf(t_n, y_n)$$

we know, from (D.2.4), that

$$A_{1,n+1} = \phi(t_n + h) + Kh^2 + O(h^3)$$

The problem, of course, is that we don't know what the error is, even approximately, because we don't know what the constant K is. But we can estimate K simply by redoing the step from t_n to $t_n + h$ using a judiciously chosen second algorithm. There are a number of different second algorithms that will work. We call the simple algorithm

1 We know that the error will get larger the further we go in t. So it makes sense to try to limit the error per unit increase in t.

2 Better still, accept $A_{2,n+1}$ minus the computed approximate error in $A_{2,n+1}$ as the approximate value for $y(t_{n+1})$.

that we use in this subsection Euler-2step[3]. One step of Euler-2step with step size h just consists of doing two steps of Euler with step size $h/2$:

$$A_{2,n+1} = y_n + \tfrac{h}{2} f(t_n, y_n) + \tfrac{h}{2} f\big(t_n + \tfrac{h}{2}, y_n + \tfrac{h}{2} f(t_n, y_n)\big)$$

Here, the first half-step took us from y_n to $y_{\rm mid} = y_n + \frac{h}{2} f(t_n, y_n)$ and the second half-step took us from $y_{\rm mid}$ to $y_{\rm mid} + \frac{h}{2} f\big(t_n + \frac{h}{2}, y_{\rm mid}\big)$. The local truncation error introduced in the first half-step is $K(h/2)^2 + O(h^3)$. That for the second half-step is $K(h/2)^2 + O(h^3)$ with the same[4] K, though with a different $O(h^3)$. All together

$$\begin{aligned} A_{2,n+1} &= \phi(t_n + h) + \big[K\big(\tfrac{h}{2}\big)^2 + O(h^3)\big] + \big[K\big(\tfrac{h}{2}\big)^2 + O(h^3)\big] \\ &= \phi(t_n + h) + \frac{1}{2}Kh^2 + O(h^3) \end{aligned}$$

The difference is[5]

$$\begin{aligned} A_{1,n+1} - A_{2,n+1} &= \big[\phi(t_n + h) + Kh^2 + O(h^3)\big] - \big[\phi(t_n + h) - \frac{1}{2}Kh^2 - O(h^3)\big] \\ &= \frac{1}{2}Kh^2 + O(h^3) \end{aligned}$$

So if we do one step of both Euler and Euler-2step, we can estimate

$$\frac{1}{2}Kh^2 = A_{1,n+1} - A_{2,n+1} + O(h^3)$$

We now know that in the step just completed Euler-2step introduced an error of about $\frac{1}{2}Kh^2 \approx A_{1,n+1} - A_{2,n+1}$. That is, the current error rate is about $r = \frac{|A_{1,n+1} - A_{2,n+1}|}{h} \approx \frac{1}{2}|K|h$ per unit increase of t.

- If this $r = \frac{|A_{1,n+1} - A_{2,n+1}|}{h} > \varepsilon$, we reject[6] $A_{2,n+1}$ and repeat the current step with a new trial step size chosen so that $\frac{1}{2}|K|(\text{new } h) < \varepsilon$, i.e. $\frac{r}{h}(\text{new } h) < \varepsilon$. To give ourselves a small safety margin, we could use[7]

$$\text{new } h = 0.9\,\frac{\varepsilon}{r}\,h$$

3 This name is begging for a dance related footnote and we invite the reader to supply their own.

4 Because the two half-steps start at values of t only $h/2$ apart, and we are thinking of h as being very small, it should not be surprising that we can use the same value of K in both. In case you don't believe us, we have included a derivation of the local truncation error for Euler-2step later in this appendix.

5 Recall that every time the symbol $O(h^3)$ is used it can stand for a different function that is bounded by some constant times h^3 for small h. Thus $O(h^3) - O(h^3)$ need not be zero, but is $O(h^3)$. What is important here is that if K is not zero and if h is very small, then $O(h^3)$ is much smaller than $\frac{1}{2}Kh^2$.

6 The measured error rate, r, is bigger than the desired error rate ε. That means that it is harder to get the accuracy we want than we thought. So we have to take smaller steps.

7 We don't want to make the new h too close to $\frac{\varepsilon}{r}h$ since we are only estimating things and we might end up with an error rate bigger that ε. On the other hand, we don't want to make the new h too small because that means too much work — so we choose it to be just a little smaller than $\frac{\varepsilon}{r}h$ … say $0.9\frac{\varepsilon}{r}h$.

- If $r = \frac{|A_{1,n+1}-A_{2,n+1}|}{h} < \varepsilon$ we can accept[8] $A_{2,n+1}$ as an approximate value for $y(t_{n+1})$, with $t_{n+1} = t_n + h$, and move on to the next step, starting with the new trial step size[9]
$$\text{new } h = 0.9\,\frac{\varepsilon}{r}\,h$$

That is our preliminary version of the Euler/Euler-2step variable step size method. We call it the preliminary version, because we will shortly tweak it to get a much more efficient procedure.

Example D.3.1

As a concrete example, suppose that our problem is

$$y(0) = e^{-2},\ y' = 8(1-2t)y,\ \varepsilon = 0.1$$

and that we have gotten as far as

$$t_n = 0.33,\ y_n = 0.75,\quad \text{trial } h = 0.094$$

Then, using $E = |A_{1,n+1} - A_{2,n+1}|$ to denote the magnitude of the estimated local truncation error in $A_{2,n+1}$ and r the corresponding error rate

$$\begin{aligned}
f(t_n, y_n) &= 8(1-2\times 0.33)0.75 = 2.04\\
A_{1,n+1} &= y_n + hf(t_n, y_n) = 0.75 + 0.094\times 2.04 = 0.942\\
y_{\text{mid}} &= y_n + \tfrac{h}{2}f(t_n, y_n) = 0.75 + \tfrac{0.094}{2}\times 2.04 = 0.846\\
f\big(t_n + \tfrac{h}{2}, y_{\text{mid}}\big) &= 8\Big[1 - 2\big(0.33 + \tfrac{0.094}{2}\big)\Big]0.846 = 1.66\\
A_{2,n+1} &= y_{\text{mid}} + \tfrac{h}{2}f(t_n + \tfrac{h}{2}, y_{\text{mid}}) = 0.846 + \tfrac{0.094}{2}1.66 = 0.924\\
E &= |A_{1,n+1} - A_{2,n+1}| = |0.942 - 0.924| = 0.018\\
r &= \frac{|E|}{h} = \frac{0.018}{0.094} = 0.19
\end{aligned}$$

Since $r = 0.19 > \varepsilon = 0.1$, the current step size is unacceptable and we have to recompute with the new step size

$$\text{new } h = 0.9\frac{\varepsilon}{r}(\text{old } h) = 0.9\,\frac{0.1}{0.19}\,0.094 = 0.045$$

to give

$$\begin{aligned}
f(t_n, y_n) &= 8(1-2\times 0.33)0.75 = 2.04\\
A_{1,n+1} &= y_n + hf(t_n, y_n) = 0.75 + 0.045\times 2.04 = 0.842\\
y_{\text{mid}} &= y_n + \tfrac{h}{2}f(t_n, y_n) = 0.75 + \tfrac{0.045}{2}\times 2.04 = 0.796
\end{aligned}$$

8 The measured error rate, r, is smaller than the desired error rate ε. That means that it is easier to get the accuracy we want than we thought. So we can make the next step larger.

9 Note that in this case $\frac{\varepsilon}{r} > 1$. So the new h can be bigger than the last h.

$$\begin{aligned}
f\big(t_n + \tfrac{h}{2}, y_{\rm mid}\big) &= 8\Big[1 - 2\big(0.33 + \tfrac{0.045}{2}\big)\Big]0.796 = 1.88 \\
A_{2,n+1} &= y_{\rm mid} + \tfrac{h}{2} f\big(t_n + \tfrac{h}{2}, y_{\rm mid}\big) = 0.796 + \tfrac{0.045}{2}1.88 = 0.838 \\
E &= A_{1.n+1} - A_{2.n+1} = 0.842 - 0.838 = 0.004 \\
r &= \frac{|E|}{h} = \frac{0.004}{0.045} = 0.09
\end{aligned}$$

This time $r = 0.09 < \varepsilon = 0.1$, is acceptable so we set $t_{n+1} = 0.33 + 0.045 = 0.375$ and

$$y_{n+1} = A_{2,n+1} = 0.838$$

The initial trial step size from t_{n+1} to t_{n+2} is

$$\text{new } h = 0.9\frac{\varepsilon}{r}(\text{old } h) = 0.9\,\frac{0.1}{0.09}\,.045 = .045$$

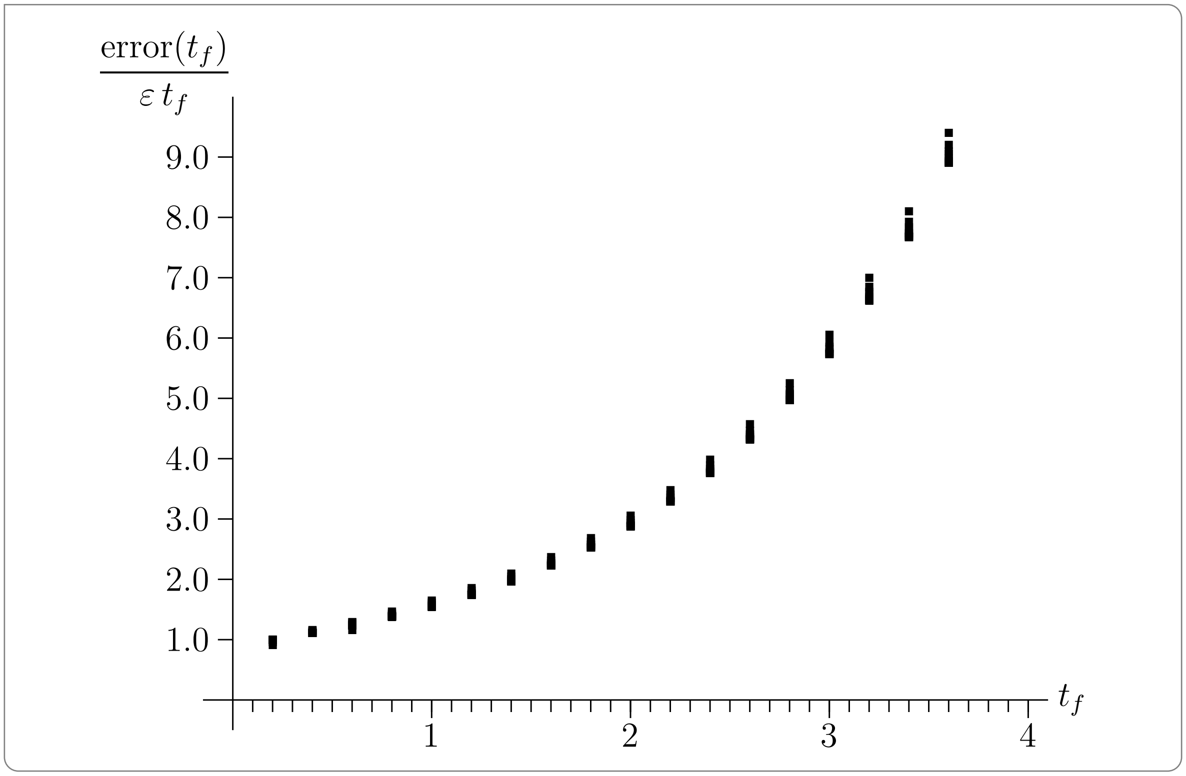
By a fluke, it has turned out that the new h is the same as the old h (to three decimal places). If r had been significantly smaller than ε, then the new h would have been signficantly bigger than the old h - indicating that it is (relatively) easy to estimate things in this region, making a larger step size sufficient.

Example D.3.1

As we said above, we will shortly upgrade the above variable step size method, that we are calling the preliminary version of the Euler/Euler-2step method, to get a much more efficient procedure. Before we do so, let's pause to investigate a little how well our preliminary procedure does at controlling the rate of error production.

We have been referring, loosely, to ε as the desired rate for introduction of error, by our variable step size method, as t advances. If the rate of increase of error were exactly ε, then at final time t_f the accumulated error would be exactly $\varepsilon(t_f - t_0)$. But our algorithm actually chooses the step size h for each step so that the estimated local truncation error in $A_{2,n+1}$ for that step is about εh. We have seen that, once some local truncation error has been introduced, its contribution to the global truncation error can grow exponentially with t_f.

Here are the results of a numerical experiment that illustrate this effect. In this experiment, the above preliminary Euler/Euler-2step method is applied to the initial value problem $y' = t - 2y,\ y(0) = 3$ for $\varepsilon = \frac{1}{16}, \frac{1}{32}, \cdots$ (ten different values) and for $t_f = 0.2,\ 0.4,\ \cdots,\ 3.8$. Here is a plot of the resulting $\frac{\text{actual error at } t=t_f}{\varepsilon t_f}$ against t_f.



If the rate of introduction of error were exactly ε, we would have $\frac{\text{actual error at } t=t_f}{\varepsilon t_f} = 1$. There is a small square on the graph for each different pair ε, t_f. So for each value of t_f there are ten (possibly overlapping) squares on the line $x = t_f$. This numerical experiment suggests that $\frac{\text{actual error at } t=t_f}{\varepsilon t_f}$ is relatively independent of ε and starts, when t_f is small, at about one, as we want, but grows (perhaps exponentially) with t_f.

D.3.2 ➤ Euler and Euler-2step (final version)

We are now ready to use a sneaky bit of arithemtic to supercharge our Euler/Euler-2step method. As in our development of the preliminary version of the method, denote by $\phi(t)$ the exact solution to $y' = f(t, y)$ that satisfies the initial condition $\phi(t_n) = y_n$. We have seen, at the beginning of §D.3.1, that applying one step of Euler with step size h, gives

$$\begin{aligned} A_{1,n+1} &= y_n + hf(t_n, y_n) \\ &= \phi(t_n + h) + Kh^2 + O(h^3) \end{aligned} \tag{E5}$$

and applying one step of Euler-2step with step size h (i.e. applying two steps of Euler with step size $h/2$) gives

$$\begin{aligned} A_{2,n+1} &= y_n + \tfrac{h}{2} f(t_n, y_n) + \tfrac{h}{2} f\big(t_n + \tfrac{h}{2}\,,\, y_n + \tfrac{h}{2} f(t_n, y_n)\big) \\ &= \phi(t_n + h) + \tfrac{1}{2} Kh^2 + O(h^3) \end{aligned} \tag{E6}$$

because the local truncation error introduced in the first half-step was $K(h/2)^2 + O(h^3)$ and that introduced in the second half-step was $K(h/2)^2 + O(h^3)$. Now here is the sneaky bit. Equations (E5) and (E6) are very similar and we can eliminate all Kh^2's by subtracting (E5) from 2 times (E6). This gives

$$2(\text{E6}) - (\text{E5}): \qquad 2A_{2,n+1} - A_{1,n+1} = \phi(t_n + h) + O(h^3)$$

(no more h^2 term!) or

$$\phi(t_n + h) = 2A_{2,n+1} - A_{1,n+1} + O(h^3) \tag{E7}$$

which tells us that choosing

$$y_{n+1} = 2A_{2,n+1} - A_{1,n+1} \tag{E8}$$

would give a local truncation error of order h^3, rather than the order h^2 of the preliminary Euler/Euler-2step method. To convert the preliminary version to the final version, we just replace $y_{n+1} = A_{2,n+1}$ by $y_{n+1} = 2A_{2,n+1} - A_{1,n+1}$:

Equation D.3.2 Euler/Euler-2step Method.

Given $\varepsilon > 0$, t_n, y_n and the current step size h

- compute

$$\begin{aligned} A_{1,n+1} &= y_n + hf(t_n, y_n) \\ A_{2,n+1} &= y_n + \tfrac{h}{2} f(t_n, y_n) + \tfrac{h}{2} f\big(t_n + \tfrac{h}{2}\,,\, y_n + \tfrac{h}{2} f(t_n, y_n)\big) \\ r &= \frac{|A_{1,n+1} - A_{2,n+1}|}{h} \end{aligned}$$

- If $r > \varepsilon$, repeat the first bullet but with the new step size

$$(\text{new } h) = 0.9\,\frac{\varepsilon}{r}\,(\text{old } h)$$

- If $r < \varepsilon$ set

$$\begin{aligned} t_{n+1} &= t_n + h \\ y_{n+1} &= 2A_{2,n+1} - A_{1,n+1} \quad \text{and the new trial step size} \\ (\text{new } h) &= 0.9\,\frac{\varepsilon}{r}\,(\text{old } h) \end{aligned}$$

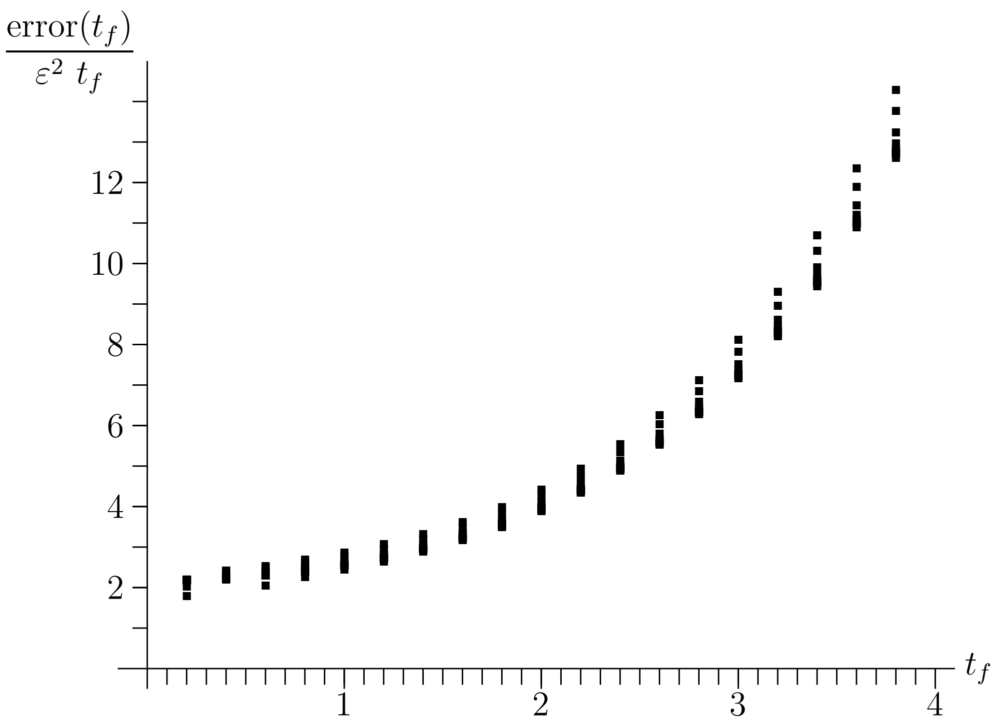
  and move on to the next step. Note that since $r < \varepsilon$, $\frac{r}{\varepsilon}h > h$ which indicates that the new h can be larger than the old h. We include the 0.9 to be careful not to make the error of the next step too big.

Let's think a bit about how our final Euler/Euler-2step method should perform.

- The step size here, as in the preliminary version, is chosen so that the local truncation error in $A_{2,n+1}$ per unit increase of t, namely $r = \frac{|A_{1,n+1} - A_{2,n+1}|}{h} \approx \frac{Kh^2/2}{h} = \frac{K}{2}h$, is approximately ε. So h is roughly proportional to ε.

- On the other hand, (E7) shows that, in the full method, local truncation error is being added to y_{n+1} at a rate of $\frac{O(h^3)}{h} = O(h^2)$ per unit increase in t.
- So one would expect that local truncation increases the error at a rate proportional to ε^2 per unit increase in t.
- If the rate of increase of error were exactly a constant time ε^2, then the error accumulated between the initial time $t = 0$ and the final time $t = t_f$ would be exactly a constant times $\varepsilon^2\, t_f$.
- However we have seen that, once some local truncation error has been introduced, its contribution to the global error can grow exponentially with t_f. So we would expect that, under the full Euler/Euler-2step method, $\frac{\text{actual error at } t=t_f}{\varepsilon^2 t_f}$ to be more or less independent of ε, but still growing exponentially in t_f.

Here are the results of a numerical experiment that illustrate this. In this experiment, the above final Euler/Euler-2step method, (D.3.2), is applied to the initial value problem $y' = t - 2y$, $y(0) = 3$ for $\varepsilon = \frac{1}{16}, \frac{1}{32}, \cdots$ (ten different values) and for $t_f = 0.2,\ 0.4,\ \cdots,\ 3.8$. In the following plot, there is a small square for the resulting $\frac{\text{actual error at } t=t_f}{\varepsilon^2 t_f}$ for each different pair ε, t_f.



It does indeed look like $\frac{\text{actual error at } t=t_f}{\varepsilon^2 t_f}$ is relatively independent of ε but grows (perhaps exponentially) with t_f. Note that $\frac{\text{actual error at } t=t_f}{\varepsilon^2 t_f}$ contains a factor of ε^2 in the denominator. The actual error rate $\frac{\text{actual error at } t=t_f}{t_f}$ is much smaller than is suggested by the graph.

D.3.3 ➡ Fehlberg's Method

Of course, in practice more efficient and more accurate methods[10] than Euler and Euler-2step are used. Fehlberg's method[11] uses improved Euler and a second more accurate method. Each step involves three calculations of f:

$$\begin{aligned} f_{1,n} &= f(t_n, y_n) \\ f_{2,n} &= f(t_n + h, y_n + hf_{1,n}) \\ f_{3,n} &= f\left(t_n + \tfrac{h}{2}, y_n + \tfrac{h}{4}[f_{1,n} + f_{2,n}]\right) \end{aligned}$$

Once these three evaluations have been made, the method generates two approximations for $y(t_n + h)$:

$$\begin{aligned} A_{1,n+1} &= y_n + \tfrac{h}{2}\left[f_{1,n} + f_{2,n}\right] \\ A_{2,n+1} &= y_n + \tfrac{h}{6}\left[f_{1,n} + f_{2,n} + 4f_{3,n}\right] \end{aligned}$$

Denote by $\phi(t)$ the exact solution to $y' = f(t, y)$ that satisfies the initial condition $\phi(t_n) = y_n$. Now $A_{1,n+1}$ is just the y_{n+1} produced by the improved Euler's method. The local truncation error for the improved Euler's method is of order h^3, one power of h smaller than that for Euler's method. So

$$A_{1,n+1} = \phi(t_n + h) + Kh^3 + O(h^4)$$

and it turns out[12] that

$$A_{2,n+1} = \phi(t_n + h) + O(h^4)$$

So the error in $A_{1,n+1}$ is

$$\begin{aligned} E &= \left|Kh^3 + O(h^4)\right| = \left|A_{1,n+1} - \phi(t_n + h)\right| + O(h^4) \\ &= \left|A_{1,n+1} - A_{2,n+1}\right| + O(h^4) \end{aligned}$$

and our estimate for rate at which error is being introduced into $A_{1,n+1}$ is

$$r = \frac{|A_{1,n+1} - A_{2,n+1}|}{h} \approx |K|h^2$$

per unit increase of t.

- If $r > \varepsilon$ we redo this step with a new trial step size chosen so that $|K|(\text{new } h)^2 < \varepsilon$, i.e. $\frac{r}{h^2}(\text{new } h)^2 < \varepsilon$. With our traditional safety factor, we take

$$\text{new } h = 0.9\sqrt{\frac{\varepsilon}{r}}\,h \qquad (\text{the new } h \text{ is smaller})$$

10 There are a very large number of such methods. We will only look briefly at a couple of the simpler ones. The interested reader can find more by search engining for such keywords as "Runge-Kutta methods" and "adaptive step size".

11 E. Fehlberg, NASA Technical Report R315 (1969) and NASA Technical Report R287 (1968).

12 The interested reader can find Fehlberg's original paper online (at NASA!) and follow the derivation. It requires careful Taylor expansions and then clever algebra to cancel the bigger error terms.

- If $r \le \varepsilon$ we set $t_{n+1} = t_n + h$ and $y_{n+1} = A_{2,n+1}$ (since $A_{2,n+1}$ should be considerably more accurate than $A_{1,n+1}$) and move on to the next step with trial step size

$$\text{new } h = 0.9\sqrt{\frac{\varepsilon}{r}}\,h \qquad \text{(the new } h \text{ is usually bigger)}$$

D.3.4 ▸▸ The Kutta-Merson Process

The Kutta-Merson process[13] uses two variations of the Runge-Kutta method. Each step involves five calculations[14] of f:

$$\begin{aligned}
k_{1,n} &= f(t_n, y_n)\\
k_{2,n} &= f\big(t_n + \tfrac{1}{3}h, y_n + \tfrac{1}{3}hk_{1,n}\big)\\
k_{3,n} &= f\big(t_n + \tfrac{1}{3}h, y_n + \tfrac{1}{6}hk_{1,n} + \tfrac{1}{6}hk_{2,n}\big)\\
k_{4,n} &= f\big(t_n + \tfrac{1}{2}h, y_n + \tfrac{1}{8}hk_{1,n} + \tfrac{3}{8}hk_{3,n}\big)\\
k_{5,n} &= f\big(t_n + h, y_n + \tfrac{1}{2}hk_{1,n} - \tfrac{3}{2}hk_{3,n} + 2hk_{4,n}\big)
\end{aligned}$$

Once these five evaluations have been made, the process generates two approximations for $y(t_n + h)$:

$$\begin{aligned}
A_{1,n+1} &= y_n + h\left[\tfrac{1}{2}k_{1,n} - \tfrac{3}{2}k_{3,n} + 2k_{4,n}\right]\\
A_{2,n+1} &= y_n + h\left[\tfrac{1}{6}k_{1,n} + \tfrac{2}{3}k_{4,n} + \tfrac{1}{6}k_{5,n}\right]
\end{aligned}$$

The (signed) error in $A_{1,n+1}$ is $\frac{1}{120}h^5K + O(h^6)$ while that in $A_{2,n+1}$ is $\frac{1}{720}h^5K + O(h^6)$ with the same constant K. So $A_{1,n+1} - A_{2,n+1} = \frac{5}{720}Kh^5 + O(h^6)$ and the unknown constant K can be determined, to within an error $O(h)$, by

$$K = \frac{720}{5\,h^5}(A_{1,n+1} - A_{2,n+1})$$

and the approximate (signed) error in $A_{2,n+1}$ and its corresponding rate per unit increase of t are

$$\begin{aligned}
E &= \frac{1}{720}Kh^5 = \frac{1}{5}(A_{1,n+1} - A_{2,n+1})\\
r = \frac{|E|}{h} &= \frac{1}{720}|K|h^4 = \frac{1}{5\,h}\big|A_{1,n+1} - A_{2,n+1}\big|
\end{aligned}$$

- If $r > \varepsilon$ we redo this step with a new trial step size chosen so that $\frac{1}{720}|K|(\text{new h})^4 < \varepsilon$, i.e. $\frac{r}{h^4}(\text{new } h)^4 < \varepsilon$. With our traditional safety factor, we take

$$\text{new } h = 0.9\left(\frac{\varepsilon}{r}\right)^{1/4} h$$

13 R.H. Merson, ``An operational method for the study of integration processes" , Proc. Symp. Data Processing , Weapons Res. Establ. Salisbury , Salisbury (1957) pp. 110–125.

14 Like the other methods described above, the coefficients 1/3, 1/6, 1/8 etc. are chosen so as to cancel larger error terms. While determining the correct choice of coefficients is not conceptually difficult, it does take some work and is beyond the scope of this appendix. The interested reader should search-engine their way to a discussion of adaptive Runge-Kutta methods.

- If $r \le \varepsilon$ we set $t_{n+1} = t_n + h$ and $y_{n+1} = A_{2,n+1} - E$ (since E is our estimate of the signed error in $A_{2,n+1}$) and move on to the next step with trial step size

$$\text{new } h = 0.9 \left(\frac{\varepsilon}{r}\right)^{1/4} h$$

D.3.5 ▸▸ The Local Truncation Error for Euler-2step

In our description of Euler/Euler-2step above we simply stated the local truncation error without an explanation. In this section, we show how it may be derived. We note that very similar calculations underpin the other methods we have described.

In this section, we will be using partial derivatives and, in particular, the chain rule for functions of two variables. That material is covered in Chapter 2 of the CLP-3 text. If you are not yet comfortable with it, you can either take our word for those bits, or you can delay reading this section until you have learned a little multivariable calculus.

Recall that, by definition, the local truncation error for an algorithm is the (signed) error generated by a single step of the algorithm, under the assumptions that we start the step with the exact solution and that there is no roundoff error[15]. Denote by $\phi(t)$ the exact solution to

$$\begin{aligned} y'(t) &= f(t, y) \\ y(t_n) &= y_n \end{aligned}$$

In other words, $\phi(t)$ obeys

$$\begin{aligned} \phi'(t) &= f\big(t, \phi(t)\big) \qquad \text{for all } t \\ \phi(t_n) &= y_n \end{aligned}$$

In particular $\phi'(t_n) = f\big(t_n, \phi(t_n)\big) = f(t_n, y_n)$ and, carefully using the chain rule, which is (2.4.2) in the CLP-3 text,

$$\begin{aligned} \phi''(t_n) &= \frac{\mathrm{d}}{\mathrm{d}t} f\big(t, \phi(t)\big)\Big|_{t=t_n} = \Big[f_t\big(t, \phi(t)\big) + f_y\big(t, \phi(t)\big)\phi'(t)\Big]_{t=t_n} \\ &= f_t(t_n, y_n) + f_y(t_n, y_n)\, f(t_n, y_n) \end{aligned} \tag{E9}$$

Remember that f_t is the partial derivative of f with respect to t, and that f_y is the partial derivative of f with respect to y. We'll need (E9) below.

By definition, the local truncation error for Euler is

$$E_1(h) = \phi(t_n + h) - y_n - hf\big(t_n, y_n\big)$$

while that for Euler-2step is

$$E_2(h) = \phi(t_n + h) - y_n - \tfrac{h}{2} f(t_n, y_n) - \tfrac{h}{2} f\big(t_n + \tfrac{h}{2}, y_n + \tfrac{h}{2} f(t_n, y_n)\big)$$

15 We should note that in serious big numerical computations, one really does have to take rounding errors into account because they can cause serious problems. The interested reader should search-engine their way to the story of Edward Lorenz's numerical simulations and the beginnings of chaos theory. Unfortunately we simply do not have space in this text to discuss all aspects of mathematics.

To understand how $E_1(h)$ and $E_2(h)$ behave for small h we can use Taylor expansions ((3.4.10) in the CLP-1 text) to write them as power series in h. To be precise, we use

$$g(h) = g(0) + g'(0)\,h + \frac{1}{2}g''(0)\,h^2 + O(h^3)$$

to expand both $E_1(h)$ and $E_2(h)$ in powers of h to order h^2. Note that, in the expression for $E_1(h)$, t_n and y_n are constants — they do not vary with h. So computing derivatives of $E_1(h)$ with respect to h is actually quite simple.

$$\begin{aligned}
E_1(h) &= \phi(t_n+h) - y_n - hf\big(t_n, y_n\big) & E_1(0) &= \phi(t_n) - y_n = 0\\
E_1'(h) &= \phi'(t_n+h) - f\big(t_n, y_n\big) & E_1'(0) &= \phi'(t_n) - f\big(t_n, y_n\big) = 0\\
E_1''(h) &= \phi''(t_n+h) & E_1''(0) &= \phi''(t_n)
\end{aligned}$$

By Taylor, the local truncation error for Euler obeys

Equation D.3.3

$$E_1(h) = \frac{1}{2}\phi''(t_n)h^2 + O(h^3) = Kh^2 + O(h^3) \qquad \text{with } K = \frac{1}{2}\phi''(t_n)$$

Computing arguments of $E_2(h)$ with respect to h is a little harder, since h now appears in the arguments of the function f. As a consequence, we have to include some partial derivatives.

$$\begin{aligned}
E_2(h) &= \phi(t_n+h) - y_n - \frac{h}{2}f(t_n, y_n) - \frac{h}{2}f\Big(t_n + \frac{h}{2}, y_n + \frac{h}{2}f(t_n, y_n)\Big)\\
E_2'(h) &= \phi'(t_n+h) - \frac{1}{2}f(t_n, y_n) - \frac{1}{2}f\Big(t_n + \frac{h}{2}, y_n + \frac{h}{2}f(t_n, y_n)\Big)\\
&\qquad - \frac{h}{2}\underbrace{\frac{\mathrm{d}}{\mathrm{d}h}f\Big(t_n + \frac{h}{2}, y_n + \frac{h}{2}f(t_n, y_n)\Big)}_{\text{leave this expression as is for now}}\\
E_2''(h) &= \phi''(t_n+h) - 2\times\frac{1}{2}\underbrace{\frac{\mathrm{d}}{\mathrm{d}h}f\Big(t_n + \frac{h}{2}, y_n + \frac{h}{2}f(t_n, y_n)\Big)}_{\text{leave this one too}}\\
&\qquad - \frac{h}{2}\underbrace{\frac{\mathrm{d}^2}{\mathrm{d}h^2}f\Big(t_n + \frac{h}{2}, y_n + \frac{h}{2}f(t_n, y_n)\Big)}_{\text{and leave this one too}}
\end{aligned}$$

Since we only need $E_2(h)$ and its derivatives at $h=0$, we don't have to compute the $\frac{\mathrm{d}^2 f}{\mathrm{d}h^2}$ term (thankfully) and we also do not need to compute the $\frac{\mathrm{d}f}{\mathrm{d}h}$ term in E_2'. We do, however, need $\frac{\mathrm{d}f}{\mathrm{d}h}\Big|_{h=0}$ for $E_2''(0)$.

$$\begin{aligned}
E_2(0) &= \phi(t_n) - y_n = 0\\
E_2'(0) &= \phi'(t_n) - \frac{1}{2}f(t_n, y_n) - \frac{1}{2}f(t_n, y_n) = 0
\end{aligned}$$

$$
\begin{aligned}
E_2''(0) &= \phi''(t_n) - \frac{\mathrm{d}}{\mathrm{d}h} f\Big(t_n + \frac{h}{2}, y_n + \frac{h}{2} f(t_n, y_n)\Big)\Big|_{h=0} \\
&= \phi''(t_n) - \frac{1}{2} f_t\Big(t_n + \frac{h}{2}, y_n + \frac{h}{2} f(t_n, y_n)\Big)\Big|_{h=0} \\
&\qquad\qquad\qquad - \frac{1}{2} f(t_n, y_n)\, f_y\Big(t_n + \frac{h}{2}, y_n + \frac{h}{2} f(t_n, y_n)\Big)\Big|_{h=0} \\
&= \phi''(t_n) - \frac{1}{2} f_t(t_n, y_n) - \frac{1}{2} f_y(t_n, y_n)\, f(t_n, y_n) \\
&= \frac{1}{2}\phi''(t_n) \qquad \text{by (E9)}
\end{aligned}
$$

By Taylor, the local truncation error for Euler-2step obeys

Equation D.3.4

$$
E_2(h) = \frac{1}{4}\phi''(t_n)\, h^2 + O(h^3) = \frac{1}{2} K h^2 + O(h^3) \qquad \text{with } K = \frac{1}{2}\phi''(t_n)
$$

Observe that the K in (D.3.4) is identical to the K in (D.3.3). This is exactly what we needed in our analysis of Sections D.3.1 and D.3.2.

Appendix E

Hints for Exercises

1 · Integration

1.1 · Definition of the Integral

1.1.8 · Exercises

Exercises — Stage 1

1.1.8.1. Hint. Draw a rectangle that *encompasses* the entire shaded area, and one that is *encompassed by* the shaded area. The shaded area is no more than the area of the bigger rectangle, and no less than the area of the smaller rectangle.

1.1.8.2. Hint. We can improve on the method of Question 1 by using three rectangles that together encompass the shaded region, and three rectangles that together are encompassed by the shaded region.

1.1.8.3. Hint. Four rectangles suffice.

1.1.8.4. Hint. Try drawing a picture.

1.1.8.5. Hint. Try an oscillating function.

1.1.8.6. Hint. The ordering of the parts is intentional: each sum can be written by changing some small part of the sum before it.

1.1.8.7. Hint. If we raise -1 to an even power, we get $+1$, and if we raise it to an odd power, we get -1.

1.1.8.8. Hint. Sometimes a little anti-simplification can make the pattern more clear.

a Re-write as $\frac{1}{3}+\frac{3}{9}+\frac{5}{27}+\frac{7}{81}+\frac{9}{243}$.

b Compare to the sum in the hint for (a).

c Re-write as $1\cdot 1000+2\cdot 100+3\cdot 10+\frac{4}{1}+\frac{5}{10}+\frac{6}{100}+\frac{7}{1000}$.

1.1.8.9. Hint.

- (a), (b) These are geometric sums.
- (c) You can write this as three separate sums.
- (d) You can write this as two separate sums. Remember that e is a constant. Don't be thrown off by the index being n instead of i.

1.1.8.10. Hint.

a Write out the terms of the two sums.

b A change of index is an easier option than expanding the cubic.

c Which terms cancel?

d Remember $2n+1$ is odd for every integer n. The index starts at $n=2$, not $n=1$.

1.1.8.11. Hint. Since the sum adds four pieces, there will be four rectangles. However, one might be extremely small.

1.1.8.12. *. Hint. Write out the general formula for the left Riemann sum from Definition 1.1.11 and choose a, b and n to make it match the given sum.

1.1.8.13. Hint. Since the sum runs from 1 to 3, there are three intervals. Suppose $2 = \Delta x = \frac{b-a}{n}$. You may assume the sum given is a right Riemann sum (as opposed to left or midpoint).

1.1.8.14. Hint. Let $\Delta x = \dfrac{\pi}{20}$. Then what is $b-a$?

1.1.8.15. *. Hint. Notice that the index starts at $k=0$, instead of $k=1$. Write out the given sum explicitly without using summation notation, and sketch where the rectangles would fall on a graph of $y=f(x)$.
Then try to identify $b-a$, and n, followed by "right", "left", or "midpoint", and finally a.

1.1.8.16. Hint. The area is a triangle.

1.1.8.17. Hint. There is one triangle of positive area, and one of negative area.

Exercises — Stage 2

1.1.8.18. *. Hint. Review Definition 1.1.11.

1.1.8.20. *. Hint. You'll want the limit as n goes to infinity of a sum with n terms. If you're having a hard time coming up with the sum in terms of n, try writing a sum with a finite number of terms of your choosing. Then, think about how that sum would change if it had n terms.

1.1.8.21. *. Hint. The main step is to express the given sum as the right

Riemann sum,

$$\sum_{i=1}^{n} f(a+i\Delta x)\Delta x.$$

Don't be afraid to guess Δx and $f(x)$ (review Definition 1.1.11). Then write out explicitly $\sum\limits_{i=1}^{n} f(a+i\Delta x)\Delta x$ with your guess substituted in, and compare the result with the given sum. Adjust your guess if they don't match.

1.1.8.22. *. Hint. The main step is to express the given sum as the right Riemann sum $\sum\limits_{k=1}^{n} f(a+k\Delta x)\Delta x$. Don't be afraid to guess Δx and $f(x)$ (review Definition 1.1.11). Then write out explicitly $\sum\limits_{k=1}^{n} f(a+k\Delta x)\Delta x$ with your guess substituted in, and compare the result with the given sum. Adjust your guess if they don't match.

1.1.8.23. *. Hint. The main step is to express the given sum in the form $\sum_{i=1}^{n} f(x_i^*)\Delta x$. Don't be afraid to guess Δx, x_i^* (for either a left or a right or a midpoint sum — review Definition 1.1.11) and $f(x)$. Then write out explicitly $\sum_{i=1}^{n} f(x_i^*)\Delta x$ with your guess substituted in, and compare the result with the given sum. Adjust your guess if they don't match.

1.1.8.24. *. Hint. The main step is to express the given sum in the form $\sum\limits_{i=1}^{n} f(x_i^*)\Delta x$. Don't be afraid to guess Δx, x_i^* (probably, based on the symbol R_n, assuming we have a right Riemann sum — review Definition 1.1.11) and $f(x)$. Then write out explicitly $\sum\limits_{i=1}^{n} f(x_i^*)\Delta x$ with your guess substituted in, and compare the result with the given sum. Adjust your guess if they don't match.

1.1.8.25. *. Hint. Try several different choices of Δx and x_i^*.

1.1.8.26. Hint. Let $x = r^3$, and re–write the sum in terms of x.

1.1.8.27. Hint. Note the sum does not start at $r^0 = 1$.

1.1.8.28. *. Hint. Draw a picture. See Example 1.1.15.

1.1.8.29. Hint. Draw a picture. Remember $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$.

1.1.8.30. Hint. Draw a picture: the area we want is a trapezoid. If you don't remember a formula for the area of a trapezoid, think of it as the difference of two triangles.

1.1.8.31. Hint. You can draw a very similar picture to Question 30, but remember the areas are negative.

1.1.8.32. Hint. If $y = \sqrt{16 - x^2}$, then y is nonnegative, and $y^2 + x^2 = 16$.

1.1.8.33. *. Hint. Sketch the graph of $f(x)$.

1.1.8.34. *. Hint. At which time in the interval, for example, $0 \le t \le 0.5$, is the car moving the fastest?

1.1.8.35. Hint. What are the possible speeds the car could have reached at time $t = 0.25$?

1.1.8.36. Hint. You need to know the speed of the plane at the midpoints of your intervals, so (for example) noon to 1pm *is not* one of your intervals.

Exercises — Stage 3

1.1.8.37. *. Hint. Sure looks like a Riemann sum.

1.1.8.38. *. Hint. For part (b): don't panic! Just take it one step at a time. The first step is to write down the Riemann sum. The second step is to evaluate the sum, using the given identity. The third step is to evaluate the limit $n \to \infty$.

1.1.8.39. *. Hint. The first step is to write down the Riemann sum. The second step is to evaluate the sum, using the given formulas. The third step is to evaluate the limit as $n \to \infty$.

1.1.8.40. *. Hint. The first step is to write down the Riemann sum. The second step is to evaluate the sum, using the given formulas. The third step is to evaluate the limit $n \to \infty$.

1.1.8.41. *. Hint. You've probably seen this hint before. It is worth repeating. Don't panic! Just take it one step at a time. The first step is to write down the Riemann sum. The second step is to evaluate the sum, using the given formula. The third step is to evaluate the limit $n \to \infty$.

1.1.8.42. Hint. Using the definition of a right Riemann sum, we can come up with an expression for $f(-5+10i)$. In order to find $f(x)$, set $x = -5+10i$.

1.1.8.43. Hint. Recall that for a positive constant a, $\frac{\mathrm{d}}{\mathrm{d}x}\left\{a^x\right\} = a^x \log a$, where $\log a$ is the natural logarithm (base e) of a.

1.1.8.44. Hint. Part (a) follows the same pattern as Question 43–there's just a little more algebra involved, since our lower limit of integration is not 0.

1.1.8.45. Hint. Your area can be divided into a section of a circle and a triangle. Then you can use geometry to find the area of each piece.

1.1.8.46. Hint.

a The difference between the upper and lower bounds is the area that is outside of the smaller rectangles but inside the larger rectangles. Drawing both sets of rectangles on one picture might make things clearer. Look for an easy way to compute the area you want.

b Use your answer from Part (a). Your answer will depend on f, a, and b.

1.1.8.47. Hint. Since $f(x)$ is linear, there exist real numbers m and c such that $f(x) = mx + c$. It's a little easier to first look at a single triangle from each sum, rather than the sums in their entirety.

1.2 · Basic properties of the definite integral

1.2.3 · Exercises

Exercises — Stage 1

1.2.3.1. Hint.

a What is the length of this figure?

b Think about cutting the area into two pieces vertically.

c Think about cutting the area into two pieces another way.

1.2.3.2. Hint. Use the identity $\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$.

1.2.3.4. Hint. Note that the limits of the integral given are in the opposite order from what we might expect: the smaller number is the top limit of integration. Recall $\Delta x = \frac{b-a}{n}$.

Exercises — Stage 2

1.2.3.5. *. Hint. Split the "target integral" up into pieces that can be evaluated using the given integrals.

1.2.3.6. *. Hint. Split the "target integral" up into pieces that can be evaluated using the given integrals.

1.2.3.7. *. Hint. Split the "target integral" up into pieces that can be evaluated using the given integrals.

1.2.3.8. Hint. For part (a), use the symmetry of the integrand. For part (b), the area $\int_0^1 \sqrt{1-x^2}\mathrm{d}x$ is easy to find–how is this useful to you?

1.2.3.9. *. Hint. The evaluation of this integral was also the subject of Question 1.2.3.9 in Section 1.1. This time try using the method of Example 1.2.7.

1.2.3.10. Hint. Use symmetry.

1.2.3.11. Hint. Check Theorem 1.2.12.

Exercises — Stage 3

1.2.3.12. *. Hint. Split the integral into a sum of two integrals. Interpret each geometrically.

1.2.3.13. *. Hint. Hmmmm. Looks like a complicated integral. It's probably a trick question. Check for symmetries.

1.2.3.14. *. Hint. Check for symmetries again.

1.2.3.15. Hint. What does the integrand look like to the left and right of $x = 3$?

1.2.3.16. Hint. In part (b), you'll have to factor a constant out through a square root. Remember the upper half of a circle looks like $\sqrt{r^2 - x^2}$.

1.2.3.17. Hint. For two functions $f(x)$ and $g(x)$, define $h(x) = f(x) \cdot g(x)$. If $h(-x) = h(x)$, then the product is even; if $h(-x) = -h(x)$, then the product is odd.
The table will *not* be the same as if we were multiplying even and odd *numbers*.

1.2.3.18. Hint. Note $f(0) = f(-0)$.

1.2.3.19. Hint. If $f(x)$ is even and odd, then $f(x) = -f(x)$ for every x.

1.2.3.20. Hint. Think about mirroring a function across an axis. What does this do to the slope?

1.3 · The Fundamental Theorem of Calculus

1.3.2 · Exercises

Exercises — Stage 1

1.3.2.2. *. Hint. First find the general antiderivative by guessing and checking.

1.3.2.3. *. Hint. Be careful. Two of these make no sense at all.

1.3.2.4. Hint. Check by differentiating.

1.3.2.5. Hint. Check by differentiating.

1.3.2.6. Hint. Use the Fundamental Theorem of Calculus Part 1.

1.3.2.7. Hint. Use the Fundamental Theorem of Calculus, Part 1.

1.3.2.8. Hint. You already know that $F(x)$ is an antiderivative of $f(x)$.

1.3.2.9. Hint. (a) Recall $\frac{\mathrm{d}}{\mathrm{d}x}\{\arccos x\} = \frac{-1}{\sqrt{1-x^2}}$.
(b) All antiderivatives of $\sqrt{1-x^2}$ differ from one another by a constant. You already know one antiderivative.

1.3.2.10. Hint. In order to apply the Fundamental Theorem of Calculus Part 2, the integrand must be continuous over the interval of integration.

1.3.2.11. Hint. Use the definition of $F(x)$ as an area.

1.3.2.12. Hint. $F(x)$ represents net *signed* area.

1.3.2.13. Hint. Note $G(x) = -F(x)$, when $F(x)$ is defined as in Question 12.

1.3.2.14. Hint. Using the definition of the derivative, $F'(x) = \lim\limits_{h\to 0} \dfrac{F(x+h) - F(x)}{h}$.
The area of a trapezoid with base b and heights h_1 and h_2 is $\frac{1}{2}b(h_1 + h_2)$.

1.3.2.15. Hint. There is only one!

1.3.2.16. Hint. If $\frac{\mathrm{d}}{\mathrm{d}x}\{F(x)\} = f(x)$, that tells us $\int f(x)\mathrm{d}x = F(x) + C$.

1.3.2.17. Hint. When you're differentiating, you can leave the e^x factored out.

1.3.2.18. Hint. After differentiation, you can simplify pretty far. Keep at it!

1.3.2.19. Hint. This derivative also simplifies considerably. You might need to add fractions by finding a common denominator.

Exercises — Stage 2

1.3.2.20. *. Hint. Guess a function whose derivative is the integrand, then use the Fundamental Theorem of Calculus Part 2.

1.3.2.21. *. Hint. Split the given integral up into two integrals.

1.3.2.22. Hint. The integrand is similar to $\dfrac{1}{1+x^2}$, so something with arctangent seems in order.

1.3.2.23. Hint. The integrand is similar to $\dfrac{1}{\sqrt{1-x^2}}$, so factoring out $\sqrt{2}$ from the denominator will make it look like some flavour of arcsine.

1.3.2.24. Hint. We know how to antidifferentiate $\sec^2 x$, and there is an identity linking $\sec^2 x$ with $\tan^2 x$.

1.3.2.25. Hint. Recall $2\sin x\cos x = \sin(2x)$.

1.3.2.26. Hint. $\cos^2 x = \dfrac{1+\cos(2x)}{2}$

1.3.2.28. *. Hint. There is a good way to test where a function is increasing, decreasing, or constant, that also has something to do with topic of this section.

1.3.2.29. *. Hint. See Example 1.3.5.

1.3.2.30. *. Hint. See Example 1.3.5.

1.3.2.31. *. Hint. See Example 1.3.5.

1.3.2.32. *. Hint. See Example 1.3.5.

1.3.2.33. *. Hint. See Example 1.3.6.

1.3.2.34. *. Hint. Apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides.

1.3.2.35. *. Hint. What is the title of this section?

1.3.2.36. *. Hint. See Example 1.3.6.

1.3.2.37. *. Hint. See Example 1.3.6.

1.3.2.38. *. Hint. See Example 1.3.6.

1.3.2.39. *. Hint. See Example 1.3.6.

1.3.2.40. *. Hint. Split up the domain of integration.

Exercises — Stage 3

1.3.2.41. *. Hint. It is possible to guess an antiderivative for $f'(x)f''(x)$ that is expressed in terms of $f'(x)$.

1.3.2.42. *. Hint. When does the car stop? What is the relation between velocity and distance travelled?

1.3.2.43. *. Hint. See Example 1.3.5. For the absolute maximum part of the question, study the sign of $f'(x)$.

1.3.2.44. *. Hint. See Example 1.3.5. For the "minimum value" part of the question, study the sign of $f'(x)$.

1.3.2.45. *. Hint. See Example 1.3.5. For the "maximum" part of the question, study the sign of $F'(x)$.

1.3.2.46. *. Hint. Review the definition of the definite integral and in particular Definitions 1.1.9 and 1.1.11.

1.3.2.47. *. Hint. Review the definition of the definite integral and in particular Definitions 1.1.9 and 1.1.11.

1.3.2.48. Hint. Carefully check the Fundamental Theorem of Calculus: as written, it only applies directly to $F(x)$ when $x \geq 0$.
Is $F(x)$ even or odd?

1.3.2.49. *. Hint. In general, the equation of the tangent line to the graph of $y = f(x)$ at $x = a$ is $y = f(a) + f'(a)\,(x - a)$.

1.3.2.50. Hint. Recall $\tan^2 x + 1 = \sec^2 x$.

1.3.2.51. Hint. Since the integration is with respect to t, the x^3 term can be moved outside the integral.

1.3.2.52. Hint. Remember that antiderivatives may have a constant term.

1.4 · Substitution

1.4.2 · Exercises

Exercises — Stage 1

1.4.2.1. Hint. One is true, the other false.

1.4.2.2. Hint. You can check whether the final answer is correct by differentiating.

1.4.2.3. Hint. Check the limits.

1.4.2.4. Hint. Check every step. Do they all make sense?

1.4.2.6. Hint. What is $\frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\}$?

Exercises — Stage 2

1.4.2.7. *. Hint. What is the derivative of the argument of the cosine?

1.4.2.8. *. Hint. What is the title of the current section?

1.4.2.9. *. Hint. What is the derivative of $x^3 + 1$?

1.4.2.10. *. Hint. What is the derivative of $\log x$?

1.4.2.11. *. Hint. What is the derivative of $1+\sin x$?

1.4.2.12. *. Hint. $\cos x$ is the derivative of what?

1.4.2.13. *. Hint. What is the derivative of the exponent?

1.4.2.14. *. Hint. What is the derivative of the argument of the square root?

1.4.2.15. Hint. What is $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sqrt{\log x}\right\}$?

Exercises — Stage 3

1.4.2.16. *. Hint. There is a short, slightly sneaky method — guess an antiderivative — and a really short, still-more-sneaky method.

1.4.2.17. *. Hint. Review the definition of the definite integral and in particular Definitions 1.1.9 and 1.1.11.

1.4.2.18. Hint. If $w=u^2+1$, then $u^2=w-1$.

1.4.2.19. Hint. Using a trigonometric identity, this is similar (though not identical) to $\int \tan\theta\cdot\sec^2\theta \mathrm{d}\theta$.

1.4.2.20. Hint. If you multiply the top and the bottom by e^x, what does this look like the antiderivative of?

1.4.2.21. Hint. You know methods other than substitution to evaluate definite integrals.

1.4.2.22. Hint. $\tan x = \dfrac{\sin x}{\cos x}$

1.4.2.23. *. Hint. Review the definition of the definite integral and in particular Definitions 1.1.9 and 1.1.11.

1.4.2.24. *. Hint. Review the definition of the definite integral and in particular Definitions 1.1.9 and 1.1.11.

1.4.2.25. Hint. Find the right Riemann sum for both definite integrals.

1.5 · Area between curves

1.5.2 · Exercises

Exercises — Stage 1

1.5.2.1. Hint. When we say "area between," we want positive area, *not* signed area.

1.5.2.2. Hint. We're taking rectangles that reach from one function to the other.

1.5.2.3. *. Hint. Draw a sketch first.

1.5.2.4. *. Hint. Draw a sketch first.

1.5.2.5. *. Hint. You can probably find the intersections by inspection.

1.5.2.6. *. Hint. To find the intersection, plug $x=4y^2$ into the equation $x+12y+5=0$.

Exercises — Stage 2

1.5.2.7. $*$. Hint. If the bottom function is the x-axis, this is a familiar question.

1.5.2.8. $*$. Hint. Part of the job is to determine whether $y = x$ lies above or below $y = 3x - x^2$.

1.5.2.9. $*$. Hint. Guess the intersection points by trying small integers.

1.5.2.10. $*$. Hint. Draw a sketch first. You can also exploit a symmetry of the region to simplify your solution.

1.5.2.11. $*$. Hint. Figure out where the two curves cross. To determine which curve is above the other, try evaluating $f(x)$ and $g(x)$ for some simple value of x. Alternatively, consider x very close to zero.

1.5.2.12. $*$. Hint. Think about whether it will easier to use vertical strips or horizontal strips.

1.5.2.13. Hint. Writing an integral for this is nasty. How can you avoid it?

Exercises — Stage 3

1.5.2.14. $*$. Hint. You are asked for the area, not the signed area. Be very careful about signs.

1.5.2.15. $*$. Hint. You are asked for the area, not the signed area. Draw a sketch of the region and be very careful about signs.

1.5.2.16. $*$. Hint. You have to determine whether

- the curve $y = f(x) = x\sqrt{25 - x^2}$ lies above the line $y = g(x) = 3x$ for all $0 \le x \le 4$ or
- the curve $y = f(x)$ lies below the line $y = g(x)$ for all $0 \le x \le 4$ or
- $y = f(x)$ and $y = g(x)$ cross somewhere between $x = 0$ and $x = 4$.

One way to do so is to study the sign of $f(x) - g(x) = x\big(\sqrt{25 - x^2} - 3\big)$.

1.5.2.17. Hint. Flex those geometry muscles.

1.5.2.18. Hint. These two functions have three points of intersection. This question is slightly messy, but uses the same concepts we've been practicing so far.

1.6 · Volumes

1.6.2 · Exercises

Exercises — Stage 1

1.6.2.1. Hint. The horizontal cross-sections were discussed in Example 1.6.1.

1.6.2.2. Hint. What are the dimensions of the cross-sections?

1.6.2.3. Hint. There are two different kinds of washers.

1.6.2.4. *. Hint. Draw sketches. The mechanically easiest way to answer part (b) uses the method of cylindrical shells, which is in the optional section 1.6. The method of washers also works, but requires you to have more patience and also to have a good idea what the specified region looks like. Look at your sketch very careful when identifying the ends of your horizontal strips.

1.6.2.5. *. Hint. Draw sketchs.

1.6.2.6. *. Hint. Draw a sketch.

1.6.2.7. Hint. If you take horizontal slices (parallel to one face), they will all be equilateral triangles.
Be careful not to confuse the height of a *triangle* with the height of the tetrahedron.

Exercises — Stage 2

1.6.2.8. *. Hint. Sketch the region.

1.6.2.9. *. Hint. Sketch the region first.

1.6.2.10. *. Hint. You can save yourself quite a bit of work by interpreting the integral as the area of a known geometric figure.

1.6.2.11. *. Hint. See Example 1.6.3.

1.6.2.12. *. Hint. See Example 1.6.5.

1.6.2.13. *. Hint. Sketch the region. To find where the curves intersect, look at where $\cos(\frac{x}{2})$ and $x^2 - \pi^2$ both have roots.

1.6.2.14. *. Hint. See Example 1.6.6.

1.6.2.15. *. Hint. See Example 1.6.6. Imagine cross-sections with shadow parallel to the y-axis, sticking straight out of the xy-plane.

1.6.2.16. *. Hint. See Example 1.6.1.

Exercises — Stage 3

1.6.2.17. Hint. (a) Don't be put off by phrases like "rotating an ellipse about its minor axis." This is the same kind of volume you've been calculating all section.
(b) Hopefully, you sketched the ellipse in part (a). What was its smallest radius? Its largest? These correspond to the polar and equitorial radii, respectively.
(c) Combine your answers from (a) and (b).
(d) Remember that the absolute error is the absolute difference of your two results–that is, you subtract them and take the absolute value. The relative error is the absolute error divided by the actual value (which we're taking, for our purposes, to be your answer from (c)). When you take the relative error, lots of terms will cancel, so it's easiest to not use a calculator till the end.

1.6.2.18. *. Hint. To find the points of intersection, set $4 - (x-1)^2 = x + 1$.

1.6.2.19. $*$. Hint. You can somewhat simplify your calculations in part (a) (but not part (b)) by using the fact that $\mathcal{R}$ is symmetric about the line $y = x$.
When you're solving an equation for x, be careful about your signs: $x-1$ is negative.

1.6.2.20. $*$. Hint. The mechanically easiest way to answer part (b) uses the method of cylindrical shells, which we have not covered. The method of washers also works, but requires you have enough patience and also to have a good idea what $\mathcal{R}$ looks like. So it is crucial to first sketch $\mathcal{R}$. Then be very careful in identifying the left end of your horizontal strips.

1.6.2.21. $*$. Hint. Note that the curves cross. The area of this region was found in Problem 1.5.2.14 of Section 1.5. It would be useful to review that problem.

1.6.2.22. Hint. You can use ideas from this section to answer the question. If you take a very thin slice of the column, the density is almost constant, so you can find the mass. Then you can add up all your little slices. It's the same idea as volume, only applied to mass.
Do be careful about units: in the problem statement, some are given in metres, others in kilometres.
If you're having a hard time with the antiderivative, try writing the exponential function with base e. Remember $2 = e^{\log 2}$.

1.7 · Integration by parts

1.7.2 · Exercises

Exercises — Stage 1

1.7.2.1. Hint. Read back over Sections 1.4 and 1.7. When these methods are introduced, they are justified using the corresponding differentiation rules.

1.7.2.2. Hint. Remember our rule: $\int u\mathrm{d}v = uv - \int v\mathrm{d}u$. So, we take u and use it to make $\mathrm{d}u$, and we take $\mathrm{d}v$ and use it to make v.

1.7.2.3. Hint. According to the quotient rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Antidifferentiate both sides of the equation, then solve for the expression in the question.

1.7.2.4. Hint. Remember all the antiderivatives differ only by a constant, so you can write them all as $v(x) + C$ for some C.

1.7.2.5. Hint. What integral do you have to evaluate, after you plug in your choices to the integration by parts formula?

Exercises — Stage 2

1.7.2.6. *. Hint. You'll probably want to use integration by parts. (It's the title of the section, after all). You'll break the integrand into two parts, integrate one, and differentiate the other. Would you rather integrate $\log x$, or differentiate it?

1.7.2.7. *. Hint. This problem is similar to Question 6.

1.7.2.8. *. Hint. Example 1.7.5 shows you how to find the antiderivative. Then the Fundamental Theorem of Calculus Part 2 gives you the definite integral.

1.7.2.9. *. Hint. Compare to Question 8. Try to do this one all the way through without peeking at another solution!

1.7.2.10. Hint. If at first you don't succeed, try using integration by parts a few times in a row. Eventually, one part will go away.

1.7.2.11. Hint. Similarly to Question 10, look for a way to use integration by parts a few times to simplify the integrand until it is antidifferentiatable.

1.7.2.12. Hint. Use integration by parts twice to get an integral with only a trigonometric function in it.

1.7.2.13. Hint. If you let $u = \log t$ in the integration by parts, then $\mathrm{d}u$ works quite nicely with the rest of the integrand.

1.7.2.14. Hint. Those square roots are a little disconcerting– get rid of them with a substitution.

1.7.2.15. Hint. This can be solved using the same ideas as Example 1.7.8 in your text.

1.7.2.16. Hint. Not every integral should be evaluated using integration by parts.

1.7.2.17. *. Hint. You know, or can easily look up, the derivative of arccosine. You can use a similar trick as the book did when antidifferentiating other inverse trigonometric functions in Example 1.7.9.

Exercises — Stage 3

1.7.2.18. *. Hint. After integrating by parts, do some algebraic manipulation to the integral until it's clear how to evaluate it.

1.7.2.19. Hint. After integration by parts, use a substitution.

1.7.2.20. Hint. This example is similar to Example 1.7.10 in the text. The functions $e^{x/2}$ and $\cos(2x)$ both do not substantially alter when we differentiate or antidifferentiate them. If we use integration by parts twice, we'll end up with an expression that includes our original integral. Then we can just solve for the original integral in the equation, without actually integrating.

1.7.2.21. Hint. This looks a bit like a substitution problem, because we have an "inside function."
It might help to review Example 1.7.11.

1.7.2.22. Hint. Start by simplifying.

1.7.2.23. Hint. $\sin(2x) = 2\sin x \cos x$

1.7.2.24. Hint. What is the derivative of xe^{-x}?

1.7.2.25. *. Hint. You'll want to do an integration by parts for (a)–check the end result to get a guess as to what your parts should be. A trig identity and some amount of algebraic manipulation will be necessary to get the final form.

1.7.2.26. *. Hint. See Examples 1.7.9 and 1.6.5 for refreshers on integrating arctangent, and using washers.
Remember $\tan^2 x + 1 = \sec^2 x$, and $\sec^2 x$ is easy to integrate.

1.7.2.27. *. Hint. Your integral can be broken into two integrals, which yield to two different integration methods.

1.7.2.28. *. Hint. Think, first, about how to get rid of the square root in the argument of f'', and, second, how to convert f'' into f'. Note that you are told that $f'(2) = 4$ and $f(0) = 1$, $f(2) = 3$.

1.7.2.29. Hint. Interpret the limit as a right Riemann sum.

1.8 · Trigonometric Integrals

1.8.4 · Exercises

Exercises — Stage 1

1.8.4.1. Hint. Go ahead and try it!

1.8.4.2. Hint. Use the substitution $u = \sec x$.

1.8.4.3. Hint. Divide both sides of the second identity by $\cos^2 x$.

Exercises — Stage 2

1.8.4.4. *. Hint. See Example 1.8.6. Note that the power of cosine is odd, and the power of sine is even (it's zero).

1.8.4.5. *. Hint. See Example 1.8.7. All you need is a helpful trig identity.

1.8.4.6. *. Hint. The power of cosine is odd, so we can reserve one cosine for $\mathrm{d}u$, and turn the rest into sines using the identity $\sin^2 x + \cos^2 x = 1$.

1.8.4.7. Hint. Since the power of sine is odd (and positive), we can reserve one sine for $\mathrm{d}u$, and turn the rest into cosines using the identity $\sin^2 + \cos^2 x = 1$.

1.8.4.8. Hint. When we have even powers of sine and cosine both, we use the identities in the last two lines of Equation 1.8.3.

1.8.4.9. Hint. Since the power of sine is odd, you can use the substitution $u = \cos x$.

1.8.4.10. Hint. Which substitution will work better: $u = \sin x$, or $u = \cos x$?

1.8.4.11. Hint. Try a substitution.

1.8.4.12. *. Hint. For practice, try doing this in two ways, with different substitutions.

1.8.4.13. *. Hint. A substitution will work. See Example 1.8.14 for a template for integrands with even powers of secant.

1.8.4.14. Hint. Try the substitution $u = \sec x$.

1.8.4.15. Hint. Compare to Question 14.

1.8.4.16. Hint. What is the derivative of tangent?

1.8.4.17. Hint. Don't be scared off by the non-integer power of secant. You can still use the strategies in the notes for an odd power of tangent.

1.8.4.18. Hint. Since there are no secants in the problem, it's difficult to use the substitution $u = \sec x$ that we've enjoyed in the past. Example 1.8.12 in the text provides a template for antidifferentiating an odd power of tangent.

1.8.4.19. Hint. Integrating even powers of tangent is surprisingly different from integrating odd powers of tangent. You'll want to use the identity $\tan^2 x = \sec^2 x - 1$, then use the substitution $u = \tan x$, $\mathrm{d}u = \sec^2 x \mathrm{d}x$ on (perhaps only a part of) the resulting integral. Example 1.8.16 show you how this can be accomplished.

1.8.4.20. Hint. Since there is an even power of secant in the integrand, we can use the substitution $u = \tan x$.

1.8.4.21. Hint. How have we handled integration in the past that involved an odd power of tangent?

1.8.4.22. Hint. Remember e is some constant. What are our strategies when the power of secant is even and positive? We've seen one such substitution in Example 1.8.15.

Exercises — Stage 3

1.8.4.23. *. Hint. See Example 1.8.16 for a strategy for integrating powers of tangent.

1.8.4.24. Hint. Write $\tan x = \dfrac{\sin x}{\cos x}$.

1.8.4.25. Hint. $\dfrac{1}{\cos\theta} = \sec\theta$

1.8.4.26. Hint. $\cot x = \dfrac{\cos x}{\sin x}$

1.8.4.27. Hint. Try substituting.

1.8.4.28. Hint. To deal with the "inside function," start with a substitution.

1.8.4.29. Hint. Try an integration by parts.

1.9 · Trigonometric Substitution

1.9.2 · Exercises

Exercises — Stage 1

1.9.2.1. $\ast$. Hint. The beginning of this section has a template for choosing a substitution. Your goal is to use a trig identity to turn the argument of the square root into a perfect square, so you can cancel $\sqrt{(\text{something})^2} = |\text{something}|$.

1.9.2.2. Hint. You want to do the same thing you did in Question 1, but you'll have to complete the square first.

1.9.2.3. Hint. Since θ is acute, you can draw it as an angle of a right triangle. The given information will let you label two sides of the triangle, and the Pythagorean Theorem will lead you to the third.

1.9.2.4. Hint. You can draw a right triangle with angle θ, and use the given information to label two of the sides. The Pythagorean Theorem gives you the third side.

Exercises — Stage 2

1.9.2.5. $\ast$. Hint. As in Question 1, choose an appropriate substitution. Your answer should be in terms of your original variable, x, which can be achieved using the methods of Question 3.

1.9.2.6. $\ast$. Hint. As in Question 1, choose an appropriate substitution. Your answer will be a number, so as long as you change your limits of integration when you substitute, you don't need to bother changing the antiderivative back into the original variable x. However, you might want to use the techniques of Question 4 to simplify your final answer.

1.9.2.7. $\ast$. Hint. Question 1 guides the way to finding the appropriate substitution. Since the integral is definite, your final answer will be a number. Your limits of integration should be common reference angles.

1.9.2.8. $\ast$. Hint. Question 1 guides the way to finding the appropriate substitution. Since you have in indefinite integral, make sure to get your answer back in terms of the original variable, x. Question 3 gives a reliable method for this.

1.9.2.9. Hint. A trig substitution is not the easiest path.

1.9.2.10. $\ast$. Hint. To antidifferentiate, change your trig functions into sines and cosines.

1.9.2.11. $\ast$. Hint. The integrand should simplify quite far after your substitution.

1.9.2.12. $\ast$. Hint. In part (a) you are asked to integrate an even power of $\cos x$. For part (b) you can use a trigonometric substitution to reduce the integral of part (b) almost to the integral of part (a).

1.9.2.13. Hint. What is the symmetry of the integrand?

1.9.2.14. $\ast$. Hint. See Example 1.9.3.

1.9.2.15. $\ast$. Hint. To integrate an even power of tangent, use the identity $\tan^2 x = \sec^2 x - 1$.

1.9.2.16. Hint. A trig substitution is not the easiest path.

1.9.2.17. *. Hint. Complete the square. Your final answer will have an inverse trig function in it.

1.9.2.18. Hint. To antidifferentiate even powers of cosine, use the formula $\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$. Then, remember $\sin(2\theta) = 2\sin\theta\cos\theta$.

1.9.2.19. Hint. After substituting, use the identity $\tan^2 x = \sec^2 x - 1$ more than once.
Remember $\displaystyle\int \sec x\,\mathrm{d}x = \log\left|\sec x + \tan x\right| + C$.

1.9.2.20. Hint. There's no square root, but we can still make use of the substitution $x = \tan\theta$.

Exercises — Stage 3

1.9.2.21. Hint. You'll probably want to use the identity $\tan^2\theta + 1 = \sec^2\theta$ more than once.

1.9.2.22. Hint. Complete the square — refer to Question 2 if you want a refresher. The constants aren't pretty, but don't let them scare you.

1.9.2.23. Hint. After substituting, use the identity $\sec^2 u = \tan^2 u + 1$. It might help to break the integral into a few pieces.

1.9.2.24. Hint. Make use of symmetry, and integrate with respect to y (rather than x).

1.9.2.25. Hint. Use the symmetry of the function to re-write your integrals without an absolute value.

1.9.2.26. Hint. Think of e^x as $\left(e^{x/2}\right)^2$, and use a trig substitution. Then, use the identity $\sec^2\theta = \tan^2\theta + 1$.

1.9.2.27. Hint.

a Use logarithm rules to simplify first.

b Think about domains.

c What went wrong in part (b)? At what point in the work was that problem introduced?

There is a subtle but important point mentioned in the introductory text to Section 1.9 that may help you make sense of things.

1.9.2.28. Hint. Consider the ranges of the inverse trigonometric functions. For (c), also consider the domain of $\sqrt{x^2-a^2}$.

1.10 · Partial Fractions

1.10.4 · Exercises

Exercises — Stage 1

1.10.4.1. Hint. If a quadratic function can be factored as $(ax+b)(cx+d)$ for some constants a,b,c,d, then it has roots $-\frac{b}{a}$ and $-\frac{d}{c}$.

1.10.4.2. *. Hint. Review Equations 1.10.7 through 1.10.11. Be careful to *fully* factor the denominator.

1.10.4.3. *. Hint. Review Example 1.10.1. Is the "Algebraic Method" or the "Sneaky Method" going to be easier?

1.10.4.4. Hint. For each part, use long division as in Example 1.10.4.

1.10.4.5. Hint. (a) Look for a pattern you can exploit to factor out a linear term.
(b) If you set $y = x^2$, this is quadratic. Remember $(x^2-a) = (x+\sqrt{a})(x-\sqrt{a})$ as long as a is positive
(c),(d) Look for integer roots, then use long division.

1.10.4.6. Hint. Why do we do partial fraction decomposition at all?

Exercises — Stage 2

1.10.4.7. *. Hint. What is the title of this section?

1.10.4.8. *. Hint. You can save yourself some work in developing your partial fraction decomposition by renaming x^2 to y and comparing the result with Question 7.

1.10.4.9. *. Hint. Review Steps 3 (particularly the "Sneaky Method") and 4 of Example 1.10.3.

1.10.4.10. *. Hint. Review Steps 3 (particularly the "Sneaky Method") and 4 of Example 1.10.3. Remember $\frac{\mathrm{d}}{\mathrm{d}x}\{\arctan x\} = \frac{1}{1+x^2}$.

1.10.4.11. *. Hint. Fill in the blank: the integrand is a   function.

1.10.4.12. *. Hint. The integrand is yet another   function.

1.10.4.13. Hint. Since the degree of the numerator is the same as the degree of the denominator, we can't do our partial fraction decomposition before we simplify the integrand.

1.10.4.14. Hint. The degree of the numerator is not smaller than the degree of the denominator.
Your final answer will have an arctangent in it.

1.10.4.15. Hint. In the partial fraction decomposition, several constants turn out to be 0.

1.10.4.16. Hint. Factor $(2x-1)$ out of the denominator to get started. You don't need long division for this step.

1.10.4.17. Hint. When it comes time to integrate, look for a convenient substitution.

Exercises — Stage 3

1.10.4.18. Hint. $\csc x = \dfrac{1}{\sin x} = \dfrac{\sin x}{\sin^2 x}$

1.10.4.19. Hint. Use the partial fraction decomposition from Queston 18 to save yourself some time.

1.10.4.20. Hint. In the final integration, complete the square to make a piece of the integrand look more like the derivative of arctangent.

1.10.4.21. Hint. Review Question 1.9.2.20 in Section 1.9 for antidifferentiation tips.

1.10.4.22. Hint. Partial fraction decomposition won't simplify this any more. Use a trig substitution.

1.10.4.23. Hint. To evaluate the antiderivative, break one of the fractions into two fractions.

1.10.4.24. Hint. $\cos^2\theta = 1 - \sin^2\theta$

1.10.4.25. Hint. If you're having a hard time making the substitution, multiply the numerator and the denominator by e^x.

1.10.4.26. Hint. Try the substitution $u = \sqrt{1+e^x}$. You'll need to do long division before you can use partial fraction decomposition.

1.10.4.27. *. Hint. The mechanically easiest way to answer part (c) uses the method of cylindrical shells, which we have not covered. The method of washers also works, but requires you have enough patience and also to have a good idea what R looks like. So look at the sketch in part (a) very carefully when identifying the left endpoints of your horizontal strips.

1.10.4.28. Hint. You'll need to use two regions, because the curves cross.

1.10.4.29. Hint. For (b), use the Fundamental Theorem of Calculus Part 1.

1.11 · Numerical Integration

1.11.6 · Exercises

Exercises — Stage 1

1.11.6.1. Hint. The absolute error is the difference of the two values; the relative error is the absolute error divided by the exact value; the percent error is one hundred times the relative error.

1.11.6.2. Hint. You should have four rectangles in one drawing, and four trapezoids in another.

1.11.6.3. Hint. Sketch the second derivative–it's quadratic.

1.11.6.4. Hint. You don't have to find the actual, exact maximum the second derivative achieves–you only have to give a reasonable "ceiling" that it never breaks through.

1.11.6.5. Hint. To compute the upper bound on the error, find an upper bound

on the fourth derivative of cosine, then use Theorem 1.11.13 in the text.
To find the actual error, you need to find the actual value of A.

1.11.6.6. Hint. Find a function with $f''(x) = 3$ for all x in $[0, 1]$.

1.11.6.7. Hint. You're allowed to use common sense for this one.

1.11.6.8. Hint. For part (b), consider Question 7.

1.11.6.9. *. Hint. Draw a sketch.

1.11.6.10. Hint. The error bound for the approximation is given in Theorem 1.11.13 in the text. You want this bound to be zero.

Exercises — Stage 2

1.11.6.11. Hint. Follow the formulas in Equations 1.11.2, 1.11.6, and 1.11.9 in the text.

1.11.6.12. *. Hint. See Section 1.11.1. You should be able to simplify your answer to an exact value (in terms of π).

1.11.6.13. *. Hint. See Section 1.11.2. To set up the volume integral, see Example 1.6.6. Note the dimensions given for the cross sections are diameters, not radii.

1.11.6.14. *. Hint. See Section 1.11.3 and compare to Question 1.11.6.13. Note the table gives diameters, not radii.

1.11.6.15. *. Hint. See §1.11.3. To set up the volume integral, see Example 1.6.6, or Question 14.
Note that the table gives the circumference, not radius, of the tree at a given height.

1.11.6.18. *. Hint. The main step is to find an appropriate value of M. It is *not necessary* to find the smallest possible M.

1.11.6.19. *. Hint. The main step is to find M. This question is unusual in that its wording requires you to find the smallest possible allowed M.

1.11.6.20. *. Hint. The main steps in part (b) are to find the smallest possible values of M and L.

1.11.6.21. *. Hint. As usual, the biggest part of this problem is finding L. Don't be thrown off by the error bound being given slightly differently from Theorem 1.11.13 in the text: these expressions are equivalent, since $\Delta x = \frac{b-a}{n}$.

1.11.6.22. *. Hint. The function $e^{-2x} = \dfrac{1}{e^{2x}}$ is positive and decreasing, so its maximum occurs when x is as small as possible.

1.11.6.23. *. Hint. Since $\dfrac{1}{x^5}$ is a decreasing function when $x > 0$, look for its maximum value when x is as small as possible.

1.11.6.24. *. Hint. The "best ... approximations that you can" means using the maximum number of intervals, given the information available.
The final sentence in part (b) is just a re-statement of the error bounds we're familiar

with from Theorem 1.11.13 in the text. The information $\left|s^{(k)}(x)\right| \leq \dfrac{k}{1000}$ gives you values of M and L when you set $k=2$ and $k=4$, respectively.

1.11.6.25. *. Hint. Set the error bound to be less than 0.001, then solve for n.

Exercises — Stage 3

1.11.6.26. *. Hint. See Section 1.11.3. To set up the volume integral, see Example 1.6.2.
Since the cross-sections of the pool are semi-circular disks, a section that is d metres across will have area $\frac{1}{2}\pi\left(\frac{d}{2}\right)^2$ square feet. Based on the drawing, you may assume the very ends of the pool have distance 0 feet across.

1.11.6.27. *. Hint. See Example 1.11.15.
Don't get caught up in the interpretation of the integral. It's nice to see how integrals can be used, but for this problem, you're still just approximating the integral given, and bounding the error.
When you find the second derivative to bound your error, pay attention to the difference between the integrand and $g(r)$.

1.11.6.28. *. Hint. See Example 1.11.16. You'll want to use a calculator for the approximation in (a), and for finding the appropriate number of intervals in (b). Remember that Simpson's rule requires an even number of intervals.

1.11.6.29. *. Hint. See Example 1.11.16.
Rather than calculating the fourth derivative of the integrand, use the graph to find the largest absolute value it attains over our interval.

1.11.6.30. *. Hint. See Example 1.11.15.
You'll have to differentiate $f(x)$. To that end, you may also want to review the fundamental theorem of calculus and, in particular, Example 1.3.5.
You don't have to find the best possible value for M. A reasonable upper bound on $|f''(x)|$ will do.
To have five decimal places of accuracy, your error must be less than 0.000005. This ensures that, if you round your approximation to five decimal places, they will all be correct.

1.11.6.31. Hint. To find the maximum value of $|f''(x)|$, check its critical points and endpoints.

1.11.6.32. Hint. In using Simpson's rule to approximate $\displaystyle\int_1^x \frac{1}{t}\mathrm{d}t$ with n intervals, $a=1$, $b=x$, and $\Delta x = \dfrac{x-1}{n}$.

1.11.6.33. Hint.

- $\int_1^2 \frac{1}{1+x^2}\mathrm{d}x = \arctan(2) - \frac{\pi}{4}$, so $\arctan(2) = \frac{\pi}{4} + \int_1^2 \frac{1}{1+x^2}\mathrm{d}x$
- If an approximation A of the integral $\int_1^2 \frac{1}{1+x^2}\mathrm{d}x$ has error at most ε, then $A-\varepsilon \leq \int_1^2 \frac{1}{1+x^2}\mathrm{d}x \leq A+\varepsilon$.

- Looking at our target interval will tell you how small ε needs to be, which in turn will tell you how many intervals you need to use.
- You can show, by considering the numerator and denominator separately, that $|f^{(4)}(x)| \le 30.75$ for every x in $[1,2]$.
- If you use Simpson's rule to approximate $\int_1^2 \frac{1}{1+x^2}\mathrm{d}x$, you won't need very many intervals to get the requisite accuracy.

1.12 · Improper Integrals

1.12.4 · Exercises

Exercises — Stage 1

1.12.4.1. Hint. There are two kinds of impropreity in an integral: an infinite discontinuity in the integrand, and an infinite limit of integration.

1.12.4.2. Hint. The integrand is continuous for all x.

1.12.4.3. Hint. What matters is which function is bigger for large values of x, not near the origin.

1.12.4.4. *. Hint. Read both the question and Theorem 1.12.17 *very* carefully.

1.12.4.5. Hint. (a) What if $h(x)$ is negative? What if it's not?
(b) What if $h(x)$ is very close to $f(x)$ or $g(x)$, rather than right in the middle?
(c) Note $|h(x)| \le 2f(x)$.

Exercises — Stage 2

1.12.4.6. *. Hint. First: is the integrand unbounded, and if so, where?
Second: when evaluating integrals, always check to see if you can use a simple substitution before trying a complicated procedure like partial fractions.

1.12.4.7. *. Hint. Is the integrand bounded?

1.12.4.8. *. Hint. See Example 1.12.21. Rather than antidifferentiating, you can find a nice comparison.

1.12.4.9. *. Hint. Which of the two terms in the denominator is more important when $x \approx 0$? Which one is more important when x is very large?

1.12.4.10. Hint. Remember to break the integral into two pieces.

1.12.4.11. Hint. Remember to break the integral into two pieces.

1.12.4.12. Hint. The easiest test in this case is limiting comparison, Theorem 1.12.22.

1.12.4.13. Hint. Not all discontinuities cause an integral to be improper–only infinite discontinuities.

1.12.4.14. *. Hint. Which of the two terms in the denominator is more important when x is very large?

1.12.4.15. *. Hint. Which of the two terms in the denominator is more important when $x \approx 0$? Which one is more important when x is very large?

1.12.4.16. *. Hint. What are the "problem x's" for this integral? Get a simple approximation to the integrand near each.

Exercises — Stage 3

1.12.4.17. Hint. To find the volume of the solid, cut it into horizontal slices, which are thin circular disks.
The true/false statement is equivalent to saying that the improper integral giving the volume of the solid when $a = 0$ diverges to infinity.

1.12.4.18. *. Hint. Review Example 1.12.8. Remember the antiderivative of $\frac{1}{x}$ looks very different from the antiderivative of other powers of x.

1.12.4.19. Hint. Compare to Example 1.12.14 in the text. You can antidifferentiate with a u-substitution.

1.12.4.20. Hint. To evaluate the integral, you can factor the denominator.
Recall $\lim\limits_{x\to\infty} \arctan x = \dfrac{\pi}{2}$. For the other limits, use logarithm rules, and beware of indeterminate forms.

1.12.4.21. Hint. Break up the integral. The absolute values give you a nice even function, so you can replace $|x-a|$ with $x-a$ if you're careful about the limits of integration.

1.12.4.22. Hint. Use integration by parts twice to find the antiderivative of $e^{-x}\sin x$, as in Example 1.7.10. Be careful with your signs–it's easy to make a mistake with all those negatives.
If you're having a hard time taking the limit at the end, review the Squeeze Theorem (see the CLP-1 text).

1.12.4.23. *. Hint. What is the limit of the integrand when $x \to 0$?

1.12.4.24. Hint. The only "source of impropriety" is the infinite domain of integration. Don't be afraid to be a little creative to make a comparison work.

1.12.4.25. *. Hint. There are two things that contribute to your error: using t as the upper bound instead of infinity, and using n intervals for the approximation. First, find a t so that the error introduced by approximating $\int_0^\infty \frac{e^{-x}}{1+x}\mathrm{d}x$ by $\int_0^t \frac{e^{-x}}{1+x}\mathrm{d}x$ is at most $\frac{1}{2}10^{-4}$. Then, find your n.

1.12.4.26. Hint. Look for a place to use Theorem 1.12.20.
Examples 1.2.10 and 1.2.11 have nice results about the area under an even/odd curve.

1.12.4.27. Hint. x should be a real number

1.13 · More Integration Examples · Exercises

Exercises — Stage 1

1.13.1. Hint. Each option in each column should be used exactly once.

Exercises — Stage 2

1.13.2. Hint. The integrand is the product of sines and cosines. See how this was handled with a substitution in Section 1.8.1.
After your substitution, you should have a polynomial expression in u—but it might take some simplification to get it into a form you can easily integrate.

1.13.3. Hint. We notice that the integrand has a quadratic polynomial under the square root. If that polynomial were a perfect square, we could get rid of the square root: try a trig substitution, as in Section 1.9.
The identity $\sin(2\theta) = 2\sin\theta\cos\theta$ might come in handy.

1.13.4. Hint. Notice the integral is improper. When you compute the limit, l'Hôpital's rule might help.
If you're struggling to think of how to antidifferentiate, try writing $\dfrac{x-1}{e^x} = (x-1)e^{-x}$.

1.13.5. Hint. Which method usually works for rational functions (the quotient of two polynomials)?

1.13.6. Hint. It would be nice to replace logarithm with its derivative, $\dfrac{1}{x}$.

1.13.7. *. Hint. The integrand is a rational function, so it is possible to use partial fractions. But there is a much easier way!

1.13.8. *. Hint. You should prepare your own personal internal list of integration techniques ordered from easiest to hardest. You should have associated to each technique your own personal list of signals that you use to decide when the technique is likely to be useful.

1.13.9. *. Hint. Despite both containing a trig function, the two integrals are easiest to evaluate using different methods.

1.13.10. *. Hint. For the integral of secant, see See Section 1.8.3 or Example 1.10.5.
In (c), notice the denominator is not yet entirely factored.

1.13.11. *. Hint. Part (a) can be done by inspection — use a little highschool geometry! Part (b) is reminiscent of the antiderivative of logarithm—how did we find that one out? Part (c) is an improper integral.

1.13.12. Hint. Use the substitution $u = \sin\theta$.

1.13.13. *. Hint. For (c), try a little algebra to split the integral into pieces that are easy to antidifferentiate.

1.13.14. *. Hint. If you're stumped, review Sections 1.8, 1.9, and 1.10.

1.13.15. *. Hint. For part (a), see Example 1.7.11. For part (d), see Example 1.10.4.

1.13.16. *. Hint. For part (b), first complete the square in the denominator. You can save some work by first comparing the derivative of the denominator with the numerator. For part (d) use a simple substitution.

1.13.17. *. Hint. For part (b), complete the square in the denominator. You can save some work by first comparing the derivative of the denominator with the numerator.

1.13.18. *. Hint. For part (a), the numerator is the derivative of a function that appears in the denominator.

1.13.19. Hint. The integral is improper.

1.13.20. *. Hint. For part (a), can you convert this into a partial fractions integral? For part (b), start by completing the square inside the square root.

1.13.21. *. Hint. For part (b), the numerator is the derivative of a function that is embedded in the denominator.

1.13.22. Hint. Try a substitution.

1.13.23. Hint. Note the quadratic function under the square root: you can solve this with trigonometric substitution, as in Section 1.9.

1.13.24. Hint. Try a u-substitution, as in Section 1.8.2.

1.13.25. Hint. What's the usual trick for evaluating a rational function (quotient of polynomials)?

1.13.26. Hint. If the denominator were $x^2 + 1$, the antiderivative would be arctangent.

1.13.27. Hint. Simplify first.

1.13.28. Hint. $x^3 + 1 = (x + 1)(x^2 - x + 1)$

1.13.29. Hint. You have the product of two quite dissimilar functions in the integrand—try integration by parts.

Exercises — Stage 3

1.13.30. Hint. Use the identity $\cos(2x) = 2\cos^2 x - 1$.

1.13.31. Hint. Using logarithm rules can make the integrand simpler.

1.13.32. Hint. What is the derivative of the function in the denominator? How could that be useful to you?

1.13.33. *. Hint. For part (a), the substitution $u = \log x$ gives an integral that you have seen before.

1.13.34. *. Hint. For part (a), split the integral in two. One part may be evaluated by interpreting it geometrically, without doing any integration at all. For part (c), multiply both the numerator and denominator by e^x and then make a substitution.

1.13.35. Hint. Let $u = \sqrt{1 - x}$.

1.13.36. Hint. Use the substitution $u = e^x$.

1.13.37. Hint. Use integration by parts. If you choose your parts well, the resulting integration will be very simple.

1.13.38. Hint. $\frac{\sin x}{\cos^2 x} = \tan x \sec x$

1.13.39. Hint. The cases $n = -1$ and $n = -2$ are different from all other values of n.

1.13.40. Hint. $x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$

2 · Applications of Integration

2.1 · Work

2.1.2 · Exercises

Exercises — Stage 1

2.1.2.1. Hint. Watch your units: $1\,\mathrm{J} = 1\,\frac{\mathrm{kg}\cdot\mathrm{m}^2}{\sec^2}$, but your mass is not given in kilograms, and your height is not given in metres.

2.1.2.2. Hint. The force of the rock on the ground is the product of its mass and the acceleration due to gravity.

2.1.2.3. Hint. Adding or subtracting two quantities of the same units doesn't change the units. For example, if I have one metre of rope, and I tie on two more metres of rope, I have $1 + 2 = 3$ metres of rope — not 3 centimetres of rope, or 3 kilograms of rope.
Multiplying or dividing quantities of some units gives rise to a quantity with the product or quotient of those units. For example, if I buy ten pounds of salmon for \$50, the price of my salmon is $\dfrac{50\text{ dollars}}{10\text{ pounds}} = \dfrac{50}{10}\frac{\text{dollars}}{\text{pound}} = 5\frac{\text{dollars}}{\text{pound}}$. (Not 5 pound-dollars, or 5 pounds.)

2.1.2.4. Hint. See Question 3.

2.1.2.5. Hint. Hooke's law says that the force required to stretch a spring x units past its natural length is proportional to x; that is, there is some constant k associated with the individual spring such that the force required to stretch it x m past its natural length is kx.

2.1.2.6. Hint. Definition 2.1.1 tells us the work done by the force from $x = 1$ to $x = b$ is $W(b) = \int_1^b F(x)\mathrm{d}x$, where $F(x)$ is the force on the object at position x. To recover the equation for $F(x)$, use the Fundamental Theorem of Calculus.

Exercises — Stage 2

2.1.2.7. *. Hint. Review Definition 2.1.1 for calculating the work done by a force over a distance.

2.1.2.8. Hint. For (a), $\frac{c}{\ell - x}$ is meausured in Newtons, while ℓ and x are in metres. For (b), notice the similarities and differences between the tube of air and a spring obeying Hooke's law.

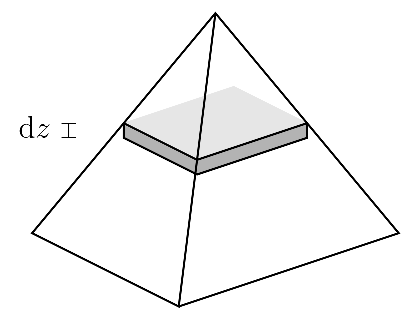
2.1.2.9. *. Hint. See Example 2.1.2. Be careful about your units.

2.1.2.10. *. Hint. Be careful about the units.

2.1.2.11. *. Hint. Suppose that the bucket is a distance y above the ground. How much work is required to raise it an additional height $\mathrm{d}y$?

2.1.2.12. Hint. Since you're given the area of the cross-section, it doesn't matter what shape it has. However, the density of water is given in cubic centimetres, while the measurements of the tank are given in metres.

2.1.2.13. *. Hint. Consider the work done to lift a horizontal plate from 2 m below the ground to a height z. You'll need to know the mass of the plate, which you can calculate from its volume, since its density is given to you.



2.1.2.14. Hint. You can find the spring constant k from the information about the hanging kilogram.

2.1.2.15. Hint. Follow the method of Example 2.1.6 and Question 11 in this section.

2.1.2.16. Hint. Calculating the work done on the rope and the weight separately makes the computation somewhat easier.

2.1.2.17. Hint. When you pull the box, the force you're exerting is exactly the same as the frictional force, but in the opposite direction. In (a), that force is constant. In (b), it changes. Check Definition 2.1.1 for how to turn force into work.

2.1.2.18. Hint. Remember that the work done on an object is equal to the change in its kinetic energy, which is $\frac{1}{2}mv^2$, where m is the mass of the object and v is its velocity. Hooke's law will tell you how much work was done stretching the spring.

2.1.2.19. Hint. As in Question 18 in this section, the change in kinetic energy of the car is equal to the work done by the compressing struts. The only added step is to calculate the spring constant, given that a car with mass 2000 kg compresses the spring 2 cm in Earth's gravity. You're not calculating *work* to find the spring constant: you're using the fact that when the car is sitting still, the force exerted upward by the struts is equal to the force exerted downward by the mass of the car under gravity.

Exercises — Stage 3

2.1.2.20. Hint. To find the radius of a horizontal layer of water, use similar triangles. Be careful with centimetres versus metres.

2.1.2.21. *. Hint. See Example 2.1.4 for a basic method for calculating the work done pumping water.
To find the area of a horizontal layer of water, use some geometry. A horizontal cross-section of a sphere is a circle, and its radius will depend on the height of the layer in the tank.

2.1.2.22. Hint. The basic ideas you've used already with "cable problems" still work, you only need to take care that the density of the cable is no longer constant. The mass of a tiny piece of cable, say of length $\mathrm{d}x$, is (density)$\times$(length) $= (10 - x)\mathrm{d}x$, where x is the distance of our piece from the bottom of the cable.
If you want more work to reference, Question 1.6.2.22 in Section 1.6 finds the mass of an object of variable density.

2.1.2.23. Hint. To calculate the force on the entire plunger, first find the force on a horizontal rectangle with height $\mathrm{d}y$ at depth y.
Checking units can be a good way to make sure your calculation makes sense.

2.1.2.24. Hint. When y metres of rope have been hauled up, what is the mass of the water?

2.1.2.25. Hint. The work you're asked for is an improper integral, moving the earth and moon infinitely far apart.

2.1.2.26. Hint. You can formulate a guess by considering the work done on the ball versus the work done on the rope in Question 16, Solution 1. But be careful — the ball in that problem did not have the same mass as the rope.

2.1.2.27. Hint. There are two things that vary with height: the density of the liquid, and the area of the cross-section of the tank. Make a formula $M(h)$ for the mass of a thin layer of liquid h metres below the top of the tank, using mass=volume$\times$density. The rest of the problem is similar to other tank-pumping problems in this section.

2.1.2.28. Hint. You can model the motion, instead of a rotation, as dividing the sand into thin horizontal slices and lifting each of them to their new position.

- In order to calculate the work involved lifting a layer of sand, you need to know the mass of the layer of sand.
- To find the mass of a layer of sand, you need its volume and the density of the sand.
- To find the density of the sand, you need to the volume of the sand: that is, the volume of half the hourglass.
- The hourglass is a solid of rotation: you can find its volume using an integral, as in Section 1.6.

2.1.2.29. Hint. Theorem 1.11.13 gives error bounds for the standard types of numerical approximations. You won't need very many intervals to achieve the desired accuracy.

2.2 · Averages

2.2.2 · Exercises

Exercises — Stage 1

2.2.2.1. Hint. See Definition 2.2.2 and the discussion following it for the link between area under the curve and averages.

2.2.2.2. Hint. Average velocity is discussed in Example 2.2.5. You don't need an integral for this.

2.2.2.3. Hint. Much like Problem 2, you don't need to do any integration here.

2.2.2.4. Hint. Part (a) is asking the length of the pieces we've cut our interval into. Part (c) should be given in terms of f. Our final answer in (d) will resemble a Riemann sum, but without some extra manipulation it won't be in exactly the form of a Riemann sum we're used to.

2.2.2.5. Hint. For (b), the value of $f(0)$ could be much, much larger than $g(0)$.

2.2.2.6. Hint. The answer is something very simple.

Exercises — Stage 2

2.2.2.7. *. Hint. Apply the definition of "average value" in Section 2.2.

2.2.2.8. *. Hint. You can antidifferentiate $x^2 \log x$ using integration by parts.

2.2.2.9. *. Hint. You can antidifferentiate an odd power of cosine with a substitution; for an even power of cosine, use the identity $\cos^2 x = \frac{1}{2}\big(1+\cos(2x)\big)$.

2.2.2.10. *. Hint. If you're not sure how to antidifferentiate, try the substitution $u = kx$, $\mathrm{d}u = k\mathrm{d}x$, keeping in mind that k is a constant. Interestingly, your final answer won't depend on k.

2.2.2.11. *. Hint. The method of partial fractions can help you antidifferentiate.

2.2.2.12. *. Hint. Try the substitution $u = \log x$, $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$.

2.2.2.13. *. Hint. Remember $\cos^2 x = \frac{1}{2}\big(1+\cos(2x)\big)$.

2.2.2.14. Hint. Notice the term $50\cos\left(\frac{t}{12}\pi\right)$ has a period of 24 hours, while the term $200\cos\left(\frac{t}{4380}\pi\right)$ has a period of one year.
If n is an approximation of c, then the relative error of n is $\frac{|n-c|}{c}$.

2.2.2.15. Hint. A cross section of S at location x is a circle with radius x^2, so area πx^4. Part (a) is asking for the average of this function on $[0,2]$.

2.2.2.16. Hint. (a) can be done without calculation

2.2.2.17. Hint. $\tan^2 x = \sec^2 x - 1$

2.2.2.18. Hint. Remember force is the product of the spring constant with the distance it's stretched past its natural length. The units given in the question are not exactly standard, but they are compatible with each other.
You can find part (b) without any calculation. For (c), remember $\sin^2 x = \frac{1}{2}\big(1 - \cos(2x)\big)$.

Exercises — Stage 3

2.2.2.19. *. Hint. The trapezoidal rule is found in Section 1.11.2.

2.2.2.20. Hint. To find a definite integral of the absolute value of a function, break up the interval of integration into regions where the function is positive, and intervals where it's negative.

2.2.2.21. Hint. This is an application of the ideas in Question 20.

2.2.2.22. Hint. Slice the solid into circular disks of radius $|f(x)|$ and thickness $\mathrm{d}x$.

2.2.2.23. Hint. The question tells you $\frac{1}{1-0}\int_0^1 f(x)\,\mathrm{d}x = \frac{f(0)+f(1)}{2}$.

2.2.2.24. Hint. Set up this question just like Question 23, but with variables for your limits of integration.
Note $(s-t)^2 = s^2 - 2st + t^2$.

2.2.2.25. Hint. What are the graphs of $f(x)$ and $f(a+b-x)$ like?

2.2.2.26. Hint. For (b), express $A(x)$ as an integral, then differentiate.

2.2.2.27. Hint. For (b), consider the cases that $f(x)$ is always bigger or always smaller than 0. Then, use the intermediate value theorem (see the CLP-1 text).

2.2.2.28. Hint. Try l'Hôpital's rule.

2.2.2.29. Hint. Use the result of Question 28.

2.3 · Centre of Mass and Torque

2.3.3 · Exercises

Exercises — Stage 1

2.3.3.1. Hint. It might help to know that $-x^2 + 2x + 1 = 2 - (x-1)^2$.

2.3.3.2. Hint. The centroid of a region doesn't have to be a point in the region.

2.3.3.3. Hint. Read over the very beginning of Section 2.3, specifically Equation 2.3.1.

2.3.3.4. Hint. Use Equation 2.3.1.

2.3.3.5. Hint. Imagine cutting out the shape and setting it on top of a pencil, so that the pencil lines up with the vertical line $x = a$. Will the figure balance, or fall to one side? Which side?

2.3.3.6. Hint. You can find the heights of the centres of mass using symmetry.

2.3.3.7. Hint. Think about whether your answers should have repetition.

2.3.3.8. Hint. The definition of a definite integral (Definition 1.1.9) will tell you how to convert your limits of sums into integrals.

2.3.3.9. Hint. In (a), the slices all have the same width, so the area of the slices is larger (and hence the density of R is higher) where $T(x) - B(x)$ is larger.

2.3.3.10. Hint. Part (a) is a significantly different model from the last question.

2.3.3.11. *. Hint. Which method involves more work: horizontal strips or vertical strips?

Exercises — Stage 2

2.3.3.12. Hint. This is a straightforward application of Equation 2.3.4.

2.3.3.13. Hint. Remember the derivative of arctangent is $\frac{1}{1+x^2}$

2.3.3.14. *. Hint. This is a straightforward application of Equations 2.3.5 and 2.3.6. Note that you're only asked for the y-coordinate of the centroid.

2.3.3.15. *. Hint. You can use a trigonometric substitution to find the area, then a partial fraction decomposition to find the y-coordinate of the centroid. Remember $\sin(1/2) = \pi/6$.

2.3.3.16. *. Hint. Vertical slices will be easier than horizontal. An integration by parts might be helpful to find $\bar{x}$, while trigonometric identities are important to finding $\bar{y}$.

2.3.3.17. *. Hint. No trigonometric substitution is necessary if you're clever with your u-substitutions, and remember the derivative of arctangent.

2.3.3.18. *. Hint. In R, the top function is $x - x^2$, and the bottom function is $x^2 - 3x$.

2.3.3.19. *. Hint. Remember $\frac{\mathrm{d}}{\mathrm{d}x}\{\arctan x\} = \frac{1}{1+x^2}$.

2.3.3.20. *. Hint. You can save quite a bit of work by, firstly, exploiting symmetry and, secondly, thinking about whether it is more efficient to use vertical strips or horizontal strips.

2.3.3.21. *. Hint. Sketch the region, being careful the domain of $\sqrt{9-4x^2}$. You can save quite a bit of work by exploiting symmetry.

2.3.3.22. Hint. Horizontal slices will be easier than vertical.

2.3.3.23. Hint. Start with a picture: whether you use vertical slices or horizontal, you'll need to break your integral into multiple pieces.

Exercises — Stage 3

2.3.3.24. *. Hint. For practice, do the computation twice — once with horizontal strips and once with vertical strips. Watch for improper integrals.

2.3.3.25. *. Hint. Draw a sketch. In part (b) be careful about the equation of the right hand boundary of A.

2.3.3.26. *. Hint. Draw a sketch. Rotating about a horizontal line is similar to rotating about the x-axis, but for the radius of a slice, you'll need to know $|y-(-1)|$: the distance from the outer edge of the region (the boundary function's y-value) to $y=-1$.

2.3.3.27. Hint. Go back to the derivation of Equation 2.3.5 (centroid for a region) to figure out what to do when your surface does not have uniform density. We will consider a rod R that reaches from $x=0$ to $x=4$, and the mass of the section of the rod along $[a,b]$ is equal to the mass of the strip of our rectangle along $[a,b]$.

2.3.3.28. Hint. Horizontal slices will help you, where symmetry doesn't, to set up a rod R whose centre of mass is the same as one coordinate of the centre of mass of the circle. When you're integrating, trigonometric substitutions are sometimes the easiest way, and sometimes not.
The equation of a circle of radius 3, centred at $(0,3)$, is $x^2+(y-3)^2=9$.

2.3.3.29. Hint. The model in the question gives you the setup to solve this problem. You know how to find the centre of mass of a rod — that's Equation 2.3.4 — so all you need to find is $\rho(y)$, the density of the rod at position y. To find this, consider a thin slice of the cone at position y with thickness $\mathrm{d}y$. Its volume $V(y)$ is the same as the mass of the small section of the rod at position y with thickness $\mathrm{d}y$. So, the density of the rod at position y is $\rho(y)=\frac{V(y)}{\mathrm{d}y}$.

2.3.3.30. Hint. Use similar triangles to show that the shape of the lower (also upper) half of the hourglass is a truncated cone, where the untruncated cone would have had a height 10 cm.
To calculate the centre of mass of the upturned sand using the result of Question 29, you should find $h=9.8$ (*not* $h=10$ — think carefully about our model from Question 29) and $k=8.8$. For the centre of mass of the sand before turning, $h=10$ and $k=6$.

2.3.3.31. Hint. The techniques of Section 2.1 get pretty complicated here, so it's easiest to use the techniques we developed in Questions 6, 29, and 30 in this section. That is, (1) find the height of the centre of mass of the water in its starting and ending positions, and then (2) model the work done as the work moving a point mass with the weight of the water from the first centre of mass to the second.
The height change of the centre of mass is all that matters to calculate the work done against gravity, so you only have to worry about the *height* of the centres of mass.

2.3.3.32. Hint. The area of R is precisely one, so the error in your approximation is the error involved in approximating $\int_0^{\sqrt{\pi/2}} 2x^2\sin(x^2)\,\mathrm{d}x$.

2.4 · Separable Differential Equations

2.4.7 · Exercises

Exercises — Stage 1

2.4.7.1. Hint. You don't need to solve the differential equation from scratch, only verify whether the given function $y = f(x)$ makes it true. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and plug it into the differential equation.

2.4.7.2. Hint. For (d), note the equation given is quadratic in the variable $\frac{\mathrm{d}y}{\mathrm{d}x}$.

2.4.7.3. Hint. The step $\displaystyle\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$ shows up whether we're using our mnemonic or not.

2.4.7.4. Hint. Note $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)\} = \frac{\mathrm{d}}{\mathrm{d}x}\{f(x)+C\}$. Plug in $y = f(x)+C$ to the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = xy$ to see whether it makes the equation is true.

2.4.7.5. Hint. If a function is differentiable at a point, it is also continuous at that point.

2.4.7.6. Hint. Let $Q(t)$ be the quantity of morphine in a patient's bloodstream at time t, where t is measured in minutes.
Using the definition of a derivative,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \lim_{h\to 0}\frac{Q(t+h)-Q(t)}{h} \approx \frac{Q(t+1)-Q(t)}{1}$$

So, $\frac{\mathrm{d}Q}{\mathrm{d}t}$ is roughly the change in the amount of morphine in one minute, from t to $t+1$.

2.4.7.7. Hint. If $p(t)$ is the proportion of the new form, then $1-p(t)$ is the proportion of the old form.
When we say two quantities are proportional, we mean that one is a constant multiple of the other.

2.4.7.8. Hint. The red marks show the slope $y(x)$ would have at a point *if* it crosses that point. So, pick a value of $y(0)$; based on the red marks, you can see how fast $y(x)$ is increasing or decreasing at that point, which leads you roughly to a value of $y(1)$; again, the red marks tell you how fast $y(x)$ is increasing or decreasing, which leads you to a value of $y(2)$, etc (unless you're already off the graph).

2.4.7.9. Hint. To draw the sketch similar to Question 8(d), don't actually calculate every single slope; find a few (for instance, where the slope is zero, or where it's negative), and use a pattern (for instance, the slope increases as y increases) to approximate most of the points.

Exercises — Stage 2

2.4.7.10. *. Hint. Start by multiplying both sides of the equation by e^y and $\mathrm{d}x$, pretending that $\frac{\mathrm{d}y}{\mathrm{d}x}$ is a fraction, according to our mnemonic.

2.4.7.11. *. Hint. You need to solve for your function $y(x)$ explicitly. Be careful with absolute values: if $|y| = F$, then $y = F$ or $y = -F$. However, $y = \pm F$ *is not a function*. You have to choose one: $y = F$ or $y = -F$.

2.4.7.12. *. Hint. If your answer doesn't quite look like the answer given, try manipulating it with logarithm rules: $\log a + \log b = \log(ab)$, and $a \log b = \log(b^a)$.

2.4.7.13. *. Hint. Simplify the equation.

2.4.7.14. *. Hint. Be careful with the arbitrary constant.

2.4.7.15. *. Hint. Start by cross-multiplying.

2.4.7.16. *. Hint. Be careful about signs. If $y^2 = F$, then possibly $y = \sqrt{F}$, and possibly $y = -\sqrt{F}$. However, $y = \pm\sqrt{F}$ is not a function.

2.4.7.17. *. Hint. Be careful about signs.

2.4.7.18. *. Hint. Be careful about signs. If $\log |y| = F$, then $|y| = e^F$. Since you should give your answer as an explicit function $y(x)$, you need to decide whether $y = e^F$ or $y = -e^F$.

2.4.7.19. *. Hint. Move the y from the left hand side to the right hand side, then use partial fractions to integrate.
Be careful about the signs. Remember that we need $y = -1$ when $x = 1$. This suggests how to deal with absolute values.

2.4.7.20. *. Hint. The unknown function $f(x)$ satisfies an equation that involves the derivative of f.

2.4.7.21. *. Hint. Try guessing the partial fractions expansion of $\dfrac{1}{x(x+1)}$.
Since $x = 1$ is in the domain and $x = 0$ is not, you may assume $x > 0$ for all x in the domain.

2.4.7.22. *. Hint. $\dfrac{\mathrm{d}}{\mathrm{d}x}\{\sec x\} = \sec x \tan x$

2.4.7.23. *. Hint. The general solution to the differential equation will contain the constant k and one other constant. They are determined by the data given in the question.

2.4.7.24. *. Hint.

- When you're solving the differential equation, you should have an integral that you can massage to look something like arctangent.
- What is the velocity of the object at its highest point?
- Your final answer will depend on the (unspecified) constants v_0, m, g and k.

2.4.7.25. *. Hint. The general solution to the differential equation will contain the constant k and one other constant. They are determined by the data given in the question.

2.4.7.26. *. Hint. The method of partial fractions will help you integrate.
To solve $\frac{x-a}{x-b} = Y$ for x, move the terms containing x out of the denominator, then

gather them on one side of the equals sign and factor out the x.

$$\begin{aligned}
\frac{x-a}{x-b} &= Y \\
x-a &= Y(x-b) = Yx - Yb \\
x - Yx &= a - Yb \\
x(1-Y) &= a - Yb \\
x &= \frac{a-Yb}{1-Y}
\end{aligned}$$

To find the limit, you can avoid l'Hôpital's rule using some clever algebra–but you can also just use l'Hôpital's rule.

2.4.7.27. *. Hint. Be careful about signs.
Part (a) has some algebraic similarities to Question 26.

2.4.7.28. *. Hint. The general solution to the differential equation will contain a constant of proportionality and one other constant. They are determined by the data given in the question.

Exercises — Stage 3

2.4.7.29. *. Hint. You do not need to know anything about investing or continuous compounding to do this problem. You are given the differential equation explicitly. The whole first sentence is just window dressing.

2.4.7.30. *. Hint. Again, you do not need to know anything about investing to do this problem. You are given the differential equation explicitly.

2.4.7.31. *. Hint. Differentiate the given integral equation. Plugging in $x=0$ gives you $y(0)$.

2.4.7.32. *. Hint. Suppose that in a very short time interval $\mathrm{d}t$, the height of water in the tank changes by $\mathrm{d}h$ (which is negative). Express in two different ways the volume of water that has escaped during this time interval. Equating the two gives the needed differential equation.
As the water escapes, it forms a cylinder of radius 1 cm.

2.4.7.33. *. Hint. Sketch the mercury in the tank at time t, when it has height h, and also at time $t+\mathrm{d}t$, when it has height $h+\mathrm{d}h$ (with $\mathrm{d}h<0$). The difference between those two volumes is the volume of (essentially) a disk of thickness $-\mathrm{d}h$. Figure out the radius and then the volume of that disk. This volume has to be the same as the volume of mercury that left through the hole in the bottom of the sphere, which runs out in the shape of a cylinder. Toricelli's law tells you what the length of that cylinder is, and from there you can find its volume. Setting the two volumes equal to each other gives the differential equation that determines $h(t)$.

2.4.7.34. *. Hint. The fundamental theorem of calculus will be useful in part (b).

2.4.7.35. *. Hint. For any $p > 0$, determine first $y(t)$ (in terms of p and c) and then the times (also depending on p and c) at which $y = 2$, $y = 1$ and $y = 0$. The condition that "the top half takes exactly the same amount of time to drain as the bottom half" then gives an equation that determines p.

2.4.7.36. Hint. For (a), think of a very simple function.
The equation in the question statement is equivalent to the equation

$$\frac{1}{\sqrt{x-a}}\int_a^x f(t)\,\mathrm{d}(t) = \sqrt{\int_a^x f^2(t)\,\mathrm{d}t}$$

which is, in some cases, easier to use.
For (d), you'll want to let $Y(x) = \int_a^x f(t)\,\mathrm{d}t$, and use the quadratic equation.

2.4.7.37. Hint. Start by antidifferentiating both sides of the equation with respect to x.

3 · Sequence and series

3.1 · Sequences

3.1.2 · Exercises

Exercises — Stage 1

3.1.2.1. Hint. Not every limit exists.

3.1.2.2. Hint. 100 isn't all that big when you're contemplating infinity. (Neither is any other number.)

3.1.2.3. Hint. $\displaystyle\lim_{n\to\infty} a_{2n+5} = \lim_{n\to\infty} a_n$

3.1.2.4. Hint. The sequence might be defined by different functions when n is large than when n is small.

3.1.2.5. Hint. Recall $(-1)^n$ is positive when n is even, and negative when n is odd.

3.1.2.6. Hint. Modify your answer from Question 5, but make the terms approach zero.

3.1.2.7. Hint. $(-n)^{-n} = \dfrac{(-1)^n}{n^n}$

3.1.2.8. Hint. What might cause your answers in (a) and (b) to differ? Carefully read Theorem 3.1.6 about convergent functions and their corresponding sequences.

3.1.2.9. Hint. You can use the fact that π is somewhat close to $\dfrac{22}{7}$, or you can use trial and error.

Exercises — Stage 2

3.1.2.10. Hint. You can compare the leading terms, or factor a high power of n from the numerator and denominator.

3.1.2.11. Hint. This isn't a rational expression, but you can treat it in a similar way. Recall $e < 3$.

3.1.2.12. Hint. The techniques of evaluating limits of rational sequences are again useful here.

3.1.2.13. Hint. Use the squeeze theorem.

3.1.2.14. Hint. $\dfrac{1}{n} \le n^{\sin n} \le n$

3.1.2.15. Hint. $e^{-1/n} = \dfrac{1}{e^{1/n}}$; what happens to $\dfrac{1}{n}$ as n grows?

3.1.2.16. Hint. Use the squeeze theorem.

3.1.2.17. Hint. L'Hôpital's rule might help you decide what happens if you are unsure.

3.1.2.18. *. Hint. Simplify a_k.

3.1.2.19. *. Hint. What happens to $\dfrac{1}{n}$ as n gets very big?

3.1.2.20. *. Hint. $\cos 0 = 1$

Exercises — Stage 3

3.1.2.21. *. Hint. This is trickier than it looks. Write $\dfrac{1}{n} = x$ and look at the limit as $x \to 0$.

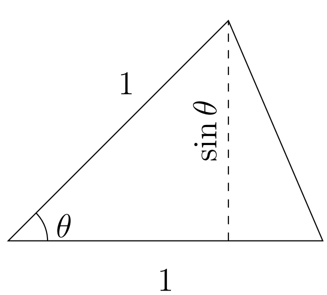
3.1.2.22. Hint. Multiply and divide by the conjugate.

3.1.2.23. Hint. Compared to Question 22, there's an easier path.

3.1.2.24. Hint. Consider $f'(x)$, when $f(x) = x^{100}$.

3.1.2.25. Hint. Look to Question 24 for inspiration.

3.1.2.26. Hint. The area of an isosceles triangle with two sides of length 1, meeting at an angle θ, is $\frac{1}{2}\sin\theta$.



3.1.2.27. Hint. Every term of A_n is the same, and $g(x)$ is a constant function.

3.1.2.28. Hint. You'll need to use a logarithm before you can apply l'Hôpital's rule.

3.1.2.29. Hint. (a) Write out the first few terms of the sequence.
(c) Consider how $a_{n+1} - L$ relates to $a_n - L$. What should happen to these numbers if a_n converges to L?

3.1.2.30. Hint. Your answer from (b) will help you a lot with the subsequent parts.

3.2 · Series

3.2.2 · Exercises

Exercises — Stage 1

3.2.2.1. Hint. S_N is the sum of the terms corresponding to $n = 1$ through $n = N$.

3.2.2.2. Hint. Note C_k is the cumulative number of cookies.

3.2.2.3. Hint. How is (a) related to Question 2?

3.2.2.4. Hint. You'll have to calculate a_1 separately from the other terms.

3.2.2.5. Hint. When does adding a number decrease the total sum?

3.2.2.6. Hint. For (b), imagine cutting up the triangle into its black and white parts, then sharing it equally among a certain number of friends. What is the easiest number of friends to share with, making sure each has the same area in their pile?

3.2.2.7. Hint. Compare to Question 6.

3.2.2.8. Hint. Iteratively divide a shape into thirds.

3.2.2.9. Hint. Lemma 3.2.5 tells us $\displaystyle\sum_{n=0}^{N} ar^n = a\frac{1-r^{N+1}}{1-r}$, for $r \neq 1$.

3.2.2.10. Hint. Note C_k is the cumulative number of cookies.

3.2.2.11. Hint. To adjust the starting index, either factor out the first term in the series, or subtract two series. For the subtraction option, consider Question 10.

3.2.2.12. Hint. Express your gains in (a) and (c) as series.

3.2.2.13. Hint. To find the difference between $\displaystyle\sum_{n=1}^{\infty} c_n$ and $\displaystyle\sum_{n=1}^{\infty} c_{n+1}$, try writing out the first few terms.

3.2.2.14. Hint. You might want to first consider a simpler true or false: $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{b_n} \stackrel{?}{=} \frac{A}{B}$.

Exercises — Stage 2

3.2.2.15. *. Hint. What kind of a series is this?

3.2.2.16. *. Hint. This is a special kind of series, that you should recognize.

3.2.2.17. *. Hint. When you see $\sum_k \left(\cdots k \cdots \quad - \quad \cdots k+1 \cdots \right)$, you should think "telescoping series."

3.2.2.18. *. Hint. When you see $\sum_n \left(\cdots n \cdots \quad - \quad \cdots n+1 \cdots \right)$, you should immediately think "telescoping series". But be careful not to jump to conclusions — evaluate the n^{th} partial sum explicitly.

3.2.2.19. *. Hint. Review Definition 3.2.3.

3.2.2.20. *. Hint. This is a special case of a general series whose sum we know.

3.2.2.21. *. Hint. Review Example 3.2.6. To write the number as a geometric series, the first few terms might not fit the pattern of the rest of the terms.

3.2.2.22. *. Hint. Start by writing it as a geometric series.

3.2.2.23. *. Hint. Review Example 3.2.6. Since the pattern repeats every three decimals, your common ratio r will be $\frac{1}{10^3}$.

3.2.2.24. *. Hint. Split the series into two parts.

3.2.2.25. *. Hint. Split the series into two parts.

3.2.2.26. *. Hint. Split the series into two parts.

3.2.2.27. Hint. Use logarithm rules to turn this into a more obvious telescoping series.

3.2.2.28. Hint. This is a telescoping series.

Exercises — Stage 3

3.2.2.29. Hint. The stone at position x has mass $\frac{1}{4^x}$ kg, and we have to pull it a distance of 2^x metres. From this, you can find the work involved in pulling up a single stone. Then, add up the work involved in pulling up all the stones.

3.2.2.30. Hint. The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

3.2.2.31. Hint. Use the properties of a telescoping series to simplify the terms. Recall $\sin^2\theta + \cos^2\theta = 1$.

3.2.2.32. Hint. Review Question 3 for using the sequence of partial sums.

3.2.2.33. Hint. What is the ratio of areas between the outermost (red) ring and the next (blue) ring?

3.3 · Convergence Tests

3.3.11 · Exercises

Exercises — Stage 1

3.3.11.1. Hint. That is, which series have terms whose limit is not zero?

3.3.11.2. Hint. That is, if $f(x)$ is a function with $f(n) = a_n$ for all whole numbers n, is $f(x)$ nonnegative and decreasing?

3.3.11.3. Hint. This isn't a trick. It's meant to give you intuition to the direct comparison test.

3.3.11.4. Hint. The comparison test is Theorem 3.3.8. However, rather than trying to memorize which way the inequalities go in all cases, you can use the same reasoning as Question 3.

3.3.11.5. Hint. Think about Question 4 to remind yourself which way the inequalities have to go for direct comparison.
Note that all the comparison series have positive terms, so we don't need to worry about that part of the limit comparison test.

3.3.11.6. Hint. The divergence test is Theorem 3.3.1.

3.3.11.7. Hint. The limit is calculated correctly.

3.3.11.8. Hint. It is true that $f(x)$ is positive. What else has to be true of $f(x)$ for the integral test to apply?

3.3.11.9. Hint. Refer to Question 4.

3.3.11.10. Hint. The definition of an alternating series is given in the start of Section 3.3.4.

3.3.11.11. Hint. For the ratio test to be inconclusive, $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|$ should be 1 or nonexistent.

3.3.11.12. Hint. By the divergence test, for a series $\sum a_n$ to *converge*, we need $\lim\limits_{n\to\infty} a_n = 0$. That is, the magnitude (absolute value) of the terms needs to be getting smaller.

3.3.11.13. Hint. If $f(x)$ is positive and decreasing, then the integral test tells you that the integral and the series either both increase or both decrease. So, in order to find an example with the properties required in the question, you need $f(x)$ to *not* be both positive and decreasing.

3.3.11.14. *. Hint. Review Theorem 3.3.11 and Example 3.3.12.

3.3.11.15. *. Hint. Don't jump to conclusions about properties of the a_n's.

Exercises — Stage 2

3.3.11.16. *. Hint. Always try the divergence test first (in your head).

3.3.11.17. *. Hint. Which test should you always try first (in your head)?

3.3.11.18. *. Hint. Review the integral test, which is Theorem 3.3.5.

3.3.11.19. Hint. A comparison might be helpful — try some algebraic manipulation to find a likely series to compare it to.

3.3.11.20. Hint. This is a geometric series.

3.3.11.21. Hint. Notice that the series is geometric, but it doesn't start at $n = 0$.

3.3.11.22. Hint. Note n only takes integer values: what's $\sin(\pi n)$ when n is an integer?

3.3.11.23. Hint. Note n only takes integer values: what's $\cos(\pi n)$ when n is an integer?

3.3.11.24. Hint. What's the test that you should always think of when you see a factorial?

3.3.11.25. Hint. This is a geometric series, but you'll need to do a little algebra to figure out r.

3.3.11.26. Hint. Which test fits most often with factorials?

3.3.11.27. Hint. Try finding a nice comparison.

3.3.11.28. *. Hint. With the substitution $u = \log x$, the function $\dfrac{1}{x(\log x)^{3/2}}$ is easily integrable.

3.3.11.29. *. Hint. Combine the integral test with the results about p-series, Example 3.3.6.

3.3.11.30. *. Hint. Try the substitution $u = \sqrt{x}$.

3.3.11.31. *. Hint. Review Example 3.3.9 for developing intuition about comparisons, and Example 3.3.10 for an example where finding an appropriate comparison series calls for some creativity.

3.3.11.32. *. Hint. What does the summand look like when k is very large?

3.3.11.33. *. Hint. What does the summand look like when n is very large?

3.3.11.34. *. Hint. $\cos(n\pi)$ is a sneaky way to write $(-1)^n$.

3.3.11.35. *. Hint. What is the behaviour for large k?

3.3.11.36. *. Hint. When m is large, $3m + \sin\sqrt{m} \approx 3m$.

3.3.11.37. Hint. This is a geometric series, but it doesn't start at $n = 0$.

3.3.11.38. *. Hint. The series is geometric.

3.3.11.39. *. Hint. The first series can be written as $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$.

3.3.11.41. *. Hint. What does the summand look like when n is very large?

3.3.11.42. *. Hint. Review the alternating series test, which is given in Theorem 3.3.14.

3.3.11.43. *. Hint. Review the alternating series test, which is given in Theorem 3.3.14.

3.3.11.44. *. Hint. Review the alternating series test, which is given in Theorem 3.3.14.

Exercises — Stage 3

3.3.11.46. *. Hint. For part (a), see Example 1.12.23.
For part (b), review Theorem 3.3.5.
For part (c), see Example 3.3.12.

3.3.11.47. *. Hint. The truncation error arising from the approximation $\displaystyle\sum_{n=1}^{\infty}\frac{e^{-\sqrt{n}}}{\sqrt{n}} \approx \sum_{n=1}^{N}\frac{e^{-\sqrt{n}}}{\sqrt{n}}$ is precisely $E_N = \displaystyle\sum_{n=N+1}^{\infty}\frac{e^{-\sqrt{n}}}{\sqrt{n}}$. You'll want to find a bound on this sum using the integral test.

A key observation is that, since $f(x) = \dfrac{e^{-\sqrt{x}}}{\sqrt{x}}$ is decreasing, we can show that

$$\frac{e^{-\sqrt{n}}}{\sqrt{n}} \le \int_{n-1}^{n}\frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$$

for every $n \ge 1$.

3.3.11.48. *. Hint. What does the fact that the series $\sum\limits_{n=0}^{\infty} a_n$ converges guarantee about the behavior of a_n for large n?

3.3.11.49. *. Hint. What does the fact that the series $\sum\limits_{n=0}^{\infty}(1-a_n)$ converges guarantee about the behavior of a_n for large n?

3.3.11.50. *. Hint. What does the fact that the series $\displaystyle\sum_{n=1}^{\infty}\frac{na_n-2n+1}{n+1}$ converges guarantee about the behavior of a_n for large n?

3.3.11.51. *. Hint. What does the fact that the series $\sum_{n=1}^{\infty} a_n$ converges guarantee about the behavior of a_n for large n? When is $x^2 \le x$?

3.3.11.52. Hint. If we add together the frequencies of all the words, they should amount to 100%. We can approximate this sum using ideas from Example 3.3.4.

3.3.11.53. Hint. We are approximating a finite sum — not an infinite series. To get greater accuracy, use exact values for the first several terms in the sum, and use an integral to approximate the rest.

3.4 · Absolute and Conditional Convergence

3.4.3 · Exercises

Exercises — Stage 1

3.4.3.1. ∗. Hint. What is conditional convergence?

3.4.3.2. Hint. If $\sum |a_n|$ converges, then $\sum a_n$ is guaranteed to converge as well. (That's Theorem 3.4.2.) So, one of the blank spaces describes an impossible sequence.

Exercises — Stage 2

3.4.3.4. ∗. Hint. Be careful about the signs.

3.4.3.5. ∗. Hint. Does the alternating series test really apply?

3.4.3.6. ∗. Hint. What does the summand look like when n is very large?

3.4.3.7. ∗. Hint. What does the summand look like when n is very large?

3.4.3.8. ∗. Hint. This is a trick question. Be sure to verify *all* of the hypotheses of any convergence test you apply.

3.4.3.9. ∗. Hint. Try the substitution $u = \log x$.

3.4.3.10. Hint. Show that it converges absolutely.

3.4.3.11. Hint. Use a similar method to Queston 10.

3.4.3.12. Hint. Show it converges absolutely using a direct comparison test.

Exercises — Stage 3

3.4.3.13. ∗. Hint. For part (a), replace n by x in the absolute value of the summand. Can you integrate the resulting function?

3.4.3.14. Hint. You don't need to add up very many terms for this level of accuracy.

3.4.3.15. Hint. Use the direct comparison test to show that the series converges absolutely.

3.5 · Power Series

3.5.3 · Exercises

Exercises — Stage 1

3.5.3.1. Hint. $f(1)$ is the sum of a geometric series.

3.5.3.2. Hint. Calculate $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{(x-5)^n}{n!+2}\right\}$ when n is a constant.

3.5.3.3. Hint. There is only one.

3.5.3.4. Hint. Use Theorem 3.5.9.

Exercises — Stage 2

3.5.3.5. ∗. Hint. Review the discussion immediately following Definition 3.5.1.

3.5.3.6. $*$. **Hint**. Review the discussion immediately following Definition 3.5.1.

3.5.3.7. $*$. **Hint**. Review the discussion immediately following Definition 3.5.1.

3.5.3.8. $*$. **Hint**. See Example 3.5.11.

3.5.3.9. $*$. **Hint**. See Example 3.5.11.

3.5.3.15. $*$. **Hint**. Start part (b) by computing the partial sums of $\displaystyle\sum_{k=1}^{\infty}\left(\frac{a_k}{a_{k+1}}-\frac{a_{k+1}}{a_{k+2}}\right)$

3.5.3.16. $*$. **Hint**. You should know a power series representation for $\dfrac{1}{1-x}$. Use it.

3.5.3.17. **Hint**. You can safely ignore one of the given equations, but not the other.

Exercises — Stage 3

3.5.3.18. $*$. **Hint**. $n \geq \log n$ for all $n \geq 1$.

3.5.3.19. $*$. **Hint**. See Example 3.5.21. For part (b), review §3.3.4.

3.5.3.20. $*$. **Hint**. You know the geometric series expansion of $\frac{1}{1-x}$. What (calculus) operation(s) can you apply to that geometric series to convert it into the given series?

3.5.3.21. $*$. **Hint**. First show that the fact that the series $\sum_{n=0}^{\infty}(1-b_n)$ converges guarantees that $\lim_{n\to\infty} b_n = 1$.

3.5.3.22. $*$. **Hint**. What does a_n look like for large n?

3.5.3.23. **Hint**. Equation 2.3.1 tells us the centre of mass of a rod with weights $\{m_n\}$ at positions $\{x_n\}$ is $\bar{x} = \dfrac{\sum m_n x_n}{\sum m_n}$.

3.5.3.24. **Hint**. Use the second derivative test.

3.5.3.25. **Hint**. What function has $\displaystyle\sum_{n=1}^{\infty} nx^{n-1}$ as its power series representation?

3.5.3.26. **Hint**. The power series representation in Example 3.5.20 is an alternating series when x is positive.

3.5.3.27. **Hint**. The power series representation in Example 3.5.21 is an alternating series when x is nonzero.

3.6 · Taylor Series

3.6.8 · Exercises

Exercises — Stage 1

3.6.8.1. **Hint**. Which of the functions are constant, linear, and quadratic?

3.6.8.2. Hint. You don't have to actually calculate the entire series $T(x)$ to answer the question.

3.6.8.3. Hint. If you don't have these memorized, it's good to be able to derive them. For instance, $\log(1+x)$ is the antiderivative of $\frac{1}{1+x}$, whose Taylor series can be found by modifying the geometric series $\sum x^n$.

3.6.8.4. Hint. See Example 3.6.18.

Exercises — Stage 2

3.6.8.5. Hint. The series will bear some resemblance to the Maclaurin series for $\log(1+x)$.

3.6.8.6. Hint. The terms $f^{(n)}(\pi)$ are going to be similar to the terms $f^{(n)}(0)$ that we used in the Maclaurin series for sine.

3.6.8.7. Hint. The Taylor series will look similar to a geometric series.

3.6.8.8. Hint. Your answer will depend on a.

3.6.8.9. *. Hint. You should know the Maclaurin series for $\frac{1}{1-x}$. Use it.

3.6.8.10. *. Hint. You should know the Maclaurin series for $\frac{1}{1-x}$. Use it.

3.6.8.11. *. Hint. You should know the Maclaurin series for e^x. Use it.

3.6.8.12. *. Hint. Review Example 3.5.20.

3.6.8.13. *. Hint. You should know the Maclaurin series for $\sin x$. Use it.

3.6.8.14. *. Hint. You should know the Maclaurin series for e^x. Use it.

3.6.8.15. *. Hint. You should know the Maclaurin series for $\arctan(x)$. Use it.

3.6.8.16. *. Hint. You should know the Maclaurin series for $\frac{1}{1-x}$. Use it.

3.6.8.17. *. Hint. Set $(-1)^n\frac{x^{2n+1}}{2n+1} = C\frac{(-1)^n}{(2n+1)3^n}$, for some constant C. What are x and C?

3.6.8.18. *. Hint. There is an important Taylor series, one of the series in Theorem 3.6.7, that looks a lot like the given series.

3.6.8.19. *. Hint. There is an important Taylor series, one of the series in Theorem 3.6.7, that looks a lot like the given series.

3.6.8.20. *. Hint. There is an important Taylor series, one of the series in Theorem 3.6.7, that looks a lot like the given series. Be careful about the limits of summation.

3.6.8.21. *. Hint. There is an important Taylor series, one of the series in Theorem 3.6.7, that looks a lot like the given series.

3.6.8.22. *. Hint. Split the series into a sum of two series. There is an important Taylor series, one of the series in Theorem 3.6.7, that looks a lot like each of the two series.

3.6.8.23. Hint. Try the ratio test.

3.6.8.24. Hint. Write it as the sum of two Taylor series.

3.6.8.25. *. Hint. Can you think of a way to eliminate the odd terms from $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$?

3.6.8.26. Hint. The series you're adding up are alternating, so it's simple to bound the error using a partial sum.

3.6.8.27. Hint. The Taylor Series is alternating, so bounding the error in a partial-sum approximation is straightforward.

3.6.8.28. Hint. The Taylor Series is not alternating, so use Theorem 3.6.3 to bound the error in a partial-sum approximation.

3.6.8.29. Hint. The Taylor Series is not alternating, so use Theorem 3.6.3 to bound the error in a partial-sum approximation.

3.6.8.30. Hint. Use Theorem 3.6.3 to bound the error in a partial-sum approximation. This theorem requires you to consider values of c between x and $x = 0$; since x could be anything from -2 to 1, you should think about values of c between -2 and 1.

3.6.8.31. Hint. Use Theorem 3.6.3 to bound the error in a partial-sum approximation.
To bound the derivative over the appropriate range, remember how to find absolute extrema.

Exercises — Stage 3

3.6.8.32. *. Hint. See Example 3.6.23.

3.6.8.33. *. Hint. See Example 3.6.23.

3.6.8.34. Hint. Set $f(x) = (1 + x + x^2)^{2/x}$, and find $\lim\limits_{x\to0} \log(f(x))$.

3.6.8.35. Hint. Use the substitution $y = \dfrac{1}{x}$, and compare to Question 34.

3.6.8.36. Hint. Start by differentiating $\displaystyle\sum_{n=0}^{\infty} x^n$.

3.6.8.37. Hint. The series bears a resemblance to the Taylor series for arctangent.

3.6.8.38. Hint. For simplification purposes, note $(1)(3)(5)(7)\cdots(2n-1) =$

$\frac{(2n)!}{2^n\, n!}$.

3.6.8.39. *. Hint. You know the Maclaurin series for $\log(1+y)$. Use it! Remember that you are asked for a series expansion in powers of $x-2$. So you want y to be some constant times $x-2$.

3.6.8.40. *. Hint. See Example 3.5.21. For parts (b) and (c), review § 3.3.4.

3.6.8.41. *. Hint. Look at the signs of successive terms in the series.

3.6.8.42. *. Hint. The magic word is "series".

3.6.8.43. *. Hint. See Example 3.6.16. For parts (b) and (c), review § 3.3.4.

3.6.8.44. *. Hint. See Example 3.6.16. For part (b), review the fundamental theorem of calculus in § 1.3. For part (c), review § 3.3.4.

3.6.8.45. *. Hint. See Example 3.6.16. For parts (b) and (c), review § 3.3.4.

3.6.8.46. *. Hint. See Example 3.6.16. For parts (b) and (c), review § 3.3.4.

3.6.8.48. *. Hint. Use the Maclaurin series for e^x.

3.6.8.49. *. Hint. For part (c), compare two power series term-by-term.

3.6.8.50. Hint. For Newton's method, recall we approximate a root of the function $g(x)$ in iterations: given an approximation x_n, our next approximation is $x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$.
To gauge your error, note that from approximation to approximation, the first digits stabilize. Keep refining your approximation until the first two digits stop changing.

3.6.8.51. Hint. First, modify your known Maclaurin series for arctangent into a Maclaurin series for $f(x)$. This series is not hard to repeatedly differentiate, so use it to find a power series for $f^{(10)}(x)$.

3.6.8.52. Hint. Remember e^x is never negative for any real number x.

3.6.8.53. Hint. Since $f(x)$ is odd, $f(-x) = -f(x)$ for all x in its domain. Consider the even-indexed terms and odd-indexed terms of the Taylor series.

Appendix F

Answers to Exercises

1 · Integration

1.1 · Definition of the Integral

1.1.8 · Exercises

Exercises — Stage 1

1.1.8.1. Answer. The area is between 1.5 and 2.5 square units.

1.1.8.2. Answer. The shaded area is between 2.75 and 4.25 square units. (Other estimates are possible, but this is a reasonable estimate, using methods from this chapter.)

1.1.8.3. Answer. The area under the curve is a number in the interval $\left(\frac{3}{8}\left[\frac{1}{2}+\frac{1}{\sqrt{2}}\right], \frac{3}{8}\left[1+\frac{1}{\sqrt{2}}\right]\right)$.

1.1.8.4. Answer. left

1.1.8.5. Answer. Many answers are possible. One example is $f(x) = \sin x$, $[a,b] = [0,\pi]$, $n = 1$. Another example is $f(x) = \sin x$, $[a,b] = [0,5\pi]$, $n = 5$.

1.1.8.6. Answer. Some of the possible answers are given, but more exist.

a $\displaystyle\sum_{i=3}^{7} i \quad ; \quad \sum_{i=1}^{5}(i+2)$

b $\displaystyle\sum_{i=3}^{7} 2i \quad ; \quad \sum_{i=1}^{5}(2i+4)$

c $\displaystyle\sum_{i=3}^{7}(2i+1) \quad ; \quad \sum_{i=1}^{5}(2i+5)$

d $\sum_{i=1}^{8}(2i-1) \quad ; \quad \sum_{i=0}^{7}(2i+1)$

1.1.8.7. Answer. Some answers are below, but others are possible.

a $\sum_{i=1}^{4}\frac{1}{3^i}$ and $\sum_{i=1}^{4}\left(\frac{1}{3}\right)^i$

b $\sum_{i=1}^{4}\frac{2}{3^i}$ and $\sum_{i=1}^{4}2\left(\frac{1}{3}\right)^i$

c $\sum_{i=1}^{4}(-1)^i\frac{2}{3^i}$ and $\sum_{i=1}^{4}\frac{2}{(-3)^i}$

d $\sum_{i=1}^{4}(-1)^{i+1}\frac{2}{3^i}$ and $\sum_{i=1}^{4}-\frac{2}{(-3)^i}$

1.1.8.8. Answer.

a $\sum_{i=1}^{5}\frac{2i-1}{3^i}$

b $\sum_{i=1}^{5}\frac{1}{3^i+2}$

c $\sum_{i=1}^{7}i\cdot 10^{4-i}$ and $\sum_{i=1}^{7}\frac{i}{10^{i-4}}$

1.1.8.9. Answer.

a $\frac{5}{2}\left[1-\left(\frac{3}{5}\right)^{101}\right]$

b $\frac{5}{2}\left(\frac{3}{5}\right)^{50}\left[1-\left(\frac{3}{5}\right)^{51}\right]$

c 270

d $\frac{1-\left(\frac{1}{e}\right)^b}{e-1}+\frac{e}{4}\left[b(b+1)\right]^2$

1.1.8.10. Answer.

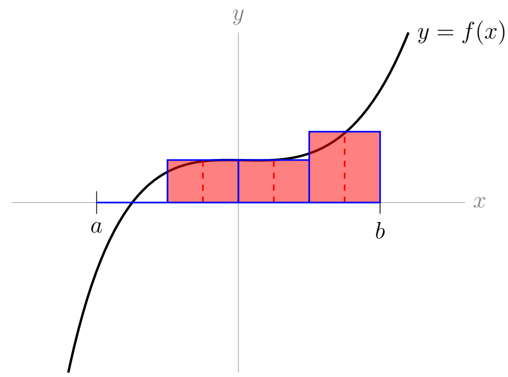
a $50\cdot 51=2550$

b $\left[\frac{1}{2}(95)(96)\right]^2-\left[\frac{1}{2}(4)(5)\right]^2=20,793,500$

c -1

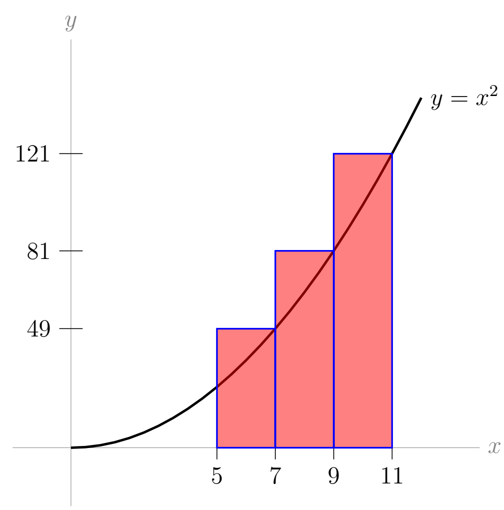
d -10

1.1.8.11. Answer.

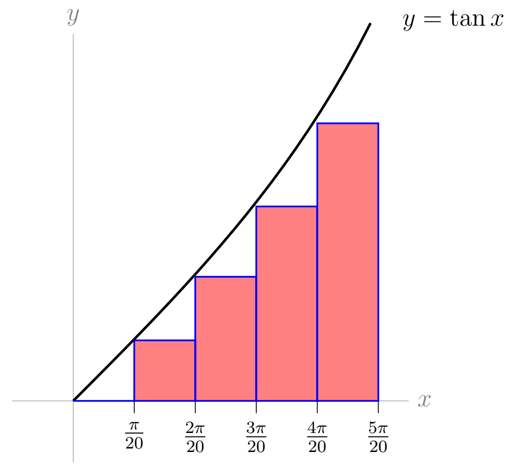


1.1.8.12. *. Answer. $n = 4$, $a = 2$, and $b = 6$

1.1.8.13. Answer. One answer is below, but other interpretations exist.



1.1.8.14. Answer. Many interpretations are possible–see the solution to Question 13 for a more thorough discussion–but the most obvious is given below.



1.1.8.15. *. Answer. Three answers are possible. It is a midpoint Riemann sum for f on the interval $[1,5]$ with $n=4$. It is also a left Riemann sum for f on the interval $[1.5, 5.5]$ with $n=4$. It is also a right Riemann sum for f on the interval $[0.5, 4.5]$ with $n=4$.

1.1.8.16. Answer. $\dfrac{25}{2}$

1.1.8.17. Answer. $\dfrac{21}{2}$

Exercises — Stage 2

1.1.8.18. *. Answer. $\sum_{i=1}^{50}\left(5+\left(i-\frac{1}{2}\right)\frac{1}{5}\right)^8 \frac{1}{5}$

1.1.8.19. *. Answer. 54

1.1.8.20. *. Answer. $\displaystyle\int_{-1}^{7} f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(-1+\frac{8i}{n}\right)\frac{8}{n}$

1.1.8.21. *. Answer. $f(x)=\sin^2(2+x)$ and $b=4$

1.1.8.22. *. Answer. $f(x)=x\sqrt{1-x^2}$

1.1.8.23. *. Answer. $\int_0^3 e^{-x/3}\cos(x)\,\mathrm{d}x$

1.1.8.24. *. Answer. $\displaystyle\int_0^1 xe^x\,\mathrm{d}x$

1.1.8.25. *. Answer. Possible answers include:

$$\int_0^2 e^{-1-x}\,\mathrm{d}x$$

$$\int_1^3 e^{-x}\ \mathrm{d}x$$

$$2\int_{1/2}^{3/2} e^{-2x}\ \mathrm{d}x \text{ and}$$

$$2\int_0^1 e^{-1-2x}\ \mathrm{d}x.$$

1.1.8.26. Answer. $\dfrac{r^{3n+3}-1}{r^3-1}$

1.1.8.27. Answer. $r^5\left(\dfrac{r^{96}-1}{r-1}\right)$

1.1.8.28. *. Answer. 5

1.1.8.29. Answer. 16

1.1.8.30. Answer. $\dfrac{b^2-a^2}{2}$

1.1.8.31. Answer. $\dfrac{b^2-a^2}{2}$

1.1.8.32. Answer. 4π

1.1.8.33. *. Answer. $\displaystyle\int_0^3 f(x)\ \mathrm{d}x = 2.5$

1.1.8.34. *. Answer. 53 m

1.1.8.35. Answer. true

1.1.8.36. Answer. 3200 km

Exercises — Stage 3

1.1.8.37. *. Answer. (a) There are many possible answers. Two are $\int_{-2}^0 \sqrt{4-x^2}\ \mathrm{d}x$ and $\int_0^2 \sqrt{4-(-2+x)^2}\ \mathrm{d}x$.
(b) π

1.1.8.38. *. Answer. (a) 30
(b) $41\frac{1}{4}$

1.1.8.39. *. Answer. $\dfrac{56}{3}$

1.1.8.40. *. Answer. 6

1.1.8.41. *. Answer. 12

1.1.8.42. Answer. $f(x) = \dfrac{3}{10}\left(\dfrac{x}{5}+8\right)^2 \sin\left(\dfrac{2x}{5}+2\right)$

1.1.8.43. Answer. $\dfrac{1}{\log 2}$

1.1.8.44. Answer. (a) $\dfrac{1}{\log 10}\left(10^b - 10^a\right)$
(b) $\dfrac{1}{\log c}\left(c^b - c^a\right)$; yes, it agrees.

1.1.8.45. Answer. $\frac{\pi}{4} - \frac{1}{2}\arccos(a) + \frac{1}{2}a\sqrt{1-a^2}$

1.1.8.46. Answer.

a $[f(b) - f(a)] \cdot \dfrac{b-a}{n}$

b Choose n to be an integer that is greater than or equal to $100\,[f(b) - f(a)]\,(b-a)$.

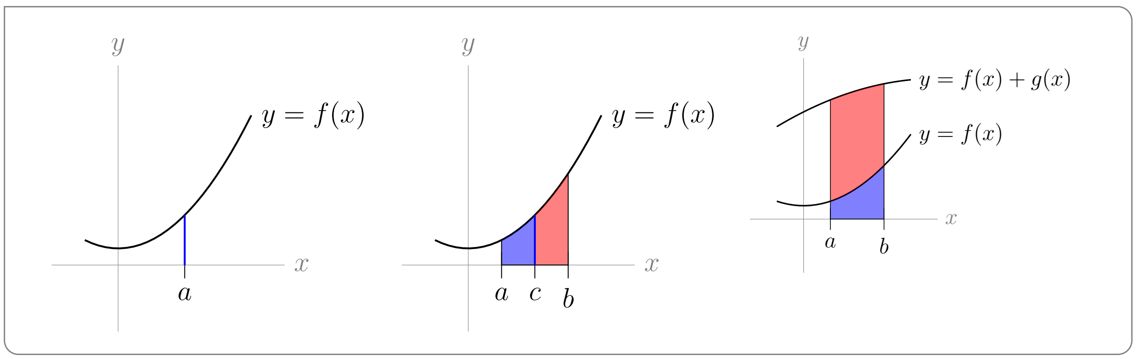
1.1.8.47. Answer. true (but note, for a non-linear function, it is possible that the midpoint Riemann sum is *not* the average of the other two)

1.2 · Basic properties of the definite integral

1.2.3 · Exercises

Exercises — Stage 1

1.2.3.1. Answer. Possible drawings:



1.2.3.2. Answer. $\sin b - \sin a$

1.2.3.3. *. Answer. (a) False. For example, the function

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \ge 0 \end{cases}$$

provides a counterexample.
(b) False. For example, the function $f(x) = x$ provides a counterexample.
(c) False. For example, the functions

$$f(x) = \begin{cases} 0 & \text{for } x < \frac{1}{2} \\ 1 & \text{for } x \ge \frac{1}{2} \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 0 & \text{for } x \ge \frac{1}{2} \\ 1 & \text{for } x < \frac{1}{2} \end{cases}$$

provide a counterexample.

1.2.3.4. Answer. (a) $-\frac{1}{20}$, (b) positive, (c) negative, (d) positive.

Exercises — Stage 2

1.2.3.5. *. Answer. -21

1.2.3.6. *. Answer. -6

1.2.3.7. *. Answer. 20

1.2.3.8. Answer.

a $\frac{\pi}{4}-\frac{1}{2}\arccos(-a)-\frac{1}{2}a\sqrt{1-a^2}=-\frac{\pi}{4}+\frac{1}{2}\arccos(a)-\frac{1}{2}a\sqrt{1-a^2}$

b $\frac{1}{2}\arccos(a)-\frac{1}{2}a\sqrt{1-a^2}$

1.2.3.9. *. Answer. 5

1.2.3.10. Answer. 0

1.2.3.11. Answer. 5

Exercises — Stage 3

1.2.3.12. *. Answer. $20+2\pi$

1.2.3.13. *. Answer. 0

1.2.3.14. *. Answer. 0

1.2.3.15. Answer. 0

1.2.3.16. Answer. (a) $y=\frac{1}{b}\sqrt{1-(ax)^2}$

(b) $\frac{a}{b}\int_{-\frac{1}{a}}^{\frac{1}{a}}\sqrt{\frac{1}{a^2}-x^2}\mathrm{d}x$

(c) $\frac{\pi}{ab}$

1.2.3.17. Answer.

$\times$	even	odd
even	even	odd
odd	odd	even

1.2.3.18. Answer. $f(0)=0$; $g(0)$ can be any real number

1.2.3.19. Answer. $f(x)=0$ for every x

1.2.3.20. Answer. The derivative of an even function is odd, and the derivative of an odd function is even.

1.3 · The Fundamental Theorem of Calculus

1.3.2 · Exercises

Exercises — Stage 1

1.3.2.1. *. Answer. $e^2 - e^{-2}$

1.3.2.2. *. Answer. $F(x) = \dfrac{x^4}{4} + \dfrac{1}{2}\cos 2x + \dfrac{1}{2}$.

1.3.2.3. *. Answer. (a) True
(b) False
(c) False, unless $\int_a^b f(x)\,\mathrm{d}x = \int_a^b xf(x)\,\mathrm{d}x = 0$.

1.3.2.4. Answer. false

1.3.2.5. Answer. false

1.3.2.6. Answer. $\sin(x^2)$

1.3.2.7. Answer. $\sqrt[3]{e}$

1.3.2.8. Answer. For any constant C, $F(x)+C$ is an antiderivative of $f(x)$. So, for example, $F(x)$ and $F(x)+1$ are both antiderivatives of $f(x)$.

1.3.2.9. Answer.

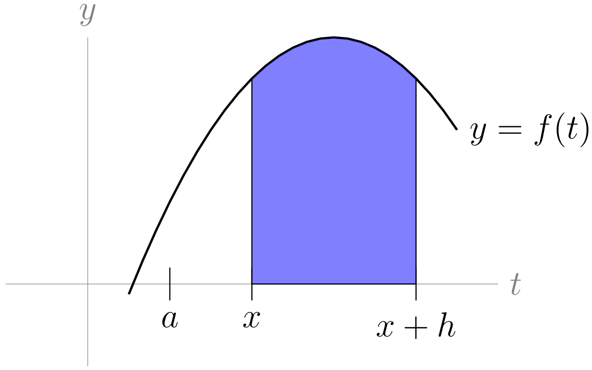
a We differentiate with respect to a. Recall $\frac{\mathrm{d}}{\mathrm{d}x}\{\arccos x\} = \frac{-1}{\sqrt{1-x^2}}$. To differentiate $\frac{1}{2}a\sqrt{1-a^2}$, we use the product and chain rules.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}a}\left\{\frac{\pi}{4} - \frac{1}{2}\arccos(a) + \frac{1}{2}a\sqrt{1-a^2}\right\}\\
&= 0 - \frac{1}{2}\cdot\frac{-1}{\sqrt{1-a^2}} + \left(\frac{1}{2}a\right)\cdot\frac{-2a}{2\sqrt{1-a^2}} + \frac{1}{2}\sqrt{1-a^2}\\
&= \frac{1}{2\sqrt{1-a^2}} - \frac{a^2}{2\sqrt{1-a^2}} + \frac{1-a^2}{2\sqrt{1-a^2}}\\
&= \frac{1-a^2+1-a^2}{2\sqrt{1-a^2}}\\
&= \frac{2(1-a^2)}{2\sqrt{1-a^2}}\\
&= \sqrt{1-a^2}
\end{aligned}$$

b $F(x) = \dfrac{5\pi}{4} - \dfrac{1}{2}\arccos(x) + \dfrac{1}{2}x\sqrt{1-x^2}$

1.3.2.10. Answer. (a) 0
(b),(c) The FTC does not apply, because the integrand is not continuous over the interval of integration.

1.3.2.11. Answer.



1.3.2.12. Answer. (a) zero
(b) increasing when $0 < x < 1$ and $3 < x < 4$; decreasing when $1 < x < 3$

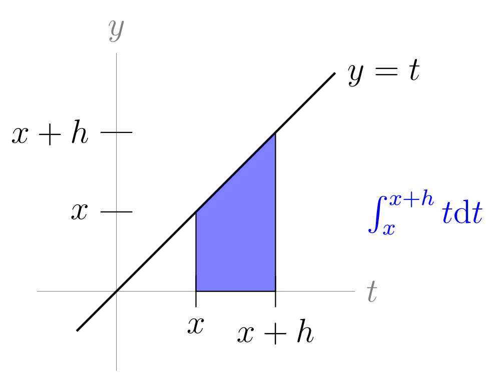
1.3.2.13. Answer. (a) zero
(b) $G(x)$ is increasing when $1 < x < 3$, and it is decreasing when $0 < x < 1$ and when $3 < x < 4$.

1.3.2.14. Answer. Using the definition of the derivative,

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0} \frac{\int_a^{x+h} t\mathrm{d}t - \int_a^x t\mathrm{d}t}{h} \\ &= \lim_{h\to 0} \frac{\int_x^{x+h} t\mathrm{d}t}{h} \end{aligned}$$

The numerator describes the area of a trapezoid with base h and heights x and $x+h$.

$$\begin{aligned} &= \lim_{h\to 0} \frac{\frac{1}{2}h(x+x+h)}{h} \\ &= \lim_{h\to 0} \left(x + \frac{1}{2}h\right) \\ &= x \end{aligned}$$



So, $F'(x) = x$.

1.3.2.15. Answer. $f(t) = 0$

1.3.2.16. Answer. $\int \log(ax)\mathrm{d}x = x\log(ax) - x + C$, where a is a given constant, and C is any constant.

1.3.2.17. Answer. $\int x^3e^x\mathrm{d}x = e^x\left(x^3 - 3x^2 + 6x - 6\right) + C$

1.3.2.18. Answer. $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\mathrm{d}x = \log\left|x + \sqrt{x^2+a^2}\right| + C$ when a is a given constant. As usual, C is an arbitrary constant.

1.3.2.19. Answer. $\displaystyle\int \frac{x}{\sqrt{x(a+x)}}\mathrm{d}x = \sqrt{x(a+x)} - a\log\left(\sqrt{x} + \sqrt{a+x}\right) + C$

Exercises — Stage 2

1.3.2.20. *. Answer. $5 - \cos 2$

1.3.2.21. *. Answer. 2

1.3.2.22. Answer. $\dfrac{1}{5}\arctan(5x) + C$

1.3.2.23. Answer. $\arcsin\left(\dfrac{x}{\sqrt{2}}\right) + C$

1.3.2.24. Answer. $\tan x - x + C$

1.3.2.25. Answer. $-\dfrac{3}{4}\cos(2x) + C$, or equivalently, $\dfrac{3}{2}\sin^2 x + C$

1.3.2.26. Answer. $\dfrac{1}{2}x + \dfrac{1}{4}\sin(2x) + C$

1.3.2.27. *. Answer. $F'\left(\frac{\pi}{2}\right) = \log(3)$
$G'\left(\frac{\pi}{2}\right) = -\log(3)$

1.3.2.28. *. Answer. $f(x)$ is increasing when $-\infty < x < 1$ and when $2 < x < \infty$.

1.3.2.29. *. Answer. $F'(x) = -\dfrac{\sin x}{\cos^3 x + 6}$

1.3.2.30. *. Answer. $4x^3e^{(1+x^4)^2}$

1.3.2.31. *. Answer. $\left(\sin^6 x + 8\right)\cos x$

1.3.2.32. *. Answer. $F'(1) = 3e^{-1}$

1.3.2.33. *. Answer. $\dfrac{\sin u}{1 + \cos^3 u}$

1.3.2.34. *. Answer. $f(x) = 2x$

1.3.2.35. *. Answer. $f(4) = 4\pi$

1.3.2.36. *. Answer. (a) $(2x+1)e^{-x^2}$
(b) $x = -1/2$

1.3.2.37. *. Answer. $e^{\sin x} - e^{\sin(x^4-x^3)}\left(4x^3-3x^2\right)$

1.3.2.38. *. Answer. $-2x\cos\left(e^{-x^2}\right) - 5x^4\cos\left(e^{x^5}\right)$

1.3.2.39. *. Answer. $e^x\sqrt{\sin(e^x)} - \sqrt{\sin(x)}$

1.3.2.40. *. Answer. 14

Exercises — Stage 3

1.3.2.41. *. Answer. $\frac{5}{2}$

1.3.2.42. *. Answer. 45 m

1.3.2.43. *. Answer. $f'(x) = (2-2x)\log\left(1+e^{2x-x^2}\right)$ and $f(x)$ achieves its absolute maximum at $x = 1$, because $f(x)$ is increasing for $x < 1$ and decreasing for $x > 1$.

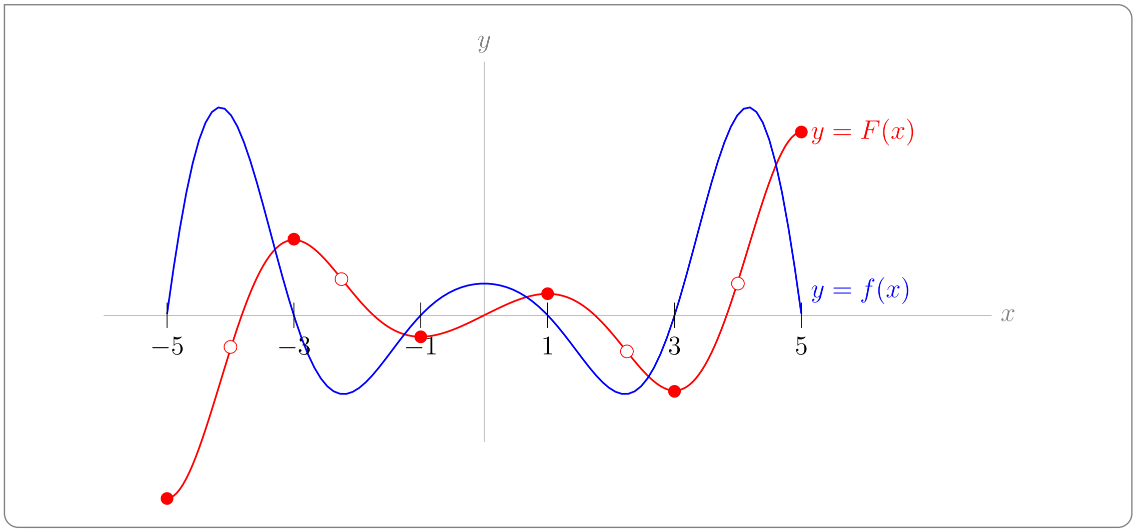
1.3.2.44. *. Answer. The minimum is $\int_0^{-1}\frac{\mathrm{d}t}{1+t^4}$. As x runs from $-\infty$ to ∞, the function $f(x) = \int_0^{x^2-2x}\frac{\mathrm{d}t}{1+t^4}$ decreases until x reaches 1 and then increases all $x > 1$. So the minimum is achieved for $x = 1$. At $x = 1$, $x^2 - 2x = -1$.

1.3.2.45. *. Answer. F achieves its maximum value at $x = \pi$.

1.3.2.46. *. Answer. 2

1.3.2.47. *. Answer. $\log 2$

1.3.2.48. Answer. In the sketch below, open dots denote inflection points, and closed dots denote extrema.



1.3.2.49. *. Answer. (a) $3x^2\int_0^{x^3+1} e^{t^3}\,\mathrm{d}t + 3x^5e^{(x^3+1)^3}$
(b) $y = -3(x+1)$

1.3.2.50. Answer. Both students.

1.3.2.51. Answer. (a) $27(1-\cos 3)$
(b) $x^3\sin(x)+3x^2[1-\cos(x)]$

1.3.2.52. Answer. If $f(x)=0$ for all x, then $F(x)$ is even and possibly also odd. If $f(x)\neq 0$ for some x, then $F(x)$ is not even. It might be odd, and it might be neither even nor odd.
(Perhaps surprisingly, every antiderivative of an odd function is even.)

1.4 · Substitution

1.4.2 · Exercises

Exercises — Stage 1

1.4.2.1. Answer. (a) true
(b) false

1.4.2.2. Answer. The reasoning is not sound: when we do a substitution, we need to take care of the differential ($\mathrm{d}x$). Remember the method of substitution comes from the chain rule: there should be a function *and its derivative.* Here's the way to do it:

Problem: Evaluate $\displaystyle\int (2x+1)^2\mathrm{d}x$.

Work: We use the substitution $u=2x+1$. Then $\mathrm{d}u=2\mathrm{d}x$, so $\mathrm{d}x=\frac{1}{2}\mathrm{d}u$:

$$\begin{aligned}\int (2x+1)^2\mathrm{d}x &= \int u^2\cdot\frac{1}{2}\mathrm{d}u\\ &= \frac{1}{6}u^3+C\\ &= \frac{1}{6}(2x+1)^3+C\end{aligned}$$

1.4.2.3. Answer. The problem is with the limits of integration, as in Question 1. Here's how it ought to go:

Problem: Evaluate $\displaystyle\int_1^\pi \frac{\cos(\log t)}{t}\mathrm{d}t$.

Work: We use the substitution $u=\log t$, so $\mathrm{d}u=\frac{1}{t}\mathrm{d}t$. When $t=1$, we have $u=\log 1=0$ and when $t=\pi$, we have $u=\log(\pi)$. Then:

$$\begin{aligned}\int_1^\pi \frac{\cos(\log t)}{t}\mathrm{d}t &= \int_{\log 1}^{\log(\pi)}\cos(u)\mathrm{d}u\\ &= \int_0^{\log(\pi)}\cos(u)\mathrm{d}u\\ &= \sin(\log(\pi))-\sin(0)=\sin(\log(\pi)).\end{aligned}$$

1.4.2.4. Answer. This one is OK.

1.4.2.5. $*$. **Answer**. $\displaystyle\int_0^1 \frac{f(u)}{\sqrt{1-u^2}}\,\mathrm{d}u$. Because the denominator $\sqrt{1-u^2}$ vanishes when $u=1$, this is what is known as an improper integral. Improper integrals will be discussed in Section 1.12.

1.4.2.6. **Answer**. some constant C

Exercises — Stage 2

1.4.2.7. $*$. **Answer**. $\dfrac{1}{2}\big(\sin(e)-\sin(1)\big)$

1.4.2.8. $*$. **Answer**. $\dfrac{1}{3}$

1.4.2.9. $*$. **Answer**. $-\dfrac{1}{300(x^3+1)^{100}}+C$

1.4.2.10. $*$. **Answer**. $\log 4$

1.4.2.11. $*$. **Answer**. $\log 2$

1.4.2.12. $*$. **Answer**. $\dfrac{4}{3}$

1.4.2.13. $*$. **Answer**. e^6-1

1.4.2.14. $*$. **Answer**. $\dfrac{1}{3}(4-x^2)^{3/2}+C$

1.4.2.15. **Answer**. $e^{\sqrt{\log x}}+C$

Exercises — Stage 3

1.4.2.16. $*$. **Answer**. 0

1.4.2.17. $*$. **Answer**. $\dfrac{1}{2}[\cos 1-\cos 2]\approx 0.478$

1.4.2.18. **Answer**. $\dfrac{1}{2}-\dfrac{1}{2}\log 2$

1.4.2.19. **Answer**. $\frac{1}{2}\tan^2\theta-\log|\sec\theta|+C$

1.4.2.20. **Answer**. $\arctan(e^x)+C$

1.4.2.21. **Answer**. $\dfrac{\pi}{4}-\dfrac{2}{3}$

1.4.2.22. **Answer**. $-\frac{1}{2}\left(\log(\cos x)\right)^2+C$

1.4.2.23. $*$. **Answer**. $\frac{1}{2}\sin(1)$

1.4.2.24. $*$. **Answer**. $\dfrac{1}{3}[2\sqrt{2}-1]\approx 0.609$

1.4.2.25. Answer. Using the definition of a definite integral with right Riemann sums:

$$\begin{aligned}
\int_a^b 2f(2x)\,\mathrm{d}x &= \lim_{n\to\infty}\sum_{i=1}^{n} \Delta x\cdot 2f(2(a+i\Delta x)) \qquad \Delta x = \frac{b-a}{n}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{b-a}{n}\right)\cdot 2f\left(2\left(a+i\left(\frac{b-a}{n}\right)\right)\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2b-2a}{n}\right)\cdot f\left(2a+i\left(\frac{2b-2a}{n}\right)\right)\\
\int_{2a}^{2b} f(x)\,\mathrm{d}x &= \lim_{n\to\infty}\sum_{i=1}^{n} \Delta x\cdot f(2a+i\Delta x) \qquad \Delta x = \frac{2b-2a}{n}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2b-2a}{n}\right)\cdot f\left(2a+i\left(\frac{2b-2a}{n}\right)\right)
\end{aligned}$$

Since the Riemann sums are exactly the same,

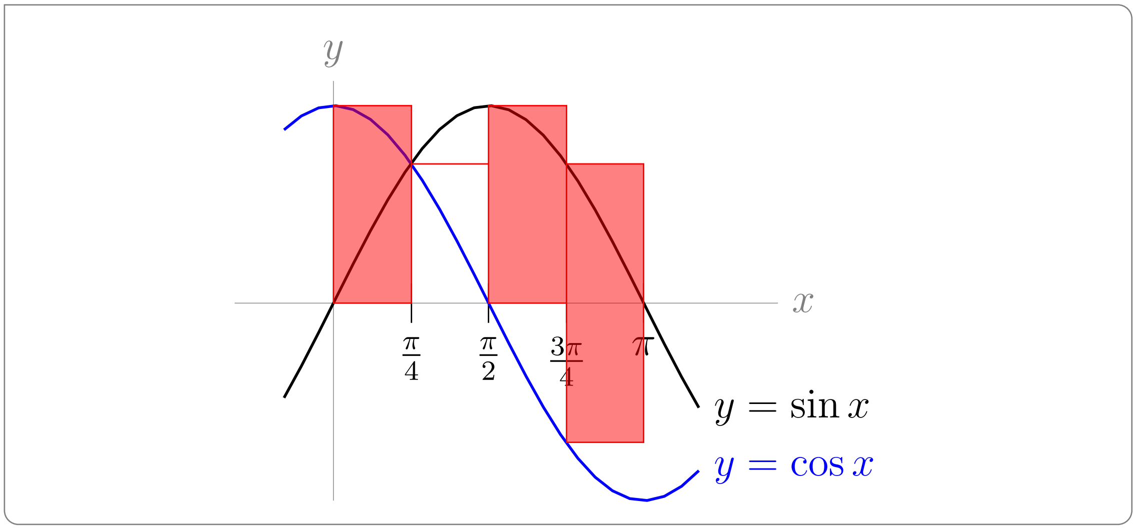
$$\int_a^b 2f(2x)\,\mathrm{d}x = \int_{2a}^{2b} f(x)\,\mathrm{d}x$$

1.5 · Area between curves

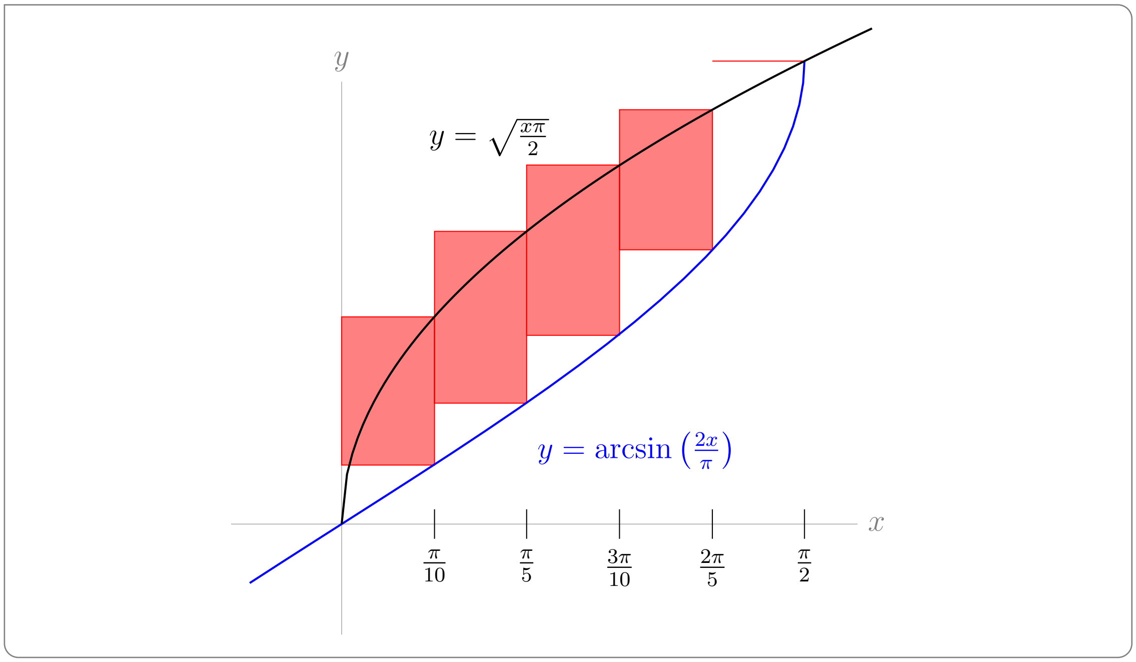
1.5.2 · Exercises

Exercises — Stage 1

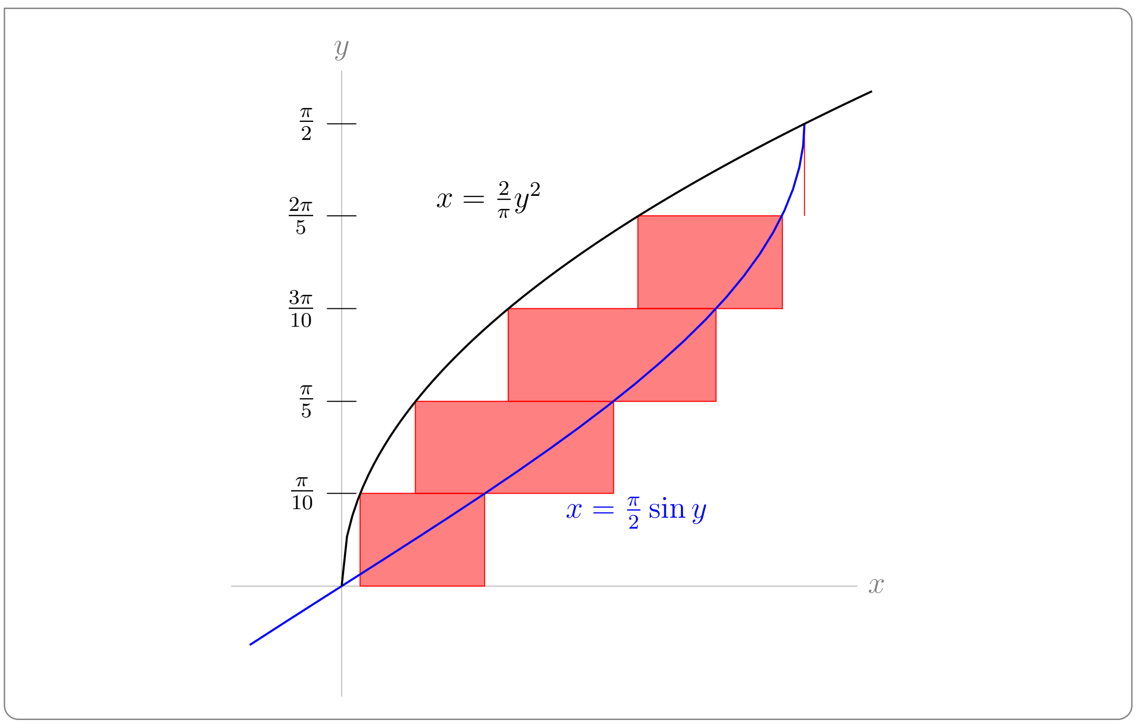
1.5.2.1. Answer. Area between curves $\approx \frac{\pi}{4}\left(2+\sqrt{2}\right)$



1.5.2.2. Answer. (a) Vertical rectangles:



(b) One possible answer:



1.5.2.3. *. Answer. $\displaystyle\int_0^{\sqrt{2}} \left[2x - x^3\right] \mathrm{d}x$

1.5.2.4. *. Answer. $\displaystyle\int_{-3/2}^{4} \left[\frac{4}{5}(6 - y^2) + 2y\right] \mathrm{d}y$

1.5.2.5. *. Answer. $\displaystyle\int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a}\right] \mathrm{d}x$

1.5.2.6. *. Answer. $\displaystyle\int_1^{25}\left[-\frac{1}{12}(x+5)+\frac{1}{2}\sqrt{x}\right]\mathrm{d}x$

Exercises — Stage 2

1.5.2.7. *. Answer. $\dfrac{1}{8}$

1.5.2.8. *. Answer. $\dfrac{4}{3}$

1.5.2.9. *. Answer. $\dfrac{5}{3}-\dfrac{1}{\log 2}$

1.5.2.10. *. Answer. $\dfrac{8}{\pi}-1$

1.5.2.11. *. Answer. $\dfrac{20}{9}$

1.5.2.12. *. Answer. $\dfrac{1}{6}$

1.5.2.13. Answer. 2π

Exercises — Stage 3

1.5.2.14. *. Answer. $2\left[\pi-\frac{1}{4}\pi^2\right]$

1.5.2.15. *. Answer. $\dfrac{31}{6}$

1.5.2.16. *. Answer. $\dfrac{26}{3}$

1.5.2.17. Answer. $\dfrac{7\pi}{8}-\dfrac{1}{2}$

1.5.2.18. Answer. $12\sqrt{2}-\dfrac{13}{4}$

1.6 · Volumes

1.6.2 · Exercises

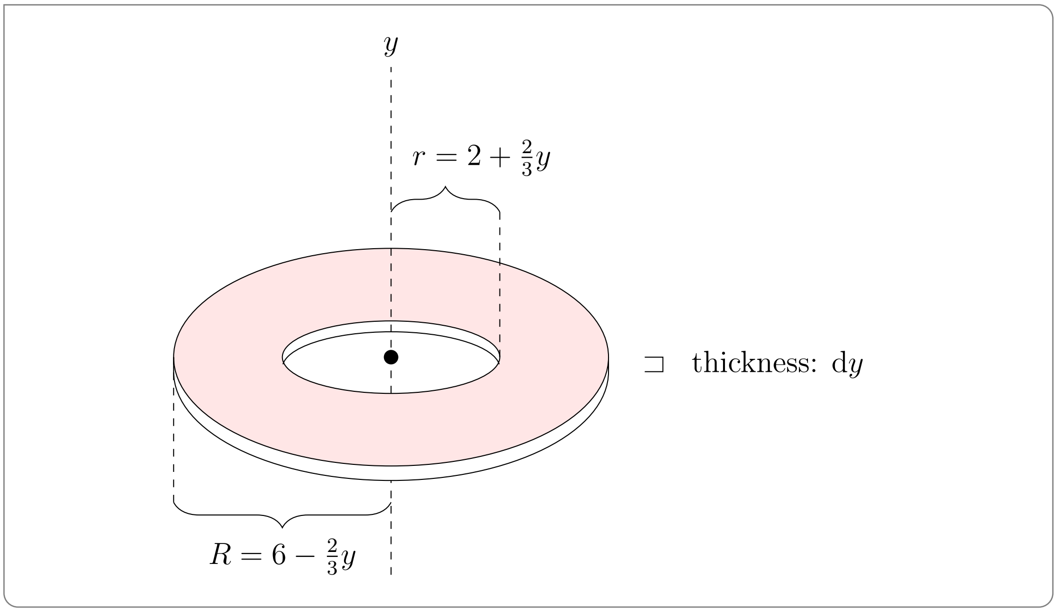
Exercises — Stage 1

1.6.2.1. Answer. The horizontal cross-sections are circles, but the vertical cross-sections are not.

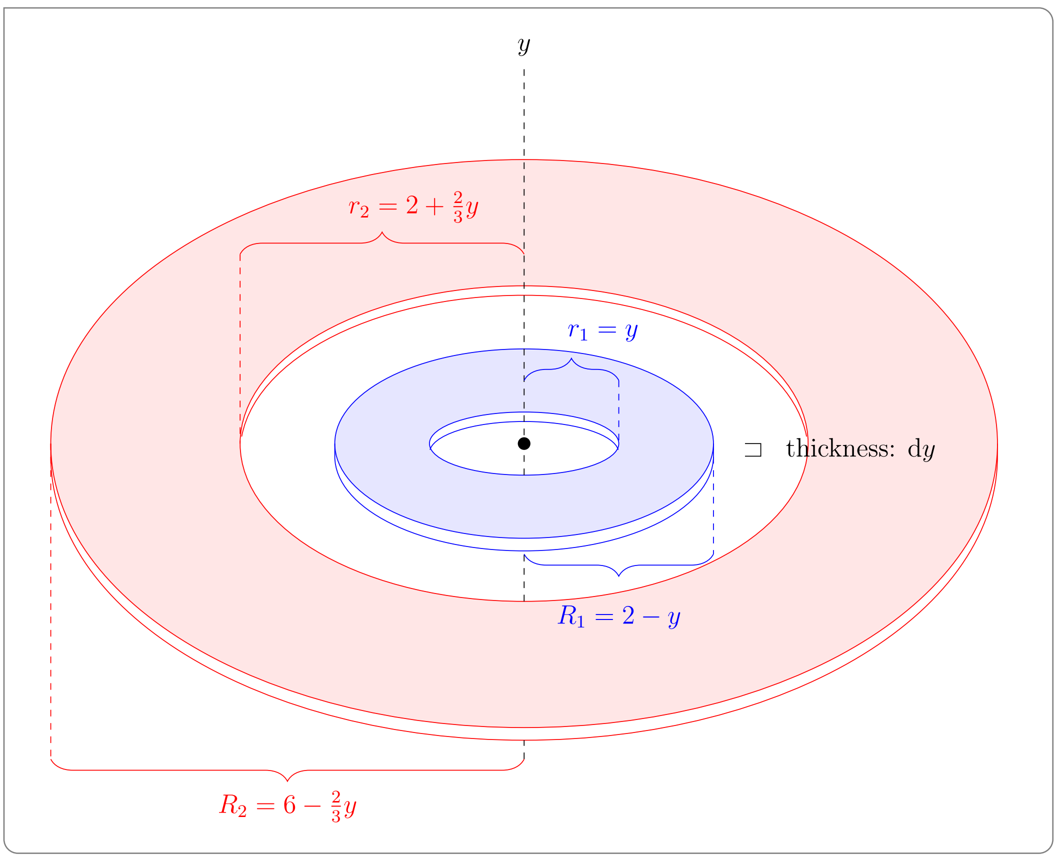
1.6.2.2. Answer. The columns have the same volume.

1.6.2.3. Answer.

- Washers when $\mathbf{1<y\leq 6}$: If $y>1$, then our washer has inner radius $2+\frac{2}{3}y$, outer radius $6-\frac{2}{3}y$, and height $\mathrm{d}y$.



- **Washers when $\mathbf{0 \le y < 1}$:** When $0 \le y < 1$, we have a "double washer," two concentric rings. The inner washer has inner radius $r_1 = y$ and outer radius $R_1 = 2 - y$. The outer washer has inner radius $r_2 = 2 + \frac{2}{3}y$ and outer radius $R_2 = 6 - \frac{2}{3}y$. The thickness of the washers is dy.



1.6.2.4. $*$. Answer. (a) $\pi\int_0^3 xe^{2x^2}\,\mathrm{d}x$

(b) $\int_0^1 \pi\left[\left(3+\sqrt{y}\right)^2-\left(3-\sqrt{y}\right)^2\right]\mathrm{d}y+\int_1^4 \pi\left[\left(5-y\right)^2-\left(3-\sqrt{y}\right)^2\right]\mathrm{d}y$

1.6.2.5. $*$. Answer. (a) $\int_{-1}^1 \pi\left[(5-4x^2)^2-(2-x^2)^2\right]\,\mathrm{d}x$

(b) $\int_{-1}^0 \pi\left[\left(5+\sqrt{y+1}\right)^2-\left(5-\sqrt{y+1}\right)^2\right]\,\mathrm{d}y$

1.6.2.6. $*$. Answer. $\pi\int_{-2}^2 \left[(9-x^2)^2-(x^2+1)^2\right]\mathrm{d}x$

1.6.2.7. Answer. $\dfrac{\sqrt{2}}{12}\ell^3$

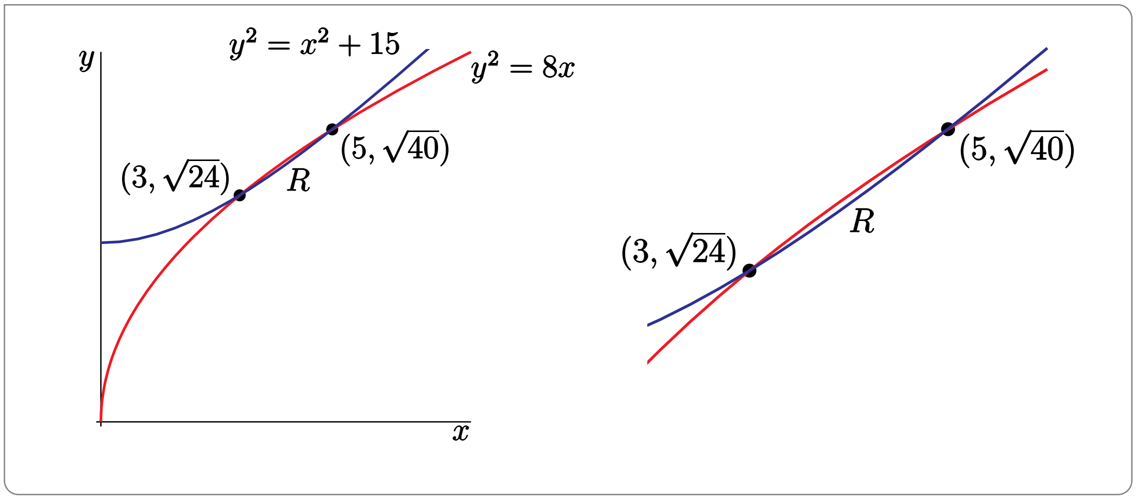
Exercises — Stage 2

1.6.2.8. $*$. Answer. $\dfrac{\pi}{4}\left(e^{2a^2}-1\right)$

1.6.2.9. $*$. Answer. $\pi\left[\dfrac{38}{3}-\dfrac{514}{3^4}\right]=\pi\dfrac{512}{81}$

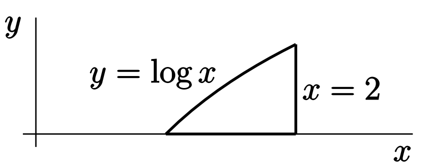
1.6.2.10. $*$. Answer. (a) $8\pi\int_{-1}^1\sqrt{1-x^2}\,\mathrm{d}x$
(b) $4\pi^2$

1.6.2.11. $*$. Answer. (a) The region R is the region between the blue and red curves, with $3\le x\le 5$, in the figures below.



(b) $\frac{4}{3}\pi\approx 4.19$

1.6.2.12. $*$. Answer. (a) The region R is sketched below.



(b) $\pi\left[4\log 2 - \frac{3}{2}\right] \approx 3.998$

1.6.2.13. *. Answer. $\pi^2 + 8\pi^3 + \frac{8\pi^6}{5}$

1.6.2.14. *. Answer. $\dfrac{8}{3}$

1.6.2.15. *. Answer. $\dfrac{256 \times 8}{15} = 136.5\dot{3}$

1.6.2.16. *. Answer. $\dfrac{28}{3}\pi h$

Exercises — Stage 3

1.6.2.17. Answer.

- (a) $\dfrac{4\pi}{3b^2a}$ cubic units,
- (b) $a = \dfrac{1}{6356.752}$ and $b = \dfrac{1}{6378.137}$,
- (c) Approximately $1.08321 \times 10^{12}\text{km}^3$, or $1.08321 \times 10^{21}\text{m}^3$,
- (d) Absolute error is about $3.64 \times 10^9\text{km}^3$, and relative error is about 0.00336, or 0.336%.

1.6.2.18. *. Answer. (a) $\dfrac{9}{2}$ (b) $\pi\displaystyle\int_{-1}^{2}\left[\big(4-x\big)^2 - \big(1+(x-1)^2\big)^2\right]\mathrm{d}x$

1.6.2.19. *. Answer. (a) $\dfrac{\pi}{2} - 1$ (b) $\dfrac{\pi^2}{2} - \pi \approx 1.793$

1.6.2.20. *. Answer. (a) $V_1 = \dfrac{4}{3}\pi c^2$ (b) $V_2 = \dfrac{\pi c}{3}\left[4\sqrt{2} - 2\right]$ (c) $c = 0$ or $c = \sqrt{2} - \frac{1}{2}$

1.6.2.21. *. Answer.

$$\int_{\pi/2}^{\pi} \pi\left[(5+\pi\sin x)^2 - (5+2\pi-2x)^2\right]\mathrm{d}x + \int_{\pi}^{3\pi/2} \pi\left[(5+2\pi-2x)^2 - (5+\pi\sin x)^2\right]\,\mathrm{d}x$$

1.6.2.22. Answer. (a) $\dfrac{6000c\pi}{\log 2}\left(1-\dfrac{1}{2^{10}}\right)$, which is close to $\dfrac{6000c\pi}{\log 2}$.
(b) 6km: that is, there is roughly the same mass of air in the lowest 6 km of the column as there is in the remaining 54 km.

1.7 · Integration by parts

1.7.2 · Exercises

Exercises — Stage 1

1.7.2.1. Answer. chain; product

1.7.2.2. Answer. The part chosen as u will be differentiated. The part chosen as $\mathrm{d}v$ will be antidifferentiated.

1.7.2.3. Answer. $\displaystyle\int \frac{f'(x)}{g(x)}\mathrm{d}x = \frac{f(x)}{g(x)} + \int \frac{f(x)g'(x)}{g^2(x)}\mathrm{d}x$

1.7.2.4. Answer. All the antiderivatives differ only by a constant, so we can write them all as $v(x)+C$ for some C. Then, using the formula for integration by parts,

$$\begin{aligned}\int u(x)\cdot v'(x)\mathrm{d}x &= \underbrace{u(x)}_{u}\underbrace{\left[v(x)+C\right]}_{v} - \int \underbrace{\left[v(x)+C\right]}_{v}\underbrace{u'(x)\mathrm{d}x}_{\mathrm{d}u}\\ &= u(x)v(x)+Cu(x) - \int v(x)u'(x)\mathrm{d}x - \int Cu'(x)\mathrm{d}x\\ &= u(x)v(x)+Cu(x) - \int v(x)u'(x)\mathrm{d}x - Cu(x) + D\\ &= u(x)v(x) - \int v(x)u'(x)\mathrm{d}x + D\end{aligned}$$

where D is any constant.
Since the terms with C cancel out, it didn't matter what we chose for C–all choices end up the same.

1.7.2.5. Answer. Suppose we choose $\mathrm{d}v = f(x)\mathrm{d}x$, $u = 1$. Then $v = \displaystyle\int f(x)\mathrm{d}x$, and $\mathrm{d}u = \mathrm{d}x$. So, our integral becomes:

$$\int \underbrace{(1)}_{u}\underbrace{f(x)\mathrm{d}x}_{\mathrm{d}v} = \underbrace{(1)}_{u}\underbrace{\int f(x)\mathrm{d}x}_{v} - \int \underbrace{\left(\int f(x)\mathrm{d}x\right)}_{v}\underbrace{\mathrm{d}x}_{\mathrm{d}u}$$

In order to figure out the first product (and the second integrand), you need to know the antiderivative of $f(x)$–but that's exactly what you're trying to figure out!

Exercises — Stage 2

1.7.2.6. *. Answer. $\dfrac{x^2\log x}{2} - \dfrac{x^2}{4} + C$

1.7.2.7. *. Answer. $-\dfrac{\log x}{6x^6} - \dfrac{1}{36x^6} + C$

1.7.2.8. *. Answer. π

1.7.2.9. *. Answer. $\dfrac{\pi}{2} - 1$

1.7.2.10. Answer. $e^x\left(x^3 - 3x^2 + 6x - 6\right) + C$

1.7.2.11. Answer. $\dfrac{x^2}{2}\log^3 x - \dfrac{3x^2}{4}\log^2 x + \dfrac{3x^2}{4}\log x - \dfrac{3x^2}{8} + C$

1.7.2.12. Answer. $(2 - x^2)\cos x + 2x\sin x + C$

1.7.2.13. Answer. $\left(t^3 - \frac{5}{2}t^2 + 6t\right)\log t - \frac{1}{3}t^3 + \frac{5}{4}t^2 - 6t + C$

1.7.2.14. Answer. $e^{\sqrt{s}}\left(2s - 4\sqrt{s} + 4\right) + C$

1.7.2.15. Answer. $x\log^2 x - 2x\log x + 2x + C$

1.7.2.16. Answer. $e^{x^2+1} + C$

1.7.2.17. *. Answer. $y\arccos y - \sqrt{1-y^2} + C$

Exercises — Stage 3

1.7.2.18. *. Answer. $2y^2\arctan(2y) - y + \frac{1}{2}\arctan(2y) + C$

1.7.2.19. Answer. $\dfrac{x^3}{3}\arctan x - \dfrac{1}{6}(1 + x^2) + \dfrac{1}{6}\log(1 + x^2) + C$

1.7.2.20. Answer. $\dfrac{2}{17}e^{x/2}\cos(2x) + \dfrac{8}{17}e^{x/2}\sin(2x) + C$

1.7.2.21. Answer. $\dfrac{x}{2}\big[\sin(\log x) - \cos(\log x)\big] + C$

1.7.2.22. Answer. $\dfrac{2^x}{\log 2}\left(x - \dfrac{1}{\log 2}\right) + C$

1.7.2.23. Answer. $2e^{\cos x}[1 - \cos x] + C$

1.7.2.24. Answer. $\dfrac{xe^{-x}}{1-x} + e^{-x} + C = \dfrac{e^{-x}}{1-x} + C$

1.7.2.25. *. Answer. (a) We integrate by parts with $u = \sin^{n-1}x$ and $\mathrm{d}v = \sin x\,\mathrm{d}x$, so that $\mathrm{d}u = (n-1)\sin^{n-2}x\cos x$ and $v = -\cos x$.

$$\int \sin^n x\,\mathrm{d}x = \underbrace{-\sin^{n-1}x\ \cos x}_{uv} + \underbrace{(n-1)\int \cos^2 x\ \sin^{n-2}x\ \mathrm{d}x}_{-\int v\mathrm{d}u}$$

Using the identity $\sin^2 x + \cos^2 x = 1$,

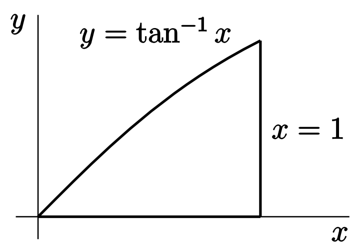
$$= -\sin^{n-1} x \;\cos x + (n-1)\int (1-\sin^2 x)\sin^{n-2} x \, dx$$
$$= -\sin^{n-1} x \;\cos x + (n-1)\int \sin^{n-2} x \, dx - (n-1)\int \sin^{n} x \, dx$$

Moving the last term on the right hand side to the left hand side gives

$$n\int \sin^n x \, dx = -\sin^{n-1} x \;\cos x + (n-1)\int \sin^{n-2} x \, dx$$

Dividing across by n gives the desired reduction formula.
(b) $\frac{35}{256}\pi \approx 0.4295$

1.7.2.26. *. Answer. (a) Area: $\frac{\pi}{4} - \frac{\log 2}{2}$



(b) Volume: $\frac{\pi^2}{2} - \pi$

1.7.2.27. *. Answer. $\pi\left(\frac{17e^{18} - 4373}{36}\right)$

1.7.2.28. *. Answer. 12

1.7.2.29. Answer. $\frac{2}{e}$

1.8 · Trigonometric Integrals

1.8.4 · Exercises

Exercises — Stage 1

1.8.4.1. Answer. (e)

1.8.4.2. Answer. $\frac{1}{n}\sec^n x + C$

1.8.4.3. Answer. We divide both sides by $\cos^2 x$, and simplify.

$$\sin^2 x + \cos^2 x = 1$$
$$\frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\begin{aligned}\frac{\sin^2 x}{\cos^2 x} + 1 &= \sec^2 x \\ \tan^2 x + 1 &= \sec^2 x\end{aligned}$$

Exercises — Stage 2

1.8.4.4. *. Answer. $\sin x - \dfrac{\sin^3 x}{3} + C$

1.8.4.5. *. Answer. $\dfrac{\pi}{2}$

1.8.4.6. *. Answer. $\dfrac{\sin^{37} t}{37} - \dfrac{\sin^{39} t}{39} + C$

1.8.4.7. Answer. $\dfrac{1}{3\cos^3 x} - \dfrac{1}{\cos x} + C$

1.8.4.8. Answer. $\dfrac{\pi}{8} - \dfrac{9\sqrt{3}}{64}$

1.8.4.9. Answer. $-\cos x + \dfrac{2}{3}\cos^3 x - \dfrac{1}{5}\cos^5 x + C$

1.8.4.10. Answer. $\dfrac{1}{2.2}\sin^{2.2} x + C$

1.8.4.11. Answer. $\dfrac{1}{2}\tan^2 x + C$, or equivalently, $\dfrac{1}{2}\sec^2 + C$

1.8.4.12. *. Answer. $\dfrac{1}{7}\sec^7 x - \dfrac{1}{5}\sec^5 x + C$

1.8.4.13. *. Answer. $\dfrac{\tan^{49} x}{49} + \dfrac{\tan^{47} x}{47} + C$

1.8.4.14. Answer. $\dfrac{1}{3.5}\sec^{3.5} x - \dfrac{1}{1.5}\sec^{1.5} x + C$

1.8.4.15. Answer. $\dfrac{1}{4}\sec^4 x - \dfrac{1}{2}\sec^2 x + C$ or $\dfrac{1}{4}\tan^4 x + C$

1.8.4.16. Answer. $\dfrac{1}{5}\tan^5 x + C$

1.8.4.17. Answer. $\dfrac{1}{1.3}\sec^{1.3} x + \dfrac{1}{0.7}\cos^{0.7} x + C$

1.8.4.18. Answer. $= \dfrac{1}{4}\sec^4 x - \sec^2 x + \log|\sec x| + C$

1.8.4.19. Answer. $\dfrac{41}{45\sqrt{3}} - \dfrac{\pi}{6}$

1.8.4.20. Answer. $\dfrac{1}{11} + \dfrac{1}{9}$

1.8.4.21. Answer. $2\sqrt{\sec x}+C$

1.8.4.22. Answer. $\tan^{e+1}\theta\left(\frac{\tan^6\theta}{7+e}+\frac{3\tan^4\theta}{5+e}+\frac{3\tan^2\theta}{3+e}+\frac{1}{1+e}\right)+C$

Exercises — Stage 3

1.8.4.23. *. Answer. (a) Using the trig identity $\tan^2 x=\sec^2 x-1$ and the substitution $y=\tan x$, $\mathrm{d}y=\sec^2 x\ \mathrm{d}x$,

$$\begin{aligned}\int \tan^n x\mathrm{d}x &= \int \tan^{n-2}x\ \tan^2 x\mathrm{d}x\\ &= \int \tan^{n-2}x\ \sec^2 x\mathrm{d}x-\int \tan^{n-2}x\mathrm{d}x\\ &= \int y^{n-2}\,\mathrm{d}y-\int \tan^{n-2}x\mathrm{d}x\\ &= \frac{y^{n-1}}{n-1}-\int \tan^{n-2}x\mathrm{d}x\\ &= \frac{\tan^{n-1}x}{n-1}-\int \tan^{n-2}x\mathrm{d}x\end{aligned}$$

(b) $\frac{13}{15}-\frac{\pi}{4}\approx 0.0813$

1.8.4.24. Answer. $\frac{1}{2\cos^2 x}+2\log|\cos x|-\frac{1}{2}\cos^2 x+C$

1.8.4.25. Answer. $\tan\theta+C$

1.8.4.26. Answer. $\log|\sin x|+C$

1.8.4.27. Answer. $\frac{1}{2}\sin^2(e^x)+C$

1.8.4.28. Answer. $(\sin^2 x+2)\cos(\cos x)+2\cos x\sin(\cos x)+C$

1.8.4.29. Answer. $\frac{x}{2}\sin^2 x-\frac{x}{4}+\frac{1}{4}\sin x\cos x+C$

1.9 · Trigonometric Substitution

1.9.2 · Exercises

Exercises — Stage 1

1.9.2.1. *. Answer. (a) $x=\frac{4}{3}\sec\theta$

(b) $x=\frac{1}{2}\sin\theta$

(c) $x=5\tan\theta$

1.9.2.2. Answer. (a) $x-2=\sqrt{3}\sec u$
(b) $x-1=\sqrt{5}\sin u$
(c) $\left(2x+\dfrac{3}{2}\right)=\dfrac{\sqrt{31}}{2}\tan u$
(d) $x-\dfrac{1}{2}=\dfrac{1}{2}\sec u$

1.9.2.3. Answer. (a) $\dfrac{\sqrt{399}}{20}$
(b) $\dfrac{5\sqrt{2}}{7}$
(c) $\dfrac{\sqrt{x-5}}{2}$

1.9.2.4. Answer. (a) $\dfrac{\sqrt{4-x^2}}{2}$
(b) $\dfrac{1}{2}$
(c) $\dfrac{1}{\sqrt{1-x}}$

Exercises — Stage 2

1.9.2.5. *. Answer. $\dfrac{1}{4}\cdot\dfrac{x}{\sqrt{x^2+4}}+C$

1.9.2.6. *. Answer. $\dfrac{1}{2\sqrt{5}}$

1.9.2.7. *. Answer. $\dfrac{\pi}{6}$

1.9.2.8. *. Answer. $\log\left|\sqrt{1+\dfrac{x^2}{25}}+\dfrac{x}{5}\right|+C$

1.9.2.9. Answer. $\dfrac{1}{2}\sqrt{2x^2+4x}+C$

1.9.2.10. *. Answer. $-\dfrac{1}{16}\dfrac{\sqrt{x^2+16}}{x}+C$

1.9.2.11. *. Answer. $\dfrac{\sqrt{x^2-9}}{9x}+C$

1.9.2.12. *. Answer. (a) We'll use the trig identity $\cos 2\theta=2\cos^2\theta-1$. It implies that

$$\cos^2\theta=\frac{\cos 2\theta+1}{2}\implies\cos^4\theta=\frac{1}{4}\big[\cos^2 2\theta+2\cos 2\theta+1\big]$$
$$=\frac{1}{4}\Big[\frac{\cos 4\theta+1}{2}+2\cos 2\theta+1\Big]$$

$$= \frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8}$$

So,

$$\begin{aligned}\int_0^{\pi/4} \cos^4\theta \mathrm{d}\theta &= \int_0^{\pi/4} \Big(\frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8}\Big)\mathrm{d}\theta \\ &= \left[\frac{\sin 4\theta}{32} + \frac{\sin 2\theta}{4} + \frac{3}{8}\theta\right]_0^{\pi/4} \\ &= \frac{1}{4} + \frac{3}{8}\cdot\frac{\pi}{4} \\ &= \frac{8+3\pi}{32}\end{aligned}$$

as required.
(b) $\dfrac{8+3\pi}{16}$

1.9.2.13. Answer. 0

1.9.2.14. *. Answer. $2\arcsin\dfrac{x}{2} + \dfrac{x}{2}\sqrt{4-x^2} + C$

1.9.2.15. *. Answer. $\sqrt{25x^2-4} - 2\,\mathrm{arcsec}\,\frac{5x}{2} + C$

1.9.2.16. Answer. $\dfrac{40}{3}$

1.9.2.17. *. Answer. $\arcsin\dfrac{x+1}{2} + C$

1.9.2.18. Answer. $\dfrac{1}{4}\left(\arccos\left(\dfrac{1}{2x-3}\right) + \dfrac{\sqrt{4x^2-12x+8}}{(2x-3)^2}\right) + C$, or equivalently, $\dfrac{1}{4}\left(\mathrm{arcsec}\,(2x-3) + \dfrac{\sqrt{4x^2-12x+8}}{(2x-3)^2}\right) + C$

1.9.2.19. Answer. $\log(1+\sqrt{2}) - \dfrac{1}{\sqrt{2}}$

1.9.2.20. Answer. $\dfrac{1}{2}\left(\arctan x + \dfrac{x}{x^2+1}\right) + C$

Exercises — Stage 3

1.9.2.21. Answer. $\dfrac{3+x}{2}\sqrt{x^2-2x+2} + \dfrac{1}{2}\log\left|\sqrt{x^2-2x+2}+x-1\right| + C$

1.9.2.22. Answer. $\dfrac{1}{\sqrt{3}}\log\left|\left(\dfrac{6}{5}x+1\right) + \dfrac{2}{5}\sqrt{9x^2+15x}\right| + C$

1.9.2.23. Answer. $\dfrac{1}{3}\sqrt{1+x^2}(4+x^2)+\log\left|\dfrac{1-\sqrt{1+x^2}}{x}\right|+C$

1.9.2.24. Answer. $\dfrac{8\pi}{3}+4\sqrt{3}$

1.9.2.25. Answer. Area: $\dfrac{4}{3}-\sqrt[4]{\dfrac{4}{3}}$

Volume: $\dfrac{\pi^2}{6}-\dfrac{\sqrt{3}\pi}{4}$

1.9.2.26. Answer. $2\sqrt{1+e^x}+2\log\left|1-\sqrt{1+e^x}\right|-x+C$

1.9.2.27. Answer.

a $\dfrac{1}{1-x^2}$

b False

c The work in the question is not correct. The most salient problem is that when we make the substitution $x=\sin\theta$, we restrict the possible values of x to $[-1,1]$, since this is the range of the sine function. However, the original integral had no such restriction.

How can we be sure we avoid this problem in the future? In the introductory text to Section 1.9 (before Example 1.9.1), the notes tell us that we are allowed to write our old variable as a function of a new variable (say $x=s(u)$) *as long as that function is invertible to recover our original variable* x. There is one very obvious reason why invertibility is necessary: after we antidifferentiate using our new variable u, we need to get it back in terms of our original variable, so we need to be able to recover x. Moreover, invertibility reconciles potential problems with domains: if an inverse function $u=s^{-1}(x)$ exists, then for any x, there exists a u with $s(u)=x$. (This was not the case in the work for the question, because we chose $x=\sin\theta$, but if $x=2$, there is no corresponding θ. Note, however, that $x=\sin\theta$ is invertible over $[-1,1]$, so the work is correct if we restrict x to those values.)

1.9.2.28. Answer. (a), (b): None.
(c): $x<-a$

1.10 · Partial Fractions
1.10.4 · Exercises

Exercises — Stage 1

1.10.4.1. Answer. (a) (iii)
(b) (ii)
(c) (ii)
(d) (i)

1.10.4.2. ∗. **Answer**. $\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x+1}+\dfrac{D}{(x+1)^2}+\dfrac{Ex+F}{x^2+1}$

1.10.4.3. ∗. **Answer**. 3

1.10.4.4. **Answer**. (a) $\dfrac{x^3+2x+2}{x^2+1}=x+\dfrac{x+2}{x^2+1}$
(b) $\dfrac{15x^4+6x^3+34x^2+4x+20}{5x^2+2x+8}=3x^2+2+\dfrac{4}{5x^2+2x+8}$
(c) $\dfrac{2x^5+9x^3+12x^2+10x+30}{2x^2+5}=x^3+2x+6$

1.10.4.5. **Answer**. (a) $5x^3-3x^2-10x+6=(x+\sqrt{2})(x-\sqrt{2})(5x-3)$
(b)

$$\begin{aligned}x^4&-3x^2-5\\&=\left(x+\sqrt{\frac{3+\sqrt{29}}{2}}\right)\left(x-\sqrt{\frac{3+\sqrt{29}}{2}}\right)\left(x^2+\frac{\sqrt{29}-3}{2}\right)\end{aligned}$$

(c) $x^4-4x^3-10x^2-11x-6=(x+1)(x-6)(x^2+x+1)$
(d)

$$\begin{aligned}2x^4&+12x^3-x^2-52x+15\\&=(x+3)(x+5)\left(x-\left(1+\frac{\sqrt{2}}{2}\right)\right)\left(x-\left(1-\frac{\sqrt{2}}{2}\right)\right)\end{aligned}$$

1.10.4.6. **Answer**. The goal of partial fraction decomposition is to write our integrand in a form that is easy to integrate. The antiderivative of (1) can be easily determined with the substitution $u=(ax+b)$. It's less clear how to find the antiderivative of (2).

Exercises — Stage 2

1.10.4.7. ∗. **Answer**. $\log\dfrac{4}{3}$

1.10.4.8. ∗. **Answer**. $-\dfrac{1}{x}-\arctan x+C$

1.10.4.9. ∗. **Answer**. $4\log|x-3|-2\log(x^2+1)+C$

1.10.4.10. ∗. **Answer**. $F(x)=\log|x-2|+\log|x^2+4|+2\arctan(x/2)+D$

1.10.4.11. ∗. **Answer**. $-2\log|x-3|+3\log|x+2|+C$

1.10.4.12. ∗. **Answer**. $-9\log|x+2|+14\log|x+3|+C$

1.10.4.13. **Answer**. $5x+\dfrac{1}{2}\log|x-1|-\dfrac{7}{2}\log|x+1|+C$

1.10.4.14. Answer. $x - \dfrac{2}{x} + \dfrac{5}{2}\arctan(2x) + C$

1.10.4.15. Answer. $\dfrac{1}{x} - \dfrac{2}{x-1} + C$

1.10.4.16. Answer. $-\dfrac{1}{2}\log|x-2| + \dfrac{1}{2}\log|x+2| + \dfrac{3}{2}\log|2x-1| + C$

1.10.4.17. Answer. $\log\left(\dfrac{4\cdot 6^3}{5^3}\right)$

Exercises — Stage 3

1.10.4.18. Answer. $\dfrac{1}{2}\log\left|\dfrac{1-\cos x}{1+\cos x}\right| + C$

1.10.4.19. Answer. $\dfrac{-\cos x}{2\sin^2 x} + \dfrac{1}{4}\log\left|\dfrac{1-\cos x}{1+\cos x}\right| + C$

1.10.4.20. Answer. $3\log 2 + \dfrac{1}{2} + \dfrac{2}{\sqrt{15}}\left(\arctan\left(\dfrac{7}{\sqrt{15}}\right) - \arctan\left(\dfrac{9}{\sqrt{15}}\right)\right)$

1.10.4.21. Answer. $= \dfrac{9}{4\sqrt{2}}\arctan\left(\dfrac{x}{\sqrt{2}}\right) - \dfrac{2+3x}{4(x^2+2)} + C$

1.10.4.22. Answer. $\dfrac{3}{8}\arctan x + \dfrac{3x^3+5x}{8(1+x^2)^2} + C$

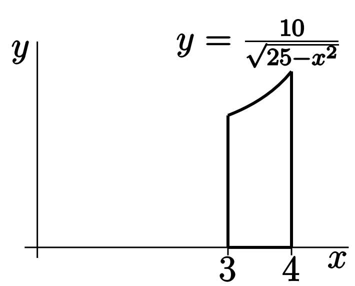
1.10.4.23. Answer. $\dfrac{3}{2}x^2 + \dfrac{1}{\sqrt{5}}\arctan\left(\dfrac{x}{\sqrt{5}}\right) + \dfrac{3}{2}\log|x^2+5| - \dfrac{3}{2x^2+10} + C$

1.10.4.24. Answer. $\log\left|\dfrac{\sin\theta - 1}{\sin\theta - 2}\right| + C$

1.10.4.25. Answer. $t - \dfrac{1}{2}\log|e^{2t}+e^t+1| - \dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{2e^t+1}{\sqrt{3}}\right) + C$

1.10.4.26. Answer. $2\sqrt{1+e^x} + \log\left|\dfrac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C$

1.10.4.27. *. Answer. (a) The region R is



(b) $10\pi \log \frac{9}{4} = 20\pi \log \frac{3}{2}$

(c) 20π

1.10.4.28. Answer. $2\log\frac{5}{3} + \frac{4}{\sqrt{3}}\arctan\frac{1}{4\sqrt{3}}$

1.10.4.29. Answer. (a) $\frac{1}{6}\left(\log\left|2\cdot\frac{x-3}{x+3}\right|\right)$

(b) $F'(x) = \frac{1}{x^2-9}$

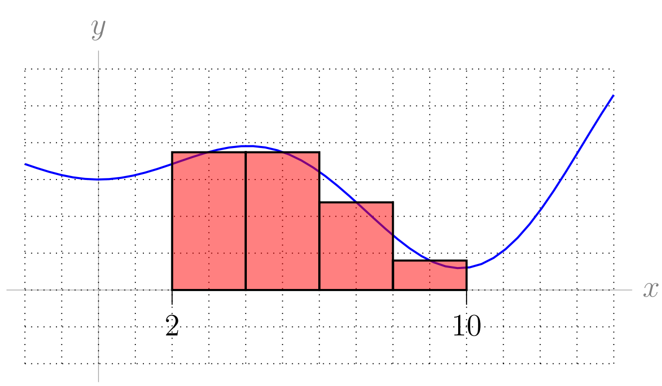
1.11 · Numerical Integration

1.11.6 · Exercises

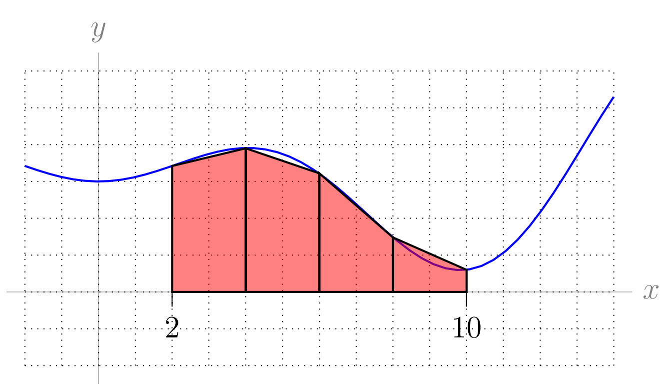
Exercises — Stage 1

1.11.6.1. Answer. Relative error: ≈ 0.08147; absolute error: 0.113; percent error: $\approx 8.147\%$.

1.11.6.2. Answer. Midpoint rule:



Trapezoidal rule:



1.11.6.3. Answer. $M = 6.25$, $L = 2$

1.11.6.4. Answer. One reasonable answer is $M = 3$.

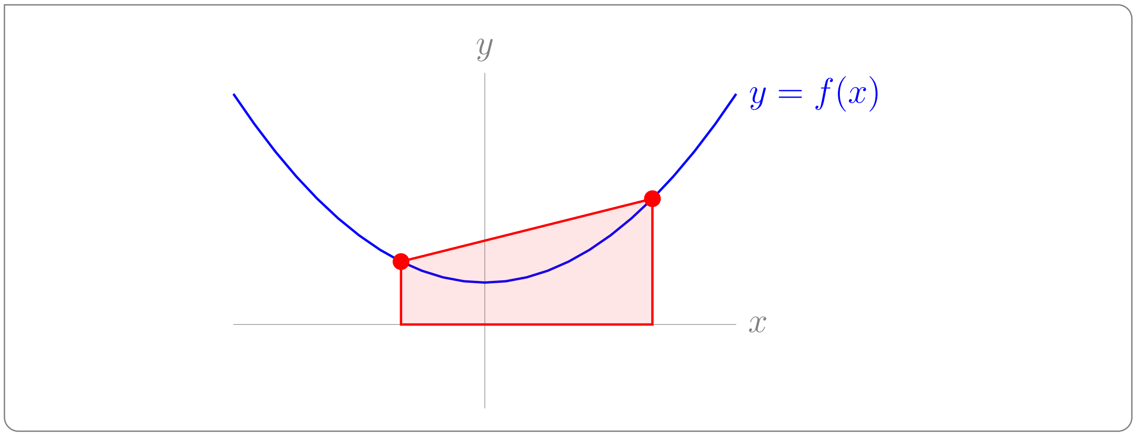
1.11.6.5. Answer. (a) $\dfrac{\pi^5}{180 \cdot 8}$
(b) 0
(c) 0

1.11.6.6. Answer. Possible answers: $f(x) = \frac{3}{2}x^2 + Cx + D$ for any constants C, D.

1.11.6.7. Answer. my mother

1.11.6.8. Answer. (a) true
(b) false

1.11.6.9. *. Answer. True. Because $f(x)$ is positive and concave up, the graph of $f(x)$ is always below the top edges of the trapezoids used in the trapezoidal rule.



1.11.6.10. Answer. Any polynomial of degree at most 3 will do. For example, $f(x) = 5x^3 - 27$, or $f(x) = x^2$.

Exercises — Stage 2

1.11.6.11. Answer. Midpoint:

$$\int_0^{30} \frac{1}{x^3+1}\,\mathrm{d}x \approx \left[\frac{1}{(2.5)^3+1} + \frac{1}{(7.5)^3+1} + \frac{1}{(12.5)^3+1} + \frac{1}{(17.5)^3+1} + \frac{1}{(22.5)^3+1} + \frac{1}{(27.5)^3+1}\right]5$$

Trapezoidal:

$$\int_0^{30} \frac{1}{x^3+1}\,\mathrm{d}x \approx \left[\frac{1/2}{0^3+1} + \frac{1}{5^3+1} + \frac{1}{10^3+1} + \frac{1}{15^3+1} + \frac{1}{20^3+1} + \frac{1}{25^3+1} + \frac{1/2}{30^3+1}\right]5$$

Simpson's:

$$\int_0^{30} \frac{1}{x^3+1}\,\mathrm{d}x \approx \left[\frac{1}{0^3+1} + \frac{4}{5^3+1} + \frac{2}{10^3+1} + \frac{4}{15^3+1} + \frac{2}{20^3+1}\right.$$

$$+\frac{4}{25^3+1}+\frac{1}{30^3+1}\bigg]\frac{5}{3}$$

1.11.6.12. $*$. Answer. $\frac{2\pi}{3}$

1.11.6.13. $*$. Answer. $1720\pi \approx 5403.5$ cm^3

1.11.6.14. $*$. Answer. $\frac{\pi}{12}(16.72) \approx 4.377$ m^3

1.11.6.15. $*$. Answer. $\frac{12.94}{6\pi} \approx 0.6865$ m^3

1.11.6.16. $*$. Answer. (a) 363,500
(b) 367,000

1.11.6.17. $*$. Answer. (a) $\frac{49}{2}$
(b) $\frac{77}{3}$

1.11.6.18. $*$. Answer. Let $f(x) = \sin(x^2)$. Then $f'(x) = 2x\cos(x^2)$ and

$$f''(x) = 2\cos(x^2) - 4x^2\sin(x^2).$$

Since $|x^2| \le 1$ when $|x| \le 1$, and $|\sin\theta| \le 1$ and $|\cos\theta| \le 1$ for all θ, we have

$$\begin{aligned}\left|2\cos(x^2) - 4x^2\sin(x^2)\right| &\le 2|\cos(x^2)| + 4x^2|\sin(x^2)|\\ &\le 2\times 1 + 4\times 1\times 1 = 2+4 = 6\end{aligned}$$

We can therefore choose $M = 6$, and it follows that the error is at most

$$\frac{M[b-a]^3}{24n^2} \le \frac{6\cdot[1-(-1)]^3}{24\cdot 1000^2} = \frac{2}{10^6} = 2\cdot 10^{-6}$$

1.11.6.19. $*$. Answer. $\frac{3}{100}$

1.11.6.20. $*$. Answer. (a)

$$\begin{aligned}\frac{1/3}{3}\bigg((-3)^5 + 4\left(\frac{1}{3}-3\right)^5 + 2\left(\frac{2}{3}-3\right)^5 + 4(-2)^5 + 2\left(\frac{4}{3}-3\right)^5\\ + 4\left(\frac{5}{3}-3\right)^5 + (-1)^5\bigg)\end{aligned}$$

(b) Simpson's Rule results in a smaller error bound.

1.11.6.21. $*$. Answer. $\frac{8}{15}$

1.11.6.22. $*$. Answer. $\frac{1}{180\times 3^4} = \frac{1}{14580}$

1.11.6.23. $*$. **Answer**. (a) $T_4 = \frac{1}{4}\left[\left(\frac{1}{2}\times 1\right)+\frac{4}{5}+\frac{2}{3}+\frac{4}{7}+\left(\frac{1}{2}\times\frac{1}{2}\right)\right]$,
(b) $S_4 = \frac{1}{12}\left[1+\left(4\times\frac{4}{5}\right)+\left(2\times\frac{2}{3}\right)+\left(4\times\frac{4}{7}\right)+\frac{1}{2}\right]$
(c) $\left|I - S_4\right| \le \frac{24}{180\times 4^4} = \frac{3}{5760}$

1.11.6.24. $*$. **Answer**. (a) $T_4 = 8.03515$, $S_4 \approx 8.03509$
(b)

$$\left|\int_a^b f(x)\,\mathrm{d}x - T_n\right| \le \frac{2}{1000}\frac{8^3}{12(4)^2} \le 0.00533$$

$$\left|\int_a^b f(x)\,\mathrm{d}x - S_n\right| \le \frac{4}{1000}\frac{8^5}{180(4)^4} \le 0.00284$$

1.11.6.25. $*$. **Answer**. Any $n \ge 68$ works.

Exercises — Stage 3

1.11.6.26. $*$. **Answer**. $\frac{472}{3} \approx 494$ ft^3

1.11.6.27. $*$. **Answer**. (a) 0.025635
(b) 1.8×10^{-5}

1.11.6.28. $*$. **Answer**. (a) ≈ 0.6931698
(b) $n \ge 12$ with n even

1.11.6.29. $*$. **Answer**. (a) 0.01345
(b) $n \ge 28$ with n even

1.11.6.30. $*$. **Answer**. $n \ge 259$

1.11.6.31. **Answer**. (a) When $0 \le x \le 1$, then $x^2 \le 1$ and $x+1 \ge 1$, so

$$|f''(x)| = \frac{x^2}{|x+1|} \le \frac{1}{1} = 1$$

(b) $\frac{1}{2}$
(c) $n \ge 65$
(d) $n \ge 46$

1.11.6.32. **Answer**. $\frac{x-1}{12}\left[1+\frac{16}{x+3}+\frac{4}{x+1}+\frac{16}{3x+1}+\frac{1}{x}\right]$

1.11.6.33. **Answer**. Note: for more detail, see the solutions.
First, we use Simpson's rule with $n = 4$ to approximate $\int_1^2 \frac{1}{1+x^2}\,\mathrm{d}x$. The choice of this method (what we're approximating, why $n = 4$, etc.) is explained in the solutions–here, we only show that it works.

$$\int_1^2 \frac{1}{1+x^2}\,\mathrm{d}x \approx \frac{1}{12}\left[\frac{1}{2}+\frac{64}{41}+\frac{8}{13}+\frac{64}{65}+\frac{1}{5}\right] \approx 0.321748$$

For ease of notation, define $A = 0.321748$.
Now, we bound the error associated with this approximation. Define $N(x) = 24(5x^4 - 10x^2 + 1)$ and $D(x) = (x^2+1)^5$, so $N(x)/D(x)$ gives the fourth derivative of $\frac{1}{1+x^2}$. When $1 \le x \le 2$, $|N(x)| \le N(2) = 984$ (because $N(x)$ is increasing over that interval) and $|D(x)| \ge D(1) = 2^5$ (because $D(x)$ is also increasing over that interval), so $\left|\frac{\mathrm{d}^4}{\mathrm{d}x^4}\left\{\frac{1}{1+x^2}\right\}\right| = \left|\frac{N(x)}{D(x)}\right| \le \frac{984}{2^5} = 30.75$. Now we find the error bound for Simpson's rule with $L = 30.75$, $b = 2$, $a = 1$, and $n = 4$.

$$\left|\int_1^2 \frac{1}{1+x^2}\,\mathrm{d}x - A\right| = |\text{error}| \le \frac{L(b-a)^5}{180 \cdot n^4} = \frac{30.75}{180 \cdot 4^4} < 0.00067$$

So,

$$\begin{aligned}
-0.00067 &< \int_1^2 \frac{1}{1+x^2}\,\mathrm{d}x - A < 0.00067\\
A - 0.0067 &< \int_1^2 \frac{1}{1+x^2}\,\mathrm{d}x < A + 0.00067\\
A - 0.00067 &< \arctan(2) - \arctan(1) < A + 0.00067\\
A - 0.00067 &< \arctan(2) - \frac{\pi}{4} < A + 0.00067\\
\frac{\pi}{4} + A - 0.00067 &< \arctan(2) < \frac{\pi}{4} + A + 0.00067\\
\frac{\pi}{4} + 0.321748 - 0.00067 &< \arctan(2) < \frac{\pi}{4} + 0.321748 + 0.00067\\
\frac{\pi}{4} + 0.321078 &< \arctan(2) < \frac{\pi}{4} + 0.322418\\
\frac{\pi}{4} + 0.321 &< \arctan(2) < \frac{\pi}{4} + 0.323
\end{aligned}$$

This was the desired bound.

1.12 · Improper Integrals

1.12.4 · Exercises

Exercises — Stage 1

1.12.4.1. Answer. Any real number in $[1, \infty)$ or $(-\infty, -1]$, and $b = \pm\infty$.

1.12.4.2. Answer. $b = \pm\infty$

1.12.4.3. Answer. The red function is $f(x)$, and the blue function is $g(x)$.

1.12.4.4. *. Answer. False. For example, the functions $f(x) = e^{-x}$ and $g(x) = 1$ provide a counterexample.

1.12.4.5. Answer.

a Not enough information to decide. For example, consider $h(x) = 0$ versus $h(x) = -1$.

b Not enough information to decide. For example, consider $h(x) = f(x)$ versus

$h(x) = g(x)$.

c $\int_0^\infty h(x)\mathrm{d}x$ converges by the comparison test, since $|h(x)| \leq 2f(x)$ and $\int_0^\infty 2f(x)\mathrm{d}x$ converges.

Exercises — Stage 2

1.12.4.6. *. Answer. The integral diverges.

1.12.4.7. *. Answer. The integral diverges.

1.12.4.8. *. Answer. The integral does not converge.

1.12.4.9. *. Answer. The integral converges.

1.12.4.10. Answer. The integral diverges.

1.12.4.11. Answer. The integral diverges.

1.12.4.12. Answer. The integral diverges.

1.12.4.13. Answer. The integral diverges.

1.12.4.14. *. Answer. The integral diverges.

1.12.4.15. *. Answer. The integral converges.

1.12.4.16. *. Answer. The integral converges.

Exercises — Stage 3

1.12.4.17. Answer. false

1.12.4.18. *. Answer. $q = \frac{1}{5}$

1.12.4.19. Answer. $p > 1$

1.12.4.20. Answer. $\dfrac{\log 3 - \pi}{4} + \dfrac{1}{2}\arctan 2$

1.12.4.21. Answer. The integral converges.

1.12.4.22. Answer. $\dfrac{1}{2}$

1.12.4.23. *. Answer. The integral converges.

1.12.4.24. Answer. The integral converges.

1.12.4.25. *. Answer. $t = 10$ and $n = 2042$ will do the job. There are many other correct answers.

1.12.4.26. Answer. (a) The integral converges.
(b) The interval converges.

1.12.4.27. Answer. false

1.13 · More Integration Examples · Exercises

Exercises — Stage 1

1.13.1. Answer. (A)–(I), (B)–(IV), (C)–(II), (D)–(III)

Exercises — Stage 2

1.13.2. Answer. $\frac{1}{5} - \frac{2}{7} + \frac{1}{9} = \frac{8}{315}$

1.13.3. Answer. $\frac{3}{2\sqrt{5}} \arcsin\left(x\sqrt{\frac{5}{3}}\right) + \frac{x}{2}\sqrt{3-5x^2} + C$

1.13.4. Answer. 0

1.13.5. Answer. $\log\left|\frac{x+1}{3x+1}\right| + C$

1.13.6. Answer. $\frac{8}{3}\log 2 - \frac{7}{9}$

1.13.7. *. Answer. $\frac{1}{2}\log\left|x^2-3\right| + C$

1.13.8. *. Answer. (a) 2
(b) $\frac{2}{15}$
(c) $\frac{3e^4}{16} + \frac{1}{16}$

1.13.9. *. Answer. (a) 1
(b) $\frac{8}{15}$

1.13.10. *. Answer. (a) e^2+1 (b) $\log(\sqrt{2}+1)$ (c) $\log\frac{15}{13} \approx 0.1431$

1.13.11. *. Answer. (a) $\frac{9}{4}\pi$
(b) $\log 2 - 2 + \frac{\pi}{2} \approx 0.264$
(c) $2\log 2 - \frac{1}{2} \approx 0.886$

1.13.12. Answer. $\frac{1}{3}\sin^3\theta - 2\sin\theta + 12\log\left|\frac{\sin\theta - 3}{\sin\theta - 2}\right| + C$

1.13.13. *. Answer. (a) $\dfrac{1}{15}$

(b) $\dfrac{1}{9}\cdot\dfrac{x}{\sqrt{x^2+9}}+C$

(c) $\dfrac{1}{2}\log|x-1|-\dfrac{1}{4}\log(x^2+1)-\dfrac{1}{2}\arctan x+C$

(d) $\dfrac{1}{2}\left[x^2\arctan x-x+\arctan x\right]+C$

1.13.14. *. Answer. (a) $\dfrac{1}{12}$

(b) $2\sin^{-1}\dfrac{x}{2}+x\sqrt{1-\dfrac{x^2}{4}}+C$

(c) $-2\log|x|+\dfrac{1}{x}+2\log|x-1|+C$

1.13.15. *. Answer. (a) $\dfrac{2}{5}$

(b) $\dfrac{1}{2\sqrt{2}}$

(c) $\log 2-\dfrac{1}{2}\approx 0.193$

(d) $\log 2-\dfrac{1}{2}\approx 0.193$

1.13.16. *. Answer. (a) $\dfrac{1}{2}x^2\log x-\dfrac{1}{4}x^2+C$

(b) $\dfrac{1}{2}\log[x^2+4x+5]-3\arctan(x+2)+C$

(c) $\dfrac{1}{2}\log|x-3|-\dfrac{1}{2}\log|x-1|+C$

(d) $\dfrac{1}{3}\arctan x^3+C$

1.13.17. *. Answer. (a) $\dfrac{\pi}{4}-\dfrac{1}{2}\log 2$ (b) $\log|x^2-2x+5|+\dfrac{1}{2}\arctan\dfrac{x-1}{2}+C$

1.13.18. *. Answer. (a) $-\dfrac{1}{300(x^3+1)^{100}}+C$

(b) $\dfrac{\sin^5 x}{5}-\dfrac{\sin^7 x}{7}+C$

1.13.19. Answer. -2

1.13.20. *. Answer. (a) $-\dfrac{1}{4}\log|e^x+1|+\dfrac{1}{4}\log|e^x-3|+C$ (b) $\dfrac{4\pi}{3}-2\sqrt{3}$

1.13.21. *. Answer. (a) $\dfrac{1}{2}\sec^2 x+\log|\cos x|+C$

(b) $\dfrac{1}{10}\arctan 8\approx 0.1446$

1.13.22. Answer. $\dfrac{2}{5}(x-1)^{5/2}+\dfrac{2}{3}(x-1)^{3/2}+C$

1.13.23. Answer. $\log\left|x+\sqrt{x^2-2}\right| - \dfrac{\sqrt{x^2-2}}{x} + C$

1.13.24. Answer. $\dfrac{7}{24}$

1.13.25. Answer. $3\log|x+1| + \dfrac{2}{x+1} - \dfrac{5}{2(x+1)^2} + C$

1.13.26. Answer. $\dfrac{2}{\sqrt{3}}\arctan\left(\dfrac{2}{\sqrt{3}}x + \dfrac{1}{\sqrt{3}}\right) + C$

1.13.27. Answer. $\dfrac{1}{2}(x - \sin x\cos x) + C$

1.13.28. Answer. $\dfrac{1}{3}\log|x+1| - \dfrac{1}{6}\log|x^2+x+1| + \dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{2x-1}{\sqrt{3}}\right) + C$

1.13.29. Answer. $3x^3\arcsin x + 3\sqrt{1-x^2} - (1-x^2)^{3/2} + C$

Exercises — Stage 3

1.13.30. Answer. 2

1.13.31. Answer. $\dfrac{1}{4}$

1.13.32. Answer. $\log\left(\dfrac{\log(\cos(0.1))}{\log(\cos(0.2))}\right)$

1.13.33. *. Answer. (a) $\dfrac{1}{2}x\big[\sin(\log x) - \cos(\log x)\big] + C$
(b) $2\log 2 - \log 3 = \log\frac{4}{3}$

1.13.34. *. Answer. (a) $\dfrac{9}{4}\pi + 9$
(b) $2\log|x-2| - \log(x^2+4) + C$ (c) $\dfrac{\pi}{2}$

1.13.35. Answer. $-\arcsin(\sqrt{1-x}) - \sqrt{1-x}\sqrt{x} + C$

1.13.36. Answer. $e^e(e-1)$

1.13.37. Answer. $\dfrac{e^x}{x+1} + C$

1.13.38. Answer. $x\sec x - \log|\sec x + \tan x| + C$

1.13.39. Answer. $$\int x(x+a)^n\mathrm{d}x = \begin{cases} \frac{(x+a)^{(n+2)}}{n+2} - a\frac{(x+a)^{n+1}}{n+1} + C & \text{if } n \neq -1,-2 \\ (x+a) - a\log|x+a| + C & \text{if } n = -1 \\ \log|x+a| + \frac{a}{x+a} + C & \text{if } n = -2 \end{cases}$$

1.13.40. Answer.

$$x\arctan(x^2)-\frac{1}{\sqrt{2}}\left(\frac{1}{2}\log\left|\tfrac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|+\arctan\left(\sqrt{2}x+1\right)\right.$$
$$\left.+\arctan\left(\sqrt{2}x-1\right)\right)+C$$

2 · Applications of Integration

2.1 · Work

2.1.2 · Exercises

Exercises — Stage 1

2.1.2.1. Answer. 0.00294 J

2.1.2.2. Answer. The rock has mass $\frac{1}{9.8}$ kg (about 102 grams); lifting it one metre takes 1 J of work.

2.1.2.3. Answer. (a) metres
(b) newtons
(c) joules

2.1.2.4. Answer. $\frac{\text{smoot}\cdot\text{barn}}{\text{megaFonzie}}$ (smoot-barns per megaFonzie)

2.1.2.5. Answer. 10 cm below the bottom of the unloaded spring

2.1.2.6. Answer. $x=2$

Exercises — Stage 2

2.1.2.7. *. Answer. $a=3$

2.1.2.8. Answer. (a) joules
(b) $c\log\left(\frac{\ell-1}{\ell-1.5}\right)$ J

2.1.2.9. *. Answer. $\frac{1}{4}$ J

2.1.2.10. *. Answer. 25 J

2.1.2.11. *. Answer. 196 J

2.1.2.12. Answer. 14700 J

2.1.2.13. *. Answer. $\displaystyle\int_0^3(9.8)(8000)(2+z)(3-z)^2\,\mathrm{d}z$ joules

2.1.2.14. Answer. 0.2352 J

2.1.2.15. Answer. $\frac{20}{49}$ kg, or about 408 grams

2.1.2.16. Answer. 294 J

2.1.2.17. Answer. (a) 117.6 J
(b) $3.92\left[30-2\sqrt{3}\right] \approx 104$ J

2.1.2.18. Answer. $\dfrac{1}{2\sqrt{5}}$ m/sec, or about 22.36 cm/sec

2.1.2.19. Answer. yes (at least, the car won't scrape the ground)

Exercises — Stage 3

2.1.2.20. Answer. ≈ 0.144 J

2.1.2.21. *. Answer. $904{,}050\pi$ J

2.1.2.22. Answer. $1020\frac{5}{6}$ J

2.1.2.23. Answer. (a) 4900 N
(b) $\frac{44100}{x^2}$ N
(c) 29 400 J

2.1.2.24. Answer. 220.5 J

2.1.2.25. Answer. About 7×10^{28} J

2.1.2.26. Answer. true

2.1.2.27. Answer. $9255\frac{5}{9}$ J

2.1.2.28. Answer. $\dfrac{7}{40} = 0.175$ J

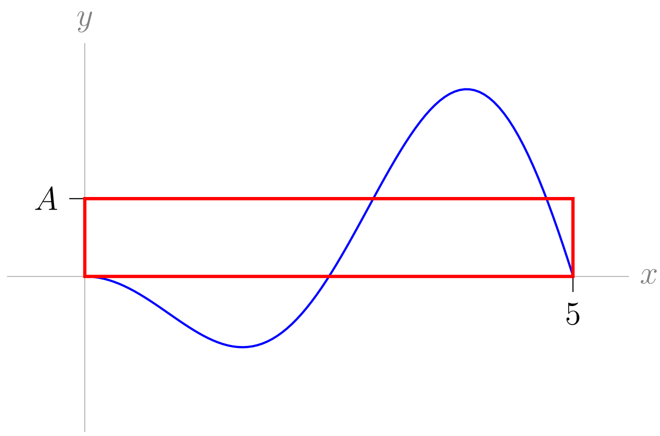
2.1.2.29. Answer. One possible answer: $\dfrac{1}{4}\left[\sqrt{1-\left(\dfrac{1}{8}\right)^4}+\sqrt{1-\left(\dfrac{3}{8}\right)^4}\right]$

2.2 · Averages

2.2.2 · Exercises

Exercises — Stage 1

2.2.2.1. Answer. The most straightforward of many possible answers is shown.



2.2.2.2. Answer. 500 km

2.2.2.3. Answer. $\dfrac{W}{b-a}$ N

2.2.2.4. Answer. (a) $\dfrac{b-a}{n}$
(b) $a+3\dfrac{b-a}{n}$
(c) $f\left(a+3\dfrac{b-a}{n}\right)$
(d) $\dfrac{1}{n}\sum_{i=1}^{n} f\left(a+(i-1)\frac{b-a}{n}\right)$

2.2.2.5. Answer. (a) yes
(b) not enough information

2.2.2.6. Answer. 0

Exercises — Stage 2

2.2.2.7. *. Answer. 1

2.2.2.8. *. Answer. $\dfrac{1}{e-1}\left[\dfrac{2}{9}e^3+\dfrac{1}{9}\right]$

2.2.2.9. *. Answer. $\dfrac{4}{\pi}+1$

2.2.2.10. *. Answer. $\dfrac{2}{\pi}$

2.2.2.11. *. Answer. $\dfrac{10}{3}\log 7$ degrees Celsius

2.2.2.12. *. Answer. $\dfrac{1}{2(e-1)}$

2.2.2.13. *. Answer. $\dfrac{1}{2}$

2.2.2.14. Answer. (a) 400 ppm
(b) ≈ 599.99 ppm
(c) 0.125, or 12.5%

2.2.2.15. Answer. (a) $\dfrac{16\pi}{5}$
(b) $\dfrac{32\pi}{5}$
(c) $\dfrac{32\pi}{5}$

2.2.2.16. Answer. (a) 0
(b) $\sqrt{3}$

2.2.2.17. Answer. $\sqrt{\dfrac{4}{\pi}-1} \approx 0.52$

2.2.2.18. Answer. (a) $F(t) = 3f(t) = 3\sin(t\pi)$ N
(b) 0
(c) $\dfrac{3}{\sqrt{2}} \approx 2.12$

Exercises — Stage 3

2.2.2.19. *. Answer. (a) 130 km
(b) 65 km/hr

2.2.2.20. Answer. (a) $A = e - 1$
(b) 0
(c) $4 - 2e + 2(e-1)\log(e-1) \approx 0.42$

2.2.2.21. Answer. (a) neither — both are zero
(b) $|f(x) - A|$ has the larger average on $[0, 4]$

2.2.2.22. Answer. $(b-a)\pi R^2$

2.2.2.23. Answer. 0

2.2.2.24. Answer. Yes, but if $a \neq 0$, then $s = t$.

2.2.2.25. Answer. A

2.2.2.26. Answer. (a) $\dfrac{bA(b) - aA(a)}{b-a}$
(b) $f(t) = A(t) + tA'(t)$

2.2.2.27. Answer.

a One of many possible answers: $f(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$.

b No such function exists.

- Note 1: Suppose $f(x) > 0$ for all x in $[-1, 1]$. Then $\frac{1}{2}\int_{-1}^{1} f(x)\,\mathrm{d}x > \frac{1}{2}\int_{-1}^{1} 0\,\mathrm{d}x = 0$. That is, the average value of $f(x)$ on the interval $[-1, 1]$

is not zero — it's something greater than zero.

- Note 2: Suppose $f(x) < 0$ for all x in $[-1,1]$. Then $\frac{1}{2}\int_{-1}^{1} f(x)\,\mathrm{d}x < \frac{1}{2}\int_{-1}^{1} 0\,\mathrm{d}x = 0$. That is, the average value of $f(x)$ on the interval $[-1,1]$ is not zero — it's something less than zero.

So, if the average value of $f(x)$ is zero, then $f(x) \geq 0$ for some x in $[-1,1]$, and $f(y) \leq 0$ for some $y \in [-1,1]$. Since f is a continuous function, and 0 is between $f(x)$ and $f(y)$, by the intermediate value theorem (see the CLP-1 text) there is some value c between x and y such that $f(c) = 0$. Since x and y are both in $[-1,1]$, then c is as well. Therefore, no function exists as described in the question.

2.2.2.28. Answer. true

2.2.2.29. Answer. 0

2.3 · Centre of Mass and Torque

2.3.3 · Exercises

Exercises — Stage 1

2.3.3.1. Answer. $(1,1)$

2.3.3.2. Answer. $(0,0)$

2.3.3.3. Answer. In general, false.

2.3.3.4. Answer. 3.5 metres from the left end

2.3.3.5. Answer. (a) to the left
(b) to the left
(c) not enough information
(d) along the line $x = a$
(e) to the right

2.3.3.6. Answer. $\dfrac{39200\pi}{9}(12-\pi) \approx 121{,}212$ J

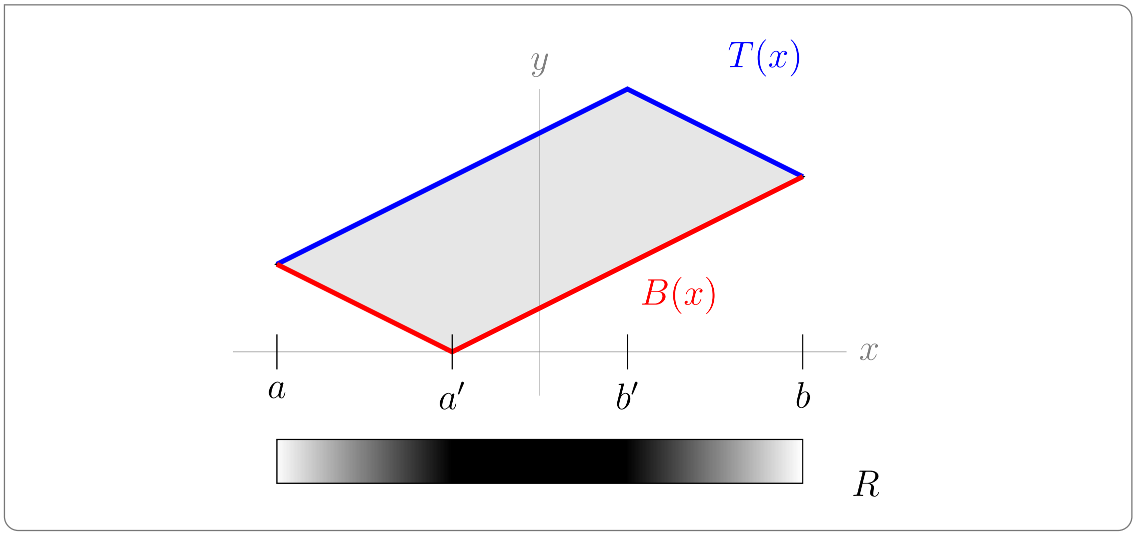
2.3.3.7. Answer. (a), (b) $\dfrac{1}{x}\,\mathrm{d}x$
(c), (d) $\log 3$
(e), (f) $\dfrac{2}{\log 3}$

2.3.3.8. Answer. (a)

$$\frac{\sum_{i=1}^{n}\left[\frac{b-a}{n}\rho\left(a+\left(i-\frac{1}{2}\right)\left(\frac{b-a}{n}\right)\right)\times\left(a+\left(i-\frac{1}{2}\right)\left(\frac{b-a}{n}\right)\right)\right]}{\sum_{i=1}^{n}\frac{b-a}{n}\rho\left(a+\left(i-\frac{1}{2}\right)\left(\frac{b-a}{n}\right)\right)}$$

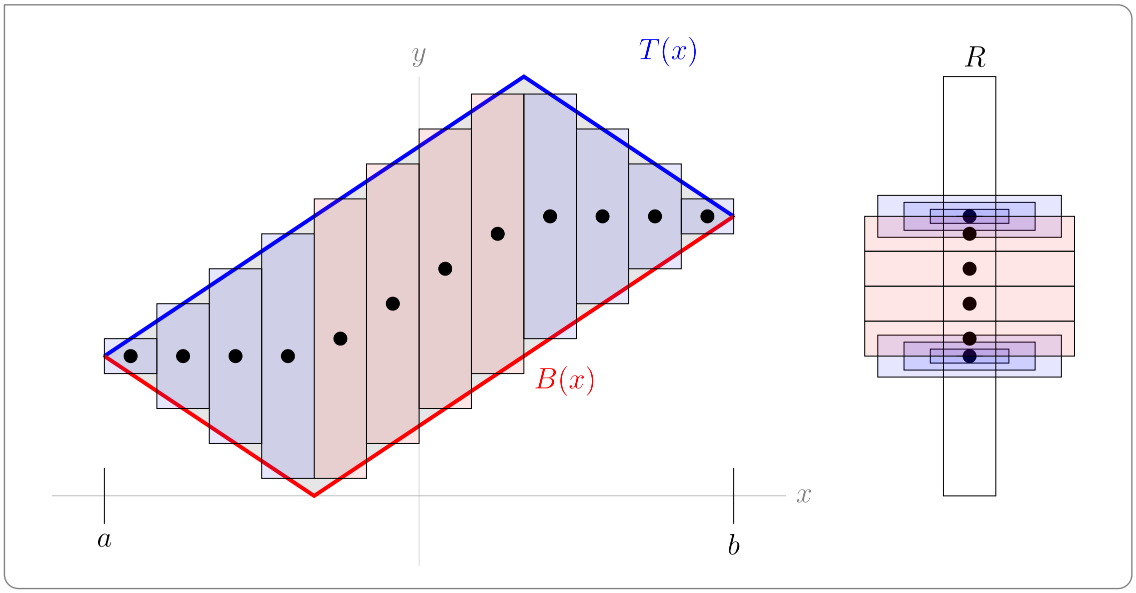
(b) $\bar{x} = \dfrac{\int_a^b x\rho(x)\,\mathrm{d}x}{\int_a^b \rho(x)\,\mathrm{d}x}$

2.3.3.9. Answer. (a)



(b) $(T(x) - B(x))\,\mathrm{d}x$
(c) $T(x) - B(x)$
(d) $\bar{x} = \dfrac{\int_a^b x(T(x) - B(x))\,\mathrm{d}x}{\int_a^b (T(x) - B(x))\,\mathrm{d}x}$

2.3.3.10. Answer. (a) The strips between $x = a$ and $x = a'$ at the left end of the figure all have the same centre of mass, which is the y-value where $T(x) = B(x)$, $x < 0$. So, there should be multiple weights of different mass piled up at that y-value.
Similarly, the strips between $x = b'$ and $x = b$ at the right end of the figure all have the same centre of mass, which is the y-value where $T(x) = B(x)$, $x > 0$. So, there should be a second pile of weights of different mass, at that (higher) y-value.
Between these two piles, there are a collection of weights with identical mass distributed fairly evenly. The top and bottom ends of R (above the uppermost pile, and below the lowermost pile) have no weights.
One possible answer (using twelve slices):



(b) The area of the strip is $(T(x) - B(x))\,\mathrm{d}x$, and its centre of mass is at height $\dfrac{T(x)+B(x)}{2}$.

(c) $\bar{y} = \dfrac{\int_a^b \big(T(x)^2 - B(x)^2\big)\,\mathrm{d}x}{2\int_a^b \big(T(x) - B(x)\big)\,\mathrm{d}x}$

2.3.3.11. *. Answer. $\bar{x} = -\dfrac{1}{3}\displaystyle\int_{-1}^{0} 6x^2\mathrm{d}x$

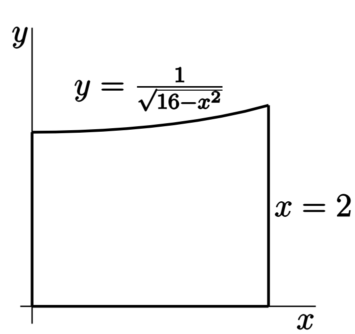
Exercises — Stage 2

2.3.3.12. Answer. $\bar{x} = \dfrac{14}{3}$

2.3.3.13. Answer. $\bar{x} = \dfrac{\log 10.1}{2(\arctan 10 + \arctan(3))} \approx 0.43$

2.3.3.14. *. Answer. $\bar{y} = \dfrac{3}{4e} - \dfrac{e}{4}$

2.3.3.15. *. Answer. (a)

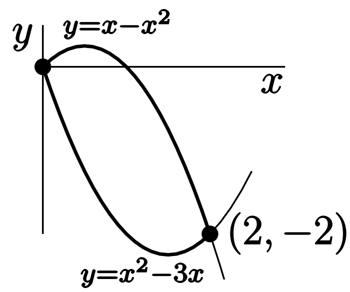


(b) $\dfrac{3\log 3}{8\pi}$

2.3.3.16. *. Answer. $\bar{x} = \dfrac{\frac{\pi}{4}\sqrt{2}-1}{\sqrt{2}-1}$ and $\bar{y} = \dfrac{1}{4(\sqrt{2}-1)}$

2.3.3.17. *. Answer. (a) $\bar{x} = \dfrac{k}{A}\big[\sqrt{2}-1\big]$, $\qquad \bar{y} = \dfrac{k^2\pi}{8A}$
(b) $k = \dfrac{8}{\pi}\big[\sqrt{2}-1\big]$

2.3.3.18. *. Answer. (a)



(b) $\dfrac{8}{3}$
(c) 1

2.3.3.19. *. Answer. $\dfrac{2}{\pi}\log 2 \approx 0.44127$

2.3.3.20. *. Answer. $\bar{x} = 0$ and $\bar{y} = \dfrac{12}{24+9\pi}$

2.3.3.21. *. Answer. (a) $\dfrac{9}{4}\pi$
(b) $\bar{x} = 0$ and $\bar{y} = \dfrac{4}{\pi}$

2.3.3.22. Answer. $(\bar{x}, \bar{y}) = \left(1, -\dfrac{2}{\pi}\right)$

2.3.3.23. Answer. $\left(\dfrac{e^2-3/2}{e^2-5/2}, \dfrac{e^4-7}{4e^2-10}\right) \approx (1.2, 2.4)$

Exercises — Stage 3

2.3.3.24. *. Answer. $\bar{y} = \dfrac{8}{5}$

2.3.3.25. *. Answer. (a) $\bar{x} = \dfrac{8}{11}$, $\bar{y} = \dfrac{166}{55}$
(b) $\pi\displaystyle\int_0^4 y\,\mathrm{d}y + \pi\int_4^6 (6-y)^2\,\mathrm{d}y$

2.3.3.26. *. Answer. (a) $\bar{y} = \dfrac{e}{4} - \dfrac{3}{4e}$
(b) $\pi\left(\dfrac{e^2}{2} + 2e - \dfrac{3}{2}\right)$

2.3.3.27. Answer. $(3, 1.5)$

2.3.3.28. Answer. $(0, 3.45)$

2.3.3.29. Answer. (a) $\dfrac{h}{4}$
(b) $\dfrac{\frac{1}{2}h^2k - \frac{2}{3}hk^2 + \frac{1}{4}k^3}{h^2 - hk + \frac{1}{3}k^2}$

2.3.3.30. Answer. about 0.833 N

2.3.3.31. Answer. (a) $17{,}150\pi$ J
(b) $\dfrac{2450}{9}\pi\,(8\pi - 9) \approx 13,797$ J
(c) about 74%

2.3.3.32. Answer. $\bar{x} = \dfrac{\pi}{162}\sqrt{\dfrac{\pi}{2}}\left[\sin\left(\dfrac{\pi}{72}\right) + 2\sin\left(\dfrac{\pi}{18}\right) + 9\sin\left(\dfrac{\pi}{8}\right) + 8\sin\left(\dfrac{2\pi}{9}\right) + 25\sin\left(\dfrac{25\pi}{72}\right) + 9\right] \approx 0.976$

2.4 · Separable Differential Equations

2.4.7 · Exercises

Exercises — Stage 1

2.4.7.1. Answer. (a) yes
(b) yes
(c) no

2.4.7.2. Answer.

a One possible answer: $f(x) = x$, $g(y) = \dfrac{\sin y}{3y}$.

b One possible answer: $f(x) = e^x$, $g(y) = e^y$.

c One possible answer: $f(x) = x - 1$, $g(y) = 1$.

d The given equation is equivalent to the equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x$, which fits the form of a separable equation with $f(x) = x$, $g(y) = 1$.

2.4.7.3. Answer. The mnemonic allows us to skip from the separable differential equation we want to solve (very first line) to the equation

$$\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$$

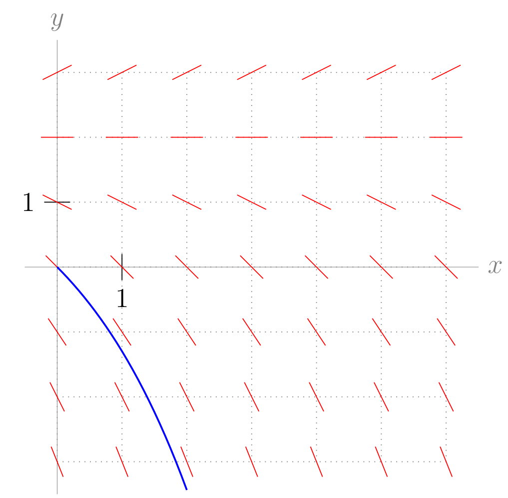
2.4.7.4. Answer. false

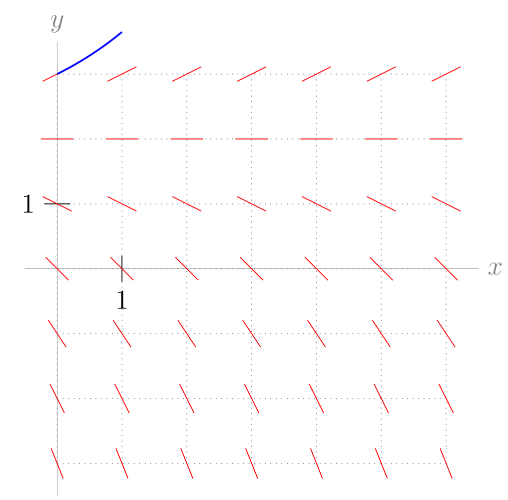
2.4.7.5. Answer. (a) $[0, \infty)$
(b) No such function exists. If $|f(x)| = Cx$ and $f(x)$ switches from $f(x) = Cx$ to $f(x) = -Cx$ at some point, then that point is a jump discontinuity. Where $f(x)$ contains a discontinuity, $\frac{dy}{dx}$ does not exist.

2.4.7.6. Answer. $\dfrac{dQ}{dt} = -0.003Q(t)$

2.4.7.7. Answer. $\frac{dp}{dt} = \alpha p(t)\big(1 - p(t)\big)$, for some constant α.

2.4.7.8. Answer. (a) -1
(b) 0
(c) 0.5
(d) Two possible answers are shown below:





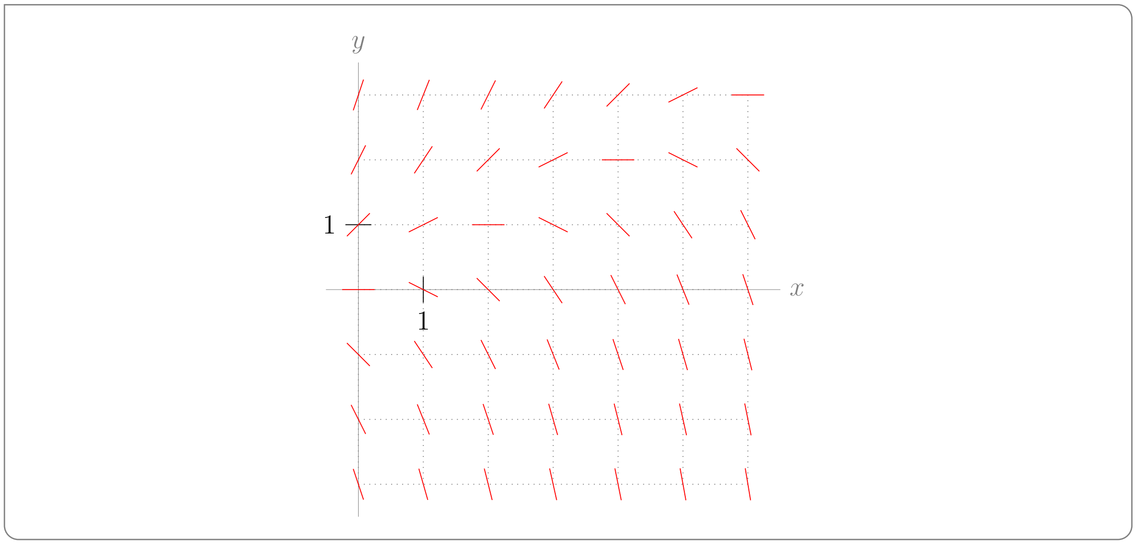
Another possible answer is the constant function $y = 2$.

2.4.7.9. Answer. (a) $-\dfrac{1}{2}$

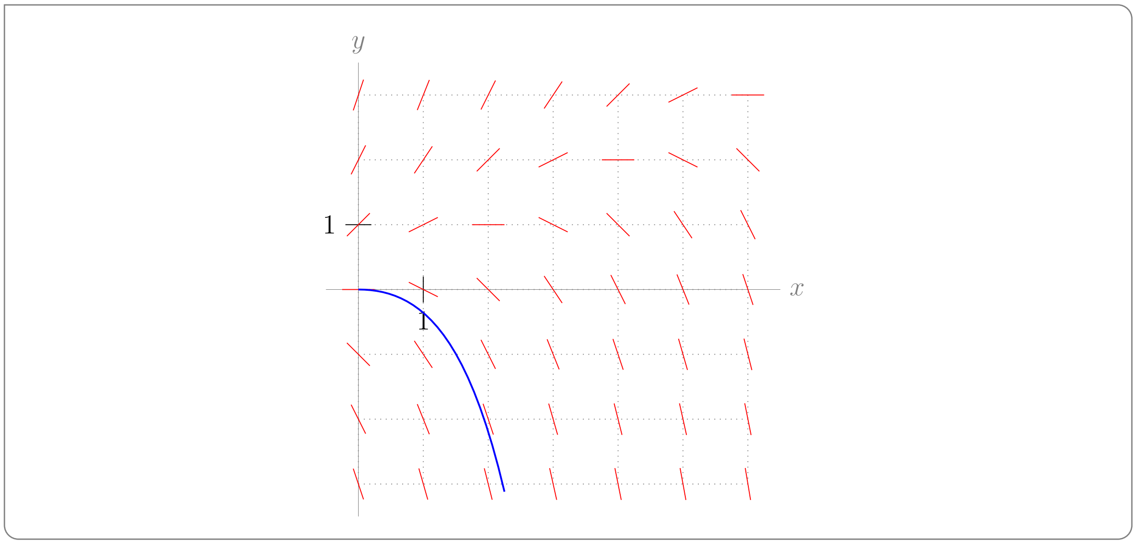
(b) $\dfrac{3}{2}$

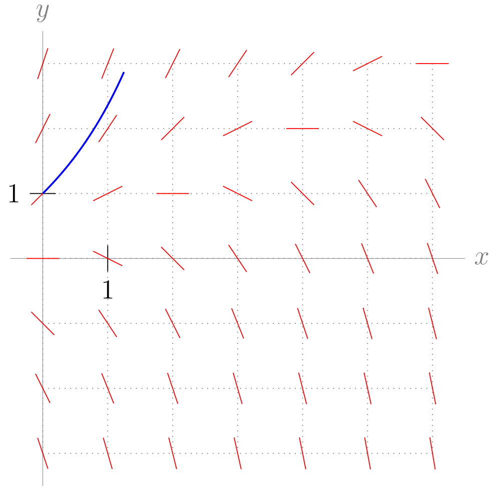
(c) $-\dfrac{5}{2}$

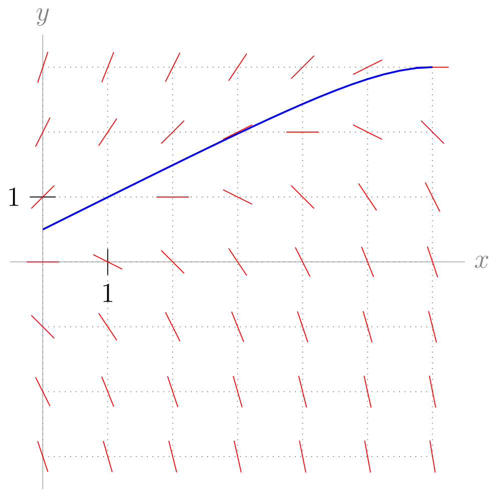
(d) Your sketch should look something like this:

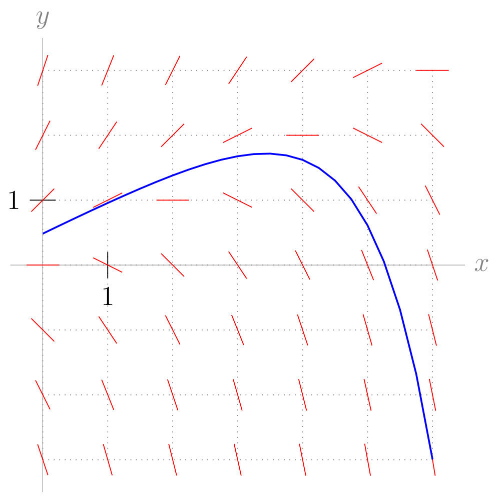


(e) There are lots of possible answers. Several are shown below.









Exercises — Stage 2

2.4.7.10. *. Answer. $y = \log(x^2+2)$

2.4.7.11. *. Answer. $y(x) = 3\sqrt{1+x^2}$

2.4.7.12. *. Answer. $y(t) = 3\log\left(\dfrac{-3}{C+\sin t}\right)$

2.4.7.13. *. Answer. $y = \sqrt[3]{\frac{3}{2}e^{x^2}+C}$.

2.4.7.14. *. Answer. $y = -\log\left(C - \dfrac{x^2}{2}\right)$

The solution only exists for $C - \frac{x^2}{2} > 0$, i.e. $C > 0$ and the function has domain $\left\{x : |x| < \sqrt{2C}\right\}$.

2.4.7.15. *. Answer. $y = (3e^x - 3x^2 + 24)^{1/3}$

2.4.7.16. *. Answer. $y = f(x) = -\dfrac{1}{\sqrt{x^2+16}}$

2.4.7.17. *. Answer. $y = \sqrt{10x^3+4x^2+6x-4}$

2.4.7.18. *. Answer. $y(x) = e^{x^4/4}$

2.4.7.19. *. Answer. $y = \frac{1}{1-2x}$

2.4.7.20. *. Answer. $f(x) = e \cdot e^{x^2/2}$

2.4.7.21. *. Answer. $y(x) = \sqrt{4 + 2\log\frac{2x}{x+1}}$. Note that, to satisfy $y(1) = 2$, we need the positive square root.

2.4.7.22. *. Answer. $y^2 + \dfrac{2}{3}(y^2-4)^{3/2} = 2\sec x + 2$

2.4.7.23. *. Answer. 12 weeks

2.4.7.24. *. Answer. $t = \sqrt{\dfrac{m}{kg}}\arctan\left(\sqrt{\dfrac{k}{mg}}\,v_0\right)$

2.4.7.25. *. Answer. (a) $k = \frac{1}{400}$
(b) $t = 70$sec

2.4.7.26. *. Answer. (a) $x(t) = \dfrac{3-4e^{kt}}{1-2e^{kt}}$
(b) As $t \to \infty$, $x \to 2$.

2.4.7.27. *. Answer. (a) $P = \dfrac{4}{1+e^{-4t}}$
(b) At $t = \dfrac{1}{2}$, $P \approx 3.523$. As $t \to \infty$, $P \to 4$.

2.4.7.28. *. Answer. (a) $\frac{\mathrm{d}v}{\mathrm{d}t} = -kv^2$
(b) $v = \frac{400}{t+1}$
(c) $t = 7$

Exercises — Stage 3

2.4.7.29. *. Answer. (a) $B(t) = C\,e^{0.06t-0.02\cos t}$ with the arbitrary constant $C \geq 0$.
(b) \$1159.89

2.4.7.30. *. Answer. (a) $B(t) = \{30000 - 50m\}\,e^{t/50} + 50m$
(b) \$600

2.4.7.31. *. Answer. $y(x) = \frac{4 - e^{1-\cos x}}{2 - e^{1-\cos x}}$. The largest allowed interval is

$$-\arccos(1 - \log 2) < x < \arccos(1 - \log 2)$$

or, roughly, $-1.259 < x < 1.259$.

2.4.7.32. *. Answer. $180,000\sqrt{\frac{3}{g}} \approx 99,591$ sec ≈ 27.66 hr

2.4.7.33. *. Answer. $t = \frac{4 \times 144}{15}\sqrt{\frac{12^5}{2g}} \approx 2,394\,\text{sec} \;\approx 0.665\,\text{hr}$

2.4.7.34. *. Answer. (a) 3
(b) $y' = (y-1)(y-2)$
(c) $f(x) = \frac{4 - e^x}{2 - e^x}$

2.4.7.35. *. Answer. $p = \frac{1}{4}$

2.4.7.36. Answer.

a One possible answer: $f(t) = 0$

b $\frac{1}{\sqrt{x-a}}\left[f(x) - \frac{1}{2(x-a)}\int_a^x f(t)\,\mathrm{d}t\right] = \frac{f^2(x)}{2\sqrt{\int_a^x f^2(t)\,\mathrm{d}t}}$

c $\frac{2}{x-a}\int_a^x f(t)\,\mathrm{d}t\left[f(x) - \frac{1}{2(x-a)}\int_a^x f(t)\,\mathrm{d}t\right] = f^2(x)$

d $Y(x) = D(x-a)$, where D is any constant

e $f(t) = D$, for any nonnegative constant D

2.4.7.37. Answer. $x = \frac{1}{4}\left(y - 1 + \frac{1}{4}\log\left|\frac{2y-1}{2y+1}\right|\right)$

3 · Sequence and series

3.1 · Sequences

3.1.2 · Exercises

Exercises — Stage 1

3.1.2.1. Answer. (a) -2 (b) 0 (c) the limit does not exist

3.1.2.2. Answer. true

3.1.2.3. Answer. (a) $\dfrac{A-B}{C}$ (b) 0 (c) $\dfrac{A}{B}$

3.1.2.4. Answer. Two possible answers, of many:

- $a_n = \begin{cases} 3000 - n & \text{if } n \le 1000 \\ -2 + \frac{1}{n} & \text{if } n > 1000 \end{cases}$
- $a_n = \dfrac{1,002,001}{n} - 2$

3.1.2.5. Answer. One possible answer is $a_n = (-1)^n = \{-1, 1, -1, 1, -1, 1, -1, \ldots\}$.
Another is $a_n = n(-1)^n = \{-1, 2, -3, 4, -5, 6, -7, \ldots\}$.

3.1.2.6. Answer. One sequence of many possible is $a_n = \dfrac{(-1)^n}{n} = \left\{-1, \dfrac{1}{2}, -\dfrac{1}{3}, \dfrac{1}{4}, -\dfrac{1}{5}, \dfrac{1}{6}, \ldots\right\}$.

3.1.2.7. Answer. Some possible answers:

a $\dfrac{-1}{n} \le \dfrac{\sin n}{n} \le \dfrac{1}{n}$

b $\dfrac{n^2}{13e^n} \le \dfrac{n^2}{e^n(7 + \sin n - 5\cos n)} \le \dfrac{n^2}{e^n}$

c $\dfrac{-1}{n^n} \le (-n)^{-n} \le \dfrac{1}{n^n}$

3.1.2.8. Answer. (a) $a_n = b_n = h(n) = i(n)$, $c_n = j(n)$, $d_n = f(n)$, $e_n = g(n)$
(b) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = \lim\limits_{x\to\infty} h(x) = 1$, $\lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty} e_n = \lim\limits_{x\to\infty} g(x) = \lim\limits_{x\to\infty} j(x) = 0$, $\lim\limits_{n\to\infty} d_n$, $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to\infty} i(x)$ do not exist.

3.1.2.9. Answer. (a) Some possible answers: $a_{22} \approx -0.99996$, $a_{66} \approx -0.99965$, and $a_{110} \approx -0.99902$.
(b) Some possible answers: $a_{11} \approx 0.0044$, $a_{33} \approx -0.0133$, and $a_{55} \approx 0.0221$.
The integers 11, 33, and 55 were found by approximating π by $\dfrac{22}{7}$ and finding when an odd multiple of $\dfrac{11}{7}$ (which is the corresponding approximation of $\pi/2$) is an

integer.

Exercises — Stage 2

3.1.2.10. Answer. (a) ∞ (b) $\frac{3}{4}$ (c) 0

3.1.2.11. Answer. ∞

3.1.2.12. Answer. 0

3.1.2.13. Answer. 0

3.1.2.14. Answer. 0

3.1.2.15. Answer. 1

3.1.2.16. Answer. 0

3.1.2.17. Answer. ∞

3.1.2.18. *. Answer. $\lim\limits_{k\to\infty} a_k = 0.$

3.1.2.19. *. Answer. The sequence converges to 0.

3.1.2.20. *. Answer. 9

Exercises — Stage 3

3.1.2.21. *. Answer. $\log 2$

3.1.2.22. Answer. 5

3.1.2.23. Answer. $-\infty$

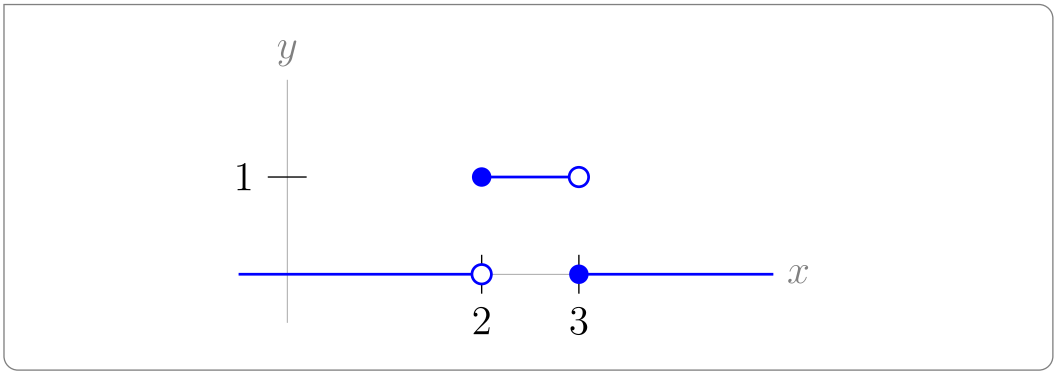
3.1.2.24. Answer. $100\cdot 2^{99}$.

3.1.2.25. Answer. Possible answers are $\{a_n\} = \left\{n\left[f\left(a+\frac{1}{n}\right) - f(a)\right]\right\}$ or $\{a_n\} = \left\{n\left[f(a) - f\left(a-\frac{1}{n}\right)\right]\right\}$.

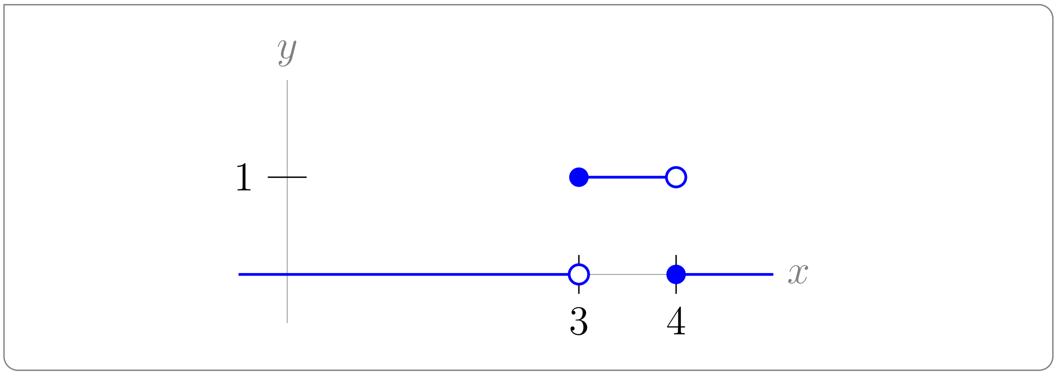
3.1.2.26. Answer. (a) $A_n = \frac{n}{2}\sin\left(\frac{2\pi}{n}\right)$ (b) π

3.1.2.27. Answer.

a



b



c $A_n = 1$ for all n

d $\lim\limits_{n\to\infty} A_n = 1.$

e $g(x) = 0$

f $\displaystyle\int_0^\infty g(x)\,\mathrm{d}x = 0.$

3.1.2.28. Answer. e^3

3.1.2.29. Answer. (a) 4 (b) $x = 4$ (c) see solution

3.1.2.30. Answer. (a) decreasing (b) $f_n = \frac{1}{n} f_1$ (c) 2% (d) 0.18%
(e) “be”: 11,019,308; “and”: 7,346,205

3.2 · Series

3.2.2 · Exercises

Exercises — Stage 1

3.2.2.1. Answer.

N	S_N
1	1
2	$1+\frac{1}{2}$
3	$1+\frac{1}{2}+\frac{1}{3}$
4	$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$
5	$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$

3.2.2.2. Answer. 3

3.2.2.3. Answer. (a) $a_n = \begin{cases} \frac{1}{2} & \text{if } n=1 \\ \dfrac{1}{n(n+1)} & \text{else} \end{cases}$ (b) 0 (c) 1

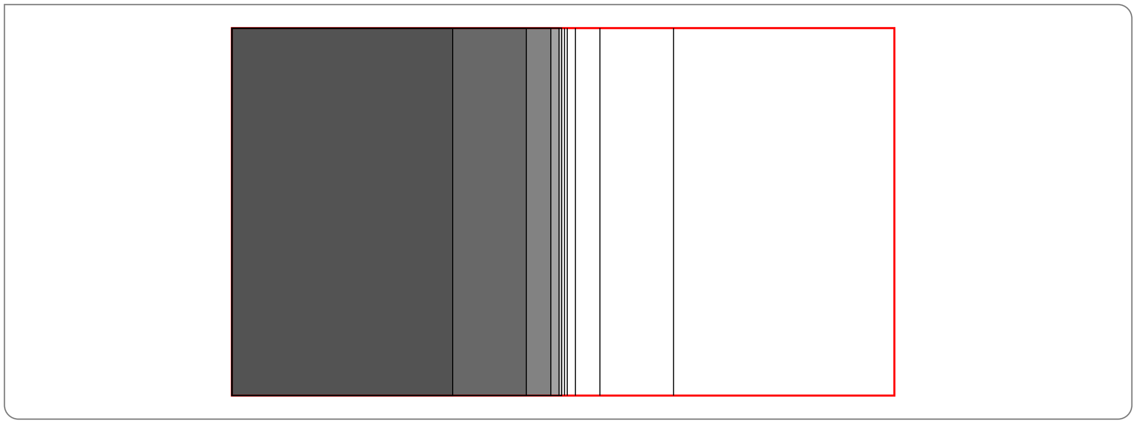
3.2.2.4. Answer. $a_n = \begin{cases} 0 & \text{if } n=1 \\ 2(-1)^n - \frac{1}{n(n-1)} & \text{else} \end{cases}$

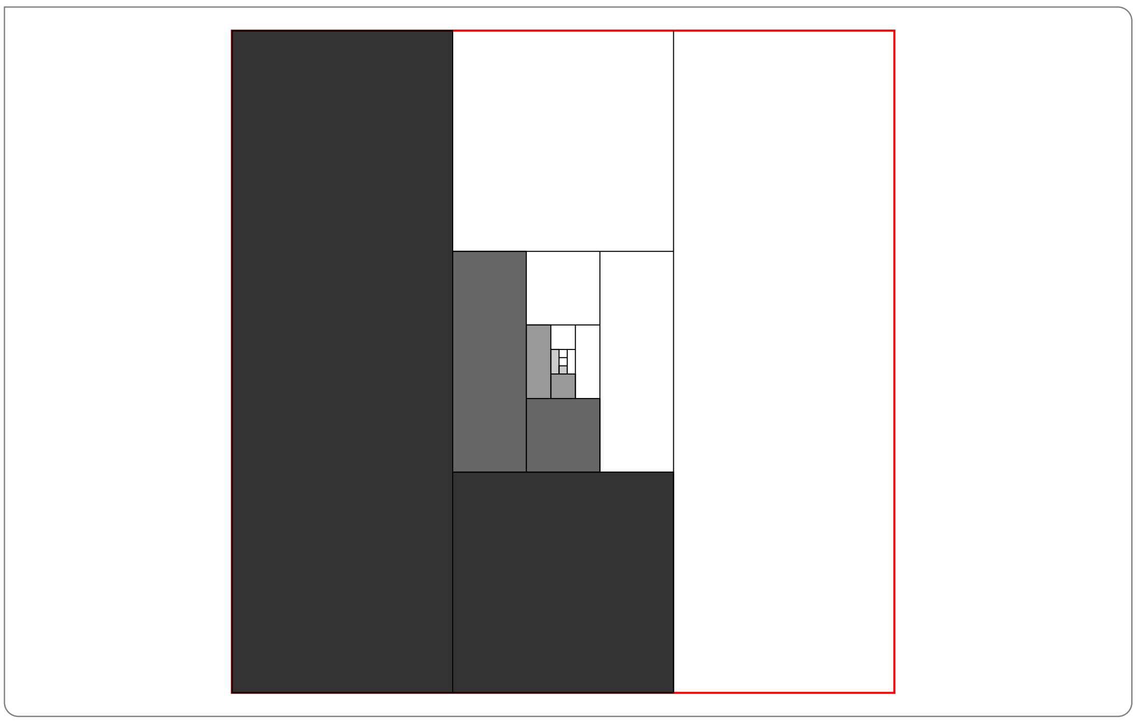
3.2.2.5. Answer. $a_n < 0$ for all $n \geq 2$

3.2.2.6. Answer. (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{4^n}$ (b) $\dfrac{2}{3}$

3.2.2.7. Answer. (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{9^n}$ (b) $\dfrac{1}{8}$

3.2.2.8. Answer. Two possible pictures:





3.2.2.9. Answer. $\dfrac{5^{101}-1}{4 \cdot 5^{100}}$

3.2.2.10. Answer. All together, there were 36 cookies brought by Student 11 through Student 20.

3.2.2.11. Answer. $\dfrac{5^{51}-1}{4\cdot 5^{100}}$

3.2.2.12. Answer. (a) As time passes, your gains increase, approaching \$1.
(b) 1
(c) As time passes, you *lose* more and more money, without bound. (d) $-\infty$

3.2.2.13. Answer. $A+B+C-c_1$

3.2.2.14. Answer. in general, false

Exercises — Stage 2

3.2.2.15. *. Answer. $\dfrac{3}{2}$

3.2.2.16. *. Answer. $\dfrac{1}{7\times 8^6}$

3.2.2.17. *. Answer. 6

3.2.2.18. *. Answer. $\cos\left(\dfrac{\pi}{3}\right)-\cos(0)=-\dfrac{1}{2}$

3.2.2.19. *. Answer. (a) $a_n=\dfrac{11}{16n^2+24n+5}$ (b) $\dfrac{3}{4}$

3.2.2.20. *. Answer. $\dfrac{24}{5}$

3.2.2.21. *. Answer. $\dfrac{7}{30}$

3.2.2.22. *. Answer. $\dfrac{263}{99}$

3.2.2.23. *. Answer. $\dfrac{321}{999}=\dfrac{107}{333}$

3.2.2.24. *. Answer. 3

3.2.2.25. *. Answer. $\dfrac{1}{2}+\dfrac{5}{7}=\dfrac{17}{14}$

3.2.2.26. *. Answer. $\dfrac{40}{3}$

3.2.2.27. Answer. The series diverges to $-\infty$.

3.2.2.28. Answer. $-\dfrac{1}{2}$

Exercises — Stage 3

3.2.2.29. Answer. 9.8 J

3.2.2.30. Answer. $\dfrac{4\pi}{3\left(\pi^3-1\right)}$

3.2.2.31. Answer. $\dfrac{\sin^2 3}{8} + 32 \approx 32.0025$

3.2.2.32. Answer. $a_n = \begin{cases} \frac{2}{n(n-1)(n-2)} & \text{if } n \geq 3, \\ -\frac{5}{2} & \text{if } n = 2, \\ 2 & \text{if } n = 1 \end{cases}$

3.2.2.33. Answer. $\dfrac{5}{8}$

3.3 · Convergence Tests

3.3.11 · Exercises

Exercises — Stage 1

3.3.11.1. Answer. (B), (C)

3.3.11.2. Answer. (A)

3.3.11.3. Answer. (a) I am old (b) not enough information to tell
(c) not enough information to tell (d) I am young

3.3.11.4. Answer.

	if $\sum a_n$ converges	if $\sum a_n$ diverges
and if $\{a_n\}$ is the red series	then $\sum b_n$ CONVERGES	inconclusive
and if $\{a_n\}$ is the blue series	inconclusive	then $\sum b_n$ DIVERGES

3.3.11.5. Answer. (a) both direct comparison and limit comparison
(b) direct comparison
(c) limit comparison (d) neither

3.3.11.6. Answer. It diverges by the divergence test, because $\lim_{n\to\infty} a_n \neq 0$.

3.3.11.7. Answer. We cannot use the divergence test to show that a series converges. It is inconclusive in this case.

3.3.11.8. Answer. The integral test does not apply because $f(x)$ is not decreasing.

3.3.11.9. Answer. The inequality goes the wrong way, so the direct comparison test (with this comparison series) is inconclusive.

3.3.11.10. Answer. (B), (D)

3.3.11.11. Answer. One possible answer: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

3.3.11.12. Answer. By the divergence test, for a series $\sum a_n$ to *converge*, we need $\lim_{n\to\infty} a_n = 0$. That is, the magnitude (absolute value) of the terms needs to be getting smaller. If $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| < 1$ or (equivalently) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, then $|a_{n+1}| > |a_n|$ for sufficiently large n, so the terms are actually *growing* in magnitude. That means the series diverges, by the divergence test.

3.3.11.13. Answer. One possible answer: $f(x) = \sin(\pi x)$, $a_n = 0$ for every n. By the integral test, any answer will use a function $f(x)$ that is *not* both positive and decreasing.

3.3.11.14. *. Answer. One possible answer: $b_n = \dfrac{2^n}{3^n}$

3.3.11.15. *. Answer. (a) In general false. The harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ provides a counterexample.
(b) In general false. If $a_n = (-1)^n\frac{1}{n}$, then $\sum_{n=1}^{\infty}(-1)^n a_n$ is again the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges.
(c) In general false. Take, for example, $a_n = 0$ and $b_n = 1$.

Exercises — Stage 2

3.3.11.16. *. Answer. No. It diverges.

3.3.11.17. *. Answer. It diverges.

3.3.11.18. *. Answer. The series diverges.

3.3.11.19. Answer. It diverges.

3.3.11.20. Answer. This is a geometric series with $r = 1.001$. Since $|r| > 1$, it is divergent.

3.3.11.21. Answer. The series converges to $-\dfrac{1}{150}$.

3.3.11.22. Answer. The series converges.

3.3.11.23. Answer. It diverges.

3.3.11.24. Answer. The series converges.

3.3.11.25. Answer. The series converges to $\dfrac{1}{3}$.

3.3.11.26. Answer. The series converges.

3.3.11.27. Answer. It converges.

3.3.11.28. *. Answer. Let $f(x) = \dfrac{5}{x(\log x)^{3/2}}$. Then $f(x)$ is positive and decreases as x increases. So the sum $\sum_{3}^{\infty} f(n)$ and the integral $\int_3^{\infty} f(x)\,dx$ either both converge or both diverge, by the integral test, which is Theorem 3.3.5. For the integral, we use the substitution $u = \log x$, $\mathrm{d}u = \frac{\mathrm{d}x}{x}$ to get

$$\int_3^\infty \frac{5\,\mathrm{d}x}{x(\log x)^{3/2}} = \int_{\log 3}^\infty \frac{5\,\mathrm{d}u}{u^{3/2}}$$

which converges by the p-test (which is Example 1.12.8) with $p = \frac{3}{2} > 1$.

3.3.11.29. *. Answer. $p > 1$

3.3.11.30. *. Answer. It converges.

3.3.11.31. *. Answer. The series $\displaystyle\sum_{n=2}^\infty \frac{\sqrt{3}}{n^2}$ converges by the p-test with $p = 2$. Note that

$$0 < a_n = \frac{\sqrt{3n^2-7}}{n^3} < \frac{\sqrt{3n^2}}{n^3} = \frac{\sqrt{3}}{n^2}$$

for all $n \geq 2$. As the series $\sum_{n=2}^\infty \frac{\sqrt{3}}{n^2}$ converges, the comparison test says that $\sum_{n=2}^\infty \frac{\sqrt{3n^2-7}}{n^3}$ converges too.

3.3.11.32. *. Answer. The series converges.

3.3.11.33. *. Answer. It diverges.

3.3.11.34. *. Answer. (a) diverges (b) converges

3.3.11.35. *. Answer. The series diverges.

3.3.11.36. *. Answer. (a) converges (b) diverges

3.3.11.37. Answer. $\dfrac{1}{e^5 - e^4}$

3.3.11.38. *. Answer. $\frac{1}{7}$

3.3.11.39. *. Answer. (a) diverges by limit comparison with the harmonic series
(b) converges by the ratio test

3.3.11.40. *. Answer. (a) Converges by the limit comparison test with $b = \frac{1}{k^{5/3}}$.
(b) Diverges by the ratio test.
(c) Diverges by the integral test.

3.3.11.41. *. Answer. It converges.

3.3.11.42. *. Answer. $N = 5$

3.3.11.43. *. Answer. $N \geq 999$

3.3.11.44. *. Answer. We need $N = 4$ and then $S_4 = \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{9^2}$

Exercises — Stage 3

3.3.11.45. *. Answer. (a) converges (b) converges

3.3.11.46. ***. Answer**. (a) See the solution.
(b) $f(x) = \dfrac{x + \sin x}{1 + x^2}$ is not a decreasing function.
(c) See the solution.

3.3.11.47. ***. Answer**. The sum is between 0.9035 and 0.9535.

3.3.11.48. ***. Answer**. Since $\lim_{n\to\infty} a_n = 0$, there must be some integer N such that $\frac{1}{2} > a_n \geq 0$ for all $n > N$. Then, for $n > N$,

$$\frac{a_n}{1 - a_n} \leq \frac{a_n}{1 - 1/2} = 2a_n$$

From the information in the problem statement, we know

$$\sum_{n=N+1}^{\infty} 2a_n = 2\sum_{n=N+1}^{\infty} a_n \qquad \text{converges.}$$

So, by the direct comparison test,

$$\sum_{n=N+1}^{\infty} \frac{a_n}{1 - a_n} \qquad \text{converges as well.}$$

Since the convergence of a series is not affected by its first N terms, as long as N is finite, we conclude

$$\sum_{n=1}^{\infty} \frac{a_n}{1 - a_n} \qquad \text{converges.}$$

3.3.11.49. ***. Answer**. It diverges.

3.3.11.50. ***. Answer**. It converges to $-\log 2 = \log \frac{1}{2}$,

3.3.11.51. ***. Answer**. See the solution.

3.3.11.52. **Answer**. About 9% to 10%

3.3.11.53. **Answer**. The total population is between 29,820,091 and 30,631,021 people.

3.4 · Absolute and Conditional Convergence

3.4.3 · Exercises

Exercises — Stage 1

3.4.3.1. ***. Answer**. False. For example, $b_n = \frac{1}{n}$ provides a counterexample.

3.4.3.2. **Answer**.

	$\sum a_n$ converges	$\sum a_n$ diverges
$\sum \lvert a_n \rvert$ converges	converges absolutely	not possible
$\sum \lvert a_n \rvert$ diverges	converges conditionally	diverges

Exercises — Stage 2

3.4.3.3. *. Answer. Conditionally convergent

3.4.3.4. *. Answer. The series diverges.

3.4.3.5. *. Answer. It diverges.

3.4.3.6. *. Answer. It converges absolutely.

3.4.3.7. *. Answer. It converges absolutely.

3.4.3.8. *. Answer. It diverges.

3.4.3.9. *. Answer. It converges absolutely.

3.4.3.10. Answer. See solution.

3.4.3.11. Answer. See solution.

3.4.3.12. Answer. See solution.

Exercises — Stage 3

3.4.3.13. *. Answer. (a) See the solution.
(b) $|S - S_5| \le 24 \times 36e^{-6^3}$

3.4.3.14. Answer. $\cos 1 \approx \frac{389}{720}$; the actual associated error (using a calculator) is about 0.000025.

3.4.3.15. Answer. See solution.

3.5 · Power Series

3.5.3 · Exercises

Exercises — Stage 1

3.5.3.1. Answer. 2

3.5.3.2. Answer. $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{n(x-5)^{n-1}}{n!+2}$

3.5.3.3. Answer. Only $x = c$

3.5.3.4. Answer. $R = 6$

Exercises — Stage 2

3.5.3.5. $\ast$. Answer. (a) $R = \frac{1}{2}$
(b) $\frac{2}{1+2x}$ for all $|x| < \frac{1}{2}$

3.5.3.6. $\ast$. Answer. $R = \infty$

3.5.3.7. $\ast$. Answer. 1

3.5.3.8. $\ast$. Answer. The interval of convergence is $-1 < x + 2 \leq 1$ or $(-3, -1]$.

3.5.3.9. $\ast$. Answer. The interval of convergence is $-4 < x \leq 2$, or simply $(-4, 2]$.

3.5.3.10. $\ast$. Answer. $-3 \leq x < 7$ or $[-3, 7)$

3.5.3.11. $\ast$. Answer. The given series converges if and only if $-3 \leq x \leq -1$. Equivalently, the series has interval of convergence $[-3, -1]$.

3.5.3.12. $\ast$. Answer. The interval of convergence is $\frac{3}{4} \leq x < \frac{5}{4}$, or $\left[\frac{3}{4}, \frac{5}{4}\right)$.

3.5.3.13. $\ast$. Answer. The radius of convergence is 2. The interval of convergence is $-1 < x \leq 3$, or $\big(-1, 3\big]$.

3.5.3.14. $\ast$. Answer. The interval of convergence is $a - 1 < x < a + 1$, or $\big(a - 1, a + 1\big)$.

3.5.3.15. $\ast$. Answer. (a) $|x + 1| \leq 9$ or $-10 \leq x \leq 8$ or $[-10, 8]$
(b) This series converges only for $x = 1$.

3.5.3.16. $\ast$. Answer. $\sum_{n=0}^{\infty} x^{n+3} = \sum_{n=3}^{\infty} x^n$

3.5.3.17. Answer. $f(x) = 3 + \sum_{n=1}^{\infty} \frac{(x-1)^n}{n(n+1)}$

Exercises — Stage 3

3.5.3.18. $\ast$. Answer. The series converges absolutely for $|x| < 9$, converges conditionally for $x = -9$ and diverges otherwise.

3.5.3.19. $\ast$. Answer. (a) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1} + C$
(b) We need to keep two terms (the $n = 0$ and $n = 1$ terms).

3.5.3.20. $\ast$. Answer. (a) See the solution.
(b) $\sum_{n=0}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$. The series converges for $-1 < x < 1$.

3.5.3.21. $\ast$. Answer. See the solution.

3.5.3.22. $\ast$. Answer. (a) 1.
(b) The series converges for $-1 \leq x < 1$, i.e. for the interval $[-1, 1)$

3.5.3.23. Answer. $\frac{5}{6}$

3.5.3.24. Answer. The point $x = c$ corresponds to a local maximum if $A_2 < 0$ and a local minimum if $A_2 > 0$.

3.5.3.25. Answer. $\frac{13}{80}$

3.5.3.26. Answer. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

3.5.3.27. Answer. $x - \frac{x^3}{3} + \frac{x^5}{5}$

3.6 · Taylor Series

3.6.8 · Exercises

Exercises — Stage 1

3.6.8.1. Answer. A: linear
B: constant
C: quadratic

3.6.8.2. Answer. $T(5) = \arctan^3\left(e^5 + 7\right)$

3.6.8.3. Answer. A - V
B - I
C - IV
D - VI
E - II
F - III

3.6.8.4. Answer. (a) $f^{(20)}(3) = 20^2\left(\frac{20!}{20!+1}\right)$
(b) $g^{(20)}(3) = 10^2\left(\frac{20!}{10!+1}\right)$
(c) $h^{(20)}(0) = 0; \quad h^{(22)}(0) = \frac{22! \cdot 5^{13}}{13}$

Exercises — Stage 2

3.6.8.5. Answer. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$

3.6.8.6. Answer. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1}$

3.6.8.7. Answer. $\displaystyle\frac{1}{10}\sum_{n=0}^{\infty}\left(\frac{10-x}{10}\right)^n$ with interval of convergence $(0, 20)$.

3.6.8.8. Answer. $\displaystyle\sum_{n=0}^{\infty}\frac{3^n e^{3a}}{n!}(x-a)^n$, with infinite radius of convergence

3.6.8.9. *. Answer. $\displaystyle -\sum_{n=0}^{\infty}2^n x^n$

3.6.8.10. *. Answer. $b_n = 3(-1)^n + 2^n$

3.6.8.11. *. Answer. $\displaystyle c_5 = \frac{3^5}{5!}$

3.6.8.12. *. Answer. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{2^{n+1}x^{n+1}}{n+1}$ for all $|x| < \frac{1}{2}$

3.6.8.13. *. Answer. $\displaystyle a = 1,\ b = -\frac{1}{3!} = -\frac{1}{6}.$

3.6.8.14. *. Answer. $\displaystyle\int\frac{e^{-x^2}-1}{x}\,\mathrm{d}x = C - \frac{x^2}{2} + \frac{x^4}{8} + \cdots.$
It is not clear from the wording of the question whether or not the arbitrary constant C is to be counted as one of the "first two nonzero terms".

3.6.8.15. *. Answer. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n+1}x^{2n+6}}{(2n+1)(2n+6)} + C = \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}x^{2n+6}}{(2n+1)(n+3)} + C$

3.6.8.16. *. Answer. $\displaystyle f(x) = 1 + \sum_{n=0}^{\infty}(-1)^n\frac{3^n}{3n+2}x^{3n+2}$

3.6.8.17. *. Answer. $\displaystyle\frac{\pi}{2\sqrt{3}}$

3.6.8.18. *. Answer. $\displaystyle\frac{1}{e}$

3.6.8.19. *. Answer. $e^{1/e}$

3.6.8.20. *. Answer. $e^{1/\pi} - 1$

3.6.8.21. *. Answer. $\log(3/2)$

3.6.8.22. *. Answer. $(e+2)e^e - 2$

3.6.8.23. Answer. The sum diverges — see the solution.

3.6.8.24. Answer. $\displaystyle\frac{1+\sqrt{2}}{\sqrt{2}}$

3.6.8.25. *. Answer. (a) See the solution.
(b) $\frac{1}{2}\left(e+\frac{1}{e}\right)$

3.6.8.26. Answer. (a) 50,000
(b) three terms ($n=0$ to $n=2$)
(c) six terms ($n=0$ to $n=5$)

3.6.8.27. Answer. 29

3.6.8.28. Answer. S_{13} or higher

3.6.8.29. Answer. S_9 or higher

3.6.8.30. Answer. S_{18} or higher

3.6.8.31. Answer. The error is in the interval $\left(\frac{-5^7}{14\cdot 3^7}\left[1+\frac{1}{3^7}\right]\quad,\quad\frac{-5^7}{7\cdot 6^7}\right)\approx$ $(-0.199,-0.040)$

Exercises — Stage 3

3.6.8.32. *. Answer. -1

3.6.8.33. *. Answer. $\frac{1}{5!}=\frac{1}{120}$

3.6.8.34. Answer. e^2

3.6.8.35. Answer. $\sqrt{e}$

3.6.8.36. Answer. $\frac{2}{(6/7)^3}=\frac{343}{108}$

3.6.8.37. Answer. $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+4}}{(2n+1)(2n+2)}=x^3\arctan x-\frac{x^2}{2}\log(1+x^2)$

3.6.8.38. Answer. (a) the Maclaurin series for $f(x)$ is $\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}\,(n!)^2}x^n$, and its radius of convergence is $R=1$.
(b) the Maclaurin series for $\arcsin x$ is $\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}\,(n!)^2(2n+1)}x^{2n+1}$, and its radius of convergence is $R=1$.

3.6.8.39. *. Answer. $\log(x)=\log 2+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\,2^n}(x-2)^n$. It converges when $0<x\le 4$.

3.6.8.40. *. Answer. (a) $\sum_{n=0}^{\infty}(-1)^n\,\frac{x^{4n+1}}{4n+1}$
(b) 0.493967
(c) The approximate value of part (b) is larger than the true value of $I(1/2)$

3.6.8.41. *. Answer. $\dfrac{1}{66}$

3.6.8.42. *. Answer. Any interval of length 0.0002 that contains 0.03592 and 0.03600 is fine.

3.6.8.43. *. Answer. (a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\,\frac{x^n}{n\;n!}$
(b)-0.80
(c) See the solution.

3.6.8.44. *. Answer. (a) $\displaystyle\Sigma(x)=\sum_{n=0}^{\infty}(-1)^n\,\frac{x^{2n+1}}{(2n+1)(2n+1)!}$
(b) $x=\pi$
(c) 1.8525

3.6.8.45. *. Answer. (a) $\displaystyle I(x)=\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n)!(2n-1)}$
(b) $I(1)=-\frac{1}{2}+\frac{1}{4!3}\pm\frac{1}{6!5}=-0.486\pm 0.001$
(c) $I(1)<-\frac{1}{2}+\frac{1}{4!3}$

3.6.8.46. *. Answer. (a) $\displaystyle I(x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{(2n)!}=\frac{1}{2!}x-\frac{1}{4!}x^3+\frac{1}{6!}x^5-\frac{1}{8!}x^8+\cdots$
(b) 0.460
(c) $I(1)<\frac{1}{2!}-\frac{1}{4!}+\frac{1}{6!}<0.460$

3.6.8.47. *. Answer. (a) See the solution.
(b) The series converges for all x.

3.6.8.48. *. Answer. See the solution.

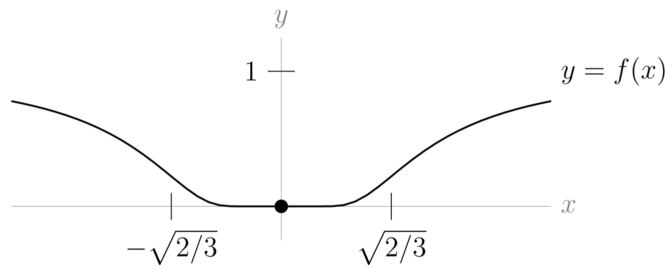
3.6.8.49. *. Answer. (a) $\displaystyle\cosh(x)=\sum_{\substack{n=0\\ n\text{ even}}}^{\infty}\frac{x^n}{n!}=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$ for all x.

(b), (c) See the solution.

3.6.8.50. Answer. (a) $\sqrt[3]{3}\approx 1.26$
(b) 12 terms (S_{11})

3.6.8.51. Answer. $\dfrac{15!}{5!\cdot 5^6}-\dfrac{21!}{7!\cdot 11!\cdot 5^{11}}+\dfrac{27!}{9!\cdot 17!\cdot 5^{17}}-\dfrac{33!}{11!\cdot 23!\cdot 5^{23}}$

3.6.8.52. Answer. (a)



(b) the constant function 0
(c) everywhere
(d) only at $x = 0$

3.6.8.53. Answer. 0

Appendix G

Solutions to Exercises

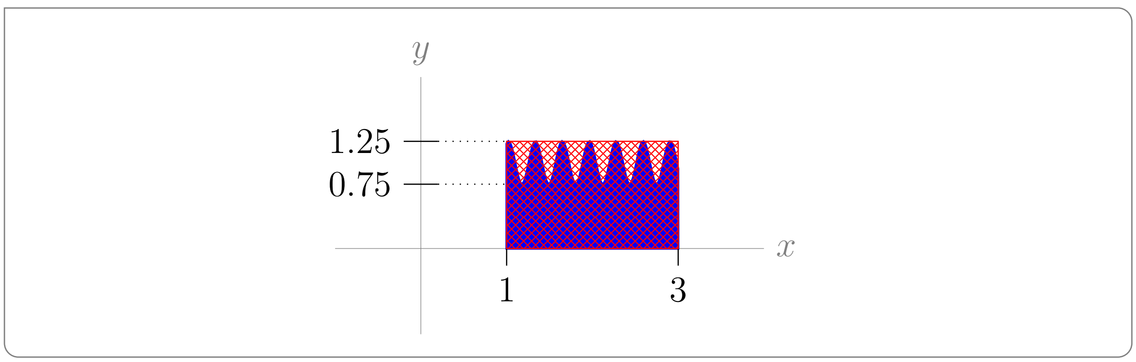
1 · Integration

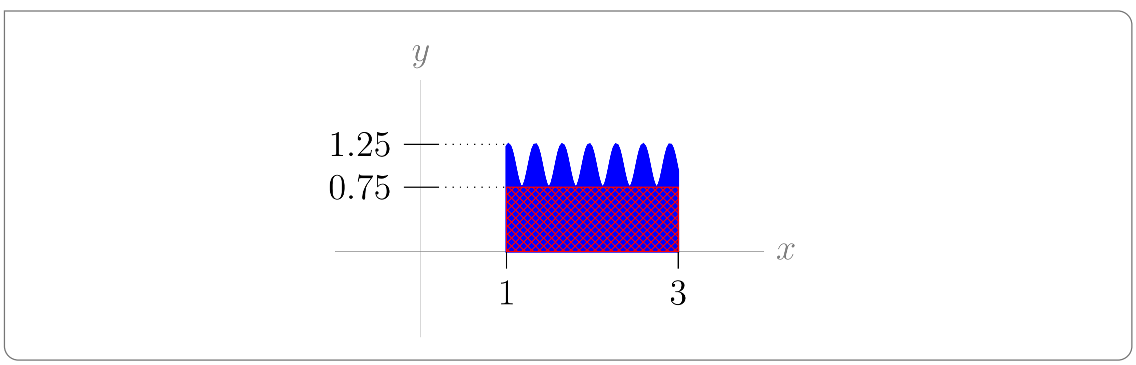
1.1 · Definition of the Integral

1.1.8 · Exercises

Exercises — Stage 1

1.1.8.1. Solution.



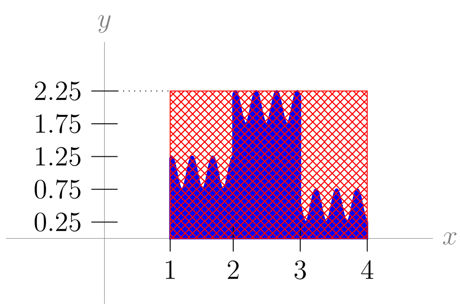


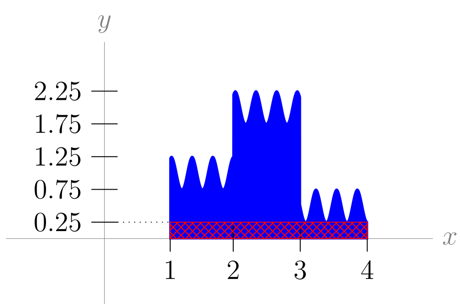
The diagram on the left shows a rectangle with area $2 \times 1.25 = 2.5$ square units. Since the blue-shaded region is entirely inside this rectangle, the area of the blue-

shaded region is no more than 2.5 square units.
The diagram on the right shows a rectangle with area $2 \times 0.75 = 1.5$ square units. Since the blue-shaded region contains this entire rectangle, the area of the blue region is no less than 1.5 square units.
So, the area of the blue-shaded region is between 1.5 and 2.5 square units.
Remark: we could also give an obvious range, like "the shaded area is between zero and one million square units." This would be true, but not very useful or interesting.

1.1.8.2. Solution.

- Solution 1: One naive way to solve this is to simply use the same method as Question 1.

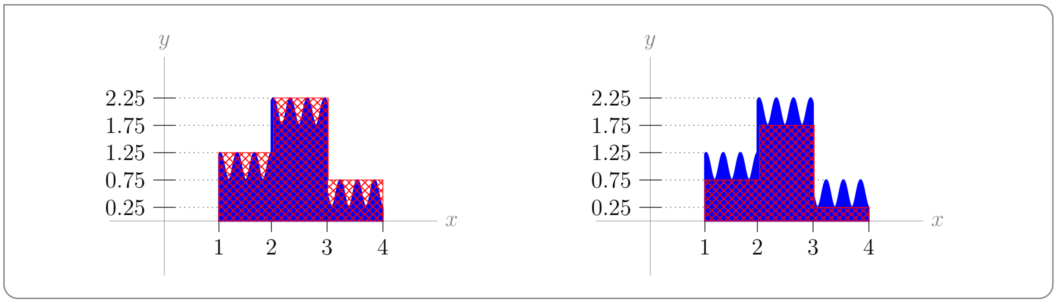




The rectangle on the left has area $3 \times 2.25 = 6.75$ square units, and encompasses the entire shaded region. The rectangle on the right has area $3 \times 0.25 = 0.75$ square units, and is entirely contained inside the blue-shaded region. So, the area of the blue-shaded region is between 0.75 and 6.75 square units.

This is a legitimate approximation, but we can easily do much better. The shape of this graph suggests that using the areas of *three* rectangles would be a natural way to improve our estimate.

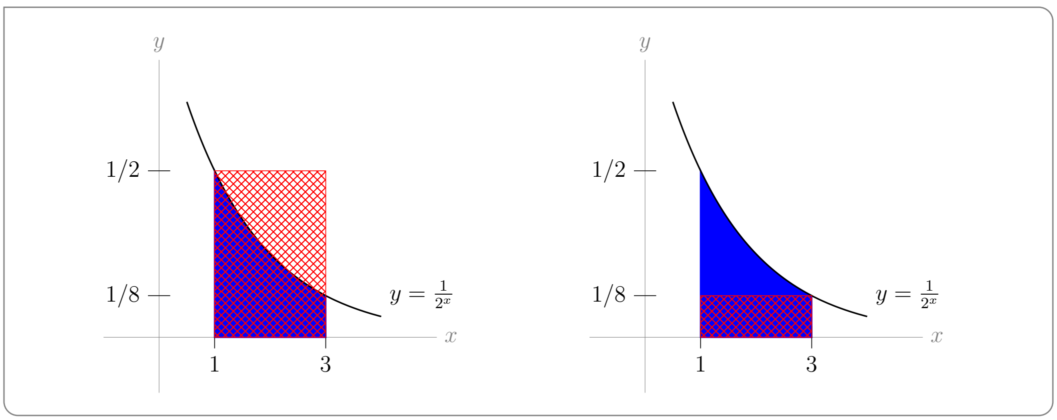
- Solution 2: Let's use these rectangles instead:



In the left picture, the red area is $(1\times1.25)+(1\times2.25)+(1\times0.75) = 4.25$ square units. In the right picture, the red area is $(1\times0.75)+(1\times1.75)+(1\times0.25) = 2.75$ square units. So, the blue shaded area is between 2.75 and 4.25 square units.

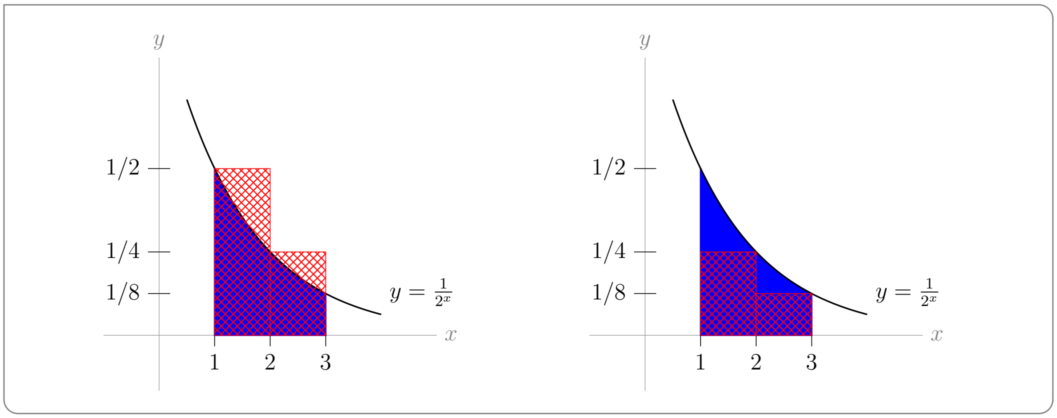
1.1.8.3. Solution. Remark: in the solution below, we find the appropriate approximation using trial and error. In Question 46, we take a more systematic approach.

- Try 1: First, we can try by using a single rectangle as an overestimate, and a single rectangle as an underestimate.



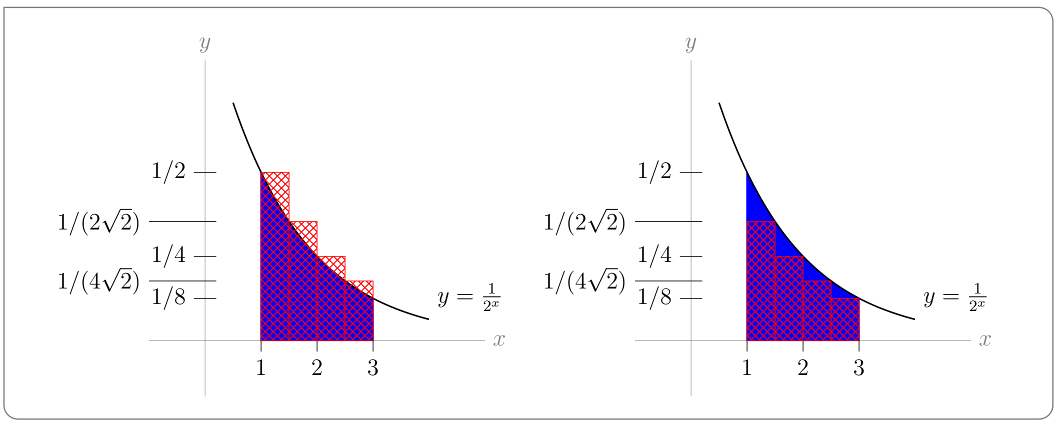
The area under the curve is less than the area of the rectangle on the left $(2\times\frac{1}{2}=1)$ and greater than the area of the rectangle on the right $(2\times\frac{1}{8}=\frac{1}{4})$. So, the area is in the range $\left(\frac{1}{4},1\right)$. Unfortunately, this range is too big–we need our range to have length at most 0.2. So, we refine our approximation by using more rectangles.

- Try 2: Let's try using two rectangles each for the upper and lower bounds.



The rectangles in the left picture have area $\left(1 \times \frac{1}{2}\right) + \left(1 \times \frac{1}{4}\right) = \frac{3}{4}$, and the rectangles in the right picture have area $\left(1 \times \frac{1}{4}\right) + \left(1 \times \frac{1}{8}\right) = \frac{3}{8}$. So, the area under the curve is in the interval $\left(\frac{3}{8}, \frac{3}{4}\right)$. The length of this interval is $\frac{3}{8}$, and $\frac{3}{8} > \frac{3}{15} = \frac{1}{5} = 0.2$. (Indeed, $\frac{3}{8} = 0.375 > 0.2$.) Since the length of our interval is still bigger than 0.2, we need even more rectangles.

- Try 3: Let's go ahead and try four rectangles each for the upper and lower estimates.



The area of the rectangles on the left is:

$$\left(\frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2\sqrt{2}}\right) + \left(\frac{1}{2} \times \frac{1}{4}\right) + \left(\frac{1}{2} \times \frac{1}{4\sqrt{2}}\right)$$
$$= \frac{3}{8}\left[1 + \frac{1}{\sqrt{2}}\right],$$

and the area of the rectangles on the right is:

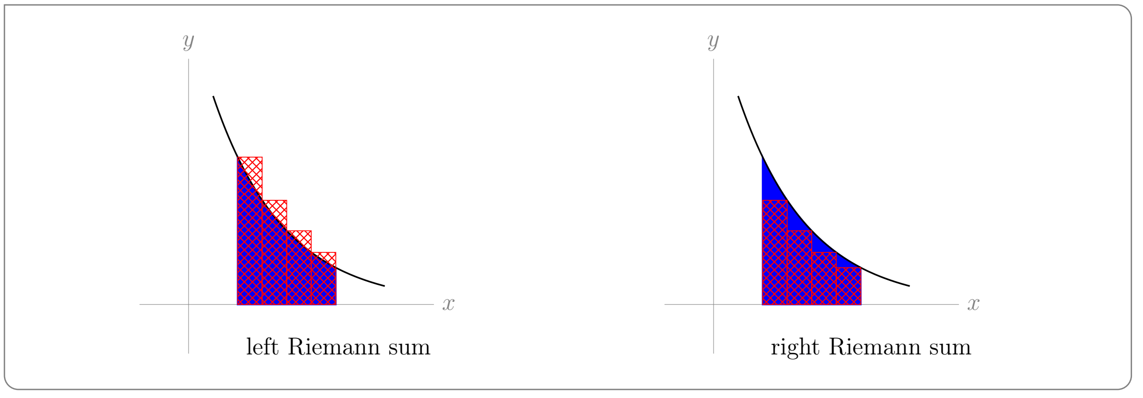
$$\left(\frac{1}{2} \times \frac{1}{2\sqrt{2}}\right) + \left(\frac{1}{2} \times \frac{1}{4}\right) + \left(\frac{1}{2} \times \frac{1}{4\sqrt{2}}\right) + \left(\frac{1}{2} \times \frac{1}{8}\right)$$
$$= \frac{3}{8}\left[\frac{1}{2} + \frac{1}{\sqrt{2}}\right].$$

So, the area under the curve is in the interval $\left(\frac{3}{8}\left[\frac{1}{2} + \frac{1}{\sqrt{2}}\right], \frac{3}{8}\left[1 + \frac{1}{\sqrt{2}}\right]\right)$. The length of this interval is $\frac{3}{16}$, and $\frac{3}{16} < \frac{3}{15} = \frac{1}{5} = 0.2$, as desired. (Indeed, $\frac{3}{16} = 0.1875 < 0.2$.)

Note, if we choose any value in the interval $\left(\frac{3}{8}\left[\frac{1}{2}+\frac{1}{\sqrt{2}}\right], \frac{3}{8}\left[1+\frac{1}{\sqrt{2}}\right]\right)$ as an approximation for the area under the curve, our error is no more than 0.2.

1.1.8.4. Solution. Since $f(x)$ is decreasing, it is larger on the left endpoint of an interval than on the right endpoint of an interval. So, a *left* Riemann sum gives a larger approximation. Notice this does not depend on n.

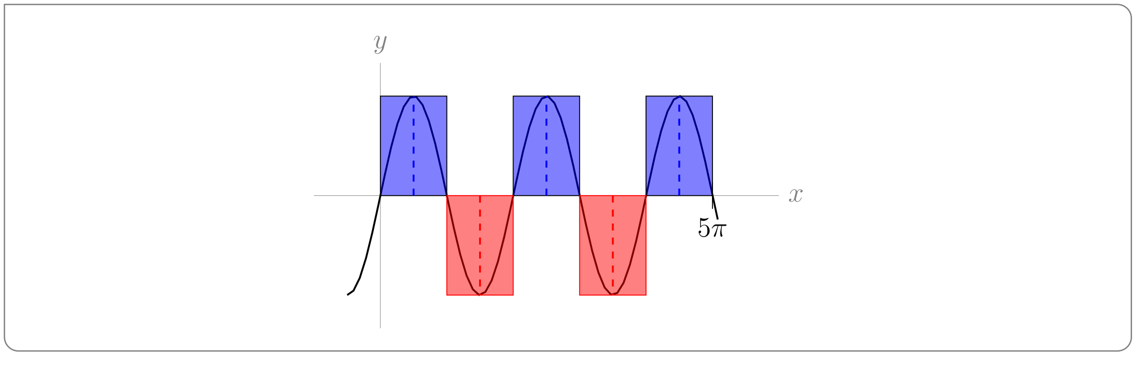
Furthermore, the actual area $\displaystyle\int_0^5 f(x)\mathrm{d}x$ is larger than its right Riemann sum, and smaller than its left Riemann sum.



1.1.8.5. Solution. If $f(x)$ is always increasing or always decreasing, then the midpoint Riemann sum will be between the left and right Riemann sums. So, we need a function that goes up and down. Many examples are possible, but let's work with a familiar one: $\sin x$.

If our intervals have endpoints that are integer multiples of π, then the left and right Riemann sums will be 0, since $\sin(0) = \sin(\pi) = \sin(2\pi) = \cdots = 0$. The midpoints of these intervals will give y-values of 1 and -1. So, for example, we can let $f(x) = \sin x$, $[a, b] = [0, \pi]$, and $n = 1$. Then the right and left Riemann sums are 0, while the midpoint Riemann sum is π.

We can extend the example of $f(x) = \sin x$ to have more intervals. As long as we have more positive terms than negative, the midpoint approximation will be a positive number, and so it will be larger than both the left and right Riemann sums. So, for example, we can let $f(x) = \sin x$, $[a, b] = [0, 5\pi]$, and $n = 5$. Then the midpoint Riemann sum is $\pi - \pi + \pi - \pi + \pi = \pi$, which is strictly larger than 0 and so it is larger than both the left and right Riemann sums.



1.1.8.6. Solution.

a Two possible answers are $\sum_{i=3}^{7} i$ and $\sum_{i=1}^{5}(i+2)$. The first has simpler terms (i versus $i+2$), while the second has simpler indices (we often like to start at $i=1$). Neither is objectively better than the other, but depending on your purposes you might find one more useful.

b The terms of this sum are each double the terms of the sum from part (a), so two possible answers are $\sum_{i=3}^{7} 2i$ and $\sum_{i=1}^{5}(2i+4)$. We often want to write a sum that involves even numbers: it will be useful for you to remember that the term $2i$ (with index i) generates evens.

c The terms of this sum are each one more than the terms of the sum from part (b), so two possible answers are $\sum_{i=3}^{7}(2i+1)$ and $\sum_{i=1}^{5}(2i+5)$.

In the last part, we used the expression $2i$ to generate even numbers; $2i+1$ will generate odds. So will the index $2i+5$, and indeed, $2i+k$ for any odd number k. The choice of what you add will depend on the bounds of i.

d This sum adds up the odd numbers from 1 to 15. From Part (c), we know that the formula $2i+1$ is a simple way of generating odd numbers. Since our first term should be 1 and our last term should be 15, if we use $\sum(2i+1)$, then i should run from 0 to 7. So, one way of expressing our sum in sigma notation is $\sum_{i=0}^{7}(2i+1)$.

Sometimes we like our sum to start at $i=1$ instead of $i=0$. If this is our desire, we can use $2i-1$ as our terms, and let i run from 1 to 8. This gives us another way of expressing our sum: $\sum_{i=1}^{8}(2i-1)$.

1.1.8.7. Solution.

a The denominators are successive powers of three, so one way of writing this is $\sum_{i=1}^{4}\frac{1}{3^i}$. Equivalently, the terms we're adding are powers of $1/3$, so we can also write $\sum_{i=1}^{4}\left(\frac{1}{3}\right)^i$.

b This sum is obtained from the sum in (a) by multiplying each term by two, so we can write $\sum_{i=1}^{4}\frac{2}{3^i}$ or $\sum_{i=1}^{4}2\left(\frac{1}{3}\right)^i$.

c The difference between this sum and the previous sum is its alternating sign, minus-plus-minus-plus. This behaviour appears when we raise a negative number to successive powers. We can multiply each term by $(-1)^i$, or we can slip a negative into the number that is already raised to the power i: $\displaystyle\sum_{i=1}^{4}(-1)^i\frac{2}{3^i}$, or $\displaystyle\sum_{i=1}^{4}\frac{2}{(-3)^i}$.

d This sum is the negative of the sum in part (c), so we can simply multiply each term by negative one: $\displaystyle\sum_{i=1}^{4}(-1)^{i+1}\frac{2}{3^i}$ or $\displaystyle\sum_{i=1}^{4}-\frac{2}{(-3)^i}$.

Be careful with the second form: a common mistake is to think that $-\dfrac{2}{(-3)^i}=\dfrac{2}{3^i}$, but these are not the same.

1.1.8.8. Solution.

a If we re-write the second term as $\frac{3}{9}$ instead of $\frac{1}{3}$, our sum becomes:

$$\frac{1}{3}+\frac{3}{9}+\frac{5}{27}+\frac{7}{81}+\frac{9}{243}$$

The numerators are the first five odd numbers, and the denominators are the first five positive powers of 3. We learned how to generate odd numbers in Question 6, and we learned how to generate powers of three in Question 7. Combining these, we can write our sum as $\displaystyle\sum_{i=1}^{5}\frac{2i-1}{3^i}$.

b The denominators of these terms differ from the denominators of part (a) by precisely two, while the numerators are simply 1. So, we can modify our previous answer: $\displaystyle\sum_{i=1}^{5}\frac{1}{3^i+2}$.

c Let's re-write the sum to make the pattern clearer.

$$\begin{array}{rcccccccc}
 & 1000 & + & 200 & + & 30 & + & 4 \\
 & & + & \frac{1}{2} & + & \frac{3}{50} & + & \frac{7}{1000} \\
= & 1\cdot 1000 & + & 2\cdot 100 & + & 3\cdot 10 & + & \frac{4}{1} \\
 & & + & \frac{5}{10} & + & \frac{6}{100} & + & \frac{7}{1000} \\
= & 1\cdot 10^3 & + & 2\cdot 10^2 & + & 3\cdot 10^1 & + & 4\cdot 10^0 \\
 & & + & 5\cdot 10^{-1} & + & 6\cdot 10^{-2} & + & 7\cdot 10^{-3} \\
= & 1\cdot 10^{4-1} & + & 2\cdot 10^{4-2} & + & 3\cdot 10^{4-3} & + & 4\cdot 10^{4-4} \\
 & & + & 5\cdot 10^{4-5} & + & 6\cdot 10^{4-6} & + & 7\cdot 10^{4-7}
\end{array}$$

If we let the red numbers be our index i, this gives us the expression $\displaystyle\sum_{i=1}^{7} i \cdot 10^{4-i}$.

Equivalently, we can write $\displaystyle\sum_{i=1}^{7} \frac{i}{10^{i-4}}$.

1.1.8.9. Solution.

a Using Theorem 1.1.6, part (a) with $a = 1$, $r = \frac{3}{5}$ and $n = 100$:

$$\sum_{i=0}^{100} \left(\frac{3}{5}\right)^i = \frac{1 - \left(\frac{3}{5}\right)^{101}}{1 - \frac{3}{5}} = \frac{5}{2}\left[1 - \left(\frac{3}{5}\right)^{101}\right]$$

b We want to use Theorem 1.1.6, part (a) again, but our sum doesn't start at $\left(\frac{3}{5}\right)^0 = 1$. We have two options: factor out the leading term, or use the difference of two sums that start where we want them to.

- Solution 1: In this solution, we'll make our sum start at 1 by factoring out the leading term. We wrote our work out the long way (expanding the sigma into "dot-dot-dot" notation) for clarity, but it's faster to do the algebra in sigma notation all the way through.

$$\begin{aligned}\sum_{i=50}^{100} \left(\frac{3}{5}\right)^i &= \left(\frac{3}{5}\right)^{50} + \left(\frac{3}{5}\right)^{51} + \left(\frac{3}{5}\right)^{52} + \cdots + \left(\frac{3}{5}\right)^{100} \\ &= \left(\frac{3}{5}\right)^{50}\left[1 + \left(\frac{3}{5}\right) + \left(\frac{3}{5}\right)^2 + \cdots + \left(\frac{3}{5}\right)^{50}\right] \\ &= \left(\frac{3}{5}\right)^{50} \frac{1 - \left(\frac{3}{5}\right)^{51}}{1 - \frac{3}{5}} \\ &= \frac{5}{2}\left(\frac{3}{5}\right)^{50}\left[1 - \left(\frac{3}{5}\right)^{51}\right].\end{aligned}$$

- Solution 2: In this solution, we write our given expression as the difference of two sums, both starting at $i = 0$.

$$\begin{aligned}\sum_{i=50}^{100} \left(\frac{3}{5}\right)^i &= \sum_{i=0}^{100} \left(\frac{3}{5}\right)^i - \sum_{i=0}^{49} \left(\frac{3}{5}\right)^i \\ &= \frac{1 - \left(\frac{3}{5}\right)^{101}}{1 - \frac{3}{5}} - \frac{1 - \left(\frac{3}{5}\right)^{50}}{1 - \frac{3}{5}} \\ &= \frac{5}{2}\left[\left(\frac{3}{5}\right)^{50} - \left(\frac{3}{5}\right)^{101}\right] \\ &= \frac{5}{2}\left(\frac{3}{5}\right)^{50}\left[1 - \left(\frac{3}{5}\right)^{51}\right].\end{aligned}$$

c Before we can use the equations in Theorem 1.1.6, we'll need to do a little simplification.

$$\begin{aligned}\sum_{i=1}^{10}\left(i^2-3i+5\right) &= \sum_{i=1}^{10} i^2 + \sum_{i=1}^{10} -3i + \sum_{i=1}^{10} 5\\ &= \sum_{i=1}^{10} i^2 - 3\sum_{i=1}^{10} i + 5\sum_{i=1}^{10} 1\\ &= \frac{1}{6}(10)(11)(21) - 3\left(\frac{1}{2}(10\cdot 11)\right) + 5\cdot 10\\ &= 270\end{aligned}$$

d As in part (c), we'll simplify first. The first part (shown here in red) is a geometric sum, but it does not start at $1 = \left(\frac{1}{e}\right)^0$.

$$\begin{aligned}\sum_{n=1}^{b}\left[\left(\frac{1}{e}\right)^n + en^3\right] &= \sum_{n=1}^{b}\left(\frac{1}{e}\right)^n + \sum_{n=1}^{b} en^3\\ &= \sum_{n=0}^{b}\left(\frac{1}{e}\right)^n - 1 + e\sum_{n=1}^{b} n^3\\ &= \frac{1-\left(\frac{1}{e}\right)^{b+1}}{1-\frac{1}{e}} - 1 + e\left[\frac{1}{2}b(b+1)\right]^2\\ &= \frac{\frac{1}{e}-\left(\frac{1}{e}\right)^{b+1}}{1-\frac{1}{e}} + e\left[\frac{1}{2}b(b+1)\right]^2\\ &= \frac{1-\left(\frac{1}{e}\right)^{b}}{e-1} + \frac{e}{4}\left[b(b+1)\right]^2\end{aligned}$$

1.1.8.10. Solution.

a The two pieces are very similar, which we can see by changing the index, or expanding them out:

$$\begin{aligned}&\sum_{i=50}^{100}(i-50) + \sum_{i=0}^{50} i\\ &= (0+1+2+\cdots+50) + (0+1+2+\cdots+50)\\ &= (1+2+\cdots+50) + (1+2+\cdots+50)\\ &= 2\left(1+2+\cdots+50\right)\\ &= 2\sum_{i=1}^{50} i\\ &= 2\left(\frac{50\cdot 51}{2}\right) = 50\cdot 51 = 2550\end{aligned}$$

b If we expand $(i-5)^3 = i^3 - 15i^2 + 75i - 125$, we can break the sum into four parts, and evaluate each separately. However, it is much simpler to change the index and make the term $(i-5)^3$ into i^3.

$$\sum_{i=10}^{100}(i-5)^3 = 5^3+6^3+7^3+\cdots+95^3$$

We have a formula to evaluate the sum of cubes if they start at 1, so we turn our expression into the difference of two sums starting at 1:

$$\begin{aligned}
&= \left[1^3+2^3+3^3+4^3+5^3+6^3+7^3+\cdots+95^3\right] \\
&\qquad - \left[1^3+2^3+3^3+4^3\right] \\
&= \sum_{i=1}^{95} i^3 - \sum_{i=1}^{4} i^3 \\
&= \left[\frac{1}{2}(95)(96)\right]^2 - \left[\frac{1}{2}(4)(5)\right]^2 \\
&= 20,793,500\,.
\end{aligned}$$

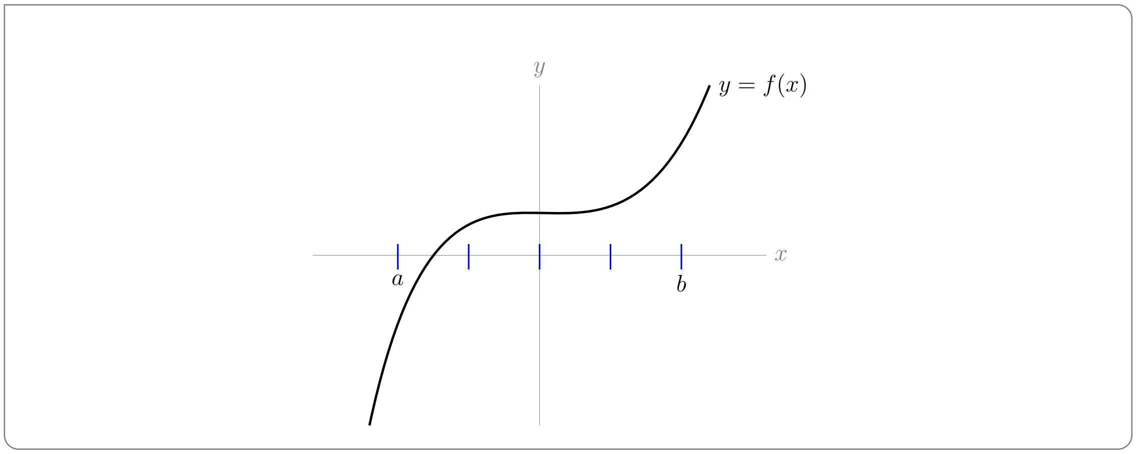
c Notice every two terms cancel with each other, since the sum is $(-1)+(+1)$, etc. Then the terms $n=1$ through $n=10$ cancel, and we're left only with the final term, $(-1)^{11}=-1$.

Written out more explicitly:

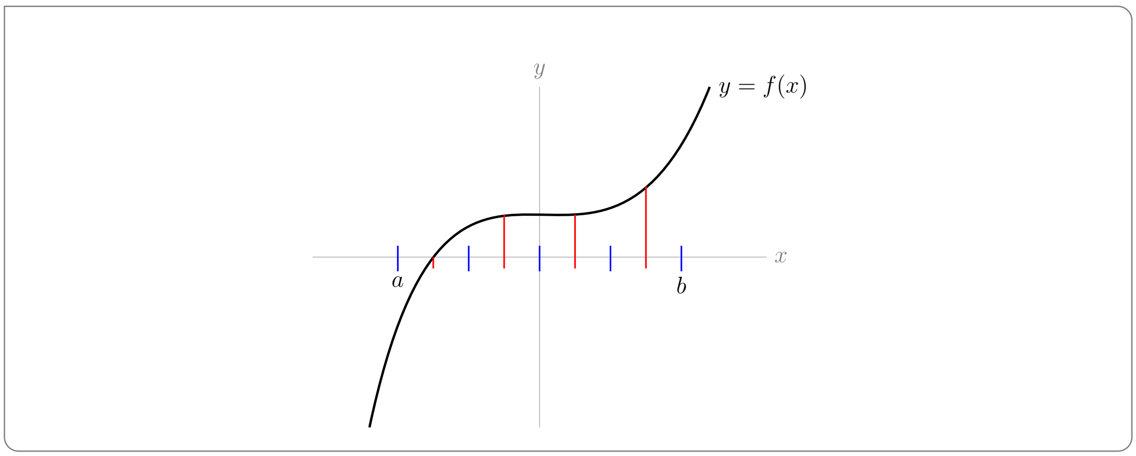
$$\begin{aligned}
&\sum_{n=1}^{11}(-1)^n = -1+1-1+1-1+1-1+1-1+1-1 \\
&= [-1+1]+[-1+1]+[-1+1]+[-1+1]+[-1+1]-1 \\
&= 0+0+0+0+0-1 = -1.
\end{aligned}$$

d For every integer n, $2n+1$ is odd, so $(-1)^{2n+1}=-1$. Then $\displaystyle\sum_{n=2}^{11}(-1)^{2n+1} = \sum_{n=2}^{11}-1 = -10.$

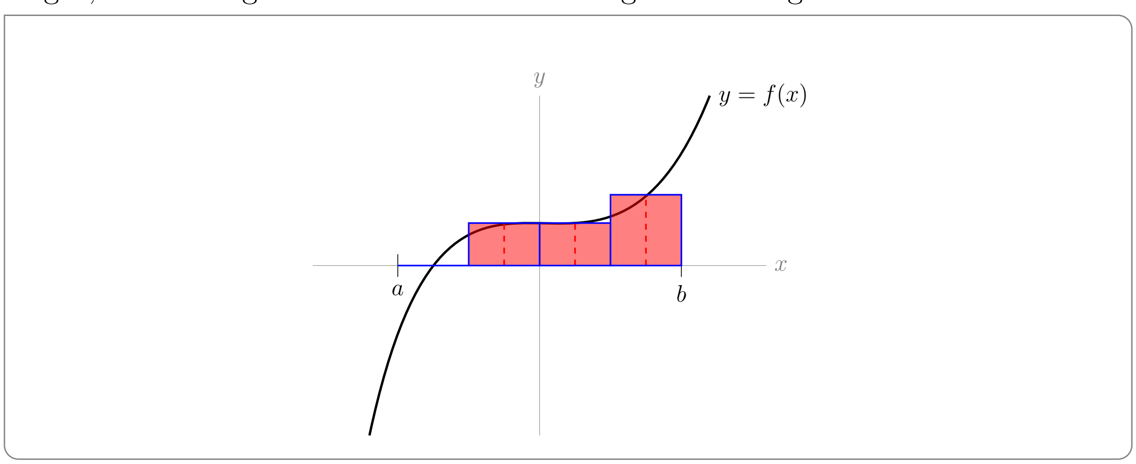
1.1.8.11. Solution. The index of the sum runs from 1 to 4: the first, second, third, and fourth rectangles. So, we have four rectangles in our Riemann sum. Let's start by drawing in the intervals along the x-axis taken up by these four rectangles. Note each has the same width: $\dfrac{b-a}{4}$.



Since this is a midpoint Riemann sum, the height of each rectangle is given by the y-value of the function in the midpoint of the interval. So, now let's find the height of the function at the midpoints of each of the four intervals.



The left-most interval has a height of about 0, so it gives a "trivial" rectangle with no height and no area. The middle two intervals have rectangles of about the same height, and the right-most interval has the highest rectangle.



1.1.8.12. $*$**.** **Solution**. In general, the {left} Riemann sum for the integral

$\int_a^b f(x)\,\mathrm{d}x$ is of the form

$$\sum_{k=1}^{n} f\left(a+(k-1)\frac{b-a}{n}\right)\frac{b-a}{n}$$

- To get the limits of summation to match the given sum, we need $n = 4$.
- Then to get the factor multiplying f to match that in the given sum, we need $\frac{b-a}{n} = 1$, so $b - a = 4$.
- Finally, to get the argument of f to match that in the given sum, we need

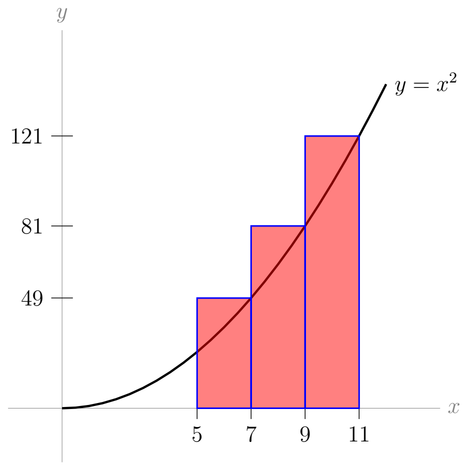
  $$a+(k-1)\frac{b-a}{n} = a - \frac{b-a}{n} + k\frac{b-a}{n} = 1+k$$

  Subbing in $n = 4$ and $b - a = 4$ gives $a - 1 + k = 1 + k$, so $a = 2$ and $b = 6$.

1.1.8.13. Solution. The general form of a Riemann sum is $\displaystyle\sum_{i=1}^{n} \Delta x \cdot f(x_i^*)$, where $\Delta x = \frac{b-a}{n}$ is the width of each rectangle, and $f(x_i^*)$ is the height.
There are different ways to interpret the given sum as a Riemann sum. The most obvious is given in Solution 1. You may notice that we make some convenient assumptions in this solution about values for Δx and a, and we assume the sum is a right Riemann sum. Other visualizations of the sum arise from making more exotic choices. Some of these are explored in Solutions 2-4.
All cases have three rectangles, and the three rectangles will have the same areas: 98, 162, and 242 square units, respectively. This is because the terms of the given sum simplify to $98 + 162 + 242$.

- Solution 1:
  - Because the index runs from 1 to 3, there are three intervals: $n = 3$.
  - Looking at our sum, it seems reasonable to interpret $\Delta x = 2$. Then, since $n = 3$, we conclude $\frac{b-a}{3} = 2$, hence $b - a = 6$.
  - If $\Delta x = 2$, then $f(x_i^*) = (5 + 2i)^2$. Recall that x_i^* is the x-coordinate we use to decide the height of the ith rectangle. In a right Riemann sum, $x_i^* = a+i\cdot\Delta x$. So, using $2 = \Delta x$, we can let $f(x_i^*) = f(a+2i) = (5 + 2i)^2$. This fits with the function $f(x) = x^2$, and $a = 5$.
  - Since $b - a = 6$, and $a = 5$, this tells us $b = 11$

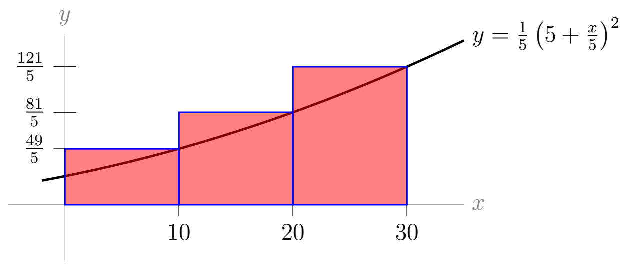
  To sum up, we can interpret the Riemann sum as a right Riemann sum, with three intervals, of the function $f(x) = x^2$ from $x = 5$ to $x = 11$.



- Solution 2: We could have chosen a different value for Δx.
  - The index of the sum runs from 1 to 3, so we have $n = 3$.
  - We didn't have to interpret Δx as 2–that was just the path of least resistance. We could have chosen it to be any other number–for the sake of argument, let's say $\Delta x = 10$. (Positive numbers are easiest to interpret, but negatives are technically allowed as well.)
  - Then $10 = \frac{b-a}{n} = \frac{b-a}{3}$, so $b - a = 30$.
  - Let's use the paradigm of a right Riemann sum, and match up the terms of the sum given in the problem to the terms in the definition:

$$\begin{aligned} \Delta x \cdot f\left(a + i \cdot \Delta x\right) &= 2 \cdot (5 + 2i)^2 \\ 10 \cdot f(a + 10i) &= 2 \cdot (5 + 2i)^2 \\ f(a + 10i) &= \frac{1}{5} \cdot (5 + 2i)^2 \\ f(a + 10i) &= \frac{1}{5} \cdot \left(5 + \frac{1}{5} \cdot 10i\right)^2 \end{aligned}$$

  - The easiest value of a in this case is $a = 0$. Then $f(\color{red}{10i}) = \frac{1}{5} \cdot \left(5 + \frac{1}{5} \cdot \color{red}{10i}\right)^2$, so $f(\color{red}{x}) = \frac{1}{5} \cdot \left(5 + \frac{1}{5} \cdot \color{red}{x}\right)^2$.
  - If $a = 0$ and $b - a = 30$, then $b = 30$.
  - To sum up: $n = 3$, $a = 0$, $b = 30$, $\Delta x = 10$, and $f(x) = \frac{1}{5} \cdot \left(5 + \frac{x}{5}\right)^2$.

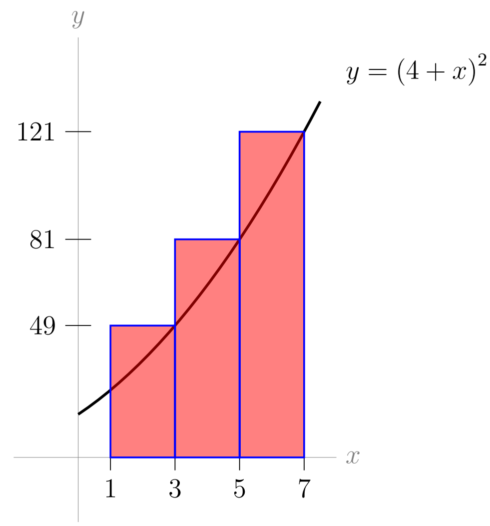


By changing Δx, we changed the *widths* of the rectangles. The rectangles in this picture are wider and shorter than the rectangles in Solution 1. Their *areas* are the same: 98, 162, and 242.

- Solution 3: We could have chosen a different value of a.
  - Suppose $\Delta x = 2$, and we interpret our sum as a right Riemann sum, but we didn't assume $a = 5$. We could have chosen a to be any number–say, $a = 1$.
  - Let's match up what we're given in the problem to what we're given as a definition:

$$\begin{aligned}
\Delta x \cdot f\left(a + i \cdot \Delta x\right) &= 2 \cdot (5 + 2i)^2 \\
2 \cdot f\left(1 + 2i\right) &= 2 \cdot (5 + 2i)^2 \\
f\left(1 + 2i\right) &= (5 + 2i)^2 \\
f\left(1 + 2i\right) &= (4 + 1 + 2i)^2
\end{aligned}$$

  - Since $f({\color{red}1 + 2i}) = (4 + {\color{red}1 + 2i})^2$, we have $f({\color{red}x}) = (4 + {\color{red}x})^2$
  - Since $a = 1$ and $\frac{b-a}{3} = 2$, in this case $b = 7$.
  - To sum up: $n = 3$, $a = 1$, $b = 7$, $\Delta x = 2$, and $f(x) = (4 + x)^2$.

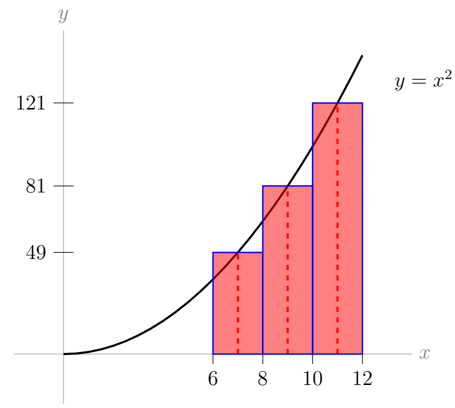


This picture is a lot like the picture in Solution 1, but shifted to the left. By changing a, we changed the *left endpoint* of our region.

- Solution 4: We could have chosen a different kind of Riemann sum.
  - We didn't have to assume that we were dealing with a *right* Riemann sum. Suppose $\Delta x = 2$, and we have a *midpoint* Riemann sum.
  - Let's match up what we're given in the problem with what we're given in the definition:

$$\begin{aligned}
\Delta x \cdot f\left(a + \left(i - \tfrac{1}{2}\right)\Delta x\right) &= 2 \cdot (5 + 2i)^2 \\
2 \cdot f\left(a + \left(i - \tfrac{1}{2}\right)2\right) &= 2 \cdot (5 + 2i)^2 \\
f\left(a + \left(i - \tfrac{1}{2}\right)2\right) &= (5 + 2i)^2 \\
f\left(a + 2i - 1\right) &= (5 + 2i)^2 \\
f\left((a - 1) + 2i\right) &= (5 + 2i)^2
\end{aligned}$$

  - It is now convenient to set $a - 1 = 5$, hence $a = 6$.
  - Then $f(5 + 2i) = (5 + 2i)^2$, so $f(x) = x^2$
  - Since $2 = \frac{b-a}{3}$ and $a = 6$, we see $b = 12$.
  - To sum up: $n = 3$, $a = 6$, $b = 12$, $\Delta x = 2$, and $f(x) = x^2$.



By choosing to interpret our sum as a midpoint Riemann sum instead of a right Riemann sum, we changed where our rectangles intersect the graph $y = f(x)$: instead of the graph hitting the right corner of the rectangle, it hits in the middle.

1.1.8.14. Solution. Many interpretations are possible–see the solution to Question 13 for a more thorough discussion–but the most obvious is given below. Recall the definition of a left Riemann sum:

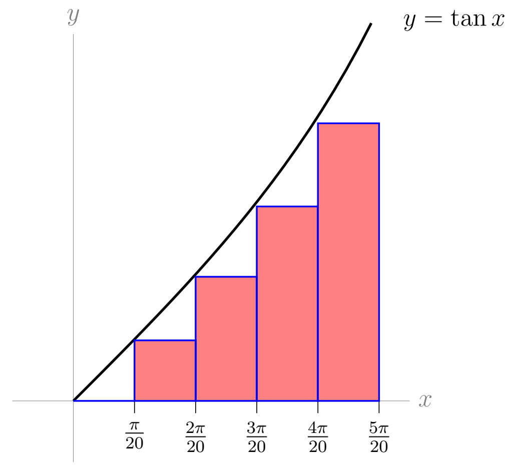
$$\sum_{i=1}^{n} \Delta x \cdot f\left(a + (i-1)\Delta x\right)$$

We chose a left Riemann sum instead of right or midpoint because our given sum has $(i-1)$ in it, rather than $(i - \frac{1}{2})$ or simply i.

- Since the sum has five terms (i runs from 1 to 5), there are 5 rectangles. That is, $n = 5$.
- In the definition of the Riemann sum, note that the term Δx appears twice: once multiplied by the entire term, and once multiplied by $i-1$. So, a convenient choice for Δx is $\frac{\pi}{20}$, because this is the constant that is both multiplied at the start of the term, and multiplied by $i-1$.
- Since $\dfrac{\pi}{20} = \Delta x = \dfrac{b-a}{n} = \dfrac{b-a}{5}$, we see $b - a = \dfrac{5\pi}{20} = \dfrac{\pi}{4}$.
- We match the terms in the definition with the terms in the problem:

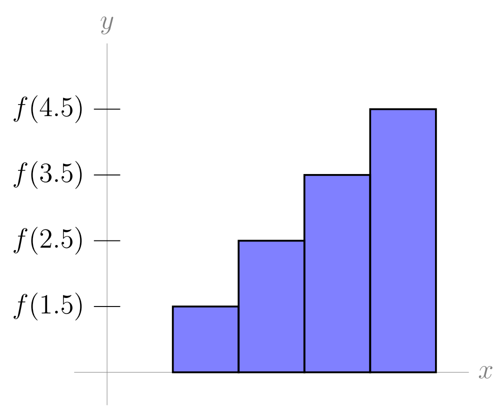
  $$f(a + (i-1)\Delta x) = \tan\left(\frac{\pi(i-1)}{20}\right)$$
  $$f\left(a + (i-1)\frac{\pi}{20}\right) = \tan\left((i-1)\frac{\pi}{20}\right)$$

  So, we choose $a = 0$ and $f(x) = \tan x$.
- Since $a = 0$ and $b - a = \frac{\pi}{4}$, we see $b = \frac{\pi}{4}$.

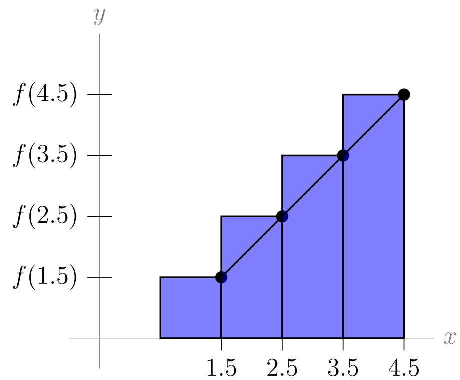


We note that the first rectangle of the five is a "trivial" rectangle, with height (and area) 0.

1.1.8.15. *. Solution. Since there are four terms in the sum, $n = 4$. (Note the sum starts at $k = 0$, instead of $k = 1$.) Since the function is multiplied by 1, $1 = \Delta x = \dfrac{b-a}{n} = \dfrac{b-a}{4}$, hence $b - a = 4$.
We can choose to view the given sum as a left, right, or midpoint Riemann sum. The choice we make determines the interval. Note that the heights of the rectangles are determined when $x = 1.5$, 2.5, 3.5, and 4.5.

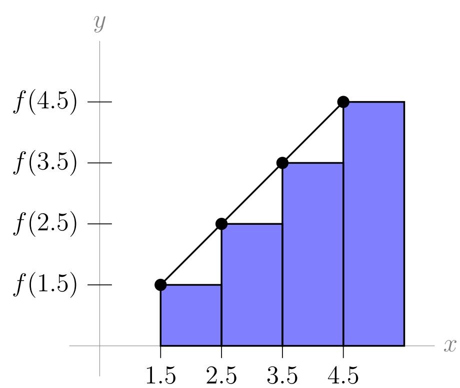


- Option 1: right Riemann sum. If our sum is a right Riemann sum, then we take the heights of the rectangles from the right endpoint of each interval.



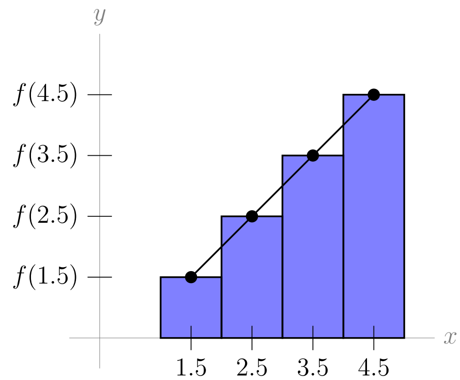
Then $a = 0.5$ and $b = 4.5$. Therefore: $\sum_{k=0}^{3} f(1.5+k)\cdot 1$ is a right Riemann sum on the interval $[0.5, 4.5]$ with $n = 4$.

- Option 2: left Riemann sum. If our sum is a left Riemann sum, then we take the heights of the rectangles from the left endpoint of each interval.



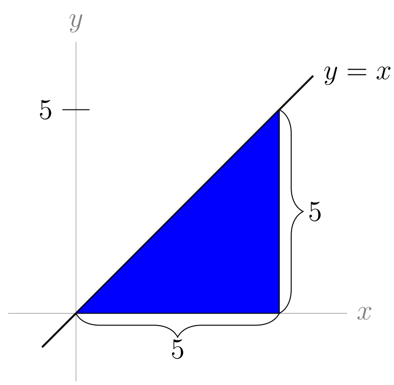
Then $a = 1.5$ and $b = 5.5$. Therefore: $\sum_{k=0}^{3} f(1.5+k)\cdot 1$ is a left Riemann sum on the interval $[1.5, 5.5]$ with $n = 4$.

- Option 3: midpoint Riemann sum. If our sum is a midpoint Riemann sum, then we take the heights of the rectangles from the midpoint of each interval.

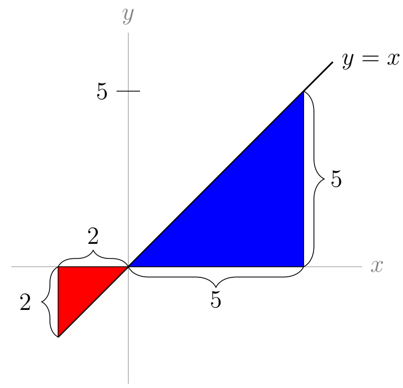


Then $a = 1$ and $b = 5$. Therefore: $\sum_{k=0}^{3} f(1.5+k)\cdot 1$ is a midpoint Riemann sum on the interval $[1,5]$ with $n = 4$.

1.1.8.16. Solution. The area in question is a triangle with base 5 and height 5, so its area is $\dfrac{25}{2}$.



1.1.8.17. Solution. There is a positive and a negative portion of this area. The positive area is a triangle with base 5 and height 5, so area $\dfrac{25}{2}$ square units. The negative area is a triangle with base 2 and height 2, so negative area $\dfrac{4}{2} = 2$ square units. So, the net area is $\dfrac{25}{2} - \dfrac{4}{2} = \dfrac{21}{2}$ square units.



Exercises — Stage 2

1.1.8.18. *. Solution. In general, the midpoint Riemann sum is given by

$$\sum_{i=1}^{n} f\Big(a + \big(i - \frac{1}{2}\big)\Delta x\Big)\ \Delta x\,, \qquad \text{where } \Delta x = \frac{b-a}{n}.$$

In this problem we are told that $f(x) = x^8$, $a = 5$, $b = 15$ and $n = 50$, so that $\Delta x = \frac{b-a}{n} = \frac{1}{5}$ and the desired Riemann sum is:

$$\sum_{i=1}^{50} \Big(5 + \big(i - \frac{1}{2}\big)\frac{1}{5}\Big)^8\ \frac{1}{5}$$

1.1.8.19. *. Solution. The given integral has interval of integration going from $a = -1$ to $b = 5$. So when we use three approximating rectangles, all of the same width, the common width is $\Delta x = \frac{b-a}{n} = 2$. The first rectangle has left endpoint $x_0 = a = -1$, the second has left hand endpoint $x_1 = a + \Delta x = 1$, and the third has left hand end point $x_2 = a + 2\Delta x = 3$. So

$$\int_{-1}^{5} x^3\ \mathrm{d}x \approx \big[f(x_0) + f(x_1) + f(x_2)\big]\Delta x = \big[(-1)^3 + 1^3 + 3^3\big] \times 2 = 54$$

1.1.8.20. *. Solution. In the given integral, the domain of integration runs from $a = -1$ to $b = 7$. So, we have $\Delta x = \frac{(b-a)}{n} = \frac{(7-(-1))}{n} = \frac{8}{n}$. The left-hand end of the first subinterval is at $x_0 = a = -1$. So, the right-hand end of the i^{th} interval is at $x_i^* = -1 + \frac{8i}{n}$. So:

$$\int_{-1}^{7} f(x)\ \mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-1 + \frac{8i}{n}\right)\frac{8}{n}$$

1.1.8.21. *. Solution. We identify the given sum as the right Riemann sum $\sum\limits_{i=1}^{n} f(a + i\Delta x)\Delta x$, with $a = 0$ (that's specified in the statement of the question). Since $\frac{4}{n}$ is multiplied in every term, and is also multiplied by i, we let $\Delta x = \frac{4}{n}$. Then $x_i^* = a + i\Delta x = \frac{4i}{n}$ and $f(x) = \sin^2(2 + x)$. So, $b = a + n\Delta x = 0 + n \cdot \frac{4}{n} = 4$.

1.1.8.22. *. Solution. The given sum is of the form

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2}\sqrt{1-\frac{k^2}{n^2}} &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{n}\right)\frac{k}{n}\sqrt{1-\left(\frac{k}{n}\right)^2}\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\Delta x f(x_k^*)\end{aligned}$$

with $\Delta x = \frac{1}{n}$, $a = 0$, $x_k^* = \frac{k}{n} = a + k\Delta x$ and $f(x) = x\sqrt{1-x^2}$. Since $x_0^* = 0$ and $x_n^* = 1$, the right hand side is the definition (using the right Riemann sum) of $\int_0^1 f(x)\,\mathrm{d}x$.

1.1.8.23. *. Solution. As i ranges from 1 to n, $3i/n$ range from $3/n$ to 3 with jumps of $\Delta x = 3/n$, so this is

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}e^{-i/n}\cos(3i/n) = \lim_{n\to\infty}\sum_{i=1}^{n}f(x_i^*)\Delta x = \int_a^b f(x)\,\mathrm{d}x$$

where $x_i^* = 3i/n$, $f(x) = e^{-x/3}\cos(x)$, $a = x_0 = 0$ and $b = x_n = 3$. Thus

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}e^{-i/n}\cos(3i/n) = \int_0^3 e^{-x/3}\cos(x)\,\mathrm{d}x$$

1.1.8.24. *. Solution. As i ranges from 1 to n, the exponent $\frac{i}{n}$ ranges from $\frac{1}{n}$ to 1 with jumps of $\Delta x = \frac{1}{n}$. So let's try $x_i^* = \frac{i}{n}$, $\Delta x = \frac{1}{n}$. Then:

$$R_n = \sum_{i=1}^{n}\frac{ie^{i/n}}{n^2} = \sum_{i=1}^{n}\frac{i}{n}e^{i/n}\frac{1}{n} = \sum_{i=1}^{n}x_i^* e^{x_i^*}\Delta x = \sum_{i=1}^{n}f(x_i^*)\Delta x$$

with $f(x) = xe^x$, and the limit

$$\lim_{n\to\infty}R_n = \lim_{n\to\infty}\sum_{i=1}^{n}f(x_i^*)\Delta x = \int_a^b f(x)\,\mathrm{d}x$$

Since we chose $x_i^* = \frac{i}{n} = 0 + i\Delta x$, we let $a = 0$. Then $\frac{1}{n} = \Delta x = \frac{b-a}{n} = \frac{b}{n}$ tells us $b = 1$. Thus,

$$\lim_{n\to\infty}R_n = \int_0^1 xe^x\,\mathrm{d}x\,.$$

1.1.8.25. *. Solution.

- Choice #1: If we set $\Delta x = \frac{2}{n}$ and $x_i^* = \frac{2i}{n}$, i.e. $x_i^* = a + i\Delta x$ with $a = 0$, then

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}e^{-1-2i/n}\cdot\frac{2}{n}\right) = \lim_{n\to\infty}\left(\sum_{i=1}^{n}e^{-1-x_i^*}\Delta x\right)$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\Delta x\right) \\
&\quad \text{with } f(x) = e^{-1-x} \\
&= \int_a^b f(x)\ \mathrm{d}x \\
&\quad \text{with } a = x_0 = 0 \text{ and } b = x_n = 2 \\
&= \int_0^2 e^{-1-x}\ \ \mathrm{d}x
\end{aligned}
$$

- Choice #2: If we set $\Delta x = \frac{2}{n}$ and $x_i^* = 1 + \frac{2i}{n}$, i.e. $x_i^* = a + i\Delta x$ with $a = 1$, then

$$
\begin{aligned}
\lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-1-2i/n}\cdot\frac{2}{n}\right) &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-x_i^*}\Delta x\right) \\
&= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\Delta x\right) \\
&\quad \text{with } f(x) = e^{-x} \\
&= \int_a^b f(x)\ \mathrm{d}x \\
&\quad \text{with } a = x_0 = 1 \text{ and } b = x_n = 3 \\
&= \int_1^3 e^{-x}\ \ \mathrm{d}x
\end{aligned}
$$

- Choice #3: If we set $\Delta x = \frac{1}{n}$ and $x_i^* = \frac{i}{n}$, i.e. $x_i^* = a + i\Delta x$ with $a = 0$, then

$$
\begin{aligned}
\lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-1-2i/n}\cdot\frac{2}{n}\right) &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-1-2x_i^*}\ 2\Delta x\right) \\
&= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\Delta x\right) \\
&\quad \text{with } f(x) = 2e^{-1-2x} \\
&= \int_a^b f(x)\ \mathrm{d}x \\
&\quad \text{with } a = x_0 = 0 \text{ and } b = x_n = 1 \\
&= 2\int_0^1 e^{-1-2x}\ \ \mathrm{d}x
\end{aligned}
$$

- Choice #4: If we set $\Delta x = \frac{1}{n}$ and $x_i^* = \frac{1}{2} + \frac{i}{n}$, i.e. $x_i = a + i\Delta x$ with $a = \frac{1}{2}$, then

$$
\lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-1-2i/n}\cdot\frac{2}{n}\right) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} e^{-2x_i}\ 2\Delta x\right)
$$

$$\begin{aligned}
&= \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\Delta x\right)\\
&\qquad \text{with } f(x) = 2e^{-2x}\\
&= \int_a^b f(x)\ \mathrm{d}x\\
&\qquad \text{with } a = x_0 = \frac{1}{2} \text{ and } b = x_n = \frac{3}{2}\\
&= 2\int_{1/2}^{3/2} e^{-2x}\ \mathrm{d}x
\end{aligned}$$

1.1.8.26. Solution. This is similar to the familiar form of a geometric sum, but the powers go up by threes. So, we make a subsitution. If $x = r^3$, then:

$$1 + r^3 + r^6 + r^9 + \cdots + r^{3n} = 1 + x + x^2 + x^3 + \cdots + x^n$$

Now, using Equation 1.1.3,

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

Substituting back in $x = r^3$, we find our sum is equal to $\dfrac{(r^3)^{n+1} - 1}{r^3 - 1}$, or $\dfrac{r^{3n+3} - 1}{r^3 - 1}$.

1.1.8.27. Solution. The sum does not start at 1, so we need to do some algebra. We can either factor out the first term, or subtract off the initial terms that are missing.

- Solution 1: If we factor out r^5, then what's left fits the form of Equation 1.1.3:

$$\begin{aligned}
r^5 + r^6 + r^7 + \cdots + r^{100} &= r^5\left[1 + r + r^2 + \cdots + r^{95}\right]\\
&= r^5\left(\frac{r^{96} - 1}{r - 1}\right)\ .
\end{aligned}$$

- Solution 2: We know how to evaluate sums of this form if they start at 1, so we re-write our sum as follows:

$$\begin{aligned}
r^5 + r^6 + r^7 + \cdots + r^{100} &= \left(1 + r + r^2 + r^3 + r^4 + r^5 + \cdots + r^{100}\right)\\
&\quad - \left(1 + r + r^2 + r^3 + r^4\right)\\
&= \frac{r^{101} - 1}{r - 1} - \frac{r^5 - 1}{r - 1}\\
&= \frac{r^{101} - 1 - r^5 + 1}{r - 1} = \frac{r^{101} - r^5}{r - 1}\\
&= r^5\left(\frac{r^{96} - 1}{r - 1}\right)\ .
\end{aligned}$$

1.1.8.28. $*$. Solution. Recall that

$$|x| = \begin{cases} -x & \text{if } x \le 0 \\ x & \text{if } x \ge 0 \end{cases}$$

so that

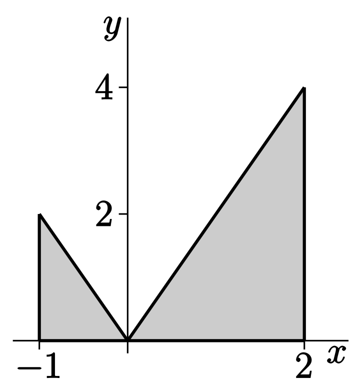
$$|2x| = \begin{cases} -2x & \text{if } x \le 0 \\ 2x & \text{if } x \ge 0 \end{cases}$$

To picture the geometric figure whose area the integral represents observe that

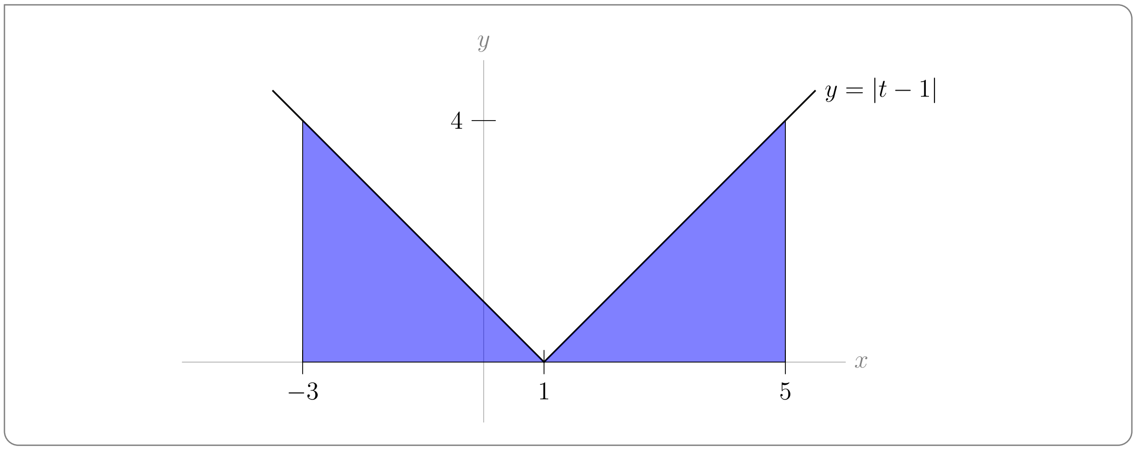
- at the left hand end of the domain of integration $x = -1$ and the integrand $|2x| = |-2| = 2$ and
- as x increases from -1 towards 0, the integrand $|2x| = -2x$ decreases linearly, until
- when x hits 0 the integrand hits $|2x| = |0| = 0$ and then
- as x increases from 0, the integrand $|2x| = 2x$ increases linearly, until
- when x hits $+2$, the right hand end of the domain of integration, the integrand hits $|2x| = |4| = 4$.

So the integral $\int_{-1}^{2} |2x| \, \mathrm{d}x$ is the area of the union of the two shaded triangles (one of base 1 and of height 2 and the other of base 2 and height 4) in the figure on the right below and

$$\int_{-1}^{2} |2x| \, \mathrm{d}x = \frac{1}{2} \times 1 \times 2 + \frac{1}{2} \times 2 \times 4 = 5$$

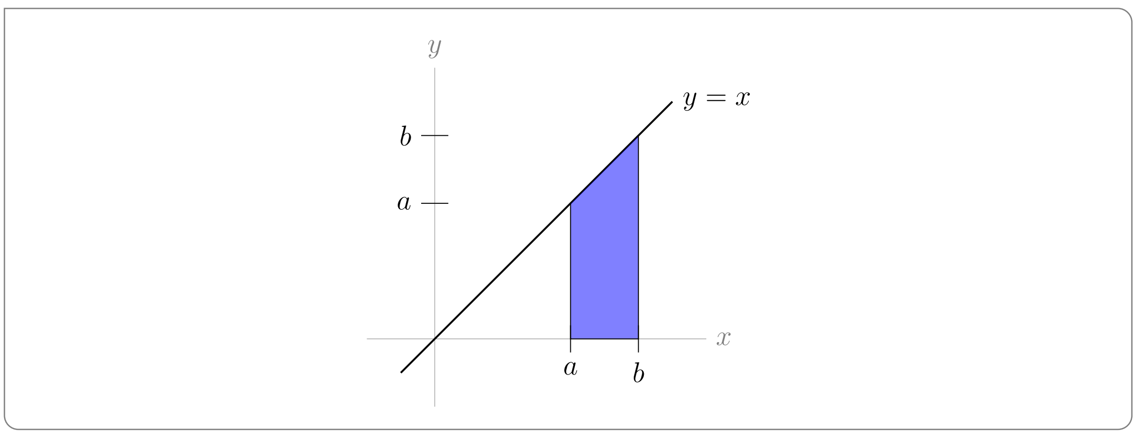


1.1.8.29. Solution. The area we want is two triangles, both above the x-axis. Each triangle has base 4 and height 4, so the total area is $2 \cdot \left(\frac{4 \cdot 4}{2}\right) = 16$.



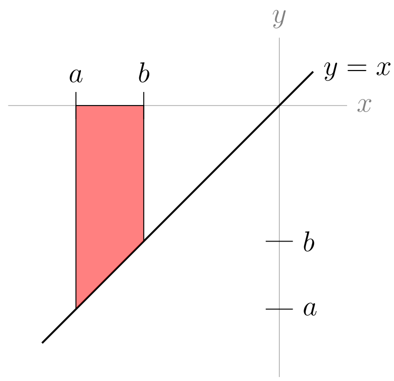
If you had a hard time sketching the function, recall that the absolute value of a number leaves it unchanged if it is positive or zero, and flips the sign if it is negative. So, when $t-1 \geq 0$ (that is, when $t \geq 1$), our function is simply $f(t) = |t-1| = t-1$. On the other hand, when $t = 1$ is negative (that is, when $t < 1$), the absolute value changes the sign, so $f(t) = |t-1| = -(t-1) = -t+1$.

1.1.8.30. Solution. The area we want is a trapezoid with base $(b-a)$ and heights a and b, so its area is $\dfrac{(b-a)(b+a)}{2} = \dfrac{b^2-a^2}{2}$.



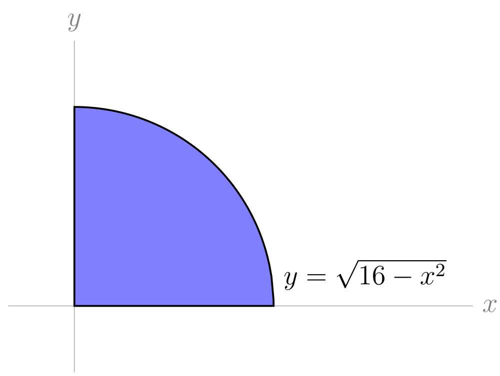
Instead of using a formula for the area of a trapezoid, you can find the blue area as the area of a triangle with base and height b, minus the area of a triangle with base and height a.

1.1.8.31. Solution. The area is negative. The shape is a trapezoid with base length $(b-a)$ and heights $0-a = -a$ and $0-b = -b$ (note: those are nonnegative numbers), so its area is $\dfrac{(b-a)(-b-a)}{2} = \dfrac{-b^2+a^2}{2}$. Since the shape is below the x-axis, we change its sign. Thus, the integral evaluates to $\dfrac{b^2-a^2}{2}$.

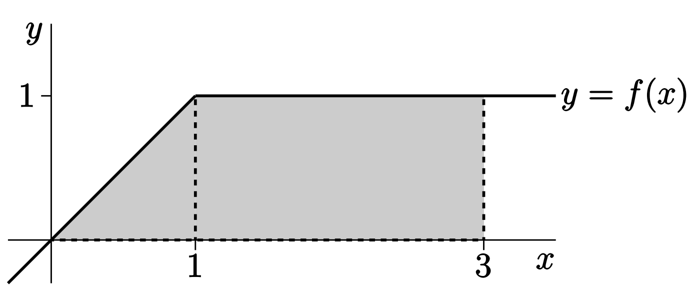


The signs can be a little hard to keep track of. The base of our trapezoid is $|a - b|$; since $b > a$, this is $b - a$. The heights of the trapezoid are $|a|$ and $|b|$; since these are both negative, $|a| = -a$ and $|b| = -b$.
We note that this is the same result as in Question 30.

1.1.8.32. Solution. If $y = \sqrt{16 - x^2}$, then y is nonnegative, and $y^2 + x^2 = 16$. So, the graph $y = \sqrt{16 - x^2}$ is the upper half of a circle of radius 4. Since x only runs from 0 to 4, we have a quarter of a circle of radius 4. Then the area under the curve is $\frac{1}{4}\left[\pi \cdot 4^2\right] = 4\pi$.



1.1.8.33. *. Solution. Here is a sketch the graph of $f(x)$.



There is a linear increase from $x = 0$ to $x = 1$, followed by a constant. Using the interpretation of $\int_0^3 f(x)\,\mathrm{d}x$ as the area between $y = f(x)$ and the x–axis with x between 0 and 3, we can break this area into:

- $\int_0^1 f(x)\,\mathrm{d}x$: a right-angled triangle of height 1 and base 1 and hence area 0.5.
- $\int_1^3 f(x)\,\mathrm{d}x$: a rectangle of height 1 and base 2 and hence area 2.

Summing up: $\int_0^3 f(x)\,\mathrm{d}x = 2.5$.

1.1.8.34. *. Solution. The car's speed increases with time. So its highest speed on any time interval occurs at the right hand end of the interval and the best possible upper estimate for the distance traveled is given by the *right* Riemann sum with $\Delta x = 0.5$, which is

$$\begin{aligned}\big[v(0.5) + v(1.0) + v(1.5) + v(2.0)\big] \times 0.5 &= \big[14 + 22 + 30 + 40\big] \times 0.5\\ &= 53 \text{ m}\end{aligned}$$

1.1.8.35. Solution. There is a key detail in the statement of Question 34: namely, that the car is continuously accelerating. So, although we don't know exactly what's going on in between our brief snippets of information, we know that the car is not going any faster during an interval than at the end of that interval. Therefore, the car certainly travelled no farther than our estimation.
We ask this question in order to point out an important detail. If we did not have the information that the car was continuously accelerating, we would not be able to give a certain upper bound on its distance travelled. It would be possible that, when the car is not being observed (for example, when $t = 0.25$), it is going much faster than when it is being observed.

1.1.8.36. Solution. First, note that the distance travelled by the plane is equal to the area under the graph of its speed.
We need to know the speed of the plane at the midpoints of our intervals. So (for example) noon to 1pm *is not* one of your intervals–we don't know the speed at 12:30. (A common idea is to average the two end values, 700 and 800. This is a fine approximation, but it is not a Riemann sum.) So, we use the two intervals 12:00 to 2:00, and 2:00 to 4:00. Then our intervals have length 2 hours, and at the midpoints of the intervals the speed of the plane is 700 kph and 900 kph, respectively. So, our midpoint Riemann sum gives us:

$$700(2) + 900(2) = 3200$$

an approximation of 3200 km travelled by the plane from noon to 4:00 pm.
Remark: if we had been asked to approximate the distance travelled from 11:30 am to 4:30 pm, then we could have used the midpoint rule with five intervals and made use of every entry in the data table. With the question as stated, however, we ignore three out of five entries in the table because they are not the midpoints of our intervals.

Exercises — Stage 3

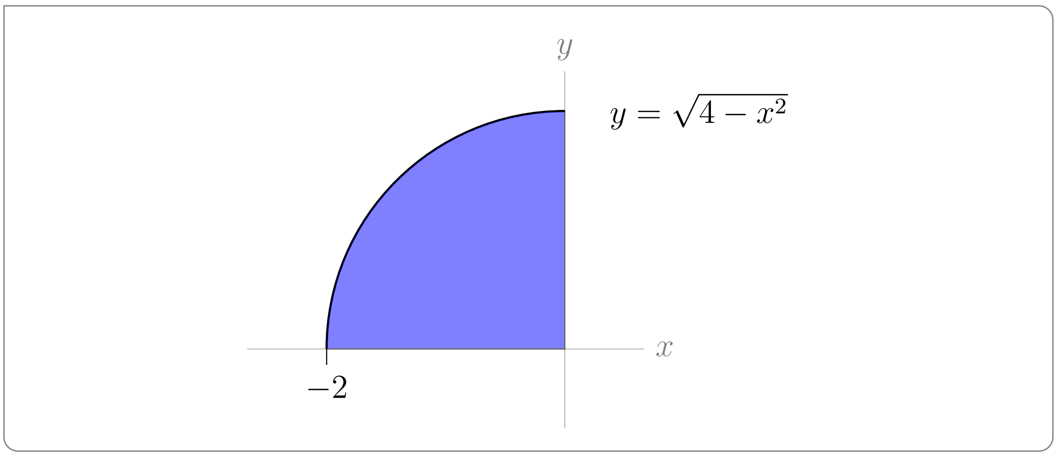
1.1.8.37. *. Solution.

- Solution #1: Set $x_i^* = -2 + \frac{2i}{n}$. Then $a = x_0 = -2$ and $b = x_n = 0$ and

$\Delta x = \frac{2}{n}$. So

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\sqrt{4-\left(-2+\frac{2i}{n}\right)^2}\\
&=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \qquad \text{with } f(x)=\sqrt{4-x^2} \text{ and } \Delta x=\frac{2}{n}\\
&=\int_{-2}^{0}\sqrt{4-x^2}\,\mathrm{d}x
\end{aligned}$$

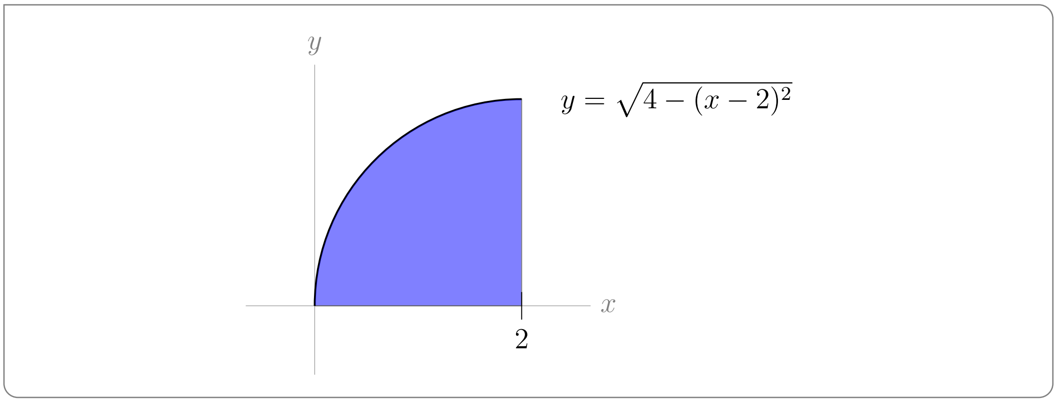
For the integral $\int_{-2}^{0}\sqrt{4-x^2}\,\mathrm{d}x$, $y=\sqrt{4-x^2}$ is equivalent to $x^2+y^2=4$, $y\ge 0$. So the integral represents the area between the upper half of the circle $x^2+y^2=4$ (which has radius 2) and the x-axis with $-2\le x\le 0$, which is a quarter circle with area $\frac{1}{4}\cdot\pi\,2^2=\pi$.



- Solution #2: Set $x_i^*=\frac{2i}{n}$. Then $a=x_0=0$ and $b=x_n=2$ and $\Delta x=\frac{2}{n}$. So

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\sqrt{4-\left(-2+\frac{2i}{n}\right)^2}\\
&=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \quad \text{with } f(x)=\sqrt{4-(-2+x)^2},\ \Delta x=\frac{2}{n}\\
&=\int_{0}^{2}\sqrt{4-(-2+x)^2}\,\mathrm{d}x
\end{aligned}$$

For the integral $\int_0^2\sqrt{4-(-2+x)^2}\,\mathrm{d}x$, $y=\sqrt{4-(x-2)^2}$ is equivalent to $(x-2)^2+y^2=4$, $y\ge 0$. So the integral represents the area between the upper half of the circle $(x-2)^2+y^2=4$ (which is centered at $(2,0)$ and has radius 2) and the x-axis with $0\le x\le 2$, which is a quarter circle with area $\frac{1}{4}\cdot\pi\,2^2=\pi$.



1.1.8.38. *. Solution. (a) The left Riemann sum is defined as

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x \qquad \text{with } x_i = a + i\Delta x$$

We subdivide into $n = 3$ intervals, so that $\Delta x = \frac{b-a}{n} = \frac{3-0}{3} = 1$, $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$. The function $f(x) = 7 + x^3$ has the values $f(x_0) = 7 + 0^3 = 7$, $f(x_1) = 7 + 1^3 = 8$, and $f(x_2) = 7 + 2^3 = 15$, from which we evaluate

$$L_3 = \big[f(x_0) + f(x_1) + f(x_2)\big]\Delta x = \big[7 + 8 + 15\big] \times 1 = 30$$

(b) We divide into n intervals so that $\Delta x = \frac{b-a}{n} = \frac{3}{n}$ and $x_i = a + i\Delta x = \frac{3i}{n}$. The right Riemann sum is therefore:

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left[7 + \frac{(3i)^3}{n^3}\right]\frac{3}{n} = \sum_{i=1}^{n} \left[\frac{21}{n} + \frac{81\, i^3}{n^4}\right]$$

To calculate the sum:

$$\begin{aligned} R_n &= \left(\frac{21}{n}\sum_{i=1}^{n} 1\right) + \left(\frac{81}{n^4}\sum_{i=1}^{n} i^3\right) \\ &= \left(\frac{21}{n} \times n\right) + \left(\frac{81}{n^4} \times \frac{n^4 + 2n^3 + n^2}{4}\right) \\ &= 21 + \frac{81}{4}(1 + 2/n + 1/n^2) \end{aligned}$$

To evaluate the limit exactly, we take $n \to \infty$. The expressions involving $1/n$ vanish leaving:

$$\int_0^3 (7 + x^3)\ \mathrm{d}x = \lim_{n\to\infty} R_n = 21 + \frac{81}{4} = 41\frac{1}{4}$$

1.1.8.39. *. Solution. In general, the right-endpoint Riemann sum approximation to the integral $\int_a^b f(x)\,\mathrm{d}x$ using n rectangles is

$$\sum_{i=1}^{n} f(a+i\Delta x)\Delta x$$

where $\Delta x = \frac{b-a}{n}$. In this problem, $a=2$, $b=4$, and $f(x)=x^2$, so that $\Delta x = \frac{2}{n}$ and the right–endpoint Riemann sum approximation becomes

$$\begin{aligned}
\sum_{i=1}^{n} f\Big(2+\frac{2i}{n}\Big)\frac{2}{n} &= \sum_{i=1}^{n} \Big(2+\frac{2i}{n}\Big)^2\frac{2}{n} \\
&= \sum_{i=1}^{n} \left(4+\frac{8i}{n}+\frac{4i^2}{n^2}\right)\frac{2}{n} \\
&= \sum_{i=1}^{n} \left(\frac{8}{n}+\frac{16i}{n^2}+\frac{8i^2}{n^3}\right) \\
&= \sum_{i=1}^{n} \frac{8}{n}+\sum_{i=1}^{n}\frac{16i}{n^2}+\sum_{i=1}^{n}\frac{8i^2}{n^3} \\
&= \frac{8}{n}\sum_{i=1}^{n} 1+\frac{16}{n^2}\sum_{i=1}^{n} i+\frac{8}{n^3}\sum_{i=1}^{n} i^2 \\
&= \frac{8}{n}n+\frac{16}{n^2}\cdot\frac{n(n+1)}{2}+\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} \\
&= 8+8\Big(1+\frac{1}{n}\Big)+\frac{4}{3}\Big(1+\frac{1}{n}\Big)\Big(2+\frac{1}{n}\Big)
\end{aligned}$$

So

$$\begin{aligned}
\int_2^4 x^2\ \mathrm{d}x &= \lim_{n\to\infty}\left[8+8\Big(1+\frac{1}{n}\Big)+\frac{4}{3}\Big(1+\frac{1}{n}\Big)\Big(2+\frac{1}{n}\Big)\right] \\
&= 8+8+\frac{4}{3}\times 2 = \frac{56}{3}
\end{aligned}$$

1.1.8.40. $*$. **Solution**. We'll use right Riemann sums with $a=0$ and $b=2$. When there are n rectangles, $\Delta x = \frac{b-a}{n} = \frac{2}{n}$ and $x_i = a+i\Delta x = 2i/n$. So we need to evaluate

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty}\sum_{i=1}^{n}\big((x_i)^3+x_i\big)\,\Delta x \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\left(\frac{2i}{n}\right)^3+\frac{2i}{n}\right)\frac{2}{n} \\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left(\frac{8i^3}{n^3}+\frac{2i}{n}\right)
\end{aligned}$$

$$= \lim_{n\to\infty}\left(\frac{16}{n^4}\sum_{i=1}^{n} i^3 + \frac{4}{n^2}\sum_{i=1}^{n} i\right)$$
$$= \lim_{n\to\infty}\left(\frac{16(n^4+2n^3+n^2)}{n^4\cdot 4} + \frac{4(n^2+n)}{n^2\cdot 2}\right)$$
$$= \lim_{n\to\infty}\left(\frac{16}{4}\left(1+\frac{2}{n}+\frac{1}{n^2}\right) + \frac{4}{2}\left(1+\frac{1}{n}\right)\right)$$
$$= \frac{16}{4} + \frac{4}{2} = 6.$$

1.1.8.41. *. Solution. We'll use right Riemann sums with $a = 1$, $b = 4$ and $f(x) = 2x - 1$. When there are n rectangles, $\Delta x = \frac{b-a}{n} = \frac{3}{n}$ and $x_i = a + i\Delta x = 1 + 3i/n$. So we need to evaluate

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x &= \lim_{n\to\infty}\sum_{i=1}^{n}(2x_i - 1)\,\Delta x\\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(2+\frac{6i}{n}-1\right)\frac{3}{n}\\
&= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left(\frac{6i}{n}+1\right)\\
&= \lim_{n\to\infty}\left(\frac{18}{n^2}\sum_{i=1}^{n} i + \frac{3}{n}\sum_{i=1}^{n} 1\right)\\
&= \lim_{n\to\infty}\left(\frac{18\cdot n(n+1)}{n^2\cdot 2} + \frac{3}{n}n\right)\\
&= \lim_{n\to\infty}\left(9\left(1+\frac{1}{n}\right)+3\right)\\
&= 9 + 3 = 12.\end{aligned}$$

1.1.8.42. Solution. Using the definition of a right Riemann sum,

$$\sum_{i=1}^{10} 3(7+2i)^2\sin(4i) = \sum_{i=1}^{10}\Delta x f(a+i\Delta x)$$

Since $\Delta x = 10$ and $a = -5$,

$$\sum_{i=1}^{10} 3(7+2i)^2\sin(4i) = \sum_{i=1}^{10} 10 f(-5+10i)$$

Dividing both expressions by 10,

$$\sum_{i=1}^{10} \frac{3}{10}(7+2i)^2\sin(4i) = \sum_{i=1}^{10} f(-5+10i)$$

So, we have an expression for $f(-5+10i)$:

$$f(-5+10i) = \frac{3}{10}(7+2i)^2\sin(4i)$$

In order to find $f(x)$, let $x = -5 + 10i$. Then $i = \frac{x}{10} + \frac{1}{2}$.

$$\begin{aligned} f(x) &= \frac{3}{10}\left(7 + 2\left(\frac{x}{10} + \frac{1}{2}\right)\right)^2 \sin\left(4\left(\frac{x}{10} + \frac{1}{2}\right)\right) \\ &= \frac{3}{10}\left(\frac{x}{5} + 8\right)^2 \sin\left(\frac{2x}{5} + 2\right) \ . \end{aligned}$$

1.1.8.43. Solution. As in the text, we'll set up a Riemann sum for the given integral. Right Riemann sums have the simplest form, so we use a right Riemann sum, but we could equally well use left or midpoint.

$$\begin{aligned} &\int_0^1 2^x \mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(a + i\Delta x) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} f\left(\frac{i}{n}\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \cdot 2^{i/n} \\ &= \lim_{n\to\infty} \frac{1}{n}\left(2^{1/n} + 2^{2/n} + 2^{3/n} + \cdots + 2^{n/n}\right) \\ &= \lim_{n\to\infty} \frac{2^{1/n}}{n}\left(1 + 2^{1/n} + 2^{2/n} + \cdots + 2^{\frac{n-1}{n}}\right) \\ &= \lim_{n\to\infty} \frac{2^{1/n}}{n}\left(1 + 2^{1/n} + \left(2^{1/n}\right)^2 + \cdots + \left(2^{1/n}\right)^{n-1}\right) \end{aligned}$$

The sum in parenthesis has the form of a geometric sum, with $r = 2^{1/n}$:

$$\begin{aligned} &= \lim_{n\to\infty} \frac{2^{1/n}}{n}\left(\frac{\left(2^{1/n}\right)^n - 1}{2^{1/n} - 1}\right) \\ &= \lim_{n\to\infty} \frac{2^{1/n}}{n}\left(\frac{2 - 1}{2^{1/n} - 1}\right) \\ &= \lim_{n\to\infty} \frac{2^{1/n}}{n(2^{1/n} - 1)} \end{aligned}$$

Note as $n \to \infty$, $1/n \to 0$, so the numerator has limit 1, while the denominator has indeterminate form $\infty \cdot 0$. So, we'll do a little algebra to get this into a l'Hôpital-style indeterminate form:

$$\begin{aligned} &= \lim_{n\to\infty} \frac{\frac{1}{n} \cdot 2^{1/n}}{2^{1/n} - 1} \\ &= \lim_{n\to\infty} \underbrace{\frac{\frac{1}{n}}{1 - 2^{-1/n}}}_{\substack{\text{num}\to 0 \\ \text{den}\to 0}} \end{aligned}$$

Now we can use l'Hôpital's rule. Recall $\frac{\mathrm{d}}{\mathrm{d}x}\{2^x\} = 2^x \log x$, where $\log x$ is the natural logarithm of x, also sometimes written $\ln x$. We'll need to use the chain rule when we differentiate the denominator.

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{\frac{-1}{n^2}}{-2^{-1/n}\log 2 \cdot \frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{2^{1/n}}{\log 2} \\
&= \frac{1}{\log 2}
\end{aligned}$$

Using a calculator, we see this is about 1.44 square units.

1.1.8.44. Solution. As in the text, we'll set up a Riemann sum for the given integral. Right Riemann sums have the simplest form:

$$\begin{aligned}
&\int_a^b 10^x \mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^n \Delta x f(a + i\Delta x) \\
&= \lim_{n\to\infty} \sum_{i=1}^n \frac{b-a}{n} f\left(a + i\frac{b-a}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^n \frac{b-a}{n} \cdot 10^{a+i\frac{b-a}{n}} \\
&= \lim_{n\to\infty} \sum_{i=1}^n \frac{b-a}{n} \cdot 10^a \cdot \left(10^{\frac{b-a}{n}}\right)^i \\
&= \lim_{n\to\infty} \frac{b-a}{n} \cdot 10^a \left(\left(10^{\frac{b-a}{n}}\right)^1 + \left(10^{\frac{b-a}{n}}\right)^2 + \left(10^{\frac{b-a}{n}}\right)^3 + \cdots + \left(10^{\frac{b-a}{n}}\right)^n\right) \\
&= \lim_{n\to\infty} \frac{b-a}{n} \cdot 10^a \cdot 10^{\frac{b-a}{n}} \left(1 + \left(10^{\frac{b-a}{n}}\right) + \left(10^{\frac{b-a}{n}}\right)^2 + \cdots + \left(10^{\frac{b-a}{n}}\right)^{n-1}\right)
\end{aligned}$$

Now the sum in parentheses has the form of a geometric sum, with $r = 10^{\frac{b-a}{n}}$:

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{b-a}{n} \cdot 10^a \cdot 10^{\frac{b-a}{n}} \left(\frac{\left(10^{\frac{b-a}{n}}\right)^n - 1}{10^{\frac{b-a}{n}} - 1}\right) \\
&= \lim_{n\to\infty} \frac{\color{blue}{b-a}}{n} \cdot \color{red}{10^a} \cdot 10^{\frac{b-a}{n}} \left(\frac{\color{red}{10^{b-a} - 1}}{10^{\frac{b-a}{n}} - 1}\right)
\end{aligned}$$

The coloured parts do not depend on n, so for simplicity we can move them outside the limit.

$$\begin{aligned}
&= \color{blue}{(b-a)} \cdot \color{red}{10^a} \left(\color{red}{10^{b-a} - 1}\right) \lim_{n\to\infty} \frac{1}{n} \cdot \left(\frac{10^{\frac{b-a}{n}}}{10^{\frac{b-a}{n}} - 1}\right) \\
&= (b-a) \cdot \left(10^b - 10^a\right) \lim_{n\to\infty} \underbrace{\left(\frac{1/n}{1 - 10^{-\frac{b-a}{n}}}\right)}_{\substack{\text{num}\to 0 \\ \text{den}\to 0}}
\end{aligned}$$

Now we can use l'Hôpital's rule. Recall $\frac{\mathrm{d}}{\mathrm{d}x}\left\{10^x\right\} = 10^x \log x$, where $\log x$ is the natural logarithm of x, also sometimes written $\ln x$. For the denominator, we will have to use the chain rule.

$$\begin{aligned}
&= (b-a)\cdot\left(10^b - 10^a\right)\lim_{n\to\infty}\left(\frac{-1/n^2}{-10^{-\frac{b-a}{n}}\cdot\log 10\cdot\frac{b-a}{n^2}}\right)\\
&= (b-a)\cdot\left(10^b - 10^a\right)\lim_{n\to\infty}\left(\frac{1}{10^{-\frac{b-a}{n}}\cdot\log 10\cdot(b-a)}\right)\\
&= (b-a)\cdot\left(10^b - 10^a\right)\left(\frac{1}{\log 10\cdot(b-a)}\right)\\
&= \frac{1}{\log 10}\left(10^b - 10^a\right)
\end{aligned}$$

For part (b), we can guess that if 10 were changed to c, our answer would be

$$\int_a^b c^x \mathrm{d}x = \frac{1}{\log c}\left(c^b - c^a\right)$$

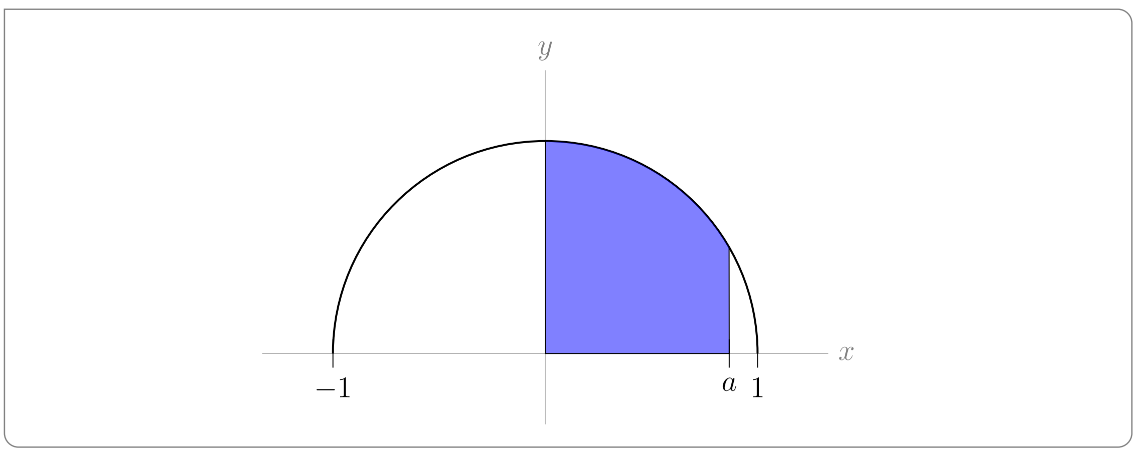
In Question 43, we had $a = 0$, $b = 1$, and $c = 2$. In this case, the formula we guessed above gives

$$\int_0^1 2^x \mathrm{d}x = \frac{1}{\log 2}\left(2^1 - 2^0\right) = \frac{1}{\log 2}$$

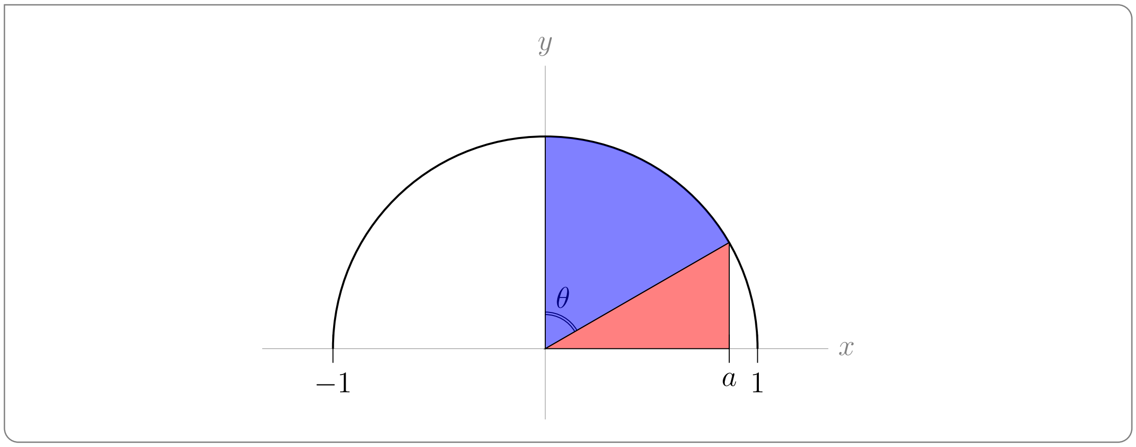
This does indeed match the answer we calculated.

(In fact, we can directly show $\displaystyle\int_a^b c^x \mathrm{d}x = \frac{1}{\log c}\left(c^b - c^a\right)$ using the method of this problem.)

1.1.8.45. Solution. First, we note $y = \sqrt{1-x^2}$ is the upper half of a circle of radius 1, centred at the origin. We're taking the area under the curve from 0 to a, so the area in question is as shown in the picture below.



In order to use geometry to find this area, we break it up into two pieces: a sector of a circle, and a triangle, shown below.



- Area of sector: The sector is a portion of a circle with radius 1, with inner angle θ. So, its area is $\frac{\theta}{2\pi}$ (area of circle) $= \frac{\theta}{2\pi}(\pi) = \frac{\theta}{2}$.

  Our job now is to find θ in terms of a. Note $\frac{\pi}{2} - \theta$ is the inner angle of the red triangle, which lies in the unit circle. So, $\cos\left(\frac{\pi}{2} - \theta\right) = a$. Then $\frac{\pi}{2} - \theta = \arccos(a)$, and so $\theta = \frac{\pi}{2} - \arccos(a)$.

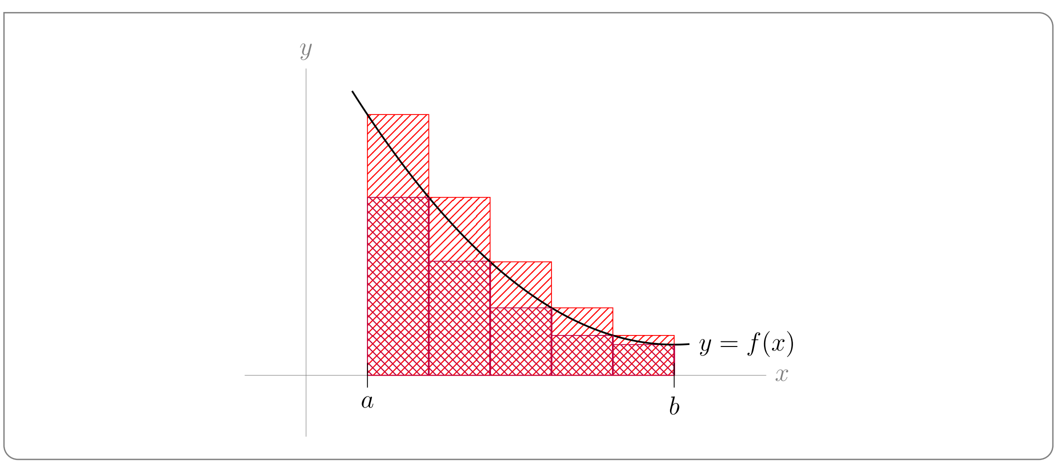
  Then the area of the sector is $\frac{\pi}{4} - \frac{1}{2}\arccos(a)$ square units.

- Area of triangle: The triangle has base a. Its height is the y-value of the function when $x = a$, so its height is $\sqrt{1-a^2}$. Then the area of the triangle is $\frac{1}{2}a\sqrt{1-a^2}$.

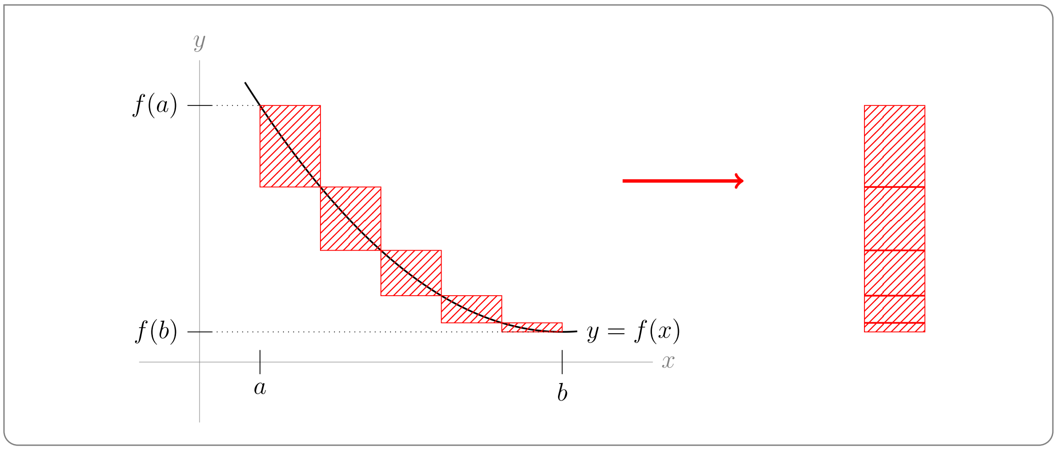
We conclude $\displaystyle\int_0^a \sqrt{1-x^2}\mathrm{d}x = \frac{\pi}{4} - \frac{1}{2}\arccos(a) + \frac{1}{2}a\sqrt{1-a^2}$.

1.1.8.46. Solution.

a The difference between our upper and lower bounds is the difference in areas between the larger set of rectangles and the smaller set of rectangles. Drawing them on a single picture makes this a little clearer.



Each of the rectangles has width $\frac{b-a}{n}$, since we took a segment of the x-axis with length $b - a$ and chopped it into n pieces. We could calculate the height of each rectangle, but it would be a little complicated, since it differs for each of them. An easier method is to notice that the area we want to calculate can be imagined as a single rectangle:



The rectangle has base $\frac{b-a}{n}$. Its highest coordinate is $f(a)$, and its lowest is $f(b)$, so its height is $f(b) - f(a)$. Therefore, the difference in area between our lower bound and our upper bound is:

$$[f(b) - f(a)] \cdot \frac{b-a}{n}$$

b We want to give a range with length at most 0.01, and guarantee that the area under the curve $y = f(x)$ is inside that range. In the previous part, we figured out that when we use n rectangles, the length of our range is $[f(b) - f(a)] \cdot \frac{b-a}{n}$. So, all we have to do is set this to be less than or equal to 0.01, and solve for n:

$$\begin{aligned}[f(b) - f(a)] \cdot \frac{b-a}{n} &\le 0.01 \\ 100\,[f(b) - f(a)] \cdot (b-a) &\le n\end{aligned}$$

We can choose n to be an integer that is greater than or equal to $100\,[f(b) - f(a)] \cdot (b-a)$. Using that many rectangles, we find an upper and lower bound for the area under the curve. If we choose any number between our upper and lower bound as an approximation for the area under the curve, our error is no more than 0.01.

Remark: this question depends on the fact that f is decreasing and positive from a to b. In general, bounding errors on approximations like this is not so straightforward.

1.1.8.47. Solution. Since $f(x)$ is linear, there exist real numbers m and c such that $f(x) = mx+c$. Now we can do some calculations. Suppose we have a rectangle in our Riemann sum that takes up the interval $[x, x+w]$.

- If we are using a left Riemann sum, our rectangle has height $f(x) = mx + c$. Then it has area $w(mx + c)$.

- If we are using a right Riemann sum, our rectangle has height $f(x+w) = m(x+w)+c = mx+c+mw$. Then it has area $w(mx+c+mw)$.
- If we are using a midpoint Riemann sum, our rectangle has height $f(x+\frac{1}{2}w) = m(x+\frac{1}{2}w)+c = mx+c+\frac{1}{2}mw$. Then it has area $w\left(mx+c+\frac{1}{2}w\right)$.

So, for each rectangle in our sums, the midpoint rectangle has the same area as the average of the left and right rectangles:

$$w\left(mx+c+\frac{1}{2}mw\right) = \frac{w(mx+c)+w(mx+c+mw)}{2}$$

It follows that the midpoint Riemann sum has a value equal to the average of the values of the left and right Riemann sums. To see this, let the rectangles in the midpoint Riemann sum have areas $M_1, M_2, \ldots, M_n$, let the rectangles in the left Riemann sum have areas $L_1, L_2, \ldots, L_n$, and let the rectangles in the right Riemann sum have areas $R_1, R_2, \ldots, R_n$. Then the midpoint Riemann sum evaluates to $M_1+M_2+\cdots+M_n$, and:

$$\begin{aligned}
&\frac{[L_1+L_2+\ldots+L_n]+[R_1+R_2+\ldots+R_n]}{2}\\
&= \frac{L_1+R_1}{2}+\frac{L_2+R_2}{2}+\cdots+\frac{L_n+R_n}{2}\\
&= M_1+M_2+\cdots+M_n
\end{aligned}$$

So, the statement is true.
(Note, however, it is false for many non-linear functions $f(x)$.)

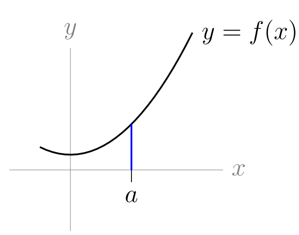
1.2 · Basic properties of the definite integral

1.2.3 · Exercises

Exercises — Stage 1

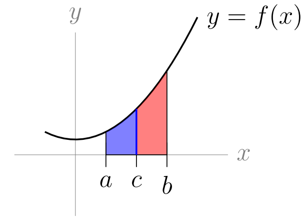
1.2.3.1. Solution.

a $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0$



The area under the curve is zero, because it's a region with no width.

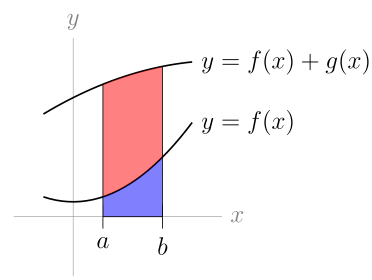
b $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$



If we assume $a \leq c \leq b$, then this identity simply tells us that if we add up the area under the curve from a to c, and from c to b, then we get the whole area under the curve from a to b.

(The situation is slightly more complicated when c is not between a and b, but it still works out.)

c $\displaystyle\int_a^b (f(x)+g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$



The blue-shaded area in the picture above is $\displaystyle\int_a^b f(x)\mathrm{d}x$. The area under the curve $f(x)+g(x)$ but above the curve $f(x)$ (shown in red) is $\displaystyle\int_a^b g(x)\mathrm{d}x$.

1.2.3.2. Solution. Using the identity

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x\ ,$$

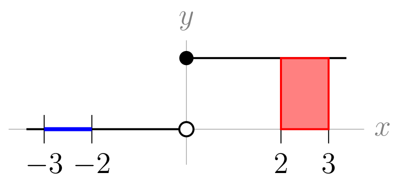
we see

$$\begin{aligned}\int_a^b \cos x\mathrm{d}x &= \int_a^0 \cos x\mathrm{d}x + \int_0^b \cos x\mathrm{d}x\\ &= -\int_0^a \cos x\mathrm{d}x + \int_0^b \cos x\mathrm{d}x\\ &= -\sin a + \sin b\\ &= \sin b - \sin a\end{aligned}$$

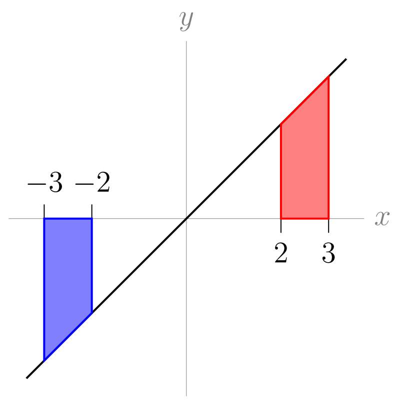
1.2.3.3. *. Solution. (a) False. For example if

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$$

then $\int_{-3}^{-2} f(x)\mathrm{d}x = 0$ and $-\int_3^2 f(x)\mathrm{d}x = -1$.



(b) False. For example, if $f(x) = x$, then $\int_{-3}^{-2} f(x)\,\mathrm{d}x$ is negative while $\int_2^3 f(x)\,\mathrm{d}x$ is positive, so they cannot be the same.



(c) False. For example, consider the functions

$$f(x) = \begin{cases} 0 & \text{for } x < \frac{1}{2} \\ 1 & \text{for } x \geq \frac{1}{2} \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 0 & \text{for } x \geq \frac{1}{2} \\ 1 & \text{for } x < \frac{1}{2} \end{cases}$$

Then $f(x) \cdot g(x) = 0$ for all x, so $\int_0^1 f(x) \cdot g(x)\mathrm{d}x = 0$. However, $\int_0^1 f(x)\mathrm{d}x = \frac{1}{2}$ and $\int_0^1 g(x)\mathrm{d}x = \frac{1}{2}$, so $\int_0^1 f(x)\mathrm{d}x \cdot \int_0^1 g(x)\mathrm{d}x = \frac{1}{4}$.

1.2.3.4. Solution.

a $\Delta x = \dfrac{b-a}{n} = \dfrac{0-5}{100} = -\dfrac{1}{20}$

Note: if we were to use the Riemann-sum definition of a definite integral, this is how we would justify the identity $\int\limits_a^b f(x)\mathrm{d}x = -\int\limits_b^a f(x)\mathrm{d}x$.

b The heights of the rectangles are given by $f(x_i)$, where $x_i = a + i\Delta x = 5 - \frac{i}{20}$. Since $f(x)$ only gives positive values, $f(x_i) > 0$, so the heights of the rectangles are positive.

c Our Riemann sum is the sum of the signed areas of individual rectangles. Each rectangle has a negative base (Δx) and a positive height ($f(x_i)$). So, each term of our sum is negative. If we add up negative numbers, the sum is negative. So, the Riemann sum is negative.

d Since $f(x)$ is always above the x-axis, $\int\limits_0^5 f(x)\mathrm{d}x$ is positive.

Exercises — Stage 2

1.2.3.5. *. Solution. The operation of integration is linear (that's part (d) of the "arithmetic of integration" Theorem 1.2.1), so that:

$$\begin{aligned}\int_2^3 [6f(x) - 3g(x)]\,\mathrm{d}x &= \int_2^3 6f(x)\,\mathrm{d}x - \int_2^3 3g(x)\,\mathrm{d}x \\ &= 6\int_2^3 f(x)\,\mathrm{d}x - 3\int_2^3 g(x)\,\mathrm{d}x \\ &= (6\times(-1)) - (3\times 5) = -21\end{aligned}$$

1.2.3.6. *. Solution. The operation of integration is linear (that's part (d) of the "arithmetic of integration" Theorem 1.2.1), so that:

$$\begin{aligned}\int_0^2 [2f(x) + 3g(x)]\,\mathrm{d}x &= \int_0^2 2f(x)\,\mathrm{d}x + \int_0^2 3g(x)\,\mathrm{d}x \\ &= 2\int_0^2 f(x)\,\mathrm{d}x + 3\int_0^2 g(x)\,\mathrm{d}x \\ &= (2\times 3) + (3\times(-4)) = -6\end{aligned}$$

1.2.3.7. *. Solution. Using part (d) of the "arithmetic of integration" Theorem 1.2.1, followed by parts (c) and (b) of the "arithmetic for the domain of integration" Theorem 1.2.3,

$$\begin{aligned}&\int_{-1}^2 \big[3g(x) - f(x)\big]\,\mathrm{d}x = 3\int_{-1}^2 g(x)\,\mathrm{d}x - \int_{-1}^2 f(x)\,\mathrm{d}x \\ &= 3\int_{-1}^0 g(x)\,\mathrm{d}x + 3\int_0^2 g(x)\,\mathrm{d}x - \int_{-1}^0 f(x)\,\mathrm{d}x - \int_0^2 f(x)\,\mathrm{d}x \\ &= 3\int_{-1}^0 g(x)\,\mathrm{d}x + 3\int_0^2 g(x)\,\mathrm{d}x + \int_0^{-1} f(x)\,\mathrm{d}x - \int_0^2 f(x)\,\mathrm{d}x \\ &= 3\times 3 + 3\times 4 + 1 - 2 = 20\end{aligned}$$

1.2.3.8. Solution.

a Since $\sqrt{1-x^2}$ is an even function,

$$\begin{aligned}\int_a^0 \sqrt{1-x^2}\mathrm{d}x &= \int_0^{|a|} \sqrt{1-x^2}\mathrm{d}x \\ &= \frac{\pi}{4} - \frac{1}{2}\arccos(|a|) + \frac{1}{2}|a|\sqrt{1-|a|^2} \\ &= \frac{\pi}{4} - \frac{1}{2}\arccos(-a) - \frac{1}{2}a\sqrt{1-a^2}\end{aligned}$$

Alternatively, since $\arccos(-a) = \pi - \arccos(a)$ we also have

$$\int_a^0 \sqrt{1-x^2}\ \mathrm{d}x = -\frac{\pi}{4} + \frac{1}{2}\arccos(a) - \frac{1}{2}a\sqrt{1-a^2}$$

b Note $\displaystyle\int_0^1 \sqrt{1-x^2}\mathrm{d}x = \frac{\pi}{4}$, since the area under the curve represents one-quarter of the unit circle. Then,

$$\begin{aligned}\int_a^1 \sqrt{1-x^2}\mathrm{d}x &= \int_0^1 \sqrt{1-x^2}\mathrm{d}x - \int_0^a \sqrt{1-x^2}\mathrm{d}x \\ &= \frac{\pi}{4} - \left(\frac{\pi}{4} - \frac{1}{2}\arccos(a) + \frac{1}{2}a\sqrt{1-a^2}\right) \\ &= \frac{1}{2}\arccos(a) - \frac{1}{2}a\sqrt{1-a^2}\end{aligned}$$

1.2.3.9. *. Solution. Recall that

$$|x| = \begin{cases} -x & \text{if } x \le 0 \\ x & \text{if } x \ge 0 \end{cases}$$

so that

$$|2x| = \begin{cases} -2x & \text{if } x \le 0 \\ 2x & \text{if } x \ge 0 \end{cases}$$

Also recall, from Example 1.2.6, that

$$\int_a^b x\mathrm{d}x = \frac{b^2-a^2}{2}$$

So

$$\begin{aligned}\int_{-1}^2 |2x|\mathrm{d}x &= \int_{-1}^0 |2x|\mathrm{d}x + \int_0^2 |2x|\mathrm{d}x = \int_{-1}^0 (-2x)\mathrm{d}x + \int_0^2 2x\mathrm{d}x \\ &= -2\int_{-1}^0 x\mathrm{d}x + 2\int_0^2 x\mathrm{d}x = -2\cdot\frac{0^2-(-1)^2}{2} + 2\cdot\frac{2^2-0^2}{2} \\ &= 1 + 4 = 5\end{aligned}$$

1.2.3.10. Solution. We note that the integrand $f(x) = x|x|$ is an odd function, because $f(-x) = -x|-x| = -x|x| = -f(x)$. Then by Theorem 1.2.12 part (b), $\displaystyle\int_{-5}^{5} x|x|\mathrm{d}x = 0$.

1.2.3.11. Solution. Using Theorem 1.2.12 part (a),

$$\begin{aligned} 10 &= \int_{-2}^{2} f(x)\mathrm{d}x = 2\int_{0}^{2} f(x)\mathrm{d}x \\ 5 &= \int_{0}^{2} f(x)\mathrm{d}x \end{aligned}$$

Also,

$$\int_{-2}^{2} f(x)\mathrm{d}x = \int_{-2}^{0} f(x)\mathrm{d}x + \int_{0}^{2} f(x)\mathrm{d}x$$

So,

$$\begin{aligned} \int_{-2}^{0} f(x)\mathrm{d}x &= \int_{-2}^{2} f(x)\mathrm{d}x - \int_{0}^{2} f(x)\mathrm{d}x \\ &= 10 - 5 = 5 \end{aligned}$$

Indeed, for any even function $f(x)$, $\int\limits_{-a}^{0} f(x)\mathrm{d}x = \int\limits_{0}^{a} f(x)\mathrm{d}x$.

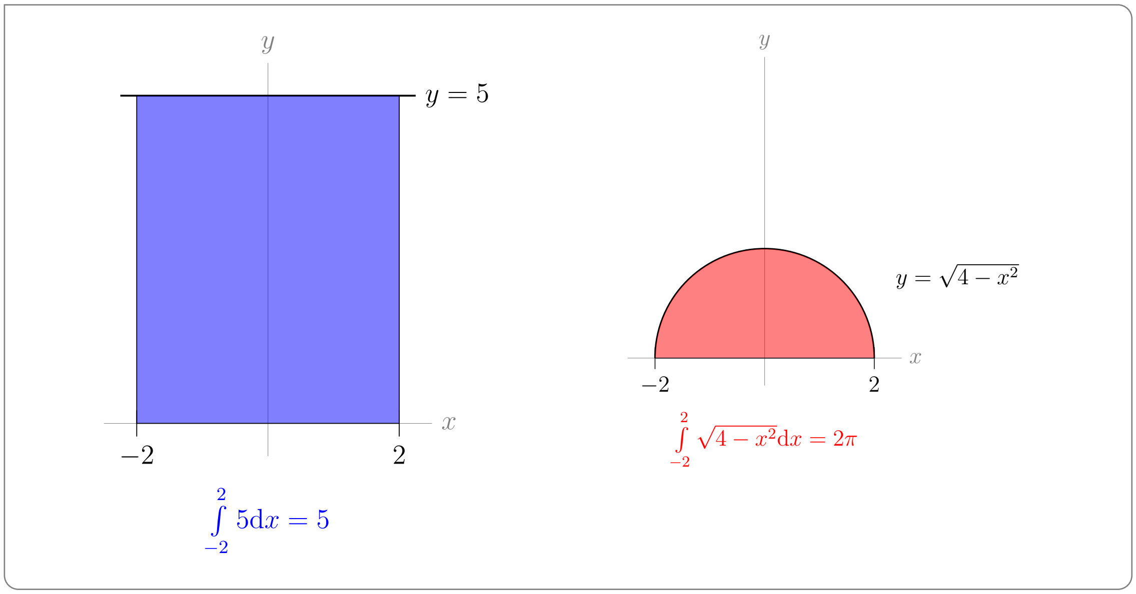
Exercises — Stage 3

1.2.3.12. *. Solution. We first use additivity:

$$\int_{-2}^{2}\left(5 + \sqrt{4-x^2}\right)\mathrm{d}x = \int_{-2}^{2} 5\,\mathrm{d}x + \int_{-2}^{2} \sqrt{4-x^2}\,\mathrm{d}x$$

The first integral represents the area of a rectangle of height 5 and width 4 and so equals 20. The second integral represents the area above the x–axis and below the curve $y = \sqrt{4-x^2}$ or $x^2 + y^2 = 4$. That is a semicircle of radius 2, which has area $\frac{1}{2}\pi 2^2$. So

$$\int_{-2}^{2}\left(5 + \sqrt{4-x^2}\right)\mathrm{d}x = 20 + 2\pi$$



1.2.3.13. $*$. **Solution**. Note that the integrand $f(x) = \frac{\sin x}{\log(3+x^2)}$ is an odd function, because:

$$f(-x) = \frac{\sin(-x)}{\log(3+(-x)^2)} = \frac{-\sin x}{\log(3+x^2)} = -f(x)$$

The domain of integration $-2012 \le x \le 2012$ is symmetric about $x = 0$. So, by Theorem 1.2.12,

$$\int_{-2012}^{+2012} \frac{\sin x}{\log(3+x^2)} \mathrm{d}x = 0$$

1.2.3.14. $*$. **Solution**. Note that the integrand $f(x) = x^{1/3}\cos x$ is an odd function, because:

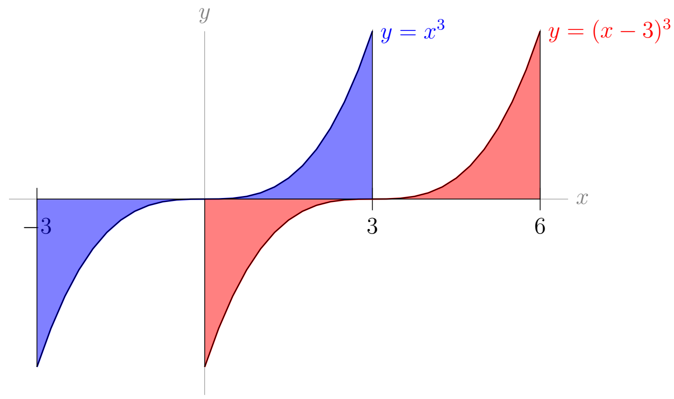
$$f(-x) = (-x)^{1/3}\cos(-x) = -x^{1/3}\cos x = -f(x)$$

The domain of integration $-2012 \le x \le 2012$ is symmetric about $x = 0$. So, by Theorem 1.2.12,

$$\int_{-2012}^{+2012} x^{1/3}\cos x \, \mathrm{d}x = 0$$

1.2.3.15. **Solution**. Our integrand $f(x) = (x-3)^3$ is neither even nor odd. However, it does have a similar symmetry. Namely, $f(3+x) = -f(3-x)$. So, f is "negatively symmetric" across the line $x = 3$. This suggests that the integral should be 0: the positive area to the right of $x = 3$ will be the same as the negative area to the left of $x = 3$.
Another way to see this is to notice that the graph of $f(x) = (x-3)^3$ is equivalent to the graph of $g(x) = x^3$ shifted three units to the right, and $g(x)$ is an odd function. So,

$$\int_0^6 (x-3)^3 \, \mathrm{d}x = \int_{-3}^{3} x^3 \, \mathrm{d}x = 0$$



1.2.3.16. Solution.

a

$$
\begin{aligned}
(ax)^2 + (by)^2 &= 1 \\
by &= \sqrt{1-(ax)^2} \\
y &= \frac{1}{b}\sqrt{1-(ax)^2}
\end{aligned}
$$

b The values of x in the domain of the function above are those that satisfy $1-(ax)^2 \geq 0$. That is, $-\frac{1}{a} \leq x \leq \frac{1}{a}$. Therefore, the upper half of the ellipse has area

$$\frac{1}{b}\int_{-\frac{1}{a}}^{\frac{1}{a}} \sqrt{1-(ax)^2}\mathrm{d}x$$

The upper half of a circle has equation $y = \sqrt{r^2 - x^2}$.

$$
\begin{aligned}
&= \frac{1}{b}\int_{-\frac{1}{a}}^{\frac{1}{a}} \sqrt{a^2\left(\frac{1}{a^2} - x^2\right)}\mathrm{d}x \\
&= \frac{1}{b}\int_{-\frac{1}{a}}^{\frac{1}{a}} a\sqrt{\frac{1}{a^2} - x^2}\mathrm{d}x \\
&= \frac{a}{b}\int_{-\frac{1}{a}}^{\frac{1}{a}} \sqrt{\frac{1}{a^2} - x^2}\mathrm{d}x
\end{aligned}
$$

c The function $y = \sqrt{\frac{1}{a^2} - x^2}$ is the upper-half of the circle centred at the origin with radius $\frac{1}{a}$. So, the expression from (b) evaluates to $\left(\frac{a}{b}\right)\frac{\pi}{2a^2} = \frac{\pi}{2ab}$.

The expression from (b) was half of the ellipse, so the area of the ellipse is $\frac{\pi}{ab}$.

Remark: this was a slightly long-winded way of getting the result. The reasoning is basically this:

- The area of the unit circle $x^2 + y^2 = 1$ is π.
- The ellipse $(ax)^2 + y^2 = 1$ is obtained by shrinking the unit circle horizontally by a factor of a. So, its area is $\frac{\pi}{a}$.
- Further, the ellipse $(ax)^2 + (by)^2 = 1$ is obtained from the previous ellipse by shrinking it vertically by a factor of b. So, its area is $\frac{\pi}{ab}$.

1.2.3.17. Solution. Let's recall the definitions of even and odd functions: $f(x)$ is *even* if $f(-x) = f(x)$ for every x in its domain, and $f(x)$ is *odd* if $f(-x) = -f(x)$ for every x in its domain.
Let $h(x) = f(x) \cdot g(x)$.

- even $\times$ even: If f and g are both even, then $h(-x) = f(-x) \cdot g(-x) = f(x) \cdot g(x) = h(x)$, so their product is even.
- odd $\times$ odd: If f and g are both odd, then $h(-x) = f(-x) \cdot g(-x) = [-f(x)] \cdot [-g(x)] = f(x) \cdot g(x) = h(x)$, so their product is even.
- even $\times$ odd: If f is even and g is odd, then $h(-x) = f(-x) \cdot g(-x) = f(x) \cdot [-g(x)] = -[f(x) \cdot g(x)] = -h(x)$, so their product is odd. Because multiplication is commutative, the order we multiply the functions in doesn't matter.

We note that the table would be the same as if we were *adding* (not multiplying) even and odd *numbers* (not functions).

1.2.3.18. Solution. Since $f(x)$ is odd, $f(0) = -f(-0) = -f(0)$. So, $f(0) = 0$. However, this restriction does not apply to $g(x)$. For example, for any constant c, let $g(x) = c$. Then $g(x)$ is even and $g(0) = c$. So, $g(0)$ can be any real number.

1.2.3.19. Solution. Let x be any real number.

- $f(x) = f(-x)$ (since $f(x)$ is even), and
- $f(x) = -f(-x)$ (since $f(x)$ is odd).
- So, $f(x) = -f(x)$.
- Then (adding $f(x)$ to both sides) we see $2f(x) = 0$, so $f(x) = 0$.

So, $f(x) = 0$ for every x.

1.2.3.20. Solution.

- Solution 1: Suppose $f(x)$ is an odd function. We investigate $f'(x)$ using the chain rule:

$$\begin{aligned} f(-x) &= -f(x) && \text{(odd function)} \\ \frac{\mathrm{d}}{\mathrm{d}x}\{f(-x)\} &= \frac{\mathrm{d}}{\mathrm{d}x}\{-f(x)\} \end{aligned}$$

$$-f'(-x) = -f'(x) \qquad \text{(chain rule)}$$
$$f'(-x) = f'(x)$$

So, when $f(x)$ is odd, $f'(x)$ is even.

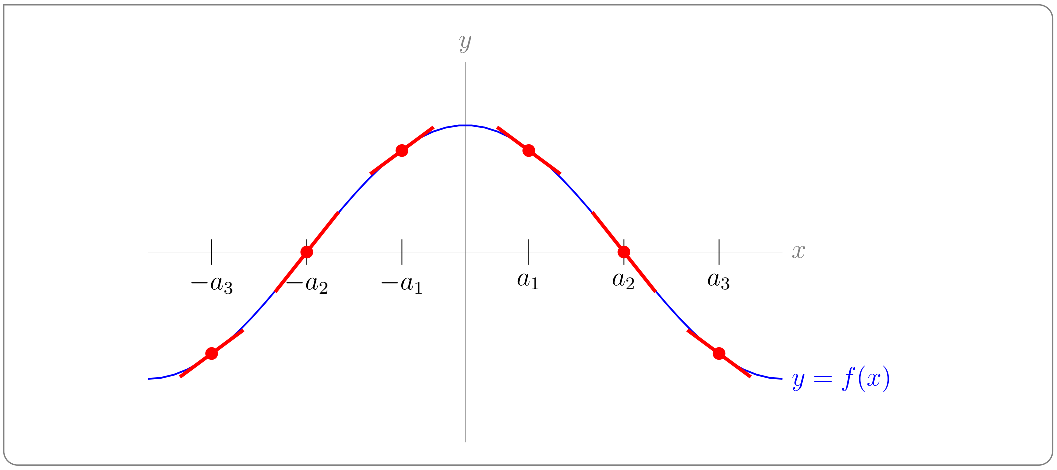
Similarly, suppose $f(x)$ is even.

$$\begin{aligned} f(-x) &= f(x) && \text{(even function)} \\ \frac{\mathrm{d}}{\mathrm{d}x}\{f(-x)\} &= \frac{\mathrm{d}}{\mathrm{d}x}\{f(x)\} \\ -f'(-x) &= f'(x) && \text{(chain rule)} \\ f'(-x) &= -f'(x) \end{aligned}$$

So, when $f(x)$ is even, $f'(x)$ is odd.

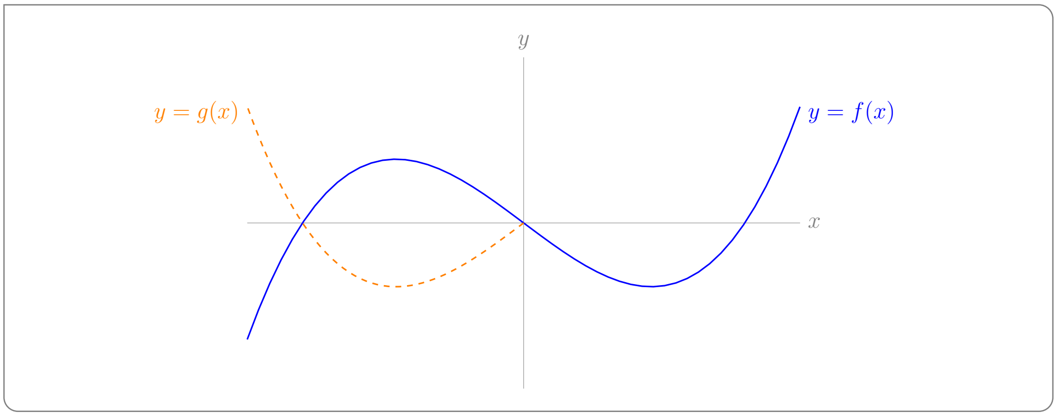
- Solution 2: Another way to think about this problem is to notice that "mirroring" a function changes the sign of its derivative. Then since an even function is "mirrored once" (across the y-axis), it should have $f'(x) = -f'(-x)$, and so the derivative of an even function should be an odd function. Since an odd function is "mirrored twice" (across the y-axis and across the x-axis), it should have $f'(x) = -(-f'(-x)) = f'(-x)$. So the derivative of an odd function should be even. These ideas are presented in more detail below.

  First, we consider the case where $f(x)$ is even, and investigate $f'(x)$.

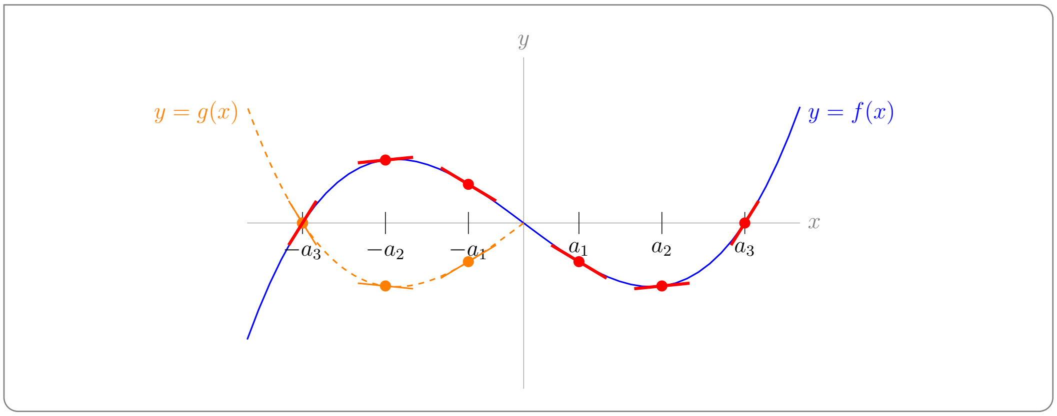


  The whole function has a mirror-like symmetry across the y-axis. So, at x and $-x$, the function will have the same "steepness," but if one is increasing then the other is decreasing. That is, $f'(-x) = -f'(x)$. (In the picture above, compare the slope at some point a_i with its corresponding point $-a_i$.) So, $f'(x)$ is odd when $f(x)$ is even.

  Second, let's consider the case where $f(x)$ is odd, and investigate $f'(x)$. Suppose the blue graph below is $y = f(x)$. If $f(x)$ were *even*, then to the left of the y-axis, it would look like the orange graph, which we'll call $y = g(x)$.



From our work above, we know that, for every $x > 0$, $-f'(x) = g'(-x)$. When $x < 0$, $f(x) = -g(x)$. So, if $x > 0$, then $-f'(x) = g'(-x) = -f'(-x)$. In other words, $f'(x) = f'(-x)$. Similarly, if $x < 0$, then $f'(x) = -g'(x) = f'(-x)$. Therefore $f'(x)$ is even. (In the graph below, you can anecdotally verify that $f'(a_i) = f'(-a_i)$.)



1.3 · The Fundamental Theorem of Calculus
1.3.2 · Exercises

Exercises — Stage 1

1.3.2.1. *. Solution. The Fundamental Theorem of Calculus Part 2 (Theorem 1.3.1) tells us that

$$\begin{aligned}\int_1^{\sqrt{5}} f(x)\,\mathrm{d}x &= F(\sqrt{5}) - F(1)\\ &= \left(e^{(\sqrt{5}^2-3)}+1\right) - \left(e^{(1^2-3)}+1\right)\\ &= e^{5-3} - e^{1-3} = e^2 - e^{-2}\end{aligned}$$

1.3.2.2. *. Solution. First, let's find a general antiderivative of $x^3 - \sin(2x)$.

- One function with derivative x^3 is $\dfrac{x^4}{4}$.
- To find an antiderivative of $\sin(2x)$, we might first guess $\cos(2x)$; checking,

we see $\frac{\mathrm{d}}{\mathrm{d}x}\{\cos(2x)\} = -2\sin(2x)$. So, we only need to multiply by $-\dfrac{1}{2}$: $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{-\dfrac{1}{2}\cos 2x\right\} = \sin(2x)$.

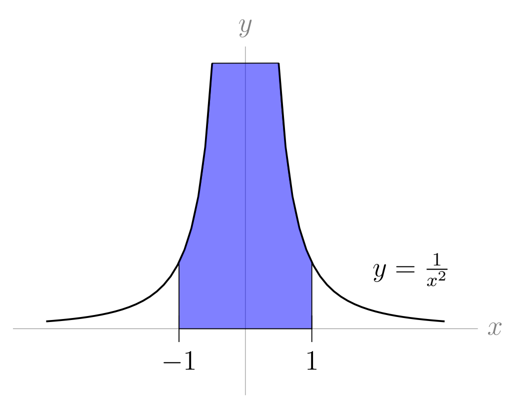
So, the general antiderivative of $f(x)$ is $\dfrac{x^4}{4} + \dfrac{1}{2}\cos 2x + C$. To satisfy $F(0) = 1$, we need [a]

$$\left[\frac{x^4}{4} + \frac{1}{2}\cos 2x + C\right]_{x=0} = 1 \iff \frac{1}{2} + C = 1 \iff C = \frac{1}{2}$$

So $F(x) = \dfrac{x^4}{4} + \dfrac{1}{2}\cos 2x + \dfrac{1}{2}$.

a The symbol $\iff$ is read "if and only if". This is used in mathematics to express the logical equivalence of two statements. To be more precise, the statement $P \iff Q$ tells us that P is true whenever Q is true and Q is true whenever P is true.

1.3.2.3. *. Solution. (a) This is true, by part 2 of the Fundamental Theorem of Calculus, Thereom 1.3.1, with $G(x) = f(x)$ and $f(x)$ replaced by $f'(x)$.
(b) This is not only false, but it makes no sense at all. The integrand is strictly positive so the integral has to be strictly positive. In fact it's $+\infty$. The Fundamental Theorem of Calculus does not apply because the integrand has an infinite discontinuity at $x = 0$.



(c) This is not only false, but it makes no sense at all, unless $\int_a^b f(x)\,\mathrm{d}x = \int_a^b xf(x)\,\mathrm{d}x = 0$. The left hand side is a number. The right hand side is a number times x.

$$\underbrace{\int_a^b xf(x)\mathrm{d}x}_{\text{area}} \qquad \text{vs} \qquad \underbrace{x}_{\text{variable}} \cdot \underbrace{\int_a^b f(x)\mathrm{d}x}_{\text{area}}$$

For example, if $a = 0$, $b = 1$ and $f(x) = 1$, then the left hand side is $\int_0^1 x\,\mathrm{d}x = \frac{1}{2}$ and the right hand side is $x\int_0^1 \mathrm{d}x = x$.

1.3.2.4. Solution. This is a tempting thought:

$$\int \frac{1}{x}\mathrm{d}x = \log|x| + C$$

so perhaps similarly

$$\int \frac{1}{x^2}\mathrm{d}x \stackrel{?}{=} \log|x^2| + C = \log(x^2) + C$$

We check by differentiating:

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\log(x^2)\} = \frac{\mathrm{d}}{\mathrm{d}x}\{2\log x\} = \frac{2}{x} \neq \frac{1}{x^2}$$

So, it wasn't so easy: false.
When we're guessing antiderivatives, we often need to adjust our original guesses a little. Changing constants works well; changing functions usually does not.

1.3.2.5. Solution. This is tempting:

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin(e^x)\} = e^x\cos(e^x)$$

so perhaps

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\sin(e^x)}{e^x}\right\} \stackrel{?}{=} \cos(e^x)$$

We check by differentiating:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\sin(e^x)}{e^x}\right\} &= \frac{e^x\left(\cos(e^x)\cdot e^x\right) - \sin(e^x)e^x}{e^{2x}} && \text{(quotient rule)}\\ &= \cos(e^x) - \frac{\sin(e^x)}{e^x}\\ &\neq \cos(e^x)\end{aligned}$$

So, the statement is false.
When we're guessing antiderivatives, we often need to adjust our original guesses a little. Dividing by constants works well; dividing by functions usually does not.

1.3.2.6. Solution. "The instantaneous rate of change of $F(x)$ with respect to x" is another way of saying "$F'(x)$". From the Fundamental Theorem of Calculus Part 1, we know this is $\sin(x^2)$.

1.3.2.7. Solution. The slope of the tangent line to $y = F(x)$ when $x = 3$ is exactly $F'(3)$. By the Fundamental Theorem of Calculus Part 1, $F'(x) = e^{1/x}$. Then $F'(3) = e^{1/3} = \sqrt[3]{e}$.

1.3.2.8. Solution. For any constant C, $F(x) + C$ is an antiderivative of $f(x)$, because $\frac{\mathrm{d}}{\mathrm{d}x}\{F(x)+C\} = \frac{\mathrm{d}}{\mathrm{d}x}\{F(x)\} = f(x)$. So, for example, $F(x)$ and $F(x)+1$ are both antiderivatives of $f(x)$.

1.3.2.9. Solution.

a We differentiate with respect to a. Recall $\frac{\mathrm{d}}{\mathrm{d}x}\{\arccos x\} = \frac{-1}{\sqrt{1-x^2}}$. To differentiate $\frac{1}{2}a\sqrt{1-a^2}$, we use the product and chain rules.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}a}\left\{\frac{\pi}{4} - \frac{1}{2}\arccos(a) + \frac{1}{2}a\sqrt{1-a^2}\right\} \\
&= 0 - \frac{1}{2}\cdot\frac{-1}{\sqrt{1-a^2}} + \left(\frac{1}{2}a\right)\cdot\frac{-2a}{2\sqrt{1-a^2}} + \frac{1}{2}\sqrt{1-a^2} \\
&= \frac{1}{2\sqrt{1-a^2}} - \frac{a^2}{2\sqrt{1-a^2}} + \frac{1-a^2}{2\sqrt{1-a^2}} \\
&= \frac{1-a^2+1-a^2}{2\sqrt{1-a^2}} \\
&= \frac{2(1-a^2)}{2\sqrt{1-a^2}} \\
&= \sqrt{1-a^2}
\end{aligned}$$

b Let $G(x) = \frac{\pi}{4} - \frac{1}{2}\arccos(x) + \frac{1}{2}x\sqrt{1-x^2}$. We showed in part (a) that $G(x)$ is an antiderivative of $\sqrt{1-x^2}$. Since $F(x)$ is also an antiderivative of $\sqrt{1-x^2}$, $F(x) = G(x) + C$ for some constant C (this is Lemma 1.3.8).

Note $G(0) = \displaystyle\int_0^0 \sqrt{1-x^2}\mathrm{d}x = 0$, so if $F(0) = \pi$, then $F(x) = G(x) + \pi$. That is,

$$F(x) = \frac{5\pi}{4} - \frac{1}{2}\arccos(x) + \frac{1}{2}x\sqrt{1-x^2}\ .$$

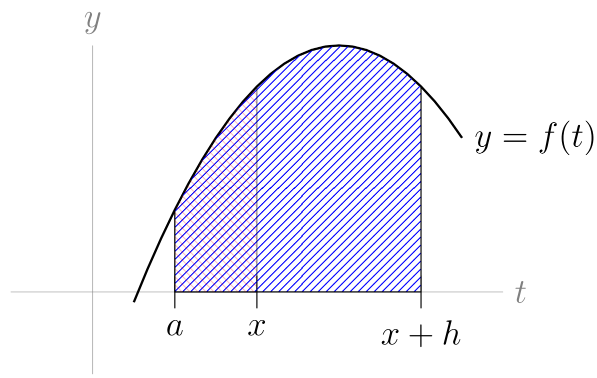
1.3.2.10. Solution.

a The antiderivative of $\cos x$ is $\sin x$, and $\cos x$ is continuous everywhere, so $\displaystyle\int_{-\pi}^{\pi} \cos x\mathrm{d}x = \sin(\pi) - \sin(-\pi) = 0$.

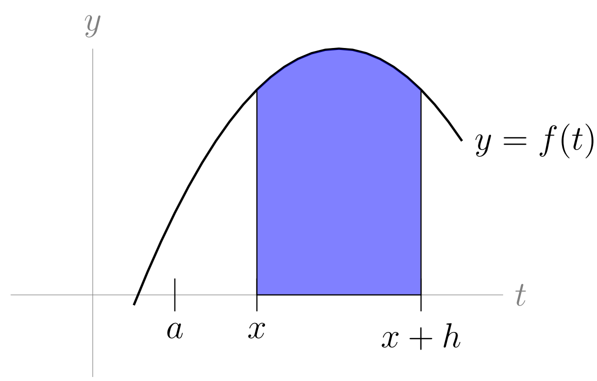
b Since $\sec^2 x$ is discontinuous at $x = \pm\frac{\pi}{2}$, the Fundamental Theorem of Calculus Part 2 does not apply to $\displaystyle\int_{-\pi}^{\pi} \sec^2 x\mathrm{d}x$.

c Since $\frac{1}{x+1}$ is discontinuous at $x = -1$, the Fundamental Theorem of Calculus Part 2 does not apply to $\displaystyle\int_{-2}^{0} \frac{1}{x+1}\mathrm{d}x$.

1.3.2.11. Solution. Using the definition of F, $F(x)$ is the area under the curve from a to x, and $F(x+h)$ is the area under the curve from a to $x+h$. These are shown on the same diagram, below.



Then the area represented by $F(x+h) - F(x)$ is the area that is outside the red, but inside the blue. Equivalently, it is $\int\limits_{x}^{x+h} f(t)\mathrm{d}t$.



1.3.2.12. Solution. We evaluate $F(0)$ using the definition: $F(0) = \int_0^0 f(t)\mathrm{d}t = 0$. Although $f(0) > 0$, the *area* from $t = 0$ to $t = 0$ is zero.
As x moves along, $F(x)$ adds bits of signed area. If it's adding positive area, it's increasing, and if it's adding negative area, it's decreasing. So, $F(x)$ is increasing when $0 < x < 1$ and $3 < x < 4$, and $F(x)$ is decreasing when $1 < x < 3$.

1.3.2.13. Solution. This question is nearly identical to Question 12, with

$$G(x) = \int_x^0 f(t)\mathrm{d}t = -\int_0^x f(t)\mathrm{d}t = -F(x).$$

So, $G(x)$ increases when $F(x)$ decreases, and vice-versa. Therefore: $G(0) = 0$, $G(x)$ is increasing when $1 < x < 3$, and $G(x)$ is decreasing when $0 < x < 1$ and when $3 < x < 4$.

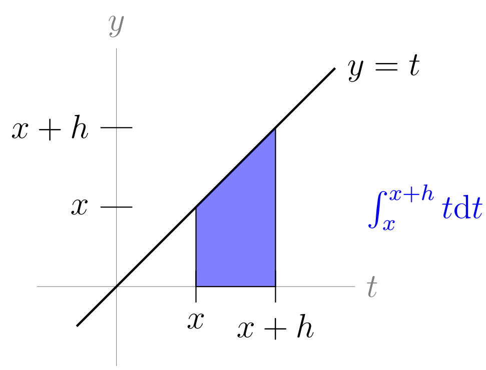
1.3.2.14. Solution. Using the definition of the derivative,

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{\int_a^{x+h} t\mathrm{d}t - \int_a^x t\mathrm{d}t}{h} \end{aligned}$$

$$= \lim_{h\to 0} \frac{\int_x^{x+h} t\mathrm{d}t}{h}$$

The numerator describes the area of a trapezoid with base h and heights x and $x+h$.

$$\begin{aligned} &= \lim_{h\to 0} \frac{\frac{1}{2}h(x+x+h)}{h} \\ &= \lim_{h\to 0}\left(x+\frac{1}{2}h\right) \\ &= x \end{aligned}$$



So, $F'(x) = x$.

1.3.2.15. Solution. If $F(x)$ is constant, then $F'(x) = 0$. By the Fundamental Theorem of Calculus Part 1, $F'(x) = f(x)$. So, the only possible continuous function fitting the question is $f(x) = 0$.

This makes intuitive sense: if moving x doesn't add or subtract area under the curve, then there must not be any area under the curve–the curve should be the same as the x-axis.

As an aside, we mention that there are other, *non-continuous* functions $f(t)$ such that $\int_0^x f(t)\mathrm{d}t = 0$ for all x. For example, $f(t) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$. These kinds of removable discontinuities will not factor heavily in our discussion of integrals.

1.3.2.16. Solution.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\{x\log(ax) - x\} &= x\left(\frac{a}{ax}\right) + \log(ax) - 1 \quad \text{(product rule, chain rule)} \\ &= \log(ax) \end{aligned}$$

So, we know

$$\int \log(ax)\mathrm{d}x = x\log(ax) - x + C$$

where a is a given constant, and C is any constant.

Remark: $\int \log(ax)\mathrm{d}x$ can be calculated using the method of Integration by Parts, which you will learn in Section 1.7.

1.3.2.17. Solution.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left\{e^x\left(x^3-3x^2+6x-6\right)\right\}\\
&=e^x\left(3x^2-6x+6\right)+e^x\left(x^3-3x^2+6x-6\right) \quad \text{(product rule)}\\
&=e^x\left(3x^2-6x+6+x^3-3x^2+6x-6\right)\\
&=x^3e^x
\end{aligned}$$

So,

$$\int x^3e^x\mathrm{d}x=e^x\left(x^3-3x^2+6x-6\right)+C$$

Remark: $\int x^3e^x\mathrm{d}x$ can be calculated using the method of Integration by Parts, which you will learn in Section 1.7.

1.3.2.18. Solution.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left\{\log\left|x+\sqrt{x^2+a^2}\right|\right\}&=\frac{1}{x+\sqrt{x^2+a^2}}\cdot\left(1+\frac{1}{2\sqrt{x^2+a^2}}\cdot 2x\right)\\
&\qquad\qquad\qquad\qquad\text{(chain rule)}\\
&=\frac{1+\frac{x}{\sqrt{x^2+a^2}}}{x+\sqrt{x^2+a^2}}=\frac{\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}}{x+\sqrt{x^2+a^2}}\\
&=\frac{1}{\sqrt{x^2+a^2}}
\end{aligned}$$

So,

$$\int\frac{1}{\sqrt{x^2+a^2}}\mathrm{d}x=\log\left|x+\sqrt{x^2+a^2}\right|+C$$

Remark: $\int\frac{1}{\sqrt{x^2+a^2}}\mathrm{d}x$ can be calculated using the method of Trigonometric Substitution, which you will learn in Section 1.9.

1.3.2.19. Solution. Using the chain rule:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sqrt{x(a+x)}-a\log\left(\sqrt{x}+\sqrt{a+x}\right)\right\}\\
&=\frac{x+(a+x)}{2\sqrt{x(a+x)}}-a\left(\frac{1}{\sqrt{x}+\sqrt{a+x}}\cdot\left(\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{a+x}}\right)\right)\\
&=\frac{2x+a}{2\sqrt{x(a+x)}}-a\left(\frac{1}{\sqrt{x}+\sqrt{a+x}}\cdot\left(\frac{\sqrt{a+x}+\sqrt{x}}{2\sqrt{x(a+x)}}\right)\right)\\
&=\frac{2x+a}{2\sqrt{x(a+x)}}-a\left(\frac{1}{2\sqrt{x(a+x)}}\right)\\
&=\frac{2x}{2\sqrt{x(a+x)}}=\frac{x}{\sqrt{x(a+x)}}
\end{aligned}$$

So,

$$\int \frac{x}{\sqrt{x(a+x)}}\mathrm{d}x = \sqrt{x(a+x)} - a\log\left(\sqrt{x}+\sqrt{a+x}\right) + C$$

Remark: $\int \frac{x}{\sqrt{x(a+x)}}\mathrm{d}x$ can be calculated using the method of Trigonometric Substitution, which you will learn in Section 1.9.

Exercises — Stage 2

1.3.2.20. *. Solution. By the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_0^2 \left(x^3+\sin x\right)\mathrm{d}x &= \left[\frac{x^4}{4} - \cos x\right]_0^2 \\ &= \left(\frac{2^4}{4} - \cos 2\right) - (0 - \cos 0) \\ &= 4 - \cos 2 + 1 = 5 - \cos 2.\end{aligned}$$

1.3.2.21. *. Solution. By part (d) of our “Arithmetic of Integration” theorem, Theorem 1.2.1,

$$\int_1^2 \frac{x^2+2}{x^2}\,\mathrm{d}x = \int_1^2 \left[1 + \frac{2}{x^2}\right]\mathrm{d}x = \int_1^2 \mathrm{d}x + 2\int_1^2 \frac{1}{x^2}\,\mathrm{d}x$$

Then by the Fundamental Theorem of Calculus Part 2,

$$\begin{aligned}\int_1^2 \mathrm{d}x + 2\int_1^2 \frac{1}{x^2}\,\mathrm{d}x &= \Big[x\Big]_1^2 + 2\Big[-\frac{1}{x}\Big]_1^2 = \Big[2-1\Big] + 2\Big[-\frac{1}{2}+1\Big] \\ &= 2\end{aligned}$$

1.3.2.22. Solution. The integrand is similar to $\frac{1}{1+x^2}$, which is the derivative of arctangent. Indeed, we have

$$\int \frac{1}{1+25x^2}\mathrm{d}x = \int \frac{1}{1+(5x)^2}\mathrm{d}x.$$

So, a reasonable first guess for the antiderivative might be

$$F(x) \stackrel{?}{=} \arctan(5x).$$

However, because of the chain rule,

$$F'(x) = \frac{5}{1+(5x)^2}.$$

In order to “fix” the numerator, we make a second guess:

$$F(x) = \frac{1}{5}\arctan(5x)$$

$$F'(x) = \frac{1}{5}\left(\frac{5}{1+(5x)^2}\right) = \frac{1}{1+25x^2}$$

So, $$\int \frac{1}{1+25x^2}\mathrm{d}x = \frac{1}{5}\arctan(5x) + C.$$

1.3.2.23. Solution. The integrand is similar to $\frac{1}{\sqrt{1-x^2}}$. In order to formulate a guess for the antiderivative, let's factor out $\sqrt{2}$ from the denominator:

$$\begin{aligned}
\int \frac{1}{\sqrt{2-x^2}}\mathrm{d}x &= \int \frac{1}{\sqrt{2\left(1-\frac{x^2}{2}\right)}}\mathrm{d}x \\
&= \int \frac{1}{\sqrt{2}\sqrt{1-\frac{x^2}{2}}}\mathrm{d}x \\
&= \int \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}\mathrm{d}x
\end{aligned}$$

At this point, we might guess that our antiderivative is something like $F(x) = \arcsin\left(\frac{x}{\sqrt{2}}\right)$. To explore this possibility, we can differentiate, and see what we get.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\arcsin\left(\frac{x}{\sqrt{2}}\right)\right\} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}$$

This is exactly what we want! So,

$$\int \frac{1}{\sqrt{2-x^2}}\mathrm{d}x = \arcsin\left(\frac{x}{\sqrt{2}}\right) + C$$

1.3.2.24. Solution. We know that $\int \sec^2 x\mathrm{d}x = \tan x + C$, and $\sec^2 x = \tan^2 x + 1$, so

$$\begin{aligned}
\int \tan^2 x\mathrm{d}x &= \int \sec^2 x - 1\mathrm{d}x \\
&= \int \sec^2 x\mathrm{d}x - \int 1\mathrm{d}x \\
&= \tan x - x + C
\end{aligned}$$

1.3.2.25. Solution.

- Solution 1: This might not obviously look like the derivative of anything familiar, but it does look like half of a familiar trig identity: $2\sin x\cos x = \sin(2x)$.

$$\int 3\sin x\cos x\mathrm{d}x = \int \frac{3}{2} \cdot 2\sin x\cos x\mathrm{d}x$$

$$= \int \frac{3}{2}\sin(2x)\mathrm{d}x$$

So, we might guess that the antiderivative is something like $-\cos(2x)$. We only need to figure out the constants.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\{-\cos(2x)\} &= 2\sin(2x) \\ \text{So,} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left\{-\frac{3}{4}\cos(2x)\right\} &= \frac{3}{2}\sin(2x) \\ \text{Therefore,} \qquad \int 3\sin x\cos x\mathrm{d}x &= -\frac{3}{4}\cos(2x) + C \end{aligned}$$

- Solution 2: You might notice that the integrand looks like it came from the chain rule, since $\cos x$ is the derivative of $\sin x$. Using this observation, we can work out the antideriative:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left\{\sin^2 x\right\} &= 2\sin x\cos x \\ \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{3}{2}\sin^2 x\right\} &= 3\sin x\cos x \\ \text{So,} \qquad \int 3\sin x\cos x\mathrm{d}x &= \frac{3}{2}\sin^2 x + C \end{aligned}$$

These two answers look different. Using the identity $\cos(2x) = 1 - 2\sin^2(x)$, we reconcile them:

$$\begin{aligned} -\frac{3}{4}\cos(2x) + C &= -\frac{3}{4}\left(1 - 2\sin^2 x\right) + C \\ &= \frac{3}{2}\sin^2 x + \left(C - \frac{3}{4}\right) \end{aligned}$$

The $\frac{3}{4}$ here is not significant. Remember that C is used to designate a constant that can take any value between $-\infty$ and $+\infty$. So $C - \frac{3}{4}$ is also just a constant that can take any value between $-\infty$ and $+\infty$. As the two answers we found differ by a constant, they are equivalent.

1.3.2.26. **Solution**. It's not immediately obvious which function has $\cos^2 x$ as its derivative, but we can make the situation a little clearer by using the identity $\cos^2 x = \dfrac{1 + \cos(2x)}{2}$:

$$\begin{aligned} \int \cos^2 x\mathrm{d}x &= \int \frac{1}{2}\cdot(1 + \cos(2x))\,\mathrm{d}x \\ &= \int \frac{1}{2}\mathrm{d}x + \int \frac{1}{2}\cos(2x)\mathrm{d}x \\ &= \frac{1}{2}x + C + \int \frac{1}{2}\cos(2x)\mathrm{d}x \end{aligned}$$

For the remaining integral, we might guess something like $F(x) = \sin(2x)$. Let's figure out the appropriate constant:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\{\sin(2x)\} &= 2\cos(2x)\\
\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{4}\sin(2x)\right\} &= \frac{1}{2}\cos(2x)\\
\text{So,}\qquad \int \frac{1}{2}\cos(2x)\mathrm{d}x &= \frac{1}{4}\sin(2x) + C\\
\text{Therefore,}\qquad \int \cos^2 x\mathrm{d}x &= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C
\end{aligned}$$

1.3.2.27. ∗. Solution. By the Fundamental Theorem of Calculus Part 1,

$$\begin{aligned}
F'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \log(2+\sin t)\,\mathrm{d}t &&= \log(2+\sin x)\\
G'(y) &= \frac{\mathrm{d}}{\mathrm{d}y}\left[-\int_0^y \log(2+\sin t)\,\mathrm{d}t\right] &&= -\log(2+\sin y)
\end{aligned}$$

So,

$$F'\left(\frac{\pi}{2}\right) = \log 3 \qquad G'\left(\frac{\pi}{2}\right) = -\log(3)$$

1.3.2.28. ∗. Solution. By the Fundamental Theorem of Calculus Part 1,

$$f'(x) = 100(x^2 - 3x + 2)e^{-x^2} = 100(x-1)(x-2)e^{-x^2}$$

As $f(x)$ is increasing whenever $f'(x) > 0$ and $100e^{-x^2}$ is always strictly bigger than 0, we have $f(x)$ increasing if and only if $(x-1)(x-2) > 0$, which is the case if and only if $(x-1)$ and $(x-2)$ are of the same sign. Both are positive when $x > 2$ and both are negative when $x < 1$. So $f(x)$ is increasing when $-\infty < x < 1$ and when $2 < x < \infty$.
Remark: even without the Fundamental Theorem of Calculus, since $f(x)$ is the area under a curve from 1 to x, $f(x)$ is increasing when the curve is above the x-axis (because we're adding positive area), and it's decreasing when the curve is below the x-axis (because we're adding negative area).

1.3.2.29. ∗. Solution. Write $G(x) = \displaystyle\int_0^x \frac{1}{t^3+6}\mathrm{d}t$. By the Fundamental Theorem of Calculus Part 1, $G'(x) = \dfrac{1}{x^3+6}$. Since $F(x) = G(\cos x)$, the chain rule gives us

$$F'(x) = G'(\cos x)\cdot(-\sin x) = -\frac{\sin x}{\cos^3 x + 6}$$

1.3.2.30. ∗. Solution. Define $g(x) = \displaystyle\int_0^x e^{t^2}\mathrm{d}t$. By the Fundamental Theorem of Calculus Part 1, $g'(x) = e^{x^2}$. As $f(x) = g(1+x^4)$ the chain rule gives us

$$f'(x) = 4x^3 g'(1+x^4) = 4x^3 e^{(1+x^4)^2}$$

1.3.2.31. *. Solution. Define $g(x) = \int_0^x (t^6+8)\mathrm{d}t$. By the fundamental theorem of calculus, $g'(x) = x^6 + 8$. We are to compute the derivative of $f(x) = g(\sin x)$. The chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_0^{\sin x}(t^6+8)\mathrm{d}t\right\} = g'(\sin x)\cdot\cos x = \big(\sin^6 x + 8\big)\cos x$$

1.3.2.32. *. Solution. Let $G(x) = \displaystyle\int_0^x e^{-t}\sin\left(\frac{\pi t}{2}\right)\mathrm{d}t$. By the Fundamental Theorem of Calculus Part 1, $G'(x) = e^{-x}\sin\left(\frac{\pi x}{2}\right)$ and, since $F(x) = G(x^3)$, $F'(x) = 3x^2G'(x^3) = 3x^2e^{-x^3}\sin\left(\frac{\pi x^3}{2}\right)$. Then $F'(1) = 3e^{-1}\sin\left(\frac{\pi}{2}\right) = 3e^{-1}$.

1.3.2.33. *. Solution. Define $G(x) = \displaystyle\int_x^0 \frac{\mathrm{d}t}{1+t^3} = -\int_0^x \frac{1}{1+t^3}\mathrm{d}t$, so that $G'(x) = -\dfrac{1}{1+x^3}$ by the Fundamental Theorem of Calculus Part 1. Then by the chain rule,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}\left\{\int_{\cos u}^0 \frac{\mathrm{d}t}{1+t^3}\right\} &= \frac{\mathrm{d}}{\mathrm{d}u}G(\cos u) = G'(\cos u)\cdot\frac{\mathrm{d}}{\mathrm{d}u}\cos u\\ &= -\frac{1}{1+\cos^3 u}\cdot(-\sin u).\end{aligned}$$

1.3.2.34. *. Solution. Applying $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides of $x^2 = 1 + \int_1^x f(t)\mathrm{d}t$ gives, by the Fundamental Theorem of Calculus Part 1, $2x = f(x)$.

1.3.2.35. *. Solution. Apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides of $x\sin(\pi x) = \int_0^x f(t)\,\mathrm{d}t$. Then, by the Fundamental Theorem of Calculus Part 1,

$$\begin{aligned} & f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f(t)\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\big\{x\sin(\pi x)\big\}\\ \implies\qquad & f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\big\{x\sin(\pi x)\big\} = \sin(\pi x) + \pi x\cos(\pi x)\\ \implies\qquad & f(4) = \sin(4\pi) + 4\pi\cos(4\pi) = 4\pi\end{aligned}$$

1.3.2.36. *. Solution. (a) Write

$$F(x) = G(x^2) - H(-x)\quad\text{with}\quad G(y) = \int_0^y e^{-t}\,\mathrm{d}t,\ H(y) = \int_0^y e^{-t^2}\,\mathrm{d}t$$

By the Fundamental Theorem of Calculus Part 1,

$$G'(y) = e^{-y}\,,\qquad H'(y) = e^{-y^2}$$

Hence, by the chain rule,

$$F'(x) = 2xG'(x^2) - (-1)H'(-x) = 2xe^{-(x^2)} + e^{-(-x)^2} = (2x+1)e^{-x^2}$$

(b) Observe that $F'(x) < 0$ for $x < -1/2$ and $F'(x) > 0$ for $x > -1/2$. Hence $F(x)$ is decreasing for $x < -1/2$ and increasing for $x > -1/2$, and $F(x)$ must take its minimum value when $x = -1/2$.

1.3.2.37. *. Solution. Define $G(y) = \int_0^y e^{\sin t} \mathrm{d}t$. Then:

$$\begin{aligned} F(x) &= \int_0^x e^{\sin t}\,\mathrm{d}t + \int_{x^4-x^3}^0 e^{\sin t}\,\mathrm{d}t = \int_0^x e^{\sin t}\,\mathrm{d}t - \int_0^{x^4-x^3} e^{\sin t}\,\mathrm{d}t \\ &= G(x) - G(x^4 - x^3) \end{aligned}$$

By the Fundamental Theorem of Calculus Part 1,

$$G'(y) = e^{\sin y}$$

Hence, by the chain rule,

$$\begin{aligned} F'(x) &= G'(x) - G'(x^4 - x^3)\ \frac{\mathrm{d}}{\mathrm{d}x}\{x^4 - x^3\} \\ &= G'(x) - G'(x^4 - x^3)\ (4x^3 - 3x^2) \\ &= e^{\sin x} - e^{\sin(x^4 - x^3)}\big(4x^3 - 3x^2\big) \end{aligned}$$

1.3.2.38. *. Solution. Define with $G(y) = \int_0^y \cos\left(e^t\right) \mathrm{d}t$. Then:

$$\begin{aligned} F(x) &= \int_{x^5}^{-x^2} \cos\left(e^t\right)\mathrm{d}t = \int_0^{-x^2} \cos\left(e^t\right)\mathrm{d}t + \int_{x^5}^0 \cos\left(e^t\right)\mathrm{d}t \\ &= \int_0^{-x^2} \cos\left(e^t\right)\mathrm{d}t - \int_0^{x^5} \cos\left(e^t\right)\mathrm{d}t \\ &= G(-x^2) - G(x^5) \end{aligned}$$

By the Fundamental Theorem of Calculus,

$$G'(y) = \cos\left(e^y\right)$$

Hence, by the chain rule,

$$\begin{aligned} F'(x) &= G'(-x^2)\ \frac{\mathrm{d}}{\mathrm{d}x}\big\{-x^2\big\} - G'(x^5)\ \frac{\mathrm{d}}{\mathrm{d}x}\big\{x^5\big\} \\ &= G'(-x^2)\ (-2x) - G'(x^5)\ (5x^4) \\ &= -2x\cos\left(e^{-x^2}\right) - 5x^4\cos\left(e^{x^5}\right) \end{aligned}$$

1.3.2.39. *. Solution. Define with $G(y) = \int_0^y \sqrt{\sin t}\,\mathrm{d}t$. Then:

$$F(x) = \int_x^{e^x} \sqrt{\sin t}\,\mathrm{d}t$$

$$\begin{aligned}&= \int_0^{e^x} \sqrt{\sin t}\,\mathrm{d}t + \int_x^0 \sqrt{\sin t}\,\mathrm{d}t = \int_0^{e^x} \sqrt{\sin t}\,\mathrm{d}t - \int_0^x \sqrt{\sin t}\,\mathrm{d}t\\ &= G(e^x) - G(x)\end{aligned}$$

By the Fundamental Theorem of Calculus Part 1,

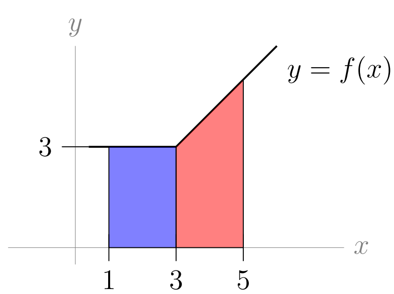
$$G'(y) = \sqrt{\sin y}$$

Hence, by the chain rule,

$$\begin{aligned}F'(x) &= G'(e^x)\,\frac{\mathrm{d}}{\mathrm{d}x}\{e^x\} - G'(x)\\ &= e^x G'(e^x) - G'(x)\\ &= e^x\sqrt{\sin(e^x)} - \sqrt{\sin(x)}\end{aligned}$$

1.3.2.40. $*$. Solution. Splitting up the domain of integration,

$$\begin{aligned}\int_1^5 f(x)\mathrm{d}x &= \int_1^3 f(x)\,\mathrm{d}x + \int_3^5 f(x)\,\mathrm{d}x\\ &= \int_1^3 3\,\mathrm{d}x + \int_3^5 x\,\mathrm{d}x\\ &= 3x\Big|_{x=1}^{x=3} + \frac{x^2}{2}\Big|_{x=3}^{x=5}\\ &= 14\end{aligned}$$



Exercises — Stage 3

1.3.2.41. $*$. Solution. By the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\,(f'(x))^2\,\right\} = 2f'(x)\,f''(x)$$

so $\frac{1}{2}f'(x)^2$ is an antiderivative for $f'(x)\,f''(x)$ and, by the Fundamental Theorem of Calculus Part 2,

$$\int_1^2 f'(x)f''(x)\,\mathrm{d}x = \left[\frac{1}{2}\,(f'(x))^2\right]_{x=1}^{x=2} = \frac{1}{2}f'(2)^2 - \frac{1}{2}f'(1)^2 = \frac{5}{2}$$

Remark: evaluating antiderivatives of this type will occupy the next section, Section 1.4.

1.3.2.42. *. Solution. The car stops when $v(t) = 30 - 10t = 0$, which occurs at time $t = 3$. The distance covered up to that time is

$$\int_0^3 v(t)\,\mathrm{d}t = (30t - 5t^2)\Big|_0^3 = (90 - 45) - 0 = 45\,\mathrm{m}.$$

1.3.2.43. *. Solution. Define $g(x) = \displaystyle\int_0^x \log\left(1 + e^t\right)\mathrm{d}t$. By the Fundamental Theorem of Calculus Part 1, $g'(x) = \log\left(1 + e^x\right)$. But $f(x) = g(2x - x^2)$, so by the chain rule,

$$f'(x) = g'(2x - x^2) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\{2x - x^2\} = (2 - 2x) \cdot \log\left(1 + e^{2x-x^2}\right)$$

Observe that $e^{2x-x^2} > 0$ for all x so that $1+e^{2x-x^2} > 1$ for all x and $\log\left(1+e^{2x-x^2}\right) > 0$ for all x. Since $2 - 2x$ is positive for $x < 1$ and negative for $x > 1$, $f'(x)$ is also positive for $x < 1$ and negative for $x > 1$. That is, $f(x)$ is increasing for $x < 1$ and decreasing for $x > 1$. So $f(x)$ achieves its absolute maximum at $x = 1$.

1.3.2.44. *. Solution. Let $f(x) = \int_0^{x^2-2x} \frac{\mathrm{d}t}{1+t^4}$ and $g(x) = \int_0^x \frac{\mathrm{d}t}{1+t^4}$. Then $g'(x) = \frac{1}{1+x^4}$ and, since $f(x) = g(x^2 - 2x)$, $f'(x) = (2x-2)g'(x^2-2x) = 2\frac{x-1}{1+(x^2-2x)^4}$. This is zero for $x = 1$, negative for $x < 1$ and positive for $x > 1$. Thus as x runs from $-\infty$ to ∞, $f(x)$ decreases until x reaches 1 and then increases all $x > 1$. So the minimum of $f(x)$ is achieved for $x = 1$. At $x = 1$, $x^2 - 2x = -1$ and $f(1) = \int_0^{-1} \frac{\mathrm{d}t}{1+t^4}$.

1.3.2.45. *. Solution. Define $G(x) = \displaystyle\int_0^x \sin(\sqrt{t})\,\mathrm{d}t$. By the Fundamental Theorem of Calculus Part 1, $G'(x) = \sin(\sqrt{x})$. Since $F(x) = G(x^2)$, and since $x > 0$, we have

$$F'(x) = 2xG'(x^2) = 2x\sin|x| = 2x\sin x.$$

Thus F increases as x runs from to 0 to π (since $F'(x) > 0$ there) and decreases as x runs from π to 4 (since $F'(x) < 0$ there). Thus F achieves its maximum value at $x = \pi$.

1.3.2.46. *. Solution. The given sum is of the form

$$\lim_{n\to\infty}\sum_{j=1}^n \frac{\pi}{n}\sin\left(\frac{j\pi}{n}\right) = \lim_{n\to\infty}\sum_{j=1}^n f(x_j^*)\Delta x$$

with $\Delta x = \frac{\pi}{n}$, $x_j^* = \frac{j\pi}{n}$ and $f(x) = \sin(x)$. Since $x_0^* = 0$ and $x_n^* = \pi$, the right hand side is the definition (using the right Riemann sum) of

$$\int_0^\pi f(x)\,\mathrm{d}x = \int_0^\pi \sin(x)\,\mathrm{d}x = [-\cos(x)]_0^\pi = 2$$

where we evaluate the definite integral using the Fundamental Theorem of Calculus Part 2.

1.3.2.47. *. Solution. The given sum is of the form

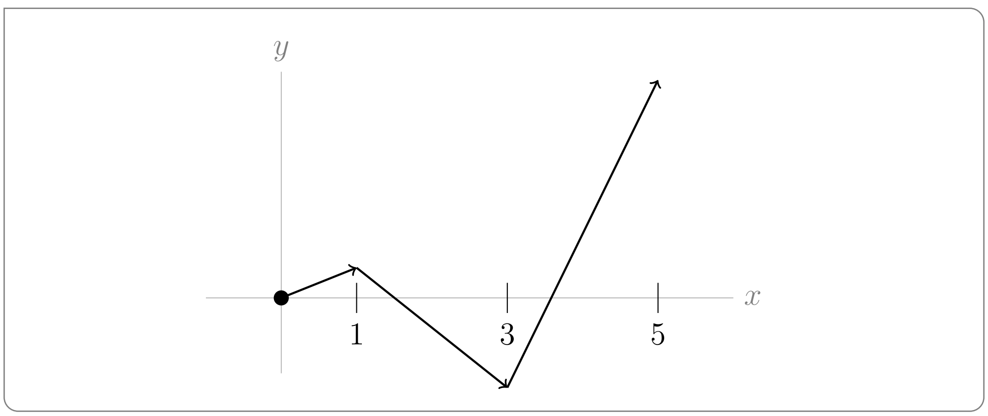
$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\frac{1}{1+\frac{j}{n}} = \lim_{n\to\infty}\sum_{j=1}^{n} f(x_j)\Delta x$$

with $\Delta x = \frac{1}{n}$, $x_j = \frac{j}{n}$ and $f(x) = \frac{1}{1+x}$. The right hand side is the definition (using the right Riemann sum) of

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 \frac{1}{1+x}\,\mathrm{d}x = \log|1+x|\Big|_0^1 = \log 2$$

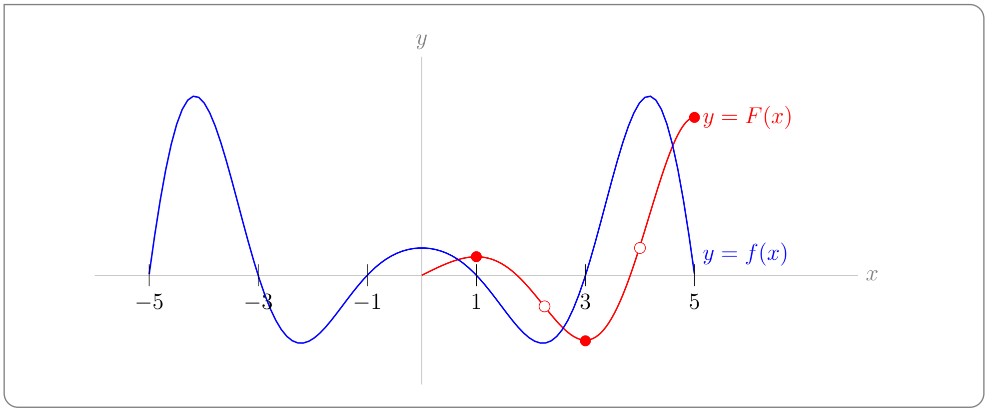
1.3.2.48. Solution.

- $\mathbf{F(x)}$, $\mathbf{x \geq 0}$: We learned quite a lot last semester about curve sketching. We can use those techniques here. We have to be quite careful about the sign of x, though. We can only directly apply the Fundamental Theorem of Calculus Part 1 (as it's written in your text) when $x \geq 0$. So first, let's graph the right-hand portion. Notice $f(x)$ has even symmetry–so, if we know one half of $F(x)$, we should be able to figure out the other half with relative ease.
  - $F(0) = \displaystyle\int_0^0 f(t)\mathrm{d}t = 0$ (so, $F(x)$ passes through the origin)
  - Using the Fundamental Theorem of Calculus Part 1, $F'(x) > 0$ when $0 < x < 1$ and when $3 < x < 5$; $F'(x) < 0$ when $1 < x < 3$. So, $F(x)$ is decreasing from 1 to 3, and increasing from 0 to 1 and also from 3 to 5. That gives us a skeleton to work with.



    We get the relative sizes of the maxes and mins by eyeballing the area under $y = f(t)$. The first lobe (from $x = 0$ to $x = 1$ has a small positive area, so $F(1)$ is a small positive number. The next lobe (from $x = 1$ to $x = 3$) has a larger absolute area than the first, so $F(3)$ is negative. Indeed, the second lobe seems to have more than twice the area of the first, so $|F(3)|$ should be larger than $F(1)$. The third lobe is larger still, and even after subtracting the area of the second lobe it looks much larger than the first or second lobe, so $|F(3)| < F(5)$.
  - We can use $F''(x)$ to get the concavity of $F(x)$. Note $F''(x) = f'(x)$. We observe $f(x)$ is decreasing on (roughly) $(0, 2.5)$ and $(4, 5)$, so $F(x)$ is

concave down on those intervals. Further, $f(x)$ is increasing on (roughly) $(2.5, 4)$, so $F(x)$ is concave up there, and has inflection points at about $x = 2.5$ and $x = 4$.

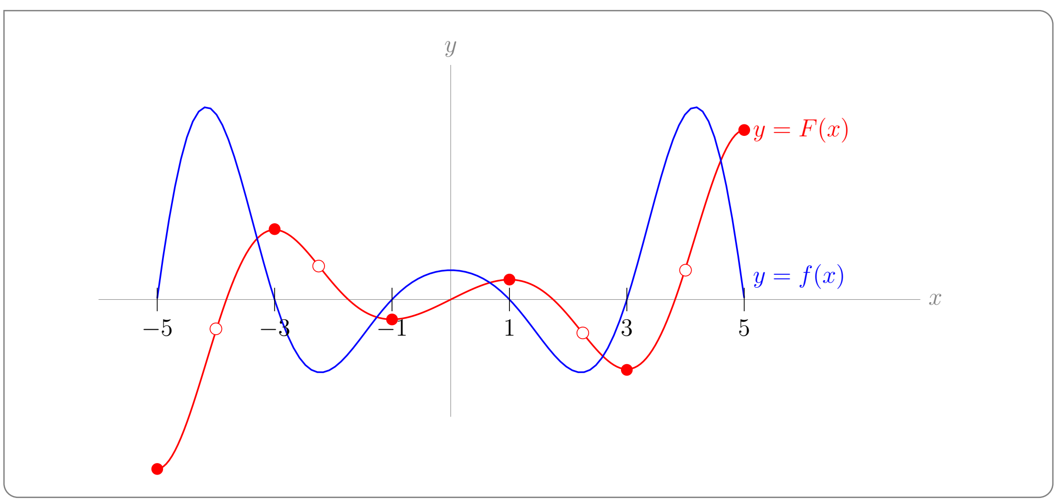


In the sketch above, closed dots are extrema, and open dots are inflection points.

- $\mathbf{F(x)}$, $\mathbf{x < 0}$: Now we can consider the left half of the graph. If you stare at it long enough, you might convince yourself that $F(x)$ is an odd function. We can also show this with the following calculation:

$$\begin{aligned} F(-x) &= \int_0^{-x} f(t)\mathrm{d}t = \int_x^0 f(t)\mathrm{d}t \\ &\qquad \text{as in Example 1.2.10, since } f(t) \text{ is even,} \\ &= -\int_0^x f(t)\mathrm{d}t \\ &= -F(x) \end{aligned}$$

Knowing that $F(x)$ is odd allows us to finish our sketch.



1.3.2.49. *. Solution. (a) Using the product rule, followed by the chain rule, followed by the Fundamental Theorem of Calculus Part 1,

$$\begin{aligned}
f'(x) &= 3x^2 \int_0^{x^3+1} e^{t^3}\,\mathrm{d}t + x^3 \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^3+1} e^{t^3}\,\mathrm{d}t \\
&= 3x^2 \int_0^{x^3+1} e^{t^3}\,\mathrm{d}t + x^3\ \left[3x^2\right]\left[\frac{\mathrm{d}}{\mathrm{d}y}\int_0^{y} e^{t^3}\,\mathrm{d}t\right]_{y=x^3+1} \\
&= 3x^2 \int_0^{x^3+1} e^{t^3}\,\mathrm{d}t + x^3\ \left[3x^2\right]\left[e^{y^3}\right]_{y=x^3+1} \\
&= 3x^2 \int_0^{x^3+1} e^{t^3}\,\mathrm{d}t + x^3\ \left[3x^2\right]e^{(x^3+1)^3} \\
&= 3x^2 \int_0^{x^3+1} e^{t^3}\,\mathrm{d}t + 3x^5 e^{(x^3+1)^3}
\end{aligned}$$

(b) In general, the equation of the tangent line to the graph of $y = f(x)$ at $x = a$ is

$$y = f(a) + f'(a)\,(x-a)$$

Substituting in the given $f(x)$ and $a = -1$:

$$\begin{aligned}
f(a) = f(-1) &= (-1)^3 \int_0^0 e^{t^3}\mathrm{d}t = 0 \\
f'(a) = f'(-1) &= 3(-1)^2 \int_0^0 e^{t^3}\mathrm{d}t + 3(-1)^5 e^0 \\
&= 0 - 3 = -3 \\
(x-a) = x - (-1) &= x + 1
\end{aligned}$$

So, the equation of the tangent line is

$$y = -3(x+1)\ .$$

1.3.2.50. Solution. Recall that "$+C$" means that we can add any constant to the function. Since $\tan^2 x = \sec^2 x - 1$, Students A and B have equivalent answers: they only differ by a constant.
So, if one is right, both are right; if one is wrong, both are wrong. We check Student A's work:

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\tan^2 x + x + C\} = \frac{\mathrm{d}}{\mathrm{d}x}\{\tan^2 x\} + 1 + 0 = f(x) - 1 + 1 = f(x)$$

So, Student A's answer is indeed an anditerivative of $f(x)$. Therefore, *both students* ended up with the correct answer.
Remark: it is a frequent occurrence that equivalent answers might look quite different. As you are comparing your work to others', this is a good thing to keep in mind!

1.3.2.51. Solution.

a When $x = 3$,

$$F(3) = \int_0^3 3^3 \sin(t)\mathrm{d}t = 27 \int_0^3 \sin t\mathrm{d}t$$

Using the Fundamental Theorem of Calculus Part 2,

$$\begin{aligned} &= 27\left[-\cos t\right]_{t=0}^{t=3} = 27\left[-\cos 3 - (-\cos 0)\right] \\ &= 27(1 - \cos 3) \end{aligned}$$

b Since the integration is with respect to t, the x^3 term can be moved outside the integral. That is: for the purposes of the integral, x^3 is a constant (although for the purposes of the derivative, it certainly is not).

$$F(x) = \int_0^x x^3 \sin(t)\mathrm{d}t = x^3 \int_0^x \sin(t)\mathrm{d}t$$

Using the product rule and the Fundamental Theorem of Calculus Part 1,

$$\begin{aligned} F'(x) &= x^3 \cdot \sin(x) + 3x^2 \int_0^x \sin(t)\mathrm{d}t \\ &= x^3 \sin(x) + 3x^2\left[-\cos(t)\right]_{t=0}^{t=x} \\ &= x^3 \sin(x) + 3x^2[-\cos(x) - (-\cos(0))] \\ &= x^3 \sin(x) + 3x^2[1 - \cos(x)] \end{aligned}$$

Remark: Since x and t play different roles in our problem, it's crucial that they have different names. This is one reason why we should avoid the common mistake of writing $\int_a^x f(x)\mathrm{d}x$ when we mean $\int_a^x f(t)\mathrm{d}t$.

1.3.2.52. Solution. If $F(x)$ is even, then $f(x)$ is odd (by the result of Question 1.2.3.20 in Section 1.2). So, $F(x)$ can only be even if $f(x)$ is both even and odd. By the result in Question 1.2.3.19, Section 1.2, this means $F(x)$ is only even if $f(x) = 0$ for all x. Note if $f(x) = 0$, then $F(x)$ is a constant function. So, it is certainly even, and it might be odd as well if $F(x) = f(x) = 0$.
Therefore, if $f(x) \neq 0$ for some x, then $F(x)$ is not even. It could be odd, or it could be neither even nor odd. We can come up with examples of both types: if $f(x) = 1$, then $F(x) = x$ is an odd antiderivative, and $F(x) = x + 1$ is an antiderivative that is neither even nor odd.
Interestingly, the antiderivative of an odd function is always even. The proof is a little beyond what we might ask you, but is given below for completeness. The proof goes like this: First, we'll show that if $g(x)$ is odd, then there is *some* antiderivative of $g(x)$ that is even. Then, we'll show that *every* antiderivative of $g(x)$ is even.
So, suppose $g(x)$ is odd and define $G(x) = \displaystyle\int_0^x g(t)\mathrm{d}t$. By the Fundamental Theorem of Calculus Part 1, $G'(x) = g(x)$, so $G(x)$ is an antiderivative of $g(x)$. Since $g(x)$ is

odd, for any $x \geq 0$, the net signed area under the curve along $[0, x]$ is the *negative* of the net signed area under the curve along $[-x, 0]$. So,

$$\begin{aligned} \int_0^x g(t)\mathrm{d}t &= -\int_{-x}^0 g(t)\mathrm{d}t && \text{(See Example 1.2.11)} \\ &= \int_0^{-x} g(t)\mathrm{d}t \end{aligned}$$

By the definition of $G(x)$,

$$G(x) = G(-x)$$

That is, $G(x)$ is even. We've shown that there exists *some* antiderivative of $g(x)$ that is even; it remains to show that *all of them* are even.
Recall that every antiderivative of $g(x)$ differs from $G(x)$ by some constant. So, any antiderivative of $g(x)$ can be written as $G(x)+C$, and $G(-x)+C = G(x)+C$. So, every antiderivative of an odd function is even.

1.4 · Substitution
1.4.2 · Exercises

Exercises — Stage 1
1.4.2.1. Solution. (a) This is true: it is an application of Theorem 1.4.2 with $f(x) = \sin x$ and $u(x) = e^x$.
(b) This is false: the upper limit of integration is incorrect. Using Theorem 1.4.6, the correct form is

$$\begin{aligned} \int_0^1 \sin(e^x) \cdot e^x \mathrm{d}x = \int_1^e \sin(u)\mathrm{d}u &= -\cos(e) + \cos(1) \\ &= \cos(1) - \cos(e). \end{aligned}$$

Alternately, we can use the Fundamental Theorem of Calculus Part 2, and our answer from (a):

$$\int_0^1 \sin(e^x) \cdot e^x \mathrm{d}x = \left[-\cos(e^x) + C\right]_0^1 = \cos(1) - \cos(e)\ .$$

1.4.2.2. Solution. The reasoning is not sound: when we do a substitution, we need to take care of the differential ($\mathrm{d}x$). Remember the method of substitution comes from the chain rule: there should be a function *and its derivative*. Here's the way to do it:

Problem: Evaluate $\displaystyle\int (2x+1)^2 \mathrm{d}x$.

Work: We use the substitution $u = 2x+1$. Then $\mathrm{d}u = 2\mathrm{d}x$, so $\mathrm{d}x = \frac{1}{2}\mathrm{d}u$:

$$\int (2x+1)^2 \mathrm{d}x = \int u^2 \cdot \frac{1}{2}\mathrm{d}u$$

$$= \frac{1}{6}u^3 + C$$
$$= \frac{1}{6}(2x+1)^3 + C$$

1.4.2.3. Solution. The problem is with the limits of integration, as in Question 1. Here's how it ought to go:

Problem: Evaluate $\displaystyle\int_1^\pi \frac{\log(\log t)}{t}\mathrm{d}t$.

Work: We use the substitution $u = \log t$, so $\mathrm{d}u = \frac{1}{t}\mathrm{d}t$. When $t = 1$, we have $u = \log 1 = 0$ and when $t = \pi$, we have $u = \log(\pi)$. Then:

$$\begin{aligned}\int_1^\pi \frac{\cos(\log t)}{t}\mathrm{d}t &= \int_{\log 1}^{\log(\pi)} \cos(u)\mathrm{d}u \\ &= \int_0^{\log(\pi)} \cos(u)\mathrm{d}u \\ &= \sin(\log(\pi)) - \sin(0) = \sin(\log(\pi)).\end{aligned}$$

1.4.2.4. Solution. Perhaps shorter ways exist, but the reasoning here is valid.

Problem: Evaluate $\displaystyle\int_0^{\pi/4} x\tan(x^2)\mathrm{d}x$.

Work: We begin with the substitution $u = x^2$, $\mathrm{d}u = 2x\mathrm{d}x$: If $u = x^2$, then $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$, so indeed $\mathrm{d}u = 2x\mathrm{d}x$.

$$\begin{aligned}\int_0^{\pi/4} x\tan(x^2)\mathrm{d}x &= \int_0^{\pi/4} \frac{1}{2}\tan(x^2)\cdot 2x\mathrm{d}x && \text{algebra} \\ &= \int_0^{\pi^2/16} \frac{1}{2}\tan u\mathrm{d}u\end{aligned}$$

Note that every piece was changed from x to u: integrand, differential, limits. So

$$\int_0^{\pi/4} x\tan(x^2)\mathrm{d}x = \frac{1}{2}\int_0^{\pi^2/16} \frac{\sin u}{\cos u}\mathrm{d}u$$

since $\tan u = \frac{\sin u}{\cos u}$. Now we use the substitution $v = \cos u$, $\mathrm{d}v = -\sin u\mathrm{d}u$:

$$\frac{1}{2}\int_0^{\pi^2/16} \frac{\sin u}{\cos u}\mathrm{d}u = \frac{1}{2}\int_{\cos 0}^{\cos(\pi^2/16)} -\frac{1}{v}\mathrm{d}v$$

Note that every piece was changed from u to v: integrand, differential, limits. So

$$\int_0^{\pi/4} x\tan(x^2)\mathrm{d}x = -\frac{1}{2}\int_1^{\cos(\pi^2/16)} \frac{1}{v}\mathrm{d}v$$

$$\text{since } \cos(0) = 1$$

$$= -\frac{1}{2}\left[\log|v|\right]_{1}^{\cos(\pi^2/16)} \qquad \text{FTC Part 2}$$

$$= -\frac{1}{2}\left(\log\left(\cos(\pi^2/16)\right) - \log(1)\right)$$

$$= -\frac{1}{2}\log\left(\cos(\pi^2/16)\right) \qquad \text{since } \log(1) = 0$$

1.4.2.5. *. Solution. We substitute:

$$\begin{aligned} u &= \sin x, \\ \mathrm{d}u &= \cos x\,\mathrm{d}x, \\ \cos x &= \sqrt{1-\sin^2 x} = \sqrt{1-u^2}, \\ \mathrm{d}x &= \frac{\mathrm{d}u}{\cos x} = \frac{\mathrm{d}u}{\sqrt{1-u^2}} \\ u(0) &= \sin 0 = 0 \\ u\left(\frac{\pi}{2}\right) &= \sin\left(\frac{\pi}{2}\right) = 1 \end{aligned}$$

So,

$$\int_{x=0}^{x=\pi/2} f(\sin x)\,\mathrm{d}x = \int_{u=0}^{u=1} f(u)\,\frac{\mathrm{d}u}{\sqrt{1-u^2}}$$

Because the denominator $\sqrt{1-u^2}$ vanishes when $u=1$, this is what is known as an improper integral. Improper integrals will be discussed in Section 1.12.

1.4.2.6. Solution. Using the chain rule, we see that

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\} = f'(g(x))g'(x)$$

So, $f(g(x))$ is an antiderivative of $f'(g(x))g'(x)$. All antiderivatives of $f'(g(x))g'(x)$ differ by only a constant, so:

$$\begin{aligned} \int f'(g(x))g'(x)\mathrm{d}x - f(g(x)) &= f(g(x)) + C - f(g(x)) \\ &= C \end{aligned}$$

That is, our expression simplifies to some constant C.
Remark: since

$$\int f'(g(x))g'(x)\mathrm{d}t - f(g(x)) = C$$

we conclude

$$\int f'(g(x))g'(x)\mathrm{d}t = f(g(x)) + C$$

which is precisely how we perform substitution on integrals.

Exercises — Stage 2

1.4.2.7. *. Solution. We write $u(x) = e^{x^2}$ and find $\mathrm{d}u = u'(x)\,\mathrm{d}x = 2xe^{x^2}\mathrm{d}x$. Note that $u(1) = e^{1^2} = e$ when $x = 1$, and $u(0) = e^{0^2} = 1$ when $x = 0$. Therefore:

$$\begin{aligned}
\int_0^1 xe^{x^2}\cos(e^{x^2})\,\mathrm{d}x &= \frac{1}{2}\int_{x=0}^{x=1}\cos(u(x))u'(x)\,\mathrm{d}x \\
&= \frac{1}{2}\int_{u=1}^{u=e}\cos(u)\,\mathrm{d}u \\
&= \frac{1}{2}\Big[\sin(u)\Big]_1^e = \frac{1}{2}\big(\sin(e) - \sin(1)\big).
\end{aligned}$$

1.4.2.8. *. Solution. Substituting $y = x^3$, $\mathrm{d}y = 3x^2\mathrm{d}x$:

$$\int_1^2 x^2 f(x^3)\,\mathrm{d}x = \frac{1}{3}\int_1^8 f(y)\,\mathrm{d}y = \frac{1}{3}$$

1.4.2.9. *. Solution. Setting $u = x^3 + 1$, we have $\mathrm{d}u = 3x^2\,\mathrm{d}x$ and so

$$\begin{aligned}
\int \frac{x^2\,\mathrm{d}x}{(x^3+1)^{101}} &= \int \frac{\mathrm{d}u/3}{u^{101}} \\
&= \frac{1}{3}\int u^{-101}\mathrm{d}u \\
&= \frac{1}{3}\cdot\frac{u^{-100}}{-100} \\
&= -\frac{1}{3\times 100u^{100}} + C \\
&= -\frac{1}{300(x^3+1)^{100}} + C
\end{aligned}$$

1.4.2.10. *. Solution. Setting $u = \log x$, we have $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$ and so

$$\int_e^{e^4}\frac{\mathrm{d}x}{x\cdot\log x} = \int_{x=e}^{x=e^4}\frac{1}{\log x}\cdot\frac{1}{x}\,\mathrm{d}x = \int_{u=1}^{u=4}\frac{1}{u}\,\mathrm{d}u,$$

since $u = \log(e) = 1$ when $x = e$ and $u = \log(e^4) = 4$ when $x = e^4$. Then, by the Fundamental Theorem of Calculus Part 2,

$$\int_1^4 \frac{1}{u}\,\mathrm{d}u = \Big[\log|u|\Big]_1^4 = \log 4 - \log 1 = \log 4.$$

1.4.2.11. *. Solution. Setting $u = 1 + \sin x$, we have $\mathrm{d}u = \cos x\mathrm{d}x$ and so

$$\int_0^{\pi/2}\frac{\cos x}{1+\sin x}\mathrm{d}x = \int_{x=0}^{x=\pi/2}\frac{1}{1+\sin x}\cos x\mathrm{d}x = \int_{u=1}^{u=2}\frac{\mathrm{d}u}{u}$$

since $u = 1 + \sin 0 = 1$ when $x = 0$ and $u = 1 + \sin(\pi/2) = 2$ when $x = \pi/2$. Then, by the Fundamental Theorem of Calculus Part 2,

$$\int_{u=1}^{u=2} \frac{\mathrm{d}u}{u} = \Big[\log|u|\Big]_1^2 = \log 2$$

1.4.2.12. *. Solution. Setting $u = \sin x$, we have $\mathrm{d}u = \cos x \mathrm{d}x$ and so

$$\begin{aligned}\int_0^{\pi/2} \cos x \cdot (1+\sin^2 x)\mathrm{d}x &= \int_{x=0}^{x=\pi/2} (1+\sin^2 x)\cdot \cos x \mathrm{d}x\\ &= \int_{u=0}^{u=1} (1+u^2)\,\mathrm{d}u,\end{aligned}$$

since $u = \sin 0 = 0$ when $x = 0$ and $u = \sin(\pi/2) = 1$ when $x = \pi/2$. Then, by the Fundamental Theorem of Calculus Part 2,

$$\int_0^1 (1+u^2)\,\mathrm{d}u = \left[u + \frac{u^3}{3}\right]_0^1 = \left(1+\frac{1}{3}\right) - 0 = \frac{4}{3}.$$

1.4.2.13. *. Solution. Substituting $t = x^2 - x$, $\mathrm{d}t = (2x-1)\,\mathrm{d}x$ and noting that $t = 0$ when $x = 1$ and $t = 6$ when $x = 3$,

$$\int_1^3 (2x-1)e^{x^2-x}\mathrm{d}x = \int_0^6 e^t\ \mathrm{d}t = \left[e^t\right]_0^6 = e^6 - 1$$

1.4.2.14. *. Solution. We use the substitution $u = 4 - x^2$, for which $\mathrm{d}u = -2x\,\mathrm{d}x$:

$$\begin{aligned}\int \frac{x^2-4}{\sqrt{4-x^2}}\,x\,\mathrm{d}x &= \int \frac{1}{2}\cdot\frac{4-x^2}{\sqrt{4-x^2}}(-2x)\,\mathrm{d}x\\ &= \frac{1}{2}\int \frac{u}{\sqrt{u}}\,\mathrm{d}u\\ &= \frac{1}{2}\int \sqrt{u}\,\mathrm{d}u\\ &= \frac{1}{2}\frac{u^{3/2}}{3/2} + C\\ &= \frac{1}{3}(4-x^2)^{3/2} + C\end{aligned}$$

1.4.2.15. Solution.

- Solution 1: If we let $u = \sqrt{\log x}$, then $\mathrm{d}u = \dfrac{1}{2x\sqrt{\log x}}\mathrm{d}x$, and:

$$\int \frac{e^{\sqrt{\log x}}}{2x\sqrt{\log x}}\ \mathrm{d}x = \int e^u\ \mathrm{d}u = e^u + C = e^{\sqrt{\log x}} + C$$

- Solution 2: In Solution 1, we made a pretty slick choice. We might have tried to work with something a little less convenient. For example, it's not unnatural to think that $u = \log x$, $\mathrm{d}u = \dfrac{1}{x}\mathrm{d}x$ would be a good choice. In that case:

$$\int \frac{e^{\sqrt{\log x}}}{2x\sqrt{\log x}}\,\mathrm{d}x = \int \frac{e^{\sqrt{u}}}{2\sqrt{u}}\mathrm{d}u$$

Now, we should be able to see that $w = \sqrt{u}$, $\mathrm{d}w = \frac{1}{2\sqrt{u}}\mathrm{d}u$ is a good choice:

$$\begin{aligned}\int \frac{e^{\sqrt{u}}}{2\sqrt{u}}\,\mathrm{d}u &= \int e^{w}\,\mathrm{d}w\\ &= e^{\sqrt{u}} + C\\ &= e^{\sqrt{\log x}} + C\end{aligned}$$

Exercises — Stage 3

1.4.2.16. *. Solution.

- ***The straightforward method:*** We use the substitution $u = x^2$, for which $\mathrm{d}u = 2x\,\mathrm{d}x$, and note that $u = 4$ for both $x = 2$ and $x = -2$:

$$\int_{-2}^{2} xe^{x^2}\,\mathrm{d}x = \int_{-2}^{2} \frac{1}{2}e^{x^2}\,2x\mathrm{d}x = \int_{4}^{4} \frac{1}{2}e^{u}\,\mathrm{d}u = 0$$

- ***The slightly sneaky method:*** We note that $\frac{\mathrm{d}}{\mathrm{d}x}\left\{e^{x^2}\right\} = 2x\,e^{x^2}$, so that $\frac{1}{2}e^{x^2}$ is a antiderivative for the integrand xe^{x^2}. So

$$\int_{-2}^{2} xe^{x^2}\,\mathrm{d}x = \left[\frac{1}{2}e^{x^2}\right]_{-2}^{2} = \frac{1}{2}e^4 - \frac{1}{2}e^4 = 0$$

- ***The really sneaky method***: The integrand $f(x) = xe^{x^2}$ is an odd function (meaning that $f(-x) = -f(x)$). So by Theorem 1.2.12 every integral of the form $\int_{-a}^{a} xe^{x^2}\,\mathrm{d}x$ is zero.

1.4.2.17. *. Solution. The given sum is of the form

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{j}{n^2}\sin\left(1+\frac{j^2}{n^2}\right) = \lim_{n\to\infty}\sum_{j=1}^{n} f(x_j^*)\Delta x$$

with $\Delta x = \frac{1}{n}$, $x_j^* = \frac{j}{n}$ and $f(x) = x\sin(1+x^2)$. Since $x_0^* = 0$ and $x_n^* = 1$, the right hand side is the definition (using the right Riemann sum) of

$$\begin{aligned}\int_0^1 f(x)\,\mathrm{d}x &= \int_0^1 x\sin(1+x^2)\,\mathrm{d}x\\ &= \frac{1}{2}\int_1^2 \sin(y)\,\mathrm{d}y \qquad \text{with } y = 1+x^2,\ \mathrm{d}y = 2x\,\mathrm{d}x\end{aligned}$$

$$= \frac{1}{2}\Big[- \cos(y) \Big]_{y=1}^{y=2}$$
$$= \frac{1}{2}[\cos 1 - \cos 2]$$

Using a calculator, we see this is close to 0.478.

1.4.2.18. Solution. Often, the denominator of a function is a good guess for the substitution. So, let's try setting $w = u^2 + 1$. Then $dw = 2udu$:

$$\int_0^1 \frac{u^3}{u^2+1}\mathrm{d}u = \frac{1}{2}\int_0^1 \frac{u^2}{u^2+1}\ 2u\mathrm{d}u$$

The numerator now is u^2, and looking at our substitution, we see $u^2 = w - 1$:

$$\begin{aligned}
&= \frac{1}{2}\int_1^2 \frac{w-1}{w}\ \mathrm{d}w \\
&= \frac{1}{2}\int_1^2 \left(1 - \frac{1}{w}\right)\mathrm{d}w \\
&= \frac{1}{2}\left[w - \log|w|\right]_{w=1}^{w=2} \\
&= \frac{1}{2}\left(2 - \log 2 - 1\right) = \frac{1}{2} - \frac{1}{2}\log 2
\end{aligned}$$

1.4.2.19. Solution. The only thing we really have to work with is a tangent, so it's worth considering what would happen if we substituted $u = \tan\theta$. Then $\mathrm{d}u = \sec^2\theta\mathrm{d}\theta$. This doesn't show up in the integrand as it's written, but we can try and bring it out by using the identity $\tan^2 = \sec^2\theta - 1$:

$$\begin{aligned}
\int \tan^3\theta\ \mathrm{d}\theta &= \int \tan\theta \cdot \tan^2\ \theta\mathrm{d}\theta \\
&= \int \tan\theta \cdot \left(\sec^2\ \theta - 1\right)\mathrm{d}\theta \\
&= \int \tan\theta \cdot \sec^2\ \theta\mathrm{d}\theta - \int \tan\theta\mathrm{d}\theta
\end{aligned}$$

In Example 1.4.17, we learned $\int \tan\theta\mathrm{d}\theta = \log|\sec\theta| + C$

$$\begin{aligned}
&= \int u\ \mathrm{d}u - \log|\sec\theta| + C \\
&= \frac{1}{2}u^2 - \log|\sec\theta| + C \\
&= \frac{1}{2}\tan^2\theta - \log|\sec\theta| + C
\end{aligned}$$

1.4.2.20. Solution. At first glance, it's not clear what substitution to use. If we try the denominator, $u = e^x + e^{-x}$, then $\mathrm{d}u = (e^x - e^{-x})\mathrm{d}x$, but it's not clear how to make this work with our integral. So, we can try something else.
If we want to tidy things up, we might think to take $u = e^x$ as a substitution. Then

$\mathrm{d}u = e^x \mathrm{d}x$, so we need an e^x in the numerator. That can be arranged.

$$\begin{aligned}\int \frac{1}{e^x + e^{-x}} \cdot \left(\frac{e^x}{e^x}\right) \mathrm{d}x &= \int \frac{e^x}{(e^x)^2 + 1}\,\mathrm{d}x \\ &= \int \frac{1}{u^2+1}\,\mathrm{d}u \\ &= \arctan(u) + C \\ &= \arctan(e^x) + C\end{aligned}$$

1.4.2.21. Solution. We often like to take the "inside" function as our substitution, in this case $u = 1 - x^2$, so $\mathrm{d}u = -2x\mathrm{d}x$. This takes care of *part* of the integral:

$$\int_0^1 (1-2x)\sqrt{1-x^2}\mathrm{d}x = \int_0^1 \sqrt{1-x^2}\mathrm{d}x + \int_0^1 (-2x)\sqrt{1-x^2}\,\mathrm{d}x$$

The left integral is tough to solve with substitution, but luckily we don't have to–it's the area of a quarter of a circle of radius 1.

$$\begin{aligned} &= \frac{\pi}{4} + \int_1^0 \sqrt{u}\,\mathrm{d}u \\ &= \frac{\pi}{4} + \left[\frac{2}{3}u^{3/2}\right]_{u=1}^{u=0} \\ &= \frac{\pi}{4} + 0 - \frac{2}{3} = \frac{\pi}{4} - \frac{2}{3}\end{aligned}$$

1.4.2.22. Solution.

- Solution 1: We often find it useful to take "inside" functions as our substitutions, so let's try $u = \cos x$, $\mathrm{d}u = -\sin x\mathrm{d}x$. In order to dig up a sine, we use the identity $\tan x = \dfrac{\sin x}{\cos x}$:

$$\begin{aligned}\int \tan x \cdot \log(\cos x)\,\mathrm{d}x &= -\int \frac{-\sin x}{\cos x} \cdot \log(\cos x)\,\mathrm{d}x \\ &= -\int \frac{1}{u}\log(u)\mathrm{d}u\end{aligned}$$

  Now, it is convenient to let $w = \log u$, $\mathrm{d}w = \frac{1}{u}\mathrm{d}u$:

$$\begin{aligned}-\int \frac{1}{u}\log(u)\mathrm{d}u &= -\int w\,\mathrm{d}w \\ &= -\frac{1}{2}w^2 + C \\ &= -\frac{1}{2}(\log u)^2 + C \\ &= -\frac{1}{2}(\log(\cos x))^2 + C\end{aligned}$$

- Solution 2: We might guess that it's useful to have $u = \log(\cos x)$,

$\mathrm{d}u = \dfrac{-\sin x}{\cos x}\mathrm{d}x = -\tan x\mathrm{d}x$:

$$\begin{aligned}\int \tan x \cdot \log(\cos x)\mathrm{d}x &= -\int -\tan x \cdot \log(\cos x)\mathrm{d}x \\ &= -\int u\ \mathrm{d}u \\ &= -\frac{1}{2}u^2 + C \\ &= -\frac{1}{2}\left(\log(\cos x)\right)^2 + C\end{aligned}$$

1.4.2.23. *. Solution. The given sum is of the form

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{j}{n^2}\cos\left(\frac{j^2}{n^2}\right) = \lim_{n\to\infty}\sum_{j=1}^{n} f(x_j^*)\Delta x$$

with $\Delta x = \frac{1}{n}$, $x_j^* = \frac{j}{n}$ and $f(x) = x\cos(x^2)$. Since $x_0^* = 0$ and $x_n^* = 1$, the right hand side is the definition (using the right Riemann sum) of

$$\begin{aligned}\int_0^1 f(x)\,\mathrm{d}x &= \int_0^1 x\cos(x^2)\,\mathrm{d}x \\ &= \frac{1}{2}\int_0^1 \cos(y)\,\mathrm{d}y \qquad \text{with } y = x^2, \mathrm{d}y = 2x\,\mathrm{d}x \\ &= \frac{1}{2}\Big[\sin(y)\Big]_0^1 \\ &= \frac{1}{2}\sin 1\end{aligned}$$

1.4.2.24. *. Solution. The given sum is of the form

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{j}{n^2}\sqrt{1+\frac{j^2}{n^2}} = \lim_{n\to\infty}\sum_{j=1}^{n} f(x_j^*)\Delta x$$

with $\Delta x = \frac{1}{n}$, $x_j^* = \frac{j}{n}$ and $f(x) = x\sqrt{1+x^2}$. Since $x_0^* = 0$ and $x_n^* = 1$, the right hand side is the definition (using the right Riemann sum) of

$$\begin{aligned}\int_0^1 f(x)\,\mathrm{d}x &= \int_0^1 x\sqrt{1+x^2}\,\mathrm{d}x \\ &= \frac{1}{2}\int_1^2 \sqrt{y}\,\mathrm{d}y \qquad \text{with } y = 1+x^2, \mathrm{d}y = 2x\,\mathrm{d}x \\ &= \frac{1}{2}\left[\frac{2}{3}y^{3/2}\right]_{y=1}^{y=2} \\ &= \frac{1}{3}[2\sqrt{2}-1]\end{aligned}$$

Using a calculator, we see this is approximately 0.609.

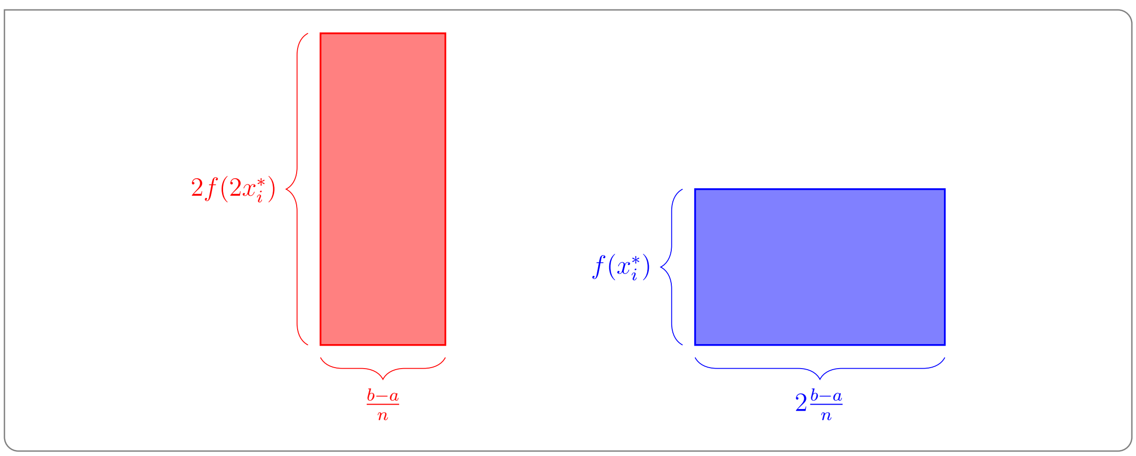
1.4.2.25. Solution. Using the definition of a definite integral with right Riemann sums:

$$\begin{aligned}
\int_a^b 2f(2x)\,\mathrm{d}x &= \lim_{n\to\infty}\sum_{i=1}^{n} \Delta x \cdot 2f(2(a+i\Delta x)) \qquad \Delta x = \frac{b-a}{n}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n} \left(\frac{b-a}{n}\right)\cdot 2f\left(2\left(a+i\left(\frac{b-a}{n}\right)\right)\right)\\
&= \lim_{n\to\infty}\sum_{i=1}^{n} \left(\frac{2b-2a}{n}\right)\cdot f\left(2a+i\left(\frac{2b-2a}{n}\right)\right)\\
\int_{2a}^{2b} f(x)\,\mathrm{d}x &= \lim_{n\to\infty}\sum_{i=1}^{n} \Delta x \cdot f(2a+i\Delta x) \qquad \Delta x = \frac{2b-2a}{n}\\
&= \lim_{n\to\infty}\sum_{i=1}^{n} \left(\frac{2b-2a}{n}\right)\cdot f\left(2a+i\left(\frac{2b-2a}{n}\right)\right)
\end{aligned}$$

Since the Riemann sums are exactly the same,

$$\int_a^b 2f(2x)\,\mathrm{d}x = \int_{2a}^{2b} f(x)\,\mathrm{d}x$$

Looking at the Riemann sum in this way is instructive, because it is very clear why the two integrals should be equal (without using substitution). The rectangles in the first Riemann sum are half as wide, but twice as tall, as the rectangles in the second Riemann sum. So, the two Riemann sums have rectangles of the same area.



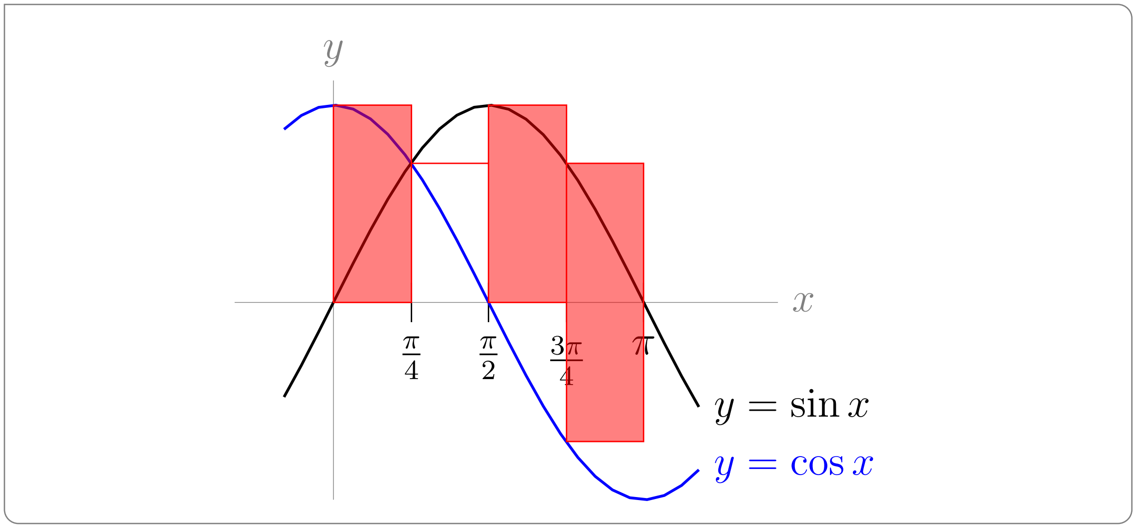
In the integral on the left, the variable is red x and in the integral on the right, the variable is blue x. Red x and blue x are not the same. In fact $2x_i^* = x_i^*$. (Not every substitution corresponds to such a simple picture.)

1.5 · Area between curves

1.5.2 · Exercises

Exercises — Stage 1

1.5.2.1. Solution.



The intervals of our rectangles are $[0, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{\pi}{2}]$, $[\frac{\pi}{2}, \frac{3\pi}{4}]$, and $[\frac{3\pi}{4}, \pi]$. Since we're taking a left Riemann sum, we find the height of the rectangles at the left endpoints of the intervals.

- $x = 0$: The distance from $\cos 0$ to $\sin 0$ is 1, so our first rectangle has height 1.
- $x = \frac{\pi}{4}$: The distance from $\cos\frac{\pi}{4}$ to $\sin\frac{\pi}{4}$ is 0, so our second rectangle has height 0.
- $x = \frac{\pi}{2}$: The distance from $\cos\frac{\pi}{2}$ to $\sin\frac{\pi}{2}$ is 1, so our third rectangle has height 1.
- $x = \frac{3\pi}{4}$: The distance from $\cos\frac{3\pi}{4}$ to $\sin\frac{3\pi}{4}$ is $\sin(3\pi/4) - \cos(3\pi/4) = \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2}$, so our fourth rectangle has height $\sqrt{2}$.

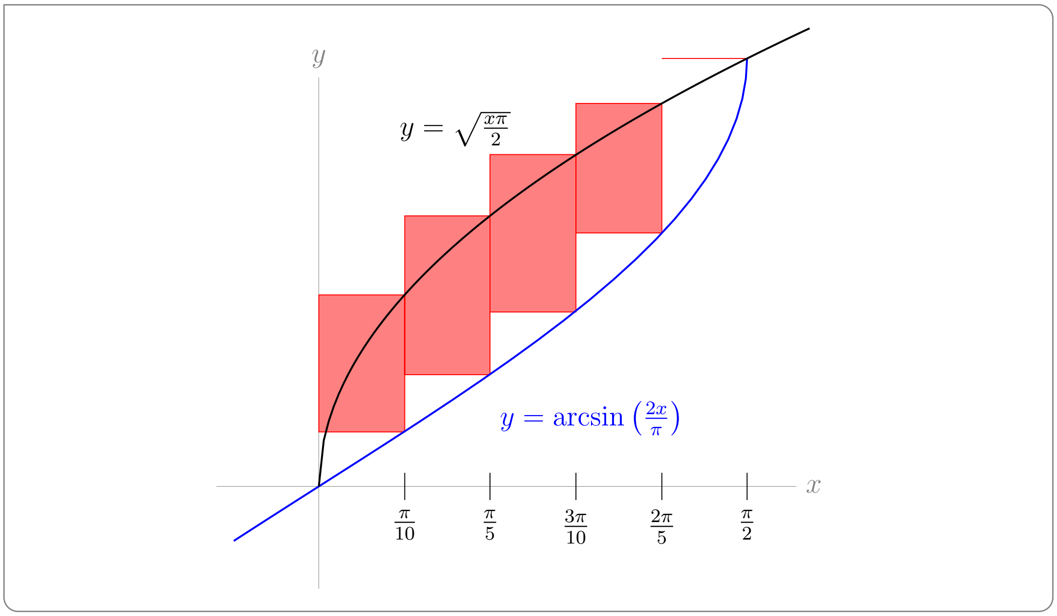
So, our approximation for the area between the two curves is

$$\frac{\pi}{4}\left(1 + 0 + 1 + \sqrt{2}\right) = \frac{\pi}{4}\left(2 + \sqrt{2}\right)$$

1.5.2.2. Solution.

a We are finding the area in the interval from $x = 0$ to $x = \frac{\pi}{2}$. Since we're taking $n = 5$ rectangles, our rectangles cover the following intervals:

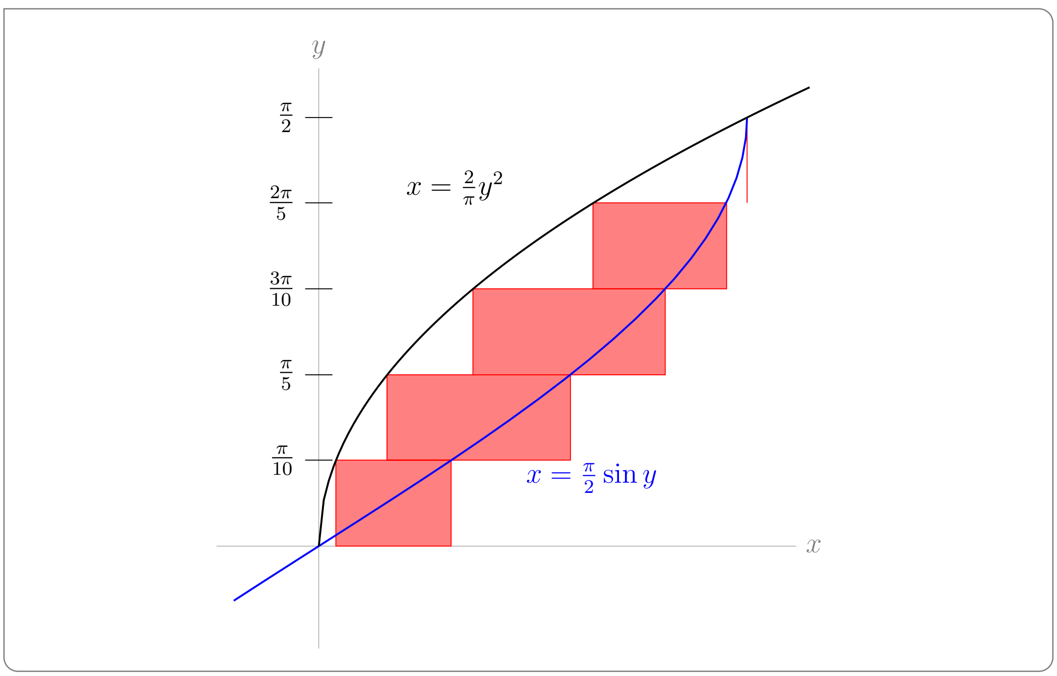
$$\left[0, \frac{\pi}{10}\right], \quad \left[\frac{\pi}{10}, \frac{\pi}{5}\right], \quad \left[\frac{\pi}{5}, \frac{3\pi}{10}\right], \quad \left[\frac{3\pi}{10}, \frac{2\pi}{5}\right], \quad \left[\frac{2\pi}{5}, \frac{\pi}{2}\right].$$



b We are finding the area in the interval from $y = 0$ to $y = \frac{\pi}{2}$. (In general, when we switch from horizontal rectangles to vertical, the limits of integration will change–it's only coincidence that they are the same in this example.) Since we're taking $n = 5$ rectangles, these rectangles cover the following intervals of the y-axis:

$$\left[0, \frac{\pi}{10}\right], \quad \left[\frac{\pi}{10}, \frac{\pi}{5}\right], \quad \left[\frac{\pi}{5}, \frac{3\pi}{10}\right], \quad \left[\frac{3\pi}{10}, \frac{2\pi}{5}\right], \quad \left[\frac{2\pi}{5}, \frac{\pi}{2}\right].$$

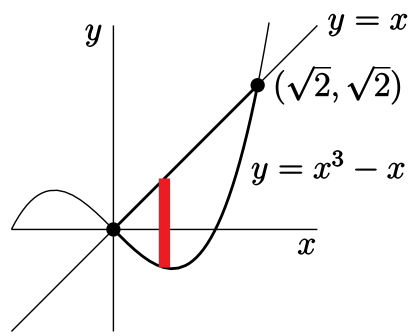
The question doesn't specify which endpoints we're using. Let's use upper endpoints, to match part (a).



1.5.2.3. *. Solution. The curves intersect when $y = x$ and $y = x^3 - x$. To find these points, we set:

$$\begin{aligned} x &= x^3 - x \\ 0 &= x^3 - 2x \\ 0 &= x(x^2 - 2) \\ 0 &= x \quad \text{or} \quad 0 = x^2 - 2 \end{aligned}$$

For $x \geq 0$, the curves intersect at $(0, 0)$ and $(\sqrt{2}, \sqrt{2})$.
A handy observation is that, since both curves are continuous and they do not meet each other between $x = 0$ and $x = \sqrt{2}$, we don't have to worry about dividing our area into two regions: one of the functions is always on the top, and the other is always on the bottom.
Using vertical strips:



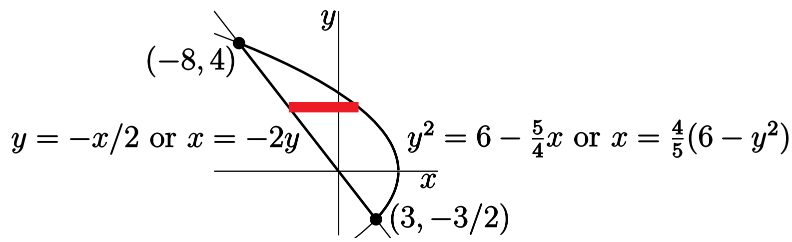
The top and bottom boundaries of the specified region are $y = T(x) = x$ and $y = B(x) = x^3 - x$, respectively. So,

$$\begin{aligned} \text{Area} &= \int_0^{\sqrt{2}} \big[T(x) - B(x)\big]\,\mathrm{d}x = \int_0^{\sqrt{2}} \big[x - (x^3 - x)\big]\,\mathrm{d}x \\ &= \int_0^{\sqrt{2}} \big[2x - x^3\big]\,\mathrm{d}x \end{aligned}$$

1.5.2.4. *. Solution. We need to find where the curves intersect.

$$\begin{aligned} \frac{x^2}{4} &= y^2 = 6 - \frac{5x}{4} \\ \frac{1}{4}x^2 + \frac{5}{4}x - 6 &= 0 \\ x^2 + 5x - 24 &= 0 \\ (x+8)(x-3) &= 0 \\ x = -8, &\quad x = 3 \end{aligned}$$

The curves intersect at $(-8, 4)$ and $(3, -\frac{3}{2})$. Using horizontal strips:



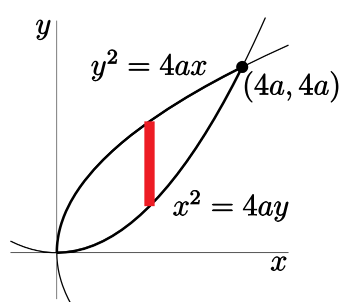
we have

$$\text{Area} = \int_{-3/2}^{4} \left[\frac{4}{5}(6-y^2) + 2y\right] \mathrm{d}y$$

1.5.2.5. *. Solution. If the curves intersect at (x, y), then

$$\begin{aligned}
\left(x^2\right)^2 &= (4a)^2\, y^2 = (4a)^2 4ax \\
x^4 &= (4a)^3 x \\
x^4 - (4a)^3 x &= 0 \\
x(x^3 - (4a)^3) &= 0 \\
x = 0 &\quad \text{or} \quad x^3 = (4a)^3
\end{aligned}$$

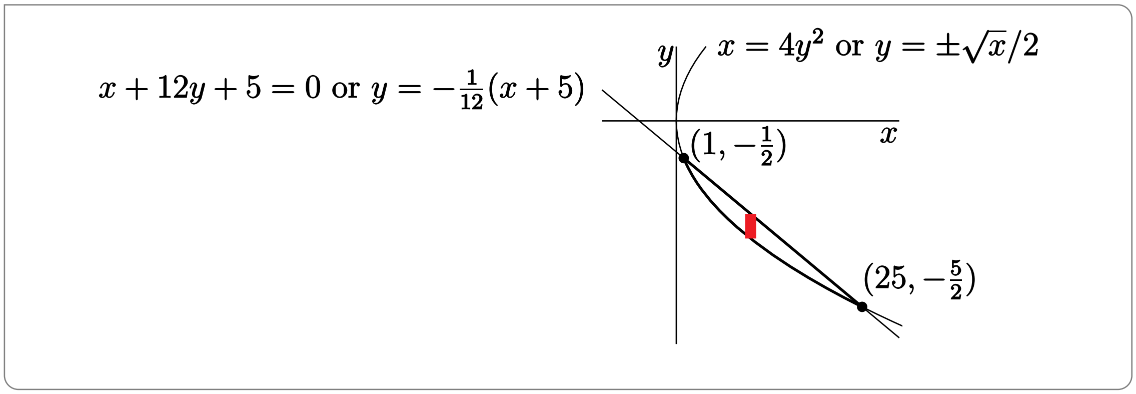
The curves intersect at $(0, 0)$ and $(4a, 4a)$. (It is also possible to find these points by inspection.) Using vertical strips:



We want the y-values of the functions. We write the top function as $y = \sqrt{4ax}$ (we care about the positive square root, not the negative one) and we write the bottom function as $y = \frac{x^2}{4a}$. Then we have

$$\text{Area} = \int_0^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a}\right] \mathrm{d}x$$

1.5.2.6. *. Solution. The curves intersect when $x = 4y^2$ and $0 = 4y^2 + 12y + 5 = (2y+5)(2y+1)$. So, the curves intersect at $(1, -\frac{1}{2})$ and $(25, -\frac{5}{2})$. Using vertical strips:

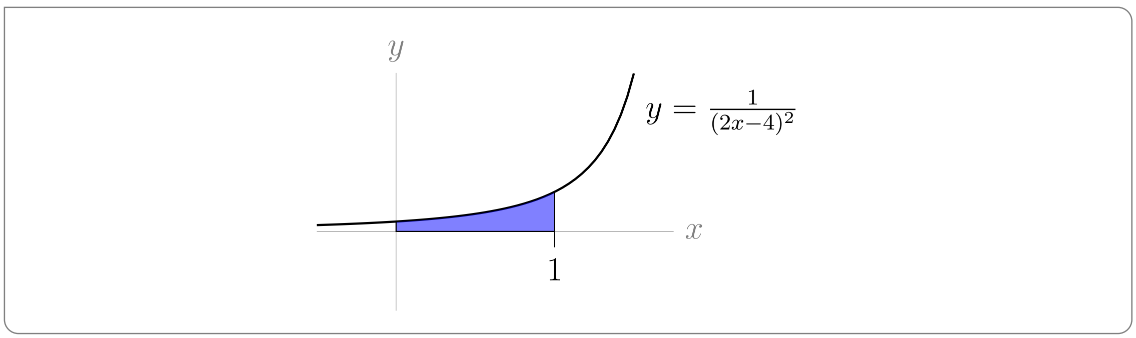


we have

$$\text{Area} = \int_1^{25} \left[-\frac{1}{12}(x+5) + \frac{1}{2}\sqrt{x} \right] \mathrm{d}x$$

Exercises — Stage 2

1.5.2.7. *. Solution.



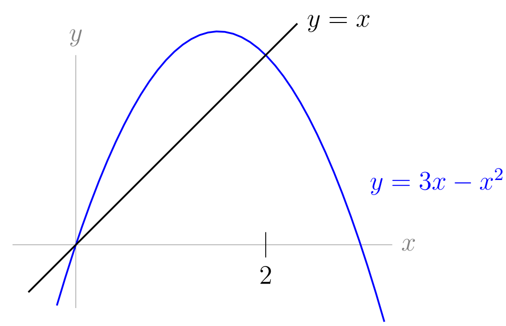
The area between the curve $y = \frac{1}{(2x-4)^2}$ and the x-axis, with x running from $a = 0$ to $b = 1$, is exactly the definite integral of $\frac{1}{(2x-4)^2}$ with limits 0 and 1.

$$\begin{aligned} \text{Area} &= \int_0^1 \frac{\mathrm{d}x}{(2x-4)^2} & u = 2x-4, \quad \mathrm{d}u = 2\mathrm{d}x \\ &= \frac{1}{2}\int_{-4}^{-2} \frac{1}{u^2}\mathrm{d}u = \frac{1}{2}\left[\frac{-1}{u}\right]_{u=-4}^{u=-2} \\ &= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{1}{8} \end{aligned}$$

1.5.2.8. *. Solution. If the curves $y = f(x) = x$ and $y = g(x) = 3x - x^2$ intersect at (x, y), then

$$\begin{aligned} 3x - x^2 &= y = x \\ x^2 - 2x &= 0 \\ x(x-2) &= 0 \\ x = 0 &\quad \text{or} \quad x = 2 \end{aligned}$$

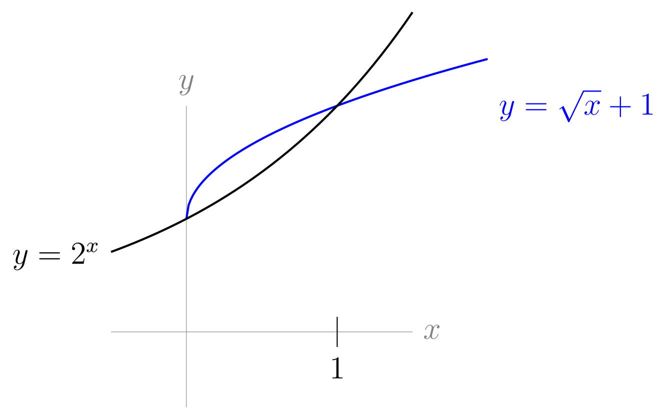
Furthermore, $g(x) - f(x) = 2x - x^2 = x(2-x)$ is positive for all $0 \le x \le 2$. That is, the curve $y = 3x - x^2$ lies above the line $y = x$ for all $0 \le x \le 2$.



We therefore evaluate the integral:

$$\int_0^2 \left[(3x - x^2) - x\right] \mathrm{d}x = \int_0^2 \left[2x - x^2\right] \mathrm{d}x = \left[x^2 - \frac{x^3}{3}\right]_0^2$$
$$= \left[4 - \frac{8}{3}\right] - 0 = \frac{4}{3}$$

1.5.2.9. $*$. Solution. The following sketch contains the graphs of $y = 2^x$ and $y = \sqrt{x} + 1$.

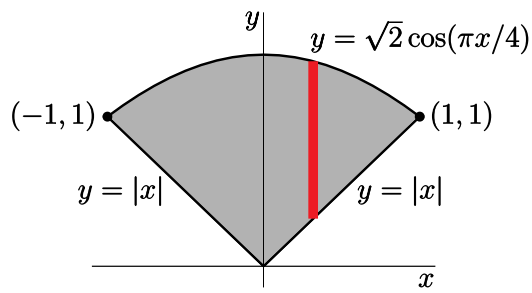


From the sketch, it looks like the two curves cross when $x = 0$ and when $x = 1$ and nowhere[a] else. Indeed, when $x = 0$ we have $2^x = \sqrt{x} + 1 = 1$ and when $x = 1$ we have $2^x = \sqrt{x} + 1 = 2$.
To antidifferentiate 2^x, we write $2^x = \left(e^{\log 2}\right)^x = e^{x \log 2}$.

$$\text{Area} = \int_0^1 \left[(\sqrt{x} + 1) - e^{x \log 2}\right] \mathrm{d}x = \left[\frac{2}{3}x^{3/2} + x - \frac{1}{\log 2} 2^x\right]_0^1$$
$$= \frac{2}{3} + 1 - \frac{1}{\log 2}[2 - 1] = \frac{5}{3} - \frac{1}{\log 2}$$

a To verify analytically that the curves have no other crossings, write $f(x) = \sqrt{x} + 1 - 2^x$ and compute $f'(x) = \frac{1}{2\sqrt{x}} - (\log 2)2^x$. Notice that $f'(x)$ decreases as x increases and so can take the value 0 for at most a single value of x. Then, by the mean value theorem (or Rolle's theorem, which is Theorem 2.13.1 in the CLP-1 text), $f(x)$ can take the value 0 for at most two distinct values of x.

1.5.2.10. *. Solution. Here is a sketch of the specified region.



Both functions are even, so the region is symmetric about the y–axis. So, we will compute the area of the part with $x \geq 0$ and multiply by 2. The curves $y = \sqrt{2}\cos(\pi x/4)$ and $y = x$ intersect when $x = \sqrt{2}\cos(\pi x/4)$ or $\cos(\pi x/4) = \frac{x}{\sqrt{2}}$, which is the case [a] when $x = 1$. So, using vertical strips as in the figure above, the area (including the multiplication by 2) is

$$\begin{aligned} 2\int_0^1 \left[\sqrt{2}\cos(\pi x/4) - x\right] \mathrm{d}x &= 2\left[\sqrt{2}\,\frac{4}{\pi}\sin(\pi x/4) - \frac{x^2}{2}\right]_0^1 \\ &= 2\left[\frac{4}{\pi} - \frac{1}{2}\right] = \frac{8}{\pi} - 1 \end{aligned}$$

a The solution $x = 1$ was found by guessing. To guess a solution to $\cos(\pi x/4) = \frac{x}{\sqrt{2}}$ just ask yourself what simple angle has a cosine that involves $\sqrt{2}$. This guessing strategy is essentially useless in the real world, but works great on problem sets and exams.

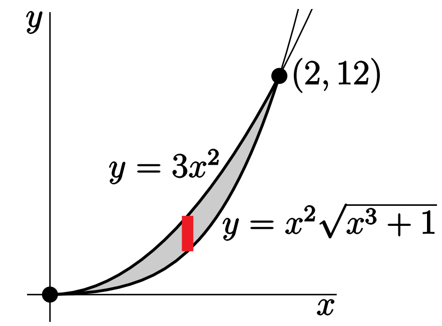
1.5.2.11. *. Solution. For our computation, we will need an antiderivative of $x^2\sqrt{x^3+1}$, which can be found using the substitution $u = x^3 + 1$, $\mathrm{d}u = 3x^2\,\mathrm{d}x$:

$$\begin{aligned} \int x^2\sqrt{x^3+1}\,\mathrm{d}x &= \int \sqrt{u}\cdot\frac{1}{3}\,\mathrm{d}u = \frac{1}{3}\int u^{1/2}\,\mathrm{d}u \\ &= \frac{1}{3}\cdot\frac{u^{3/2}}{3/2} + C = \frac{2}{9}(x^3+1)^{3/2} + C. \end{aligned}$$

The two functions $f(x)$ and $g(x)$ are clearly equal at $x = 0$. If $x \neq 0$, then the functions are equal when

$$\begin{aligned} 3x^2 &= x^2\sqrt{x^3+1} \\ 3 &= \sqrt{x^3+1} \\ 9 &= x^3 + 1 \\ 8 &= x^3 \\ 2 &= x. \end{aligned}$$

The function $g(x) = 3x^2$ is the larger of the two on the interval $[0, 2]$, as can be seen by plugging in $x = 1$, say, or by observing that when x is very small $f(x) = x^2\sqrt{x^3+1} \approx x^2$ and $g(x) = 3x^2$.



The area in question is therefore:

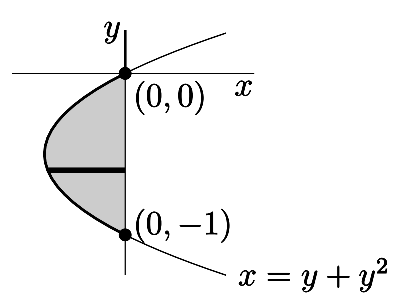
$$\begin{aligned}\int_0^2 \left(3x^2 - x^2\sqrt{x^3+1}\right) \mathrm{d}x &= \left(x^3 - \frac{2}{9}(x^3+1)^{3/2}\right)\bigg|_0^2 \\ &= \left(2^3 - \frac{2}{9}(2^3+1)^{3/2}\right) - \left(0^3 - \frac{2}{9}(0^3+1)^{3/2}\right) \\ &= \left(8-6\right) - \left(0 - \frac{2}{9}\right) = \frac{20}{9}.\end{aligned}$$

1.5.2.12. *. Solution. First, let's figure out what our curve $x = y^2+y = y(y+1)$ looks like.

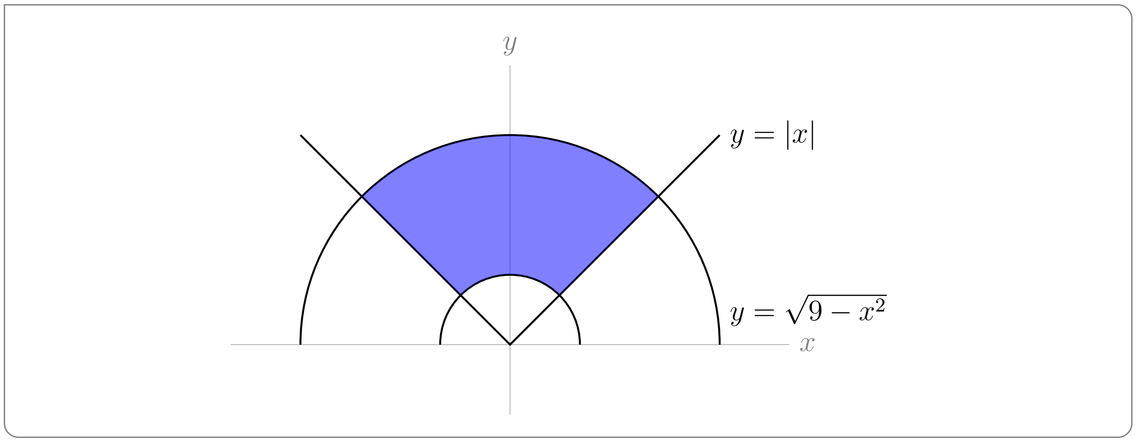
- The curve intercepts the y-axis when $y = 0$ and $y = -1$.
- The x-values of the curve are negative when $-1 < y < 0$, and positive elsewhere.

This leads to the figure below. We're evaluating the area from $y = -1$ to $y = 0$. Since $y^2 + y$ is negative there, the length of our (horizontal) slices are $0 - (y^2 + y)$.

$$\text{Area} = \int_{-1}^{0} \left(0 - (y^2+y)\right) \mathrm{d}y = -\left[\frac{y^3}{3} + \frac{y^2}{2}\right]_{-1}^{0} = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}$$



1.5.2.13. Solution. Let's begin by sketching our region. Note that $y = \sqrt{1-x^2}$ and $y = \sqrt{9-x^2}$ are the top halves of circles centred at the origin with radii 1 and 3, respectively.

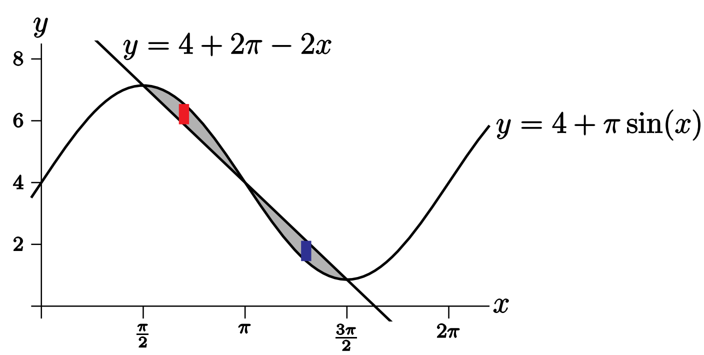


Our region is the difference of two quarter-circles, so we find its area using geometry:

$$\text{Area} = \frac{1}{4}\left(\pi \cdot 3^2\right) - \frac{1}{4}\left(\pi \cdot 1^2\right) = 2\pi$$

Exercises — Stage 3

1.5.2.14. *. Solution. We will compute the area by using thin vertical strips, as in the sketch below:



By looking at the sketch above, we guess the line $y = 4 + 2\pi - 2x$ intersects the curve $y = 4 + \pi \sin x$ when $x = \frac{\pi}{2}$, $x = \pi$, and $x = \frac{3\pi}{2}$. Let's make sure these are correct by plugging them into the two equations, and making sure the y-values match:

x	$4 + 2\pi - 2x$	$4 + \pi\sin(x)$	match?
$\frac{\pi}{2}$	$4 + \pi$	$4 + \pi$	✓
π	4	4	✓
$\frac{3\pi}{2}$	$4 - \pi$	$4 - \pi$	✓

Also from the sketch, we see that:

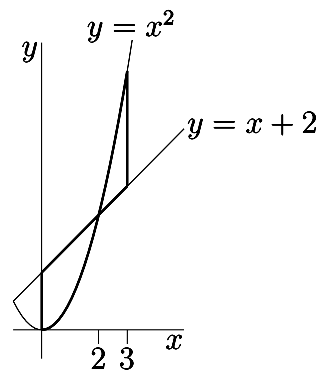
- When $\frac{\pi}{2} \le x \le \pi$, the top of the strip is at $y = 4+\pi \sin x$ and the bottom of the strip is at $y = 4+2\pi-2x$. So the strip has height $\big[(4+\pi \sin x)-(4+2\pi-2x)\big]$ and width $\mathrm{d}x$, and hence area $\big[(4 + \pi \sin x) - (4 + 2\pi - 2x)\big]\mathrm{d}x$.

- When $\pi \le x \le \frac{3\pi}{2}$, the top of the strip is at $y = 4+2\pi-2x$ and the bottom of the strip is at $y = 4+\pi\sin x$. So the strip has height $\big[(4+2\pi-2x)-(4+\pi\sin x)\big]$ and width $\mathrm{d}x$, and hence area $\big[(4+2\pi-2x)-(4+\pi\sin x)\big]\mathrm{d}x$.

Now we can calculate:

$$\begin{aligned}
\text{Area} &= \int_{\pi/2}^{\pi} \big[(4+\pi\sin x)-(4+2\pi-2x)\big]\mathrm{d}x \\
&\qquad + \int_{\pi}^{3\pi/2} \big[(4+2\pi-2x)-(4+\pi\sin x)\big]\mathrm{d}x \\
&= \int_{\pi/2}^{\pi} \big[\pi\sin x - 2\pi + 2x\big]\mathrm{d}x + \int_{\pi}^{3\pi/2} \big[2\pi - 2x - \pi\sin x\big]\mathrm{d}x \\
&= \Big[-\pi\cos x - 2\pi x + x^2\Big]_{\pi/2}^{\pi} + \Big[2\pi x - x^2 + \pi\cos x\Big]_{\pi}^{3\pi/2} \\
&= \left[\pi - \pi^2 + \frac{3}{4}\pi^2\right] + \left[\pi^2 - \frac{5}{4}\pi^2 + \pi\right] \\
&= 2\left[\pi - \frac{1}{4}\pi^2\right]
\end{aligned}$$

1.5.2.15. *. Solution. First, here is a sketch of the region. We are not asked for it, but it is crucial for understanding the question.



The two curves $y = x+2$ and $y = x^2$ cross at $(2,4)$. The area of the part between them with $0 \le x \le 2$ is:

$$\int_0^2 \big[x+2-x^2\big]\,\mathrm{d}x = \left[\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3\right]_0^2 = 2+4-\frac{8}{3} = \frac{10}{3}$$

The area of the part between the two curves with $2 \le x \le 3$ is:

$$\begin{aligned}
\int_2^3 \big[x^2-(x+2)\big]\,\mathrm{d}x &= \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x\right]_2^3 \\
&= 9 - \frac{9}{2} - 6 - \frac{8}{3} + 2 + 4 = \frac{11}{6}
\end{aligned}$$

The total area is $\dfrac{10}{3} + \dfrac{11}{6} = \dfrac{31}{6}$.

1.5.2.16. $*$. Solution. We need to figure out which curve is on top, when. To do this, set $h(x) = 3x - x\sqrt{25-x^2}$. If $h(x) > 0$, then $y = 3x$ is the top curve; if $h(x) < 0$, then $y = x\sqrt{25-x^2}$ is the top curve.

$$h(x) = 3x - x\sqrt{25-x^2} = x\left[3 - \sqrt{25-x^2}\right]$$

We only care about values of x in $[0, 4]$, so x is nonnegative. Then $h(x)$ is positive when:

$$\begin{aligned} 3 &> \sqrt{25-x^2} \\ 9 &> 25 - x^2 \\ x^2 &> 16 \\ x &> 4 \end{aligned}$$

That is, $h(x)$ is never positive over the interval $[0, 4]$. So, $y = x\sqrt{25-x^2}$ lies above $y = 3x$ for all $0 \le x \le 4$.
The area we need to calculate is therefore:

$$\begin{aligned} A &= \int_0^4 \left[x\sqrt{25-x^2} - 3x\right] \, \mathrm{d}x \\ &= \int_0^4 x\sqrt{25-x^2}\, \mathrm{d}x - \int_0^4 3x \, \mathrm{d}x \\ &= A_1 - A_2. \end{aligned}$$

To evaluate A_1, we use the substitution $u(x) = 25 - x^2$, for which $\mathrm{d}u = u'(x)\,\mathrm{d}x = -2x\,\mathrm{d}x$; and $u(4) = 25 - 4^2 = 9$ when $x = 4$, while $u(0) = 25 - 0^2 = 25$ when $x = 0$. Therefore

$$\begin{aligned} A_1 &= \int_{x=0}^{x=4} x\sqrt{25-x^2}\,\mathrm{d}x = -\frac{1}{2}\int_{u=25}^{u=9} \sqrt{u}\,\mathrm{d}u \\ &= \left[-\frac{1}{3}u^{3/2}\right]_{25}^{9} = \frac{125-27}{3} = \frac{98}{3} \end{aligned}$$

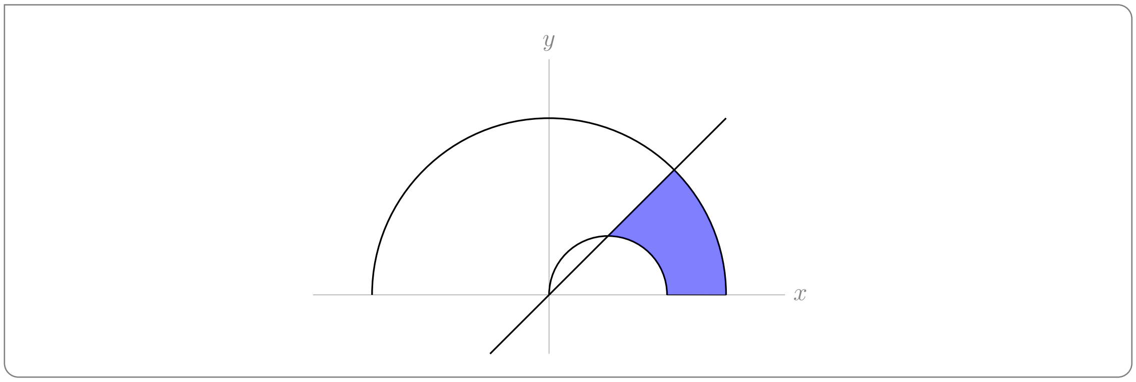
For A_2 we use the antiderivative directly:

$$A_2 = \int_0^4 3x\,\mathrm{d}x = \left[\frac{3x^2}{2}\right]_0^4 = 24$$

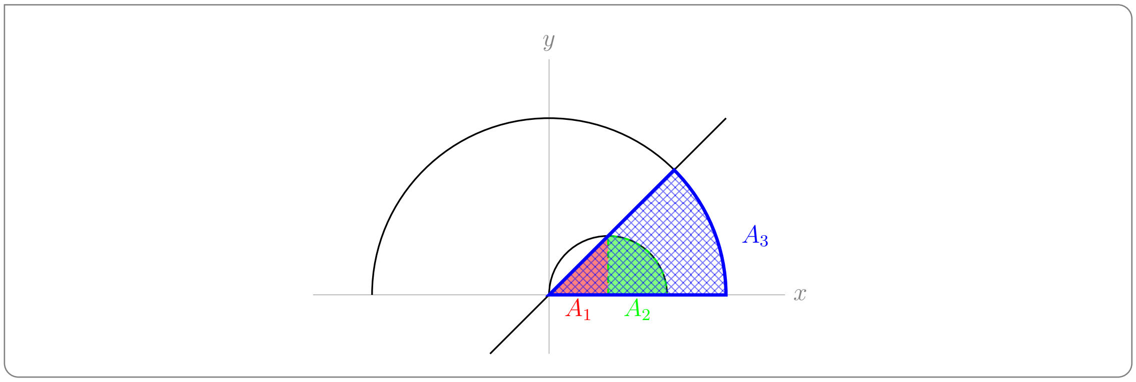
Therefore the total area is:

$$A = \frac{98}{3} - 24 = \frac{26}{3}$$

1.5.2.17. Solution. Let's begin by sketching our region. Note that $y = \sqrt{9-x^2}$ is the top half of a circle centred at the origin with radius 3, while $y = \sqrt{1-(x-1)^2}$ is the top half of a circle of radius 1 centred at $(1, 0)$.



Note $y = x$ intersects $y = \sqrt{1-(x-1)^2}$ at $(1,1)$, the highest part of the smaller half-circle.
We can easily take the area of triangles and sectors of circles. With that in mind, we cut up our region the following way:



- The desired area is $A_3 - (A_1 + A_2)$.
- A_1 is the area of right a triangle with base 1 and height 1, so $A_1 = \frac{1}{2}$.
- A_2 is the area of a quarter circle of radius 1, so $A_2 = \frac{\pi}{4}$.
- A_3 is the area of an eighth of a circle of radius 3, so $A_2 = \frac{9\pi}{8}$

So, the area of our region is $\dfrac{9\pi}{8} - \dfrac{1}{2} - \dfrac{\pi}{4} = \dfrac{7\pi}{8} - \dfrac{1}{2}$.

1.5.2.18. Solution. The first function is a cubic, with intercepts at $x = 0, \pm 2$. The second is a straight line with a positive slope.
We need to figure out what these functions look like in relation to one another, so let's find their points of intersection.

$$\begin{aligned} x(x^2-4) &= x-2 \\ x(x+2)(x-2) &= x-2 \\ \boxed{x-2=0} \quad \text{or} \quad x(x+2) &= 1 \\ x^2+2x-1 &= 0 \end{aligned}$$

$$x = \frac{-2 \pm \sqrt{4 - 4(1)(-1)}}{2}$$
$$x = \boxed{-1 \pm \sqrt{2}}$$

So, our three points of intersection are when $x = 2$ and when $x = -1 \pm \sqrt{2}$. We note

$$-1 - \sqrt{2} < -1 + \sqrt{2} < -1 + \sqrt{4} < 2\ .$$

So, we need to see which function is on top over the two intervals $\left[-1 - \sqrt{2}, -1 + \sqrt{2}\right]$ and $\left[-1 + \sqrt{2}, 2\right]$. It suffices to check points in these intervals.

x	$x(x^2-4)$	$x-2$	top function:
0	0	-2	$x(x^2-4)$
1	-3	-1	$x-2$

Since 0 is in the interval $\left[-1 - \sqrt{2}, -1 + \sqrt{2}\right]$, $x(x^2-4)$ is the top function in that interval. Since 1 is in the interval $\left[-1 + \sqrt{2}, 2\right]$, $x - 2$ is the top function in that interval. Now we can set up the integral to evaluate the area:

$$\begin{aligned}
\text{Area} &= \int_{-1-\sqrt{2}}^{-1+\sqrt{2}} \left[x(x^2-4) - (x-2)\right] \mathrm{d}x \\
&\qquad + \int_{-1+\sqrt{2}}^{2} \left[(x-2) - x(x^2-4)\right] \mathrm{d}x \\
&= \int_{-1-\sqrt{2}}^{-1+\sqrt{2}} \left[x^3 - 5x + 2\right] \mathrm{d}x \\
&\qquad + \int_{-1+\sqrt{2}}^{2} \left[-x^3 + 5x - 2\right] \mathrm{d}x \\
&= \left[\frac{1}{4}x^4 - \frac{5}{2}x^2 + 2x\right]_{-1-\sqrt{2}}^{-1+\sqrt{2}} \\
&\qquad + \left[-\frac{1}{4}x^4 + \frac{5}{2}x^2 - 2x\right]_{-1+\sqrt{2}}^{2}
\end{aligned}$$

After some taxing but rudimentary algebra:

$$= \left(8\sqrt{2}\right) + \left(4\sqrt{2} - \frac{13}{4}\right) = 12\sqrt{2} - \frac{13}{4}$$

1.6 · Volumes

1.6.2 · Exercises

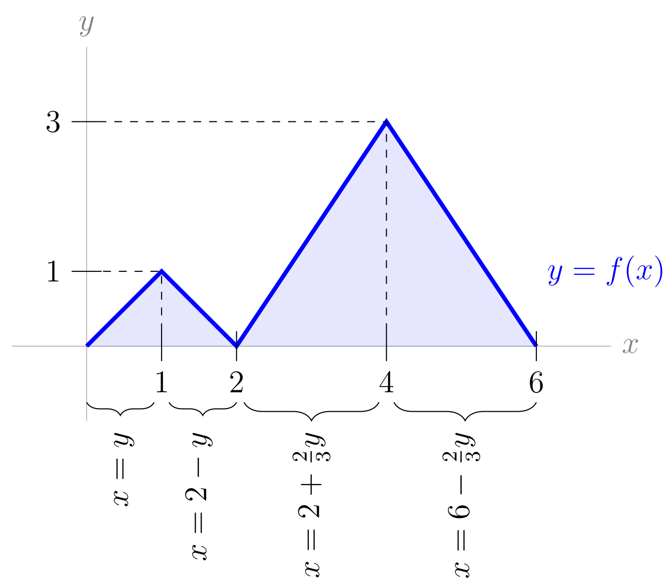
Exercises — Stage 1

1.6.2.1. Solution. If we take a horizontal slice of a cone, we get a circle. If we take a vertical cross-section, the base is flat (it's a chord on the circular base of the cone), so we know right away it isn't a circle. Indeed, if we slice down through the very centre, we get a triangle. (Other vertical slices have a curvy top, corresponding to a class of curves known as hyperbolas.)

1.6.2.2. Solution. The columns have the same volume. We can see this by chopping up the columns into horizontal cross-sections. Each cross-section has the same area as the cookie cutter, A, and height $\mathrm{d}y$. Then in both cases, the volume of the column is

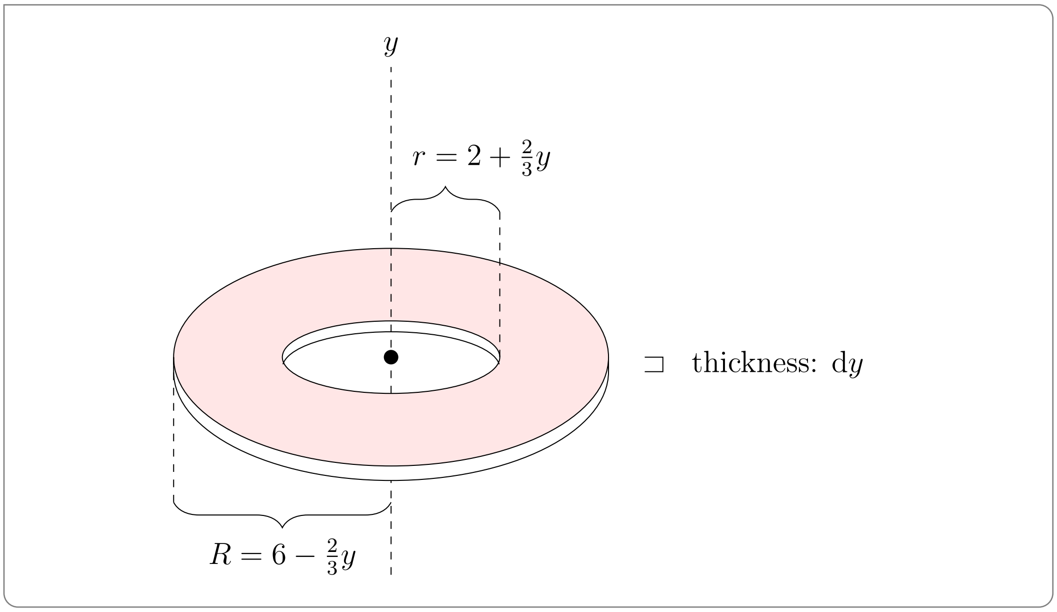
$$\int_0^h A\mathrm{d}y = hA \text{ cubic units}$$

1.6.2.3. Solution. Notice $f(x)$ is a piecewise linear function, so we can find explicit equations for each of its pieces from the graph. The radii will be determined by the x-values, so below we give the x-values as functions of y.

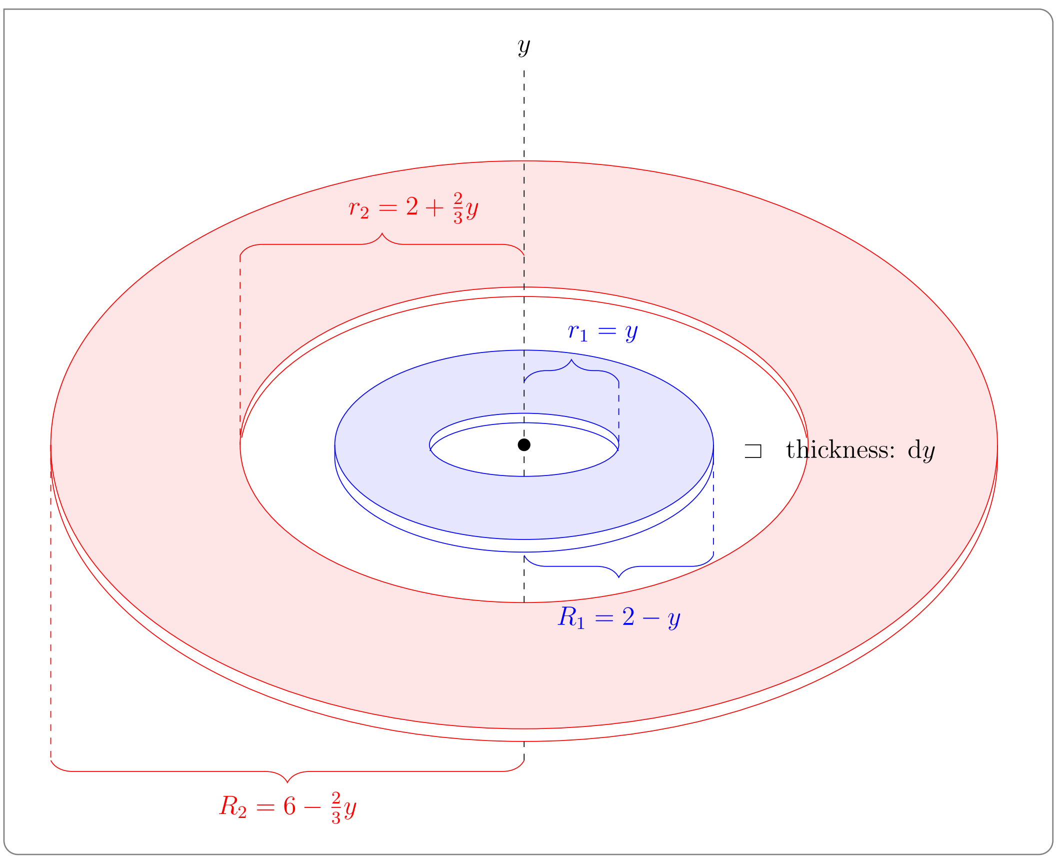


If we imagine rotating the region from the picture about the y-axis, there will be two kinds of washers formed: when $y < 1$, we have a "double washer," two concentric rings. When $y > 1$, we have a single ring.

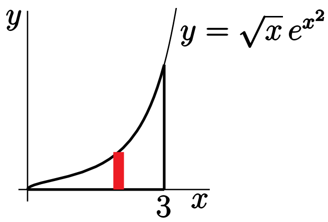
- Washers when $\mathbf{1 < y \leq 6}$: If $y > 1$, then our washer has inner radius $2 + \frac{2}{3}y$, outer radius $6 - \frac{2}{3}y$, and height $\mathrm{d}y$.



- Washers when $\mathbf{0 \leq y < 1}$: When $0 \leq y < 1$, we have a “double washer,” two concentric rings corresponding to the two “humps” in the function. The inner washer has inner radius $r_1 = y$ and outer radius $R_1 = 2 - y$. The outer washer has inner radius $r_2 = 2 + \frac{2}{3}y$ and outer radius $R_2 = 6 - \frac{2}{3}y$. The thickness of the washers is dy.



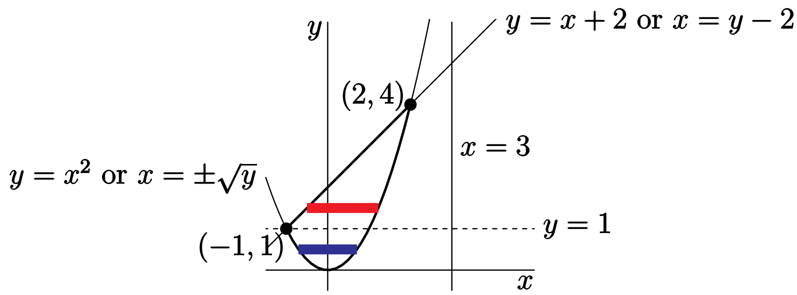
1.6.2.4. *. Solution. (a) When the strip shown in the figure



is rotated about the x–axis, it forms a thin disk of radius $\sqrt{x}e^{x^2}$ and thickness $\mathrm{d}x$ and hence of cross sectional area $\pi x e^{2x^2}$ and volume $\pi x e^{2x^2}\,\mathrm{d}x$ So the volume of the solid is

$$\pi \int_0^3 x e^{2x^2} \mathrm{d}x$$

(b) The curves intersect at $(-1, 1)$ and $(2, 4)$.

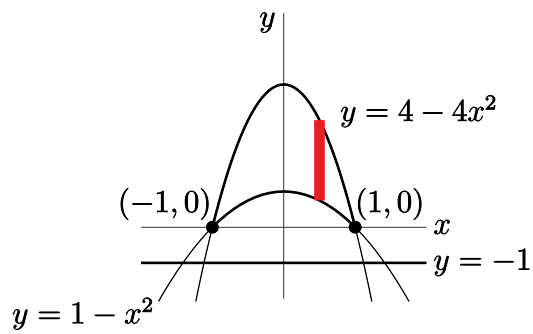


We'll use horizontal washers as in Example 1.6.5.

- We use thin horizontal strips of width $\mathrm{d}y$ as in the figure above.
- When we rotate about the line $x = 3$, each strip sweeps out a thin washer
  - whose inner radius is $r_{in} = 3 - \sqrt{y}$, and
  - whose outer radius is $r_{out} = 3 - (y - 2) = 5 - y$ when $y \geq 1$ (see the red strip in the figure on the right above), and whose outer radius is $r_{out} = 3 - (-\sqrt{y}) = 3 + \sqrt{y}$ when $y \leq 1$ (see the blue strip in the figure on the right above) and
  - whose thickness is $\mathrm{d}y$ and hence
  - whose volume is $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}y = \pi\big[(5 - y)^2 - (3 - \sqrt{y})^2\big]\mathrm{d}y$ when $y \geq 1$ and whose volume is $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}y = \pi\big[(3 + \sqrt{y})^2 - (3 - \sqrt{y})^2\big]\mathrm{d}y$ when $y \leq 1$ and
- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = 4$, the total volume is

$$\int_0^1 \pi\big[(3 + \sqrt{y})^2 - (3 - \sqrt{y})^2\big]\mathrm{d}y + \int_1^4 \pi\big[(5 - y)^2 - (3 - \sqrt{y})^2\big]\mathrm{d}y$$

1.6.2.5. *. Solution. (a) The curves intersect at $(1,0)$ and $(-1,0)$. When the strip shown in the figure



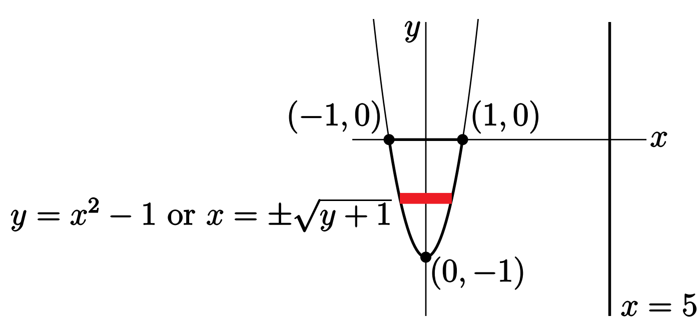
is rotated about the line $y = -1$, it forms a thin washer with:

- inner radius $(1 - x^2) - (-1) = 2 - x^2$,
- outer radius $(4 - 4x^2) - (-1) = 5 - 4x^2$ and
- thickness $\mathrm{d}x$; so, it has
- cross sectional area $\pi\big[(5-4x^2)^2 - (2-x^2)^2\big]$ and
- volume $\pi\big[(5-4x^2)^2 - (2-x^2)^2\big]\,\mathrm{d}x$.

So the volume of the solid is

$$\int_{-1}^{1} \pi\big[(5-4x^2)^2 - (2-x^2)^2\big]\,\mathrm{d}x$$

(b) The curve $y = x^2 - 1$ intersects $y = 0$ at $(1,0)$ and $(-1,0)$.



We'll use horizontal washers.

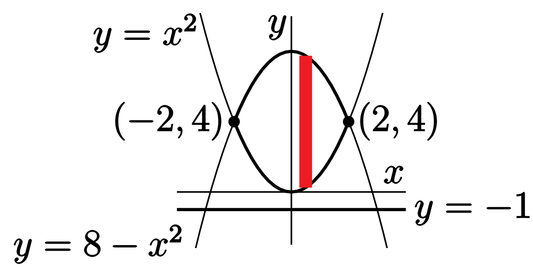
- We use thin horizontal strips of height $\mathrm{d}y$ as in the figure above.
- When we rotate about the line $x = 5$, each strip sweeps out a thin washer
  - whose inner radius is $r_{in} = 5 - \sqrt{y+1}$, and
  - whose outer radius is $r_{out} = 5 - (-\sqrt{y+1}) = 5 + \sqrt{y+1}$ and
  - whose thickness is $\mathrm{d}y$ and hence

- ∘ whose volume is $\pi(r_{out}^2 - r_{in}^2)\,\mathrm{d}y = \pi\big[\big(5+\sqrt{y+1}\big)^2 - \big(5-\sqrt{y+1}\big)^2\big]\,\mathrm{d}y$

- As our topmost strip is at $y=0$ and our bottommost strip is at $y=-1$ (when $x=0$), the total volume is

$$\int_{-1}^{0} \pi\left[\left(5+\sqrt{y+1}\right)^2 - \left(5-\sqrt{y+1}\right)^2\right]\,\mathrm{d}y$$

1.6.2.6. *. Solution. The curves intersect at $(-2,4)$ and $(2,4)$. When the strip shown in the figure



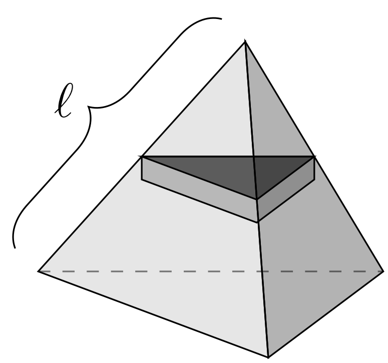
is rotated about the line $y=-1$, it forms a thin washer (punctured disc) of

- inner radius x^2+1,
- outer radius $9-x^2$ and
- thickness $\mathrm{d}x$ and hence of
- cross sectional area $\pi\big[(9-x^2)^2 - (x^2+1)^2\big]$ and
- volume $\pi\big[(9-x^2)^2 - (x^2+1)^2\big]\,\mathrm{d}x$.

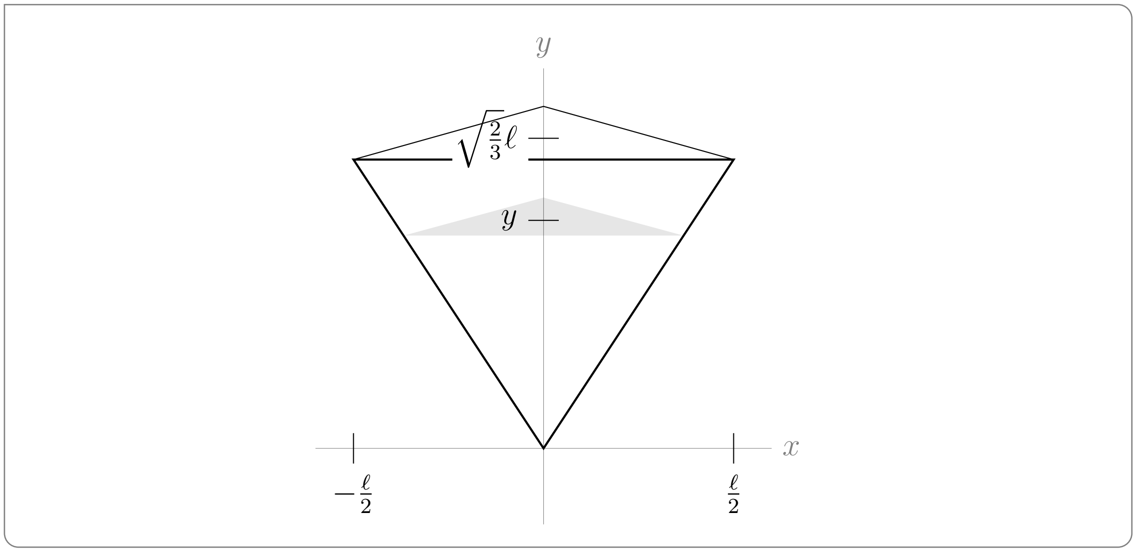
So the volume of the solid is

$$\pi\int_{-2}^{2} \big[(9-x^2)^2 - (x^2+1)^2\big]\mathrm{d}x$$

1.6.2.7. Solution. We'll make horizontal slices, parallel to one of the faces of the tetrahedron. Then our slices will be equilateral triangles, of varying sizes.



For the sake of ease, as in Example 1.6.1, we picture the tetrahedron perched on a tip, one base horizontal on top.



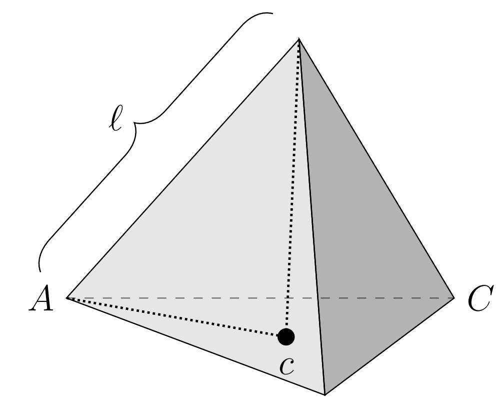
Notice our slice forms the horizontal top of a smaller tetrahedron. The horizontal top of the full tetrahedron has side length ℓ, which is $\sqrt{\frac{3}{2}}$ times the height of the full tetrahedron. Our slice is the horizontal top of a tetrahedron of height y and so has side length $\sqrt{\frac{3}{2}}y$. An equilateral triangle with side length L has base L and height $\frac{\sqrt{3}}{2}L$, and hence area $\frac{\sqrt{3}}{4}L^2$. So, the area of our slice with side length $\sqrt{\frac{3}{2}}y$ is

$$A = \frac{\sqrt{3}}{4}\left(\sqrt{\frac{3}{2}}y\right)^2 = \frac{3\sqrt{3}}{8}y^2$$

So, the volume of a tetrahedron with side length ℓ is:

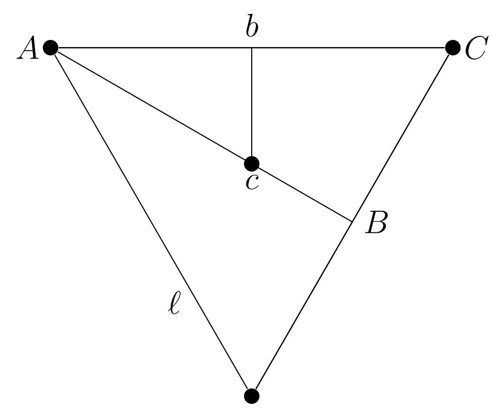
$$\begin{aligned}\text{Volume} &= \int_0^{\sqrt{\frac{2}{3}}\ell} \frac{3\sqrt{3}}{8}y^2 \mathrm{d}y \\ &= \frac{\sqrt{3}}{8}\cdot\left(\sqrt{\frac{2}{3}}\ell\right)^3 = \frac{\sqrt{2}}{12}\ell^3\end{aligned}$$

You were given the height of a tetrahedron, but for completeness we calculate it here. Draw a line starting at one tip, and dropping straight down to the middle of the opposite face. It forms a right triangle with one edge of the tetrahedron, and a line from the middle of the face to the corner.



We know the length of the hypotenuse of this right triangle (it's ℓ), so if we know the length of its base (labeled Ac in the diagram), we can figure out its third side, the height of our tetrahedron. Note by using the Pythagorean theorem, we see that the height of an equilateral triangle with edge length ℓ is $\sqrt{\frac{3}{2}}\ell$.

Here is a sketch of the base of the pyramid:



The triangles ABC and Abc are similar (since b and B are right angles, and also A has the same angle in both). Therefore,

$$\frac{Ac}{Ab} = \frac{AC}{AB}$$
$$\frac{Ac}{\ell/2} = \frac{\ell}{\sqrt{3}\ell/2}$$
$$Ac = \frac{1}{\sqrt{3}}\ell$$

With this in our pocket, we can find the height of the tetrahedron: $\sqrt{\ell^2 - \left(\frac{1}{\sqrt{3}}\ell\right)^2} = \sqrt{\frac{2}{3}}\ell$.

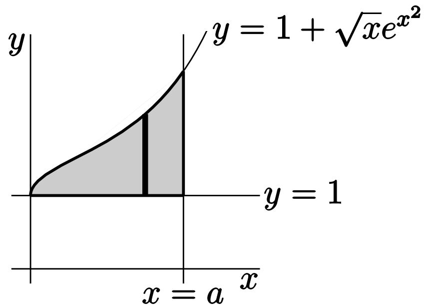
Exercises — Stage 2

1.6.2.8. $*$. **Solution**. Let $f(x) = 1 + \sqrt{x}e^{x^2}$. On the vertical slice a distance x from the y-axis, sketched in the figure below, y runs from 1 to $f(x)$. Upon rotation about the line $y = 1$, this thin slice sweeps out a thin disk of thickness $\mathrm{d}x$ and radius $f(x) - 1$ and hence of volume $\pi[f(x) - 1]^2\,\mathrm{d}x$. The full volume generated (for any fixed $a > 0$) is

$$\int_0^a \pi[f(x) - 1]^2\,\mathrm{d}x = \pi \int_0^a xe^{2x^2}\,\mathrm{d}x.$$

Using the substitution $u = 2x^2$, so that $\mathrm{d}u = 4x\,\mathrm{d}x$:

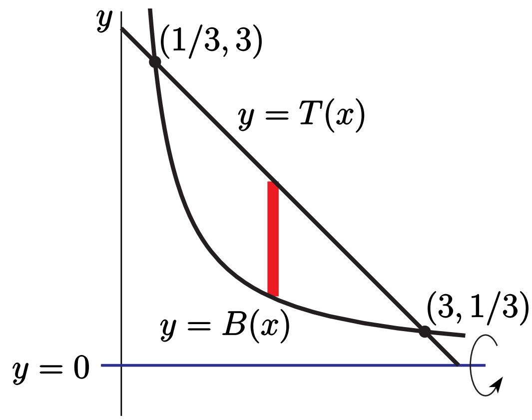
$$\text{Volume} = \pi \int_0^{2a^2} e^u\,\frac{\mathrm{d}u}{4} = \frac{\pi}{4}e^u\Big|_0^{2a^2} = \frac{\pi}{4}\Big(e^{2a^2} - 1\Big)$$



Remark: we spent a good deal of time last semester developing highly accurate but time-consuming methods for sketching common functions. For the purposes of questions like this, we don't need a detailed picture of a function–broad outlines suffice. Notice that $\sqrt{x} > 0$ whenever $x > 0$, and $e^{x^2} > 0$ for all x. Therefore, $\sqrt{x}e^{x^2}$ is nonnegative over its entire domain, and so the graph $y = 1 + \sqrt{x}e^{x^2}$ is always the top function, above the bottom function $y = 1$. That is the only information we needed to perform our calculation.

1.6.2.9. $*$. **Solution**. The curves $y = 1/x$ and $3x + 3y = 10$, i.e. $y = \frac{10}{3} - x$ intersect when

$$\begin{aligned}\frac{1}{x} = \frac{10}{3} - x &\iff 3 = 10x - 3x^2 \iff 3x^2 - 10x + 3 = 0\\ &\iff (3x - 1)(x - 3) = 0\\ &\iff x = 3\,,\ \frac{1}{3}\end{aligned}$$



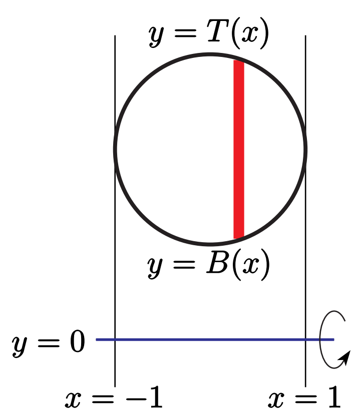
When the region is rotated about the x–axis, the vertical strip in the figure above sweeps out a washer with thickness $\mathrm{d}x$, outer radius $T(x) = \frac{10}{3} - x$ and inner radius $B(x) = \frac{1}{x}$. This washer has volume

$$\pi\big(T(x)^2 - B(x)^2\big)\,\mathrm{d}x = \pi\Big(\frac{100}{9} - \frac{20}{3}x + x^2 - \frac{1}{x^2}\Big)\,\mathrm{d}x$$

Hence the volume of the solid is

$$\begin{aligned}\pi\int_{1/3}^{3}&\Big(\frac{100}{9} - \frac{20}{3}x + x^2 - \frac{1}{x^2}\Big)\,\mathrm{d}x\\ &= \pi\Big[\frac{100x}{9} - \frac{10}{3}x^2 + \frac{1}{3}x^3 + \frac{1}{x}\Big]_{1/3}^{3}\\ &= \pi\Big[\frac{38}{3} - \frac{514}{3^4}\Big] = \pi\frac{512}{81}\end{aligned}$$

1.6.2.10. *. Solution. (a) The top and the bottom of the circle have equations $y = T(x) = 2 + \sqrt{1-x^2}$ and $y = B(x) = 2 - \sqrt{1-x^2}$, respectively.



When R is rotated about the x–axis, the vertical strip of R in the figure above sweeps out a washer with thickness $\mathrm{d}x$, outer radius $T(x)$ and inner radius $B(x)$. This washer has volume

$$\pi\big(T(x)^2 - B(x)^2\big)\,\mathrm{d}x = \pi\big(T(x) + B(x)\big)\big(T(x) - B(x)\big)\,\mathrm{d}x$$

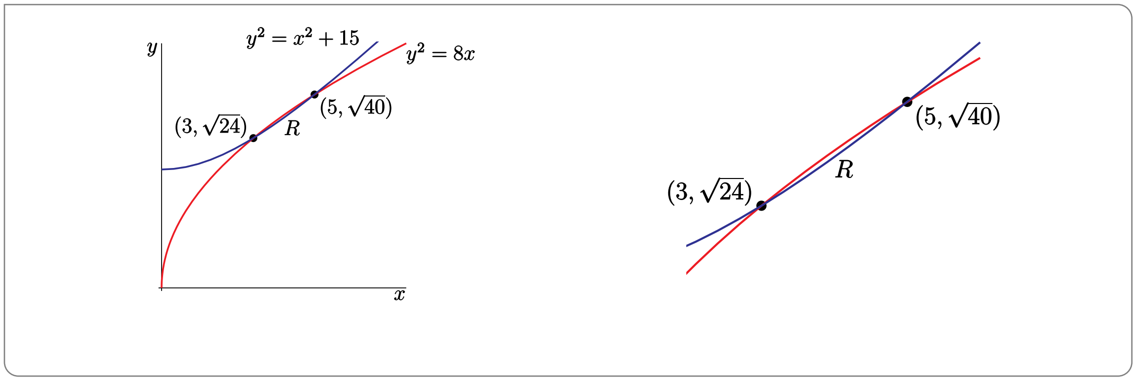
$$= \pi \times 4 \times 2\sqrt{1-x^2}\,\mathrm{d}x$$

Hence the volume of the solid is

$$8\pi \int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}x$$

(b) Since $y = \sqrt{1-x^2}$ is equivalent to $x^2 + y^2 = 1$, $y \ge 0$, the integral is 8π times the area of the upper half of the circle $x^2 + y^2 = 1$ and hence is $8\pi \times \frac{1}{2}\pi 1^2 = 4\pi^2$.

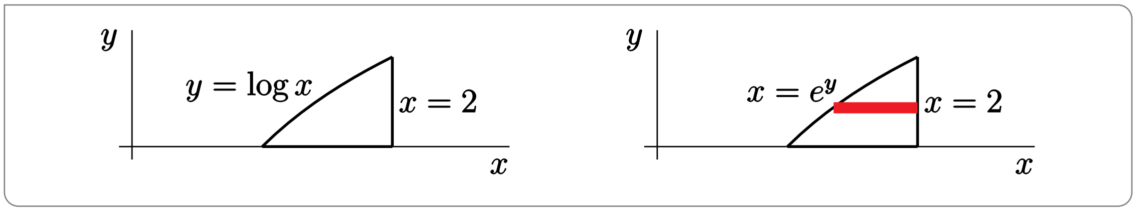
1.6.2.11. *. Solution. (a) The two curves intersect when x obeys $8x = x^2 + 15$ or $x^2 - 8x + 15 = (x-5)(x-3) = 0$. The points of intersection, in the first quadrant, are $(3, \sqrt{24})$ and $(5, \sqrt{40})$. The region R is the region between the blue and red curves, with $3 \le x \le 5$, in the figures below.



(b) The part of the solid with x coordinate between x and $x + \mathrm{d}x$ is a "washer" shaped region with inner radius $\sqrt{x^2+15}$, outer radius $\sqrt{8x}$ and thickness $\mathrm{d}x$. The surface area of the washer is $\pi(\sqrt{8x})^2 - \pi(\sqrt{x^2+15})^2 = \pi(8x - x^2 - 15)$ and its volume is $\pi(8x - x^2 - 15)\,\mathrm{d}x$. The total volume is

$$\begin{aligned}\int_3^5 \pi(8x - x^2 - 15)\,\mathrm{d}x &= \pi\Big[4x^2 - \frac{1}{3}x^3 - 15x\Big]_3^5 \\ &= \pi\Big[100 - \frac{125}{3} - 75 - 36 + 9 + 45\Big] \\ &= \frac{4}{3}\pi \approx 4.19\end{aligned}$$

1.6.2.12. *. Solution. (a) The region R is sketched in the figure on the left below. (The bound $y = 0$ renders the bound $x = 1$ unnecessary, since the graph $y = \log x$ hits the x-axis when $x = 1$.)



(b) We'll use horizontal washers as in Example 1.6.5.

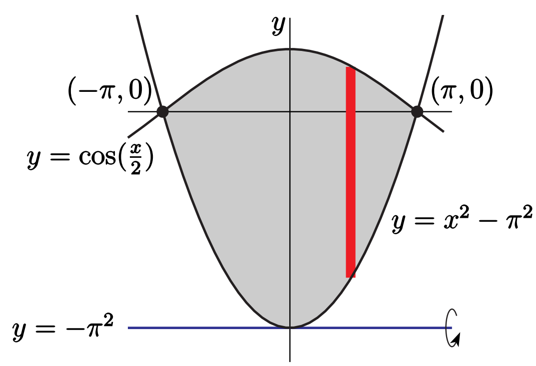
- We cut R into thin horizontal strips of height $\mathrm{d}y$ as in the figure on the right above.

- When we rotate R about the y–axis, i.e. about the line $x = 0$, each strip sweeps out a thin washer
  - whose inner radius is $r_{in} = e^y$ and outer radius is $r_{out} = 2$, and
  - whose thickness is $\mathrm{d}y$ and hence
  - whose volume $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}y = \pi\big(4 - e^{2y}\big)\mathrm{d}y$.
- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = \log 2$ (since at the top $x = 2$ and $x = e^y$), the total

$$\begin{aligned}\text{Volume} &= \int_0^{\log 2} \pi\big(4 - e^{2y}\big)\mathrm{d}y = \pi\big[4y - e^{2y}/2\big]_0^{\log 2} \\ &= \pi\Big[4\log 2 - 2 + \frac{1}{2}\Big] = \pi\Big[4\log 2 - \frac{3}{2}\Big]\end{aligned}$$

Using a calculator, we see this is approximately 3.998.

1.6.2.13. *. Solution. Here is a sketch of the curves $y = \cos(\frac{x}{2})$ and $y = x^2 - \pi^2$.



By inspection, the curves meet at $x = \pm\pi$ where both $\cos(\frac{x}{2})$ and $x^2 - \pi^2$ take the value zero. We'll use vertical washers as specified in the question.

- We cut the specified region into thin vertical strips of width $\mathrm{d}x$ as in the figure above.
- When we rotate about the line $y = -\pi^2$, each strip sweeps out a thin washer
  - whose inner radius is $r_{in} = (x^2 - \pi^2) - (-\pi^2) = x^2$ and outer radius is $r_{out} = \cos(\frac{x}{2}) - (-\pi^2) = \cos(\frac{x}{2}) + \pi^2$, and
  - whose thickness is $\mathrm{d}x$ and hence
  - whose volume $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}x = \pi\big((\cos(\frac{x}{2}) + \pi^2)^2 - (x^2)^2\big)\mathrm{d}x$.
- As our leftmost strip is at $x = -\pi$ and our rightmost strip is at $x = \pi$,

the total volume is

$$\pi\int_{-\pi}^{\pi} \big(\cos^2(\tfrac{x}{2}) + 2\pi^2\cos(\tfrac{x}{2}) + \pi^4 - x^4\big)\ \mathrm{d}x$$

$$= \pi \int_{-\pi}^{\pi} \left(\frac{1+\cos(x)}{2} + 2\pi^2 \cos(\tfrac{x}{2}) + \pi^4 - x^4 \right) \mathrm{d}x$$

Because the integrand is even,

$$\begin{aligned}
&= 2\pi \int_{0}^{\pi} \left(\frac{1+\cos(x)}{2} + 2\pi^2 \cos(\tfrac{x}{2}) + \pi^4 - x^4 \right) \mathrm{d}x \\
&= 2\pi \left[\frac{1}{2}x + \frac{1}{2}\sin(x) + 4\pi^2 \sin(\tfrac{x}{2}) + \pi^4 x - \frac{1}{5}x^5 \right]_0^{\pi} \\
&= 2\pi \left[\frac{\pi}{2} + 0 + 4\pi^2 + \pi^5 - \frac{\pi^5}{5} \right] \\
&= \pi^2 + 8\pi^3 + \frac{8\pi^6}{5}
\end{aligned}$$

We used the fact that the integrand is an even function and the interval of integration $[-\pi, \pi]$ is symmetric, but one can also compute directly.

1.6.2.14. $*$. Solution. As in Example 1.6.6, we slice V into thin horizontal "square pancakes".

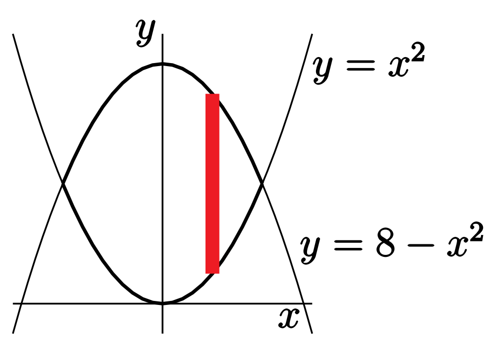
- We are told that the pancake at height x is a square of side $\frac{2}{1+x}$ and so
- has cross-sectional area $\left(\frac{2}{1+x}\right)^2$ and thickness $\mathrm{d}x$ and hence
- has volume $\left(\frac{2}{1+x}\right)^2 \mathrm{d}x$.

Hence the volume of V is

$$\int_0^2 \left[\frac{2}{1+x}\right]^2 \mathrm{d}x = \int_1^3 \frac{4}{u^2}\,\mathrm{d}u = 4\frac{u^{-1}}{-1}\bigg|_1^3 = -4\Big[\frac{1}{3} - 1\Big] = \frac{8}{3}$$

We made the change of variables $u = 1 + x$, $\mathrm{d}u = \mathrm{d}x$.

1.6.2.15. $*$. Solution. Here is a sketch of the base region.

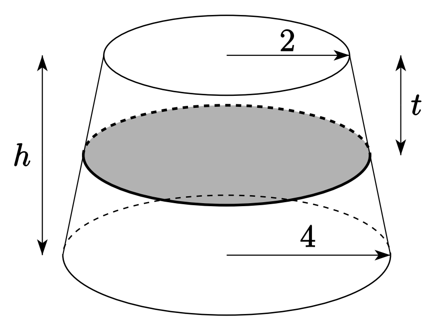


Consider the thin vertical cross–section resting on the heavy red line in the figure above. It has thickness $\mathrm{d}x$. Its face is a square whose side runs from $y = x^2$ to $y = 8 - x^2$, a distance of $8 - 2x^2$. So the face has area $(8 - 2x^2)^2$ and the slice has volume $(8-2x^2)^2\,\mathrm{d}x$. The two curves cross when $x^2 = 8-x^2$, i.e. when $x^2=4$ or $x=\pm 2$. So x runs from -2 to 2 and the total volume is

$$\begin{aligned}\int_{-2}^{2}(8-2x^2)^2\,\mathrm{d}x &= 2\int_0^2 4(4-x^2)^2\,\mathrm{d}x = 8\int_0^2\left[16-8x^2+x^4\right]\mathrm{d}x\\ &= 8\Big[16\times 2-\frac{8}{3}2^3+\frac{1}{5}2^5\Big] = \frac{256\times 8}{15} = 136.5\dot{3}\end{aligned}$$

In the first simplification step, we used the fact that our integrand was even, but we also could have finished our computation without this step.

1.6.2.16. *. Solution. Slice the frustrum into horizontal discs. When the disc is a distance t from the top of the frustrum it has radius $2+2t/h$. Note that as t runs from 0 (the top of the frustrum) to $t=h$ (the bottom of the frustrum) the radius $2+2t/h$ increases linearly from 2 to 4.



Thus the disk has volume $\pi\big(2+2t/h\big)^2\mathrm{d}t$. The total volume of the frustrum is

$$\begin{aligned}\pi\int_0^h\big(2+2t/h\big)^2\mathrm{d}t &= 4\pi\int_0^h\big(1+t/h\big)^2\mathrm{d}t = 4\pi\left[\frac{(1+t/h)^3}{3/h}\right]_0^h\\ &= \frac{4}{3}\pi h\times 7 = \frac{28}{3}\pi h\end{aligned}$$

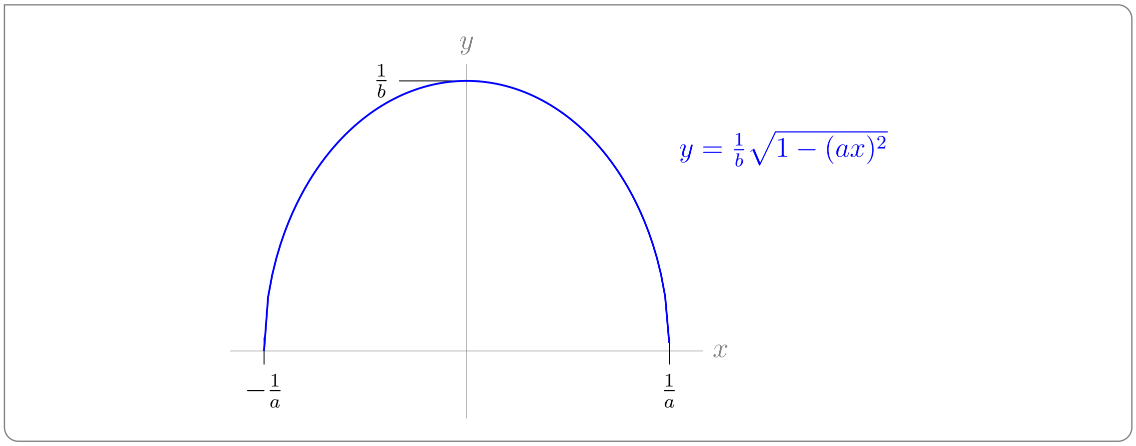
Remark: we could also solve this problem using the formula for the volume of a cone. Using similar triangles, the frustrum in question is shaped like a right circular cone of height $2h$ and base radius 4 (and hence of volume $\frac{1}{3}\pi(4^2)(2h)$), but missing its top, which is a right circular cone of height h and base radius 2 (and hence volume $\frac{1}{3}\pi(2^2)h$). So, the volume of the frustrum is

$$\frac{1}{3}\pi(4^2)(2h)-\frac{1}{3}\pi(2^2)h = \frac{28}{3}\pi h.$$

Exercises — Stage 3

1.6.2.17. Solution. (a)

We'll want to start by graphing the upper half of the ellipse $(ax)^2+(by)^2=1$. Its intercepts will be enough to get us an idea: $(0,\frac{1}{b})$ and $(\pm\frac{1}{a},0)$:



We note a few things at the outset: first, since $a \geq b$, then $\frac{1}{a} \leq \frac{1}{b}$, so indeed the x-axis is the minor axis. That is, we're rotating about the proper axis to create an oblate spheroid.
Second, if we solve our equation for y, we get $y = \frac{1}{b}\sqrt{1-(ax)^2}$. (Since we only want the upper half of the ellipse, we only need to consider the positive square root.)
Now, we have a standard volume-of-revolution problem. We make vertical slices, of width $\mathrm{d}x$ and height $y = \frac{1}{b}\sqrt{1-(ax)^2}$. When we rotate these slices about the x-axis, they form thin disks of volume $\pi\left[\frac{1}{b}\sqrt{1-(ax)^2}\right]^2\mathrm{d}x$. Since x runs from $-\frac{1}{a}$ to $\frac{1}{a}$, the volume of our oblate spheroid is:

$$\begin{aligned}
\text{Volume} &= \int_{-\frac{1}{a}}^{\frac{1}{a}} \pi\left[\frac{1}{b}\sqrt{1-(ax)^2}\right]^2 \mathrm{d}x \\
&= \frac{\pi}{b^2}\int_{-\frac{1}{a}}^{\frac{1}{a}} 1-(ax)^2\mathrm{d}x \\
&= \frac{2\pi}{b^2}\int_{0}^{\frac{1}{a}} 1-(ax)^2\mathrm{d}x && \text{(even function)} \\
&= \frac{2\pi}{b^2}\left[x - \frac{a^2x^3}{3}\right]_0^{\frac{1}{a}} \\
&= \frac{2\pi}{b^2}\left[\frac{1}{a} - \frac{1}{3a}\right] = \frac{4\pi}{3b^2a}
\end{aligned}$$

(b) As we saw in the sketch from part (a), the shortest radius of the ellipse is $\frac{1}{a}$, while the largest is $\frac{1}{b}$. So, $\frac{1}{a} = 6356.752$, and $\frac{1}{b} = 6378.137$. That is, $a = \dfrac{1}{6356.752}$ and $b = \dfrac{1}{6378.137}$.
Note $a \geq b$, as specified in part (a).
(c) Combining our answers from (a) and (b), the volume of an oblate spheroid with approximately the same dimensions as the earth is:

$$\frac{4\pi}{3b^2a} = \frac{4\pi}{3}\left(\frac{1}{b}\right)^2\left(\frac{1}{a}\right)$$

$$\begin{aligned}
&= \frac{4\pi}{3}\left(6378.137\right)^2\left(6356.752\right)\\
&\approx 1.08321\times 10^{12}\quad \mathrm{km}^3\\
&\approx 1.08321\times 10^{21}\quad \mathrm{m}^3
\end{aligned}$$

(d) A sphere of radius 6378.137 has volume

$$\frac{4}{3}\pi\left(6378.137\right)^3$$

So, our absolute error is:

$$\begin{aligned}
&\left|\frac{4\pi}{3}\left(6378.137\right)^2\left(6356.752\right)-\frac{4}{3}\pi\left(6378.137\right)^3\right|\\
=&\frac{4\pi}{3}\left(6378.137\right)^2\left|6356.752-6378.137\right|\\
=&\frac{4\pi}{3}\left(6378.137\right)^2\left(21.385\right)\\
\approx&3.64\times 10^9\mathrm{km}^3
\end{aligned}$$

And our relative error is:

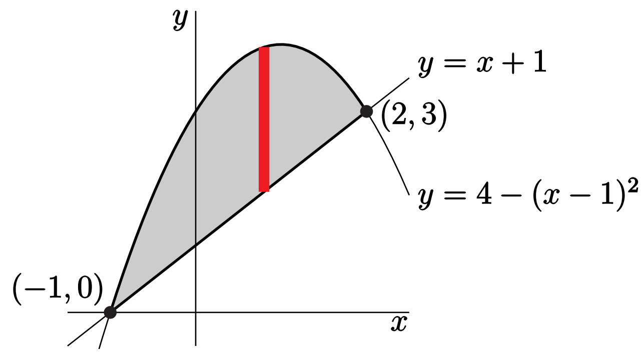
$$\begin{aligned}
\frac{\text{abs error}}{\text{actual value}} &= \frac{\frac{4\pi}{3}\left(6378.137\right)^2\left|6356.752-6378.137\right|}{\frac{4\pi}{3}\left(6378.137\right)^2\left(6356.752\right)}\\
&=\frac{\left|6356.752-6378.137\right|}{6356.752}\\
&=\frac{6378.137}{6356.752}-1\\
&\approx 0.00336
\end{aligned}$$

That is, about 0.336%, or about one-third of one percent.

1.6.2.18. $*$. **Solution**. (a) The curve $y = 4-(x-1)^2$ is an "upside down parabola" and line $y=x+1$ has slope 1. They intersect at points (x,y) which satisfy both $y=x+1$ and $y=4-(x-1)^2$. That is, when x obeys

$$\begin{aligned}
x+1 &= 4-(x-1)^2\\
x+1 &= 4-x^2+2x-1\\
x^2-x-2 &= 0\\
(x-2)(x+1) &= 0\\
x &= -1 \quad \text{or} \quad x=2
\end{aligned}$$

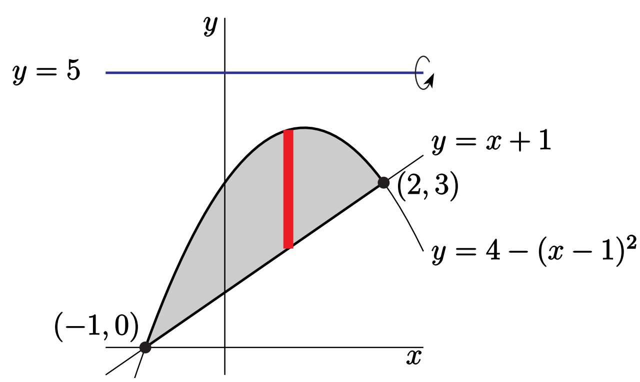
Thus the intersection points are $(-1,0)$ and $(2,3)$. Here is a sketch of R:



The red strip in the sketch above runs from $y = x + 1$ to $y = 4 - (x - 1)^2$ and so has area $[4 - (x - 1)^2 - (x + 1)]\,\mathrm{d}x = [2 + x - x^2]\,\mathrm{d}x$. All together R has

$$\begin{aligned}\text{Area} &= \int_{-1}^{2} \left[2 + x - x^2\right]\mathrm{d}x \\ &= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} \\ &= 6 + \frac{3}{2} - \frac{9}{3} = \frac{9}{2}\end{aligned}$$

(b) We'll use vertical washers as in Example 1.6.3. Note that the highest point achieved by $y = 4 - (x - 1)^2$ is $y = 4$, so rotating around the line $y = 5$ causes no unexpected problems.



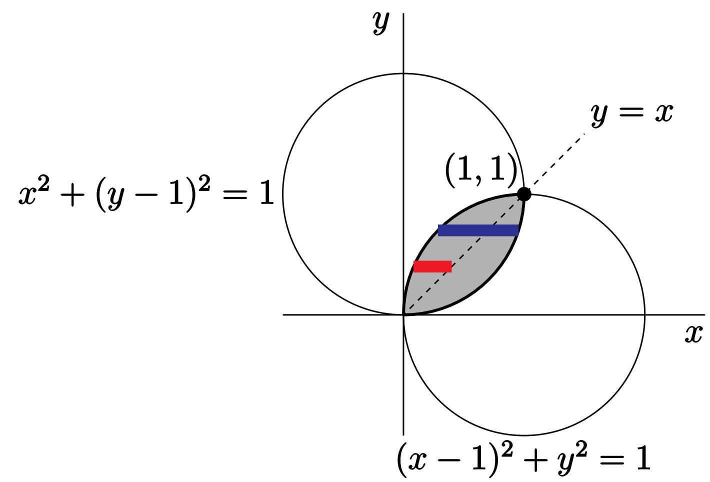
- We cut R into thin vertical strips of width $\mathrm{d}x$ like the red strip in the figure above.
- When we rotate R about the horizontal line $y = 5$, each strip sweeps out a thin washer
  - whose inner radius is $r_{in} = 5 - [4 - (x - 1)^2] = 1 + (x - 1)^2$, and
  - whose outer radius is $r_{out} = 5 - [x + 1] = 4 - x$ and
  - whose thickness is $\mathrm{d}x$ and hence

- whose volume is $\pi\left[r_{out}^2 - r_{in}^2\right]\mathrm{d}x = \pi\left[\left(4-x\right)^2 - \left(1+(x-1)^2\right)^2\right]\mathrm{d}x$

- As our leftmost strip is at $x = -1$ and our rightmost strip is at $x = 2$, the total

$$\text{Volume} = \pi\int_{-1}^{2}\left[\left(4-x\right)^2 - \left(1+(x-1)^2\right)^2\right]\mathrm{d}x$$

1.6.2.19. *. Solution. (a) The curves $(x-1)^2 + y^2 = 1$ and $x^2 + (y-1)^2 = 1$ are circles of radius 1 centered on $(1,0)$ and $(0,1)$ respectively. Both circles pass through $(0,0)$ and $(1,1)$. They are sketched below.



The region $\mathcal{R}$ is symmetric about the line $y = x$, so the area of $\mathcal{R}$ is twice the area of the part of $\mathcal{R}$ to the left of the line $y = x$. The red strip in the sketch above runs from the edge of the lower circle to $x = y$. So, given a value of y in $[0,1]$, we need to find the corresponding value of x along the circle. We solve $(x-1)^2 + y^2 = 1$ for x, keeping in mind that $0 \le x \le 1$:

$$\begin{aligned}
(x-1)^2 + y^2 &= 1\\
(x-1)^2 &= 1 - y^2\\
|x-1| &= \sqrt{1-y^2}\\
1 - x &= \sqrt{1-y^2}\\
x &= 1 - \sqrt{1-y^2}
\end{aligned}$$

Now, we calculate:

$$\begin{aligned}
\text{Area} &= 2\int_0^1 \left[y - \left(1 - \sqrt{1-y^2}\right)\right]\mathrm{d}y\\
&= 2\left\{\int_0^1 y - 1\mathrm{d}y + \int_0^1 \sqrt{1-y^2}\mathrm{d}y\right\}\\
&= 2\left\{\left[\frac{y^2}{2} - y\right]_0^1 + \int_0^1 \sqrt{1-y^2}\mathrm{d}y\right\}
\end{aligned}$$

$$= \frac{\pi}{2} - 1$$

Here the integral $\int_0^1 \sqrt{1-y^2}\mathrm{d}y$ was evaluated simply as the area of one quarter of a cicular disk of radius 1. It can also be evaluated by substituting $y = \sin\theta$, a technique we'll learn more about in Section 1.9.
(b) We'll use horizontal washers as in Example 1.6.5.

- We cut $\mathcal{R}$ into thin horizontal strips of width $\mathrm{d}y$ like the blue strip in the figure above.
- When we rotate $\mathcal{R}$ about the y–axis, each strip sweeps out a thin washer
  - whose inner radius is $r_{in} = 1 - \sqrt{1-y^2}$, and
  - whose outer radius is $r_{out} = \sqrt{1-(y-1)^2}$ and
  - whose thickness is $\mathrm{d}y$ and hence
  - whose volume is

$$\begin{aligned}&\pi\big[\big(\sqrt{1-(y-1)^2}\big)^2 - \big(1-\sqrt{1-y^2}\,\big)^2\big]\,\mathrm{d}y\\ =&\pi\big[1-(y-1)^2 - 1 + 2\sqrt{1-y^2} - (1-y^2)\big]\\ =&2\pi\big[\sqrt{1-y^2} + y - 1\big]\,\mathrm{d}y\end{aligned}$$

- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = 1$, the total

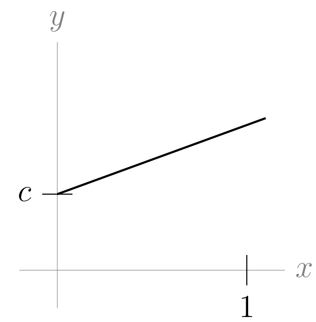
$$\begin{aligned}\text{Volume} &= 2\pi\int_0^1 \big[\sqrt{1-y^2} + y - 1\big]\mathrm{d}y = 2\pi\Big[\frac{\pi}{4} + \frac{1}{2} - 1\Big]\\ &= \frac{\pi^2}{2} - \pi \approx 1.793\end{aligned}$$

  Here, we again used that $\int_0^1 \sqrt{1-y^2}\,\mathrm{d}y$ is the area of a quarter circle of radius one, and we used a calculator to approximate the final answer.

1.6.2.20. $*$. Solution. Before we start, it will be useful to have a reasonable sketch of the graph $y = c\sqrt{1+x^2}$ over the interval $[0,1]$. Its endpoints are $(0,c)$ and $(1, c\sqrt{2})$. The function is entirely above the x-axis, which we need to know for part (a). For part (b), we need to know whether it is always increasing or not: when we're drawing horizontal strips, we need to know their endpoints, and if the function has "humps," the right endpoint will not be simply the line $x = 1$.
If you're comfortable noticing that $1+x^2$ increases as x increases because we only consider nonnegative values of x, then you can also be confident that $\sqrt{1+x^2}$ is simply increasing. Alternately, we can consider the derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{c\sqrt{1+x^2}\right\} = c\cdot\frac{1}{2\sqrt{1+x^2}}\cdot 2x = \frac{cx}{\sqrt{1+x^2}}$$

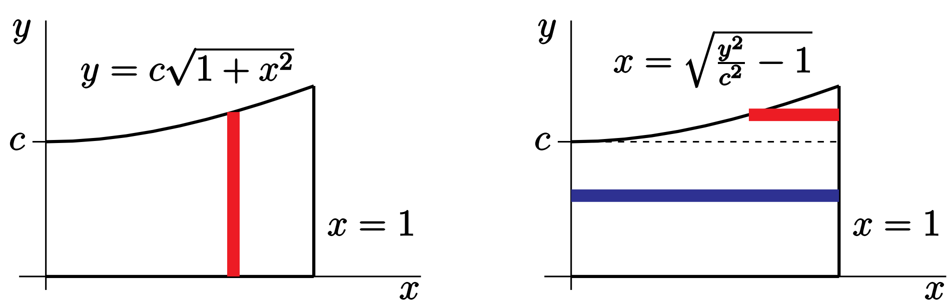
Since we only consider positive values of x, this derivative is never negative, so the function is never decreasing. This gives us the following basic sketch:



The figures in the solution below use a slightly more detailed rendering of our function, but so much accuracy is not necessary.

(a) Let $\mathcal{V}_1$ be the solid obtained by revolving $\mathcal{R}$ about the x–axis. The portion of $\mathcal{V}_1$ with x–coordinate between x and $x + \mathrm{d}x$ is obtained by rotating the red vertical strip in the figure on the left below about the x–axis. That portion is a disk of radius $c\sqrt{1+x^2}$ and thickness $\mathrm{d}x$. The volume of this disk is $\pi(c\sqrt{1+x^2})^2\mathrm{d}x = \pi c^2(1+x^2)\,\mathrm{d}x$. So the total volume of $\mathcal{V}_1$ is

$$V_1 = \int_0^1 \pi c^2(1+x^2)\,\mathrm{d}x = \pi c^2\Big[x + \frac{x^3}{3}\Big]_0^1 = \frac{4}{3}\pi c^2$$



(b) We'll use horizontal washers as in Example 1.6.5.

- We cut $\mathcal{R}$ into thin horizontal strips of width $\mathrm{d}y$ as in the figure on the right above.
- When we rotate $\mathcal{R}$ about the y–axis, i.e. about the line $x = 0$, each strip sweeps out a thin washer
    - whose outer radius is $r_{out} = 1$, and
    - whose inner radius is $r_{in} = \sqrt{\frac{y^2}{c^2} - 1}$ when $y \ge c\sqrt{1+0^2} = c$ (see the red strip in the figure on the right above), and whose inner radius is $r_{in} = 0$ when $y \le c$ (see the blue strip in the figure on the right above) and
    - whose thickness is $\mathrm{d}y$ and hence
    - whose volume is $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}y = \pi\big(2 - \frac{y^2}{c^2}\big)\mathrm{d}y$ when $y \ge c$ and whose volume is $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}y = \pi\,\mathrm{d}y$ when $y \le c$ and

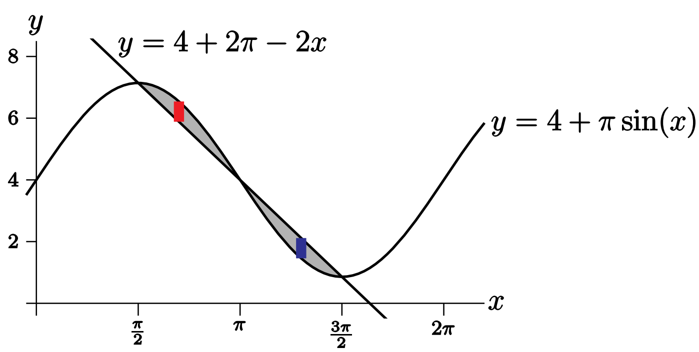
- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = \sqrt{2}\,c$ (since at the top $x = 1$ and $y = c\sqrt{1+x^2}$), the total

$$\begin{aligned}
V_2 &= \int_c^{\sqrt{2}\,c} \pi\Big(2 - \frac{y^2}{c^2}\Big)\,\mathrm{d}y + \int_0^c \pi\,\mathrm{d}y \\
&= \pi\Big[2y - \frac{y^3}{3c^2}\Big]_c^{\sqrt{2}\,c} + \pi c \\
&= \pi\,c\Big[\frac{4\sqrt{2}}{3} - \frac{5}{3}\Big] + \pi c \\
&= \frac{\pi\,c}{3}\big[4\sqrt{2} - 2\big]
\end{aligned}$$

(c) We have $V_1 = V_2$ if and only if

$$\begin{aligned}
\frac{4}{3}\pi c^2 &= \frac{\pi\,c}{3}\big[4\sqrt{2} - 2\big] \\
4c^2 &= c\left(4\sqrt{2} - 2\right) \\
4c^2 - c\left(4\sqrt{2} - 2\right) &= 0 \\
4c\left(c - \left(\sqrt{2} - \frac{1}{2}\right)\right) &= 0 \\
c = 0 \quad \text{or} \quad c &= \sqrt{2} - \frac{1}{2}
\end{aligned}$$

1.6.2.21. *. Solution. We will compute the volume by rotating thin vertical strips as in the sketch



about the line $y = -1$ to generate thin washers. We need to know when the line $y = 4+2\pi-2x$ intersects the curve $y = 4+\pi\sin x$. Looking at the graph, it appears to be at $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$. By plugging in these values of x to both functions, we see they are indeed the points of intersection.

- When $\frac{\pi}{2} \le x \le \pi$, the top of the strip is at $y = 4 + \pi\sin x$ and the bottom of the strip is at $y = 4 + 2\pi - 2x$. When the strip is rotated, we get a thin washer with outer radius $R_1(x) = 1+4+\pi\sin x = 5+\pi\sin x$ and inner radius $r_1(x) = 1 + 4 + 2\pi - 2x = 5 + 2\pi - 2x$.

- When $\pi \le x \le \frac{3\pi}{2}$, the top of the strip is at $y = 4 + 2\pi - 2x$ and the bottom of the strip is at $y = 4 + \pi \sin x$. When the strip is rotated, we get a thin washer with outer radius $R_2(x) = 1 + 4 + 2\pi - 2x = 5 + 2\pi - 2x$ and inner radius $r_2(x) = 1 + 4 + \pi \sin x = 5 + \pi \sin x$.

So, the total

$$\begin{aligned}\text{Volume} &= \int_{\pi/2}^{\pi} \pi\left[R_1(x)^2 - r_1(x)^2\right]\mathrm{d}x + \int_{\pi}^{3\pi/2} \pi\left[R_2(x)^2 - r_2(x)^2\right]\mathrm{d}x \\ &= \int_{\pi/2}^{\pi} \pi\left[(5+\pi\sin x)^2 - (5+2\pi-2x)^2\right]\mathrm{d}x \\ &+ \int_{\pi}^{3\pi/2} \pi\left[(5+2\pi-2x)^2 - (5+\pi\sin x)^2\right]\,\mathrm{d}x\end{aligned}$$

1.6.2.22. Solution. (a)
We use the same ideas for volume, and apply them to mass. We want to take slices of the column, approximate their mass, then add them up. To reconcile our units, let $k = 1000h$, so k is the height in metres. Then the density of air at height k is $c2^{-k/6000}\frac{\text{kg}}{\text{m}^3}$.
A horizontal slice of the column is a circular disk with height $\mathrm{d}k$ and radius 1 m. So, its volume is $\pi \mathrm{d}k\text{m}^3$. What we're interested in, though, is its mass. At height k, its mass is

$$\begin{aligned}(\text{volume}) \times (\text{density}) &= \left(\pi \mathrm{d}k\text{m}^3\right) \times \left(c2^{-k/6000}\frac{\text{kg}}{\text{m}^3}\right) \\ &= c\pi 2^{-k/6000}\mathrm{d}k \quad \text{kg}\end{aligned}$$

Since k runs from 0 to $60,000$, the total mass is given by

$$\int_0^{60000} c\pi 2^{-k/6000}\mathrm{d}k = c\pi \int_0^{60000} 2^{-k/6000}\mathrm{d}k$$

To facilitate integration, we can write our exponential function in terms of e, then use the substitution $u = -\frac{k}{6000}\log 2$, $\mathrm{d}u = -\frac{1}{6000}\log 2\mathrm{d}k$.

$$\begin{aligned} &= c\pi \int_0^{60000} \left(e^{\log 2}\right)^{-k/6000}\mathrm{d}k \\ &= c\pi \int_0^{60000} e^{-\frac{k}{6000}\log 2}\mathrm{d}k \\ &= -\frac{6000c\pi}{\log 2}\int_0^{-10\log 2} e^u \mathrm{d}u \\ &= \frac{6000c\pi}{\log 2}\int_{-10\log 2}^{0} e^u \mathrm{d}u \\ &= \frac{6000c\pi}{\log 2}\left(1 - \frac{1}{2^{10}}\right)\end{aligned}$$

We note this is fairly close to $\dfrac{6000c\pi}{\log 2}$.

We also remark that this is a demonstration of the usefulness of integrals. We wanted to know how much of *something* there was, but the amount of that something was different everywhere: more in some places, less in others. Integration allowed us to account for this gradient. You've seen this behaviour exploited to find distances travelled, areas, volumes, and now mass. In your studies, you will doubtless learn to use it to find still more quantities, and we will discuss other applications in Chapter 2.

(b) We want to find the value of k that gives a mass of $\dfrac{3000c\pi}{\log 2}$. By following our reasoning above, the mass of air in the column from the ground to height k is

$$\frac{6000c\pi}{\log 2}\left(1-\frac{1}{2^{k/6000}}\right)$$

So, we set this equal to the mass we want, and solve for k.

$$\begin{aligned}\frac{6000c\pi}{\log 2}\left(1-\frac{1}{2^{k/6000}}\right) &= \frac{3000c\pi}{\log 2}\\ 2\left(1-\frac{1}{2^{k/6000}}\right) &= 1\\ 1 &= \frac{2}{2^{k/6000}}\\ 2^{k/6000} &= 2^1\\ k &= 6000\\ h &= 6\end{aligned}$$

This means that there is roughly the same mass of air in the lowest 6 km of the column as there is in the remaining 54 km.

1.7 · Integration by parts

1.7.2 · Exercises

Exercises — Stage 1

1.7.2.1. Solution. Integration by substitution is just using the chain rule, backwards:

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\} = f'(g(x))g'(x)\\ \Leftrightarrow \quad & \int \frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\}\mathrm{d}x = \int f'(g(x))g'(x)\mathrm{d}x\\ \Leftrightarrow \quad & \underbrace{f(g(x))}_{f(u)} + C = \int \underbrace{f'(g(x))}_{f'(u)}\underbrace{g'(x)\mathrm{d}x}_{\mathrm{d}u}\end{aligned}$$

Similarly, integration by parts comes from the product rule:

$$\begin{aligned}
& & \frac{\mathrm{d}}{\mathrm{d}x}\{f(x)g(x)\} &= f'(x)g(x) + f(x)g'(x) \\
&\Leftrightarrow & \int \frac{\mathrm{d}}{\mathrm{d}x}\{f(x)g(x)\}\mathrm{d}x &= \int f'(x)g(x) + f(x)g'(x)\mathrm{d}x \\
&\Leftrightarrow & f(x)g(x) + C &= \int f'(x)g(x)\mathrm{d}x + \int f(x)g'(x)\mathrm{d}x \\
&\Leftrightarrow & \int \underbrace{f(x)}_{u}\underbrace{g'(x)\mathrm{d}x}_{\mathrm{d}v} &= \underbrace{f(x)}_{u}\underbrace{g(x)}_{v} - \int \underbrace{g(x)}_{v}\underbrace{f'(x)\mathrm{d}x}_{\mathrm{d}u}
\end{aligned}$$

In the last line, the "$+C$" has been absorbed into the indefinite integral on the right hand side.

1.7.2.2. Solution. Remember our rule: $\int u\mathrm{d}v = uv - \int v\mathrm{d}u$. So, we take u and use it to make $\mathrm{d}u$–that is, we differentiate it. We take $\mathrm{d}v$ and use it to make v–that is, we antidifferentiate it.

1.7.2.3. Solution. We'll use the same ideas that led to the method of integration by parts. (You can review this in your text, or see the solution to Question 1 in this section.) According to the quotient rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

Antidifferentiating both sides gives us:

$$\begin{aligned}
\int \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\}\mathrm{d}x &= \int \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}\mathrm{d}x \\
\frac{f(x)}{g(x)} + C &= \int \frac{f'(x)}{g(x)}\mathrm{d}x - \int \frac{f(x)g'(x)}{g^2(x)}\mathrm{d}x \\
\int \frac{f'(x)}{g(x)}\mathrm{d}x &= \frac{f(x)}{g(x)} + \int \frac{f(x)g'(x)}{g^2(x)}\mathrm{d}x
\end{aligned}$$

In the last line, the "$+C$" has been absorbed into the indefinite integral on the right hand side.
This is exactly the integration by parts formula for the functions $u = 1/g$ and $v = f$.

1.7.2.4. Solution. All the antiderivatives differ only by a constant, so we can write them all as $v(x) + C$ for some C. Then, using the formula for integration by parts,

$$\begin{aligned}
\int u(x)\cdot v'(x)\mathrm{d}x &= \underbrace{u(x)}_{u}\underbrace{[v(x)+C]}_{v} - \int \underbrace{[v(x)+C]}_{v}\underbrace{u'(x)\mathrm{d}x}_{\mathrm{d}u} \\
&= u(x)v(x) + Cu(x) - \int v(x)u'(x)\mathrm{d}x - \int Cu'(x)\mathrm{d}x
\end{aligned}$$

$$\begin{aligned}&= u(x)v(x) + Cu(x) - \int v(x)u'(x)\mathrm{d}x - Cu(x) + D\\&= u(x)v(x) - \int v(x)u'(x)\mathrm{d}x + D\end{aligned}$$

where D is any constant.
Since the terms with C cancel out, it didn't matter what we chose for C–all choices end up the same.

1.7.2.5. Solution. Suppose we choose $\mathrm{d}v = f(x)\mathrm{d}x$, $u = 1$. Then $v = \displaystyle\int f(x)\mathrm{d}x$, and $\mathrm{d}u = \mathrm{d}x$. So, our integral becomes:

$$\int \underbrace{(1)}_{u}\underbrace{f(x)\mathrm{d}x}_{\mathrm{d}v} = \underbrace{(1)}_{u}\underbrace{\int f(x)\mathrm{d}x}_{v} - \int \underbrace{\left(\int f(x)\mathrm{d}x\right)}_{v}\underbrace{\mathrm{d}x}_{\mathrm{d}u}$$

In order to figure out the first product (and the second integrand), you need to know the antiderivative of $f(x)$–but that's exactly what you're trying to figure out! So, using integration by parts has not eased your pain.
We note here that in certain cases, such as $\int \log x\mathrm{d}x$ (Example 1.7.8 in your text), it is useful to choose $\mathrm{d}v = 1\mathrm{d}x$. This is similar to, but crucially different from, the do-nothing method in this problem.

Exercises — Stage 2

1.7.2.6. *. Solution. For integration by parts, we want to break the integrand into two pieces, multiplied together. There is an obvious choice for how to do this: one piece is x, and the other is $\log x$. Remember that one piece will be integrated, while the other is differentiated. The question is, which choice will be more helpful. After some practice, you'll get the hang of making the choice. For now, we'll present both choices–but when you're writing a solution, you don't have to write this part down. It's enough to present your choice, and then a successful computation is justification enough.

Option 1:	$u = x$ $\mathrm{d}v = \log x\ \mathrm{d}x$	$\mathrm{d}u = 1\ \mathrm{d}x$ $v = ??$
Option 2:	$u = \log x$ $\mathrm{d}v = x\mathrm{d}x$	$\mathrm{d}u = \frac{1}{x}\ \mathrm{d}x$ $v = \frac{1}{2}x^2$

In Example 1.7.8, we found the antiderivative of logarithm, but it wasn't trivial. We might reasonably want to avoid using this complicated antiderivative. Indeed, Option 2 (differentiating logarithm, antidifferentiating x) looks promising–when we multiply the blue equations, we get something easily integrable– so let's not even bother going deeper into Option 1.
That is, we perform integration by parts with $u = \log x$ and $\mathrm{d}v = x\,\mathrm{d}x$, so that

$\mathrm{d}u = \frac{\mathrm{d}x}{x}$ and $v = \frac{x^2}{2}$.

$$\int \underbrace{x}_{u} \underbrace{\log x \,\mathrm{d}x}_{\mathrm{d}v} = \underbrace{\frac{x^2 \log x}{2}}_{uv} - \int \underbrace{\frac{x^2}{2}}_{v} \underbrace{\frac{\mathrm{d}x}{x}}_{\mathrm{d}u} = \frac{x^2 \log x}{2} - \frac{1}{2}\int x\mathrm{d}x$$
$$= \frac{x^2 \log x}{2} - \frac{x^2}{4} + C$$

1.7.2.7. *. Solution. Our integrand is the product of two functions, and there's no clear substitution. So, we might reasonably want to try integration by parts. Again, we have two obvious pieces: $\log x$, and x^{-7}. We'll consider our options for assigning these to u and $\mathrm{d}v$:

Option 1:	$u = \log x$ $\mathrm{d}v = x^{-7}\ \mathrm{d}x$	$\mathrm{d}u = \frac{1}{x}\ \mathrm{d}x$ $v = \frac{1}{-6}x^{-6}$
Option 2:	$u = x^{-7}$ $\mathrm{d}v = \log x \mathrm{d}x$	$\mathrm{d}u = -7x^{-8}\ \mathrm{d}x$ $v = ??$

Again, we remember that logarithm has some antiderivative we found in Example 1.7.8, but it was something complicated. Luckily, we don't need to bother with it: when we multiply the red equations in Option 1, we get a perfectly workable integral.

We perform integration by parts with $u = \log x$ and $\mathrm{d}v = x^{-7}\,\mathrm{d}x$, so that $\mathrm{d}u = \frac{\mathrm{d}x}{x}$ and $v = -\frac{x^{-6}}{6}$.

$$\int \frac{\log x}{x^7}\,\mathrm{d}x = \underbrace{-\log x\,\frac{x^{-6}}{6}}_{uv} + \int \underbrace{\frac{x^{-6}}{6}}_{-v} \underbrace{\frac{\mathrm{d}x}{x}}_{\mathrm{d}u} = -\frac{\log x}{6x^6} + \frac{1}{6}\int x^{-7}\mathrm{d}x$$
$$= -\frac{\log x}{6x^6} - \frac{1}{36x^6} + C$$

1.7.2.8. *. Solution. To integrate by parts, we need to decide what to use as u, and what to use as $\mathrm{d}v$. The salient parts of this integrand are x and $\sin x$, so we only need to decide which is u and which $\mathrm{d}v$. Again, this process will soon become familiar, but to help you along we show you both options below.

Option 1:	$u = x$ $\mathrm{d}v = \sin x\ \mathrm{d}x$	$\mathrm{d}u = 1\ \mathrm{d}x$ $v = -\cos x$
Option 2:	$u = \sin x$ $\mathrm{d}v = x\mathrm{d}x$	$\mathrm{d}u = \cos x\ \mathrm{d}x$ $v = \frac{1}{2}x^2$

The derivative and antiderivative of sine are almost the same, but x turns into

something simpler when we differentiate it. So, we choose Option 1.
We integrate by parts, using $u = x$, $\mathrm{d}v = \sin x\,\mathrm{d}x$ so that $v = -\cos x$ and $\mathrm{d}u = \mathrm{d}x$:

$$\int_0^\pi x\sin x\,\mathrm{d}x = \underbrace{-x\cos x}_{uv}\Big|_0^\pi - \int_0^\pi \underbrace{(-\cos x)}_{v}\underbrace{\mathrm{d}x}_{\mathrm{d}u}$$
$$= \Big[-x\cos x + \sin x\Big]_0^\pi = -\pi(-1) = \pi$$

1.7.2.9. ***.** **Solution**. When we have two functions multiplied like this, and there's no obvious substitution, our minds turn to integration by parts. We hope that our integral will be improved by differentiating one part and antidifferentiating the other. Let's see what our choices are:

Option 1:	$u = x$ $\mathrm{d}v = \cos x\ \mathrm{d}x$	$\mathrm{d}u = 1\ \mathrm{d}x$ $v = \sin x$
Option 2:	$u = \cos x$ $\mathrm{d}v = x\mathrm{d}x$	$\mathrm{d}u = -\sin x\ \mathrm{d}x$ $v = \frac{1}{2}x^2$

Option 1 seems preferable. We integrate by parts, using $u = x$, $\mathrm{d}v = \cos x\,\mathrm{d}x$ so that $v = \sin x$ and $\mathrm{d}u = \mathrm{d}x$:

$$\int_0^{\frac{\pi}{2}} x\cos x\,\mathrm{d}x = \underbrace{x}_{u}\underbrace{\sin x}_{v}\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \underbrace{\sin x}_{v}\underbrace{\mathrm{d}x}_{\mathrm{d}u}$$
$$= \Big[x\sin x + \cos x\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1$$

1.7.2.10. **Solution**. This integrand is the product of two functions, with no obvious substitution. So, let's try integration by parts, with one part e^x and one part x^3.

Option 1:	$u = e^x$ $\mathrm{d}v = x^3\ \mathrm{d}x$	$\mathrm{d}u = e^x\ \mathrm{d}x$ $v = \frac{1}{4}x^4$
Option 2:	$u = x^3$ $\mathrm{d}v = e^x\mathrm{d}x$	$\mathrm{d}u = 3x^2\ \mathrm{d}x$ $v = e^x$

At first glance, multiplying the red functions and multiplying the blue functions give largely equivalent integrands to what we started with–none of them with obvious antiderivatives. In previous questions, we were able to choose $u = x$, and then $\mathrm{d}u = \mathrm{d}x$, so the "$x$" in the integrand effectively went away. Here, we see that choosing $u = x^3$ will lead to $\mathrm{d}u = 3x^2\mathrm{d}x$, which has a lower power. If we repeatedly perform integration by parts, choosing u to be the power of x each time, then after a few iterations it should go away, because the third derivative of x^3 is a constant. So, we start with Option 2: $u = x^3$, $\mathrm{d}v = e^x\mathrm{d}x$, $\mathrm{d}u = 3x^2\mathrm{d}x$, and $v = e^x$.

$$\int x^3e^x\mathrm{d}x = \underbrace{x^3}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\cdot\underbrace{3x^2\mathrm{d}x}_{\mathrm{d}u}$$

$$= x^3e^x - 3\int e^x \cdot x^2 \mathrm{d}x$$

Now, we take $u = x^2$ and $\mathrm{d}v = e^x\mathrm{d}x$, so $\mathrm{d}u = 2x\mathrm{d}x$ and $v = e^x$. We're only using integration by parts on the actual integral–the rest of the function stays the way it is.

$$\begin{aligned} &= x^3e^x - 3\left[\underbrace{x^2e^x}_{uv} - \int \underbrace{e^x}_{v} \cdot \underbrace{2x\mathrm{d}x}_{\mathrm{d}u}\right] \\ &= x^3e^x - 3x^2e^x + 6\int xe^x\mathrm{d}x \end{aligned}$$

Continuing, we take $u = x$ and $\mathrm{d}v = e^x\mathrm{d}x$, so $\mathrm{d}u = \mathrm{d}x$ and $v = e^x$. This is the step where the polynomial part of the integrand finally disappears.

$$\begin{aligned} &= x^3e^x - 3x^2e^x + 6\left[\underbrace{xe^x}_{uv} - \int \underbrace{e^x}_{v}\underbrace{\mathrm{d}x}_{\mathrm{d}u}\right] \\ &= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C \\ &= e^x\left(x^3 - 3x^2 + 6x - 6\right) + C \end{aligned}$$

Let's check that this makes sense: the derivative of $e^x\left(x^3 - 3x^2 + 6x - 6\right) + C$ should be x^3e^x. We differentiate using the product rule.

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\left\{e^x\left(x^3 - 3x^2 + 6x - 6\right) + C\right\} \\ &\qquad = e^x\left(x^3 - 3x^2 + 6x - 6\right) + e^x\left(3x^2 - 6x + 6\right) \\ &\qquad = e^x\left(x^3 - 3x^2 + 3x^2 + 6x - 6x - 6 + 6\right) = x^3e^x \end{aligned}$$

Remark: In order to be technically correct in our antidifferentiation, we should add the $+C$ as soon as we do the first integration by parts. However, when we are using integration by parts, we usually end up evaluating an integral at the end, and we add the $+C$ at that point. Since the $+C$ comes up eventually, it is common practice to not clutter our calculations with it until the end.

1.7.2.11. Solution. Since our integrand is two functions multiplied together, and there isn't an obvious substitution, let's try integration by parts. Here are our salient options.

Option 1:	$u = x$ $\mathrm{d}v = \log^3 x\ \mathrm{d}x$	$\mathrm{d}u = 1\ \mathrm{d}x$ $v = ??$
Option 2:	$u = \log^3 x$ $\mathrm{d}v = x\mathrm{d}x$	$\mathrm{d}u = 3\log^2 x \cdot \frac{1}{x}\ \mathrm{d}x$ $v = \dfrac{1}{2}x^2$

This calls for some strategizing. Using the template of Example 1.7.8, we could probably figure out the antiderivative of $\log^3 x$. Option 1 is tempting, because our x-term goes away. So, there might be a benefit there, but on the other hand, the

antiderivative of $\log^3 x$ is probably pretty complicated.
Now let's consider Option 2. When we multiply the blue functions together, we get something similar to our original integrand, but the power of logarithm is smaller. If we were to iterate this method (using integration by parts a few times, always choosing u to be the part with a logarithm) then eventually we would end up differentiating logarithm. This seems like a safer plan: let's do Option 2.
We use integration by parts with $u = \log^3 x$, $\mathrm{d}v = x\mathrm{d}x$, $\mathrm{d}u = \frac{3}{x}\log^2 x\mathrm{d}x$, and $v = \frac{1}{2}x^2$.

$$\begin{aligned}
\int x\log^3 x\mathrm{d}x &= \underbrace{\frac{1}{2}x^2\log^3 x}_{uv} - \int \underbrace{\frac{3}{2}x\log^2 x\mathrm{d}x}_{v\mathrm{d}u} \\
&= \frac{1}{2}x^2\log^3 x - \frac{3}{2}\int x\log^2 x\mathrm{d}x
\end{aligned}$$

Continuing our quest to differentiate away the logarithm, we use integration by parts with $u = \log^2 x$, $\mathrm{d}v = x\mathrm{d}x$, $\mathrm{d}u = \dfrac{2}{x}\log x\mathrm{d}x$, and $v = \dfrac{1}{2}x^2$.

$$\begin{aligned}
&= \frac{1}{2}x^2\log^3 x - \frac{3}{2}\left[\underbrace{\frac{1}{2}x^2\log^2 x}_{uv} - \int \underbrace{x\log x\mathrm{d}x}_{v\mathrm{d}u}\right] \\
&= \frac{1}{2}x^2\log^3 x - \frac{3}{4}x^2\log^2 x + \frac{3}{2}\int x\log x\mathrm{d}x
\end{aligned}$$

One last integration by parts: $u = \log x$, $\mathrm{d}v = x\mathrm{d}x$, $\mathrm{d}u = \dfrac{1}{x}\mathrm{d}x$, and $v = \dfrac{1}{2}x^2$.

$$\begin{aligned}
&= \frac{1}{2}x^2\log^3 x - \frac{3}{4}x^2\log^2 x + \frac{3}{2}\left[\underbrace{\frac{1}{2}x^2\log x}_{uv} - \int \underbrace{\frac{1}{2}x\mathrm{d}x}_{v\mathrm{d}u}\right] \\
&= \frac{1}{2}x^2\log^3 x - \frac{3}{4}x^2\log^2 x + \frac{3}{4}x^2\log x - \frac{3}{4}\int x\mathrm{d}x \\
&= \frac{1}{2}x^2\log^3 x - \frac{3}{4}x^2\log^2 x + \frac{3}{4}x^2\log x - \frac{3}{8}x^2 + C
\end{aligned}$$

Once again, technically there is a $+C$ in the work after the first integration by parts, but we follow convention by conveniently suppressing it until the final integration.

1.7.2.12. Solution. The integrand is the product of two functions, without an obvious substitution, so let's see what integration by parts can do for us.

Option 1:	$u = x^2$ $\mathrm{d}v = \sin x\ \mathrm{d}x$	$\mathrm{d}u = 2x\ \mathrm{d}x$ $v = -\cos x$
Option 2:	$u = \sin x$ $\mathrm{d}v = x^2\mathrm{d}x$	$\mathrm{d}u = \cos x\ \mathrm{d}x$ $v = \frac{1}{3}x^3$

Neither option gives us something immediately integrable, but Option 1 replaces our x^2 term with a *lower* power of x. If we repeatedly apply integration by parts, we can reduce this power to zero. So, we start by choosing $u = x^2$ and $\mathrm{d}v = \sin x\mathrm{d}x$, so $\mathrm{d}u = 2x\mathrm{d}x$ and $v = -\cos x$.

$$\begin{aligned}\int x^2 \sin x\mathrm{d}x &= \underbrace{-x^2 \cos x}_{uv} + \underbrace{\int 2x\cos x\mathrm{d}x}_{-v\mathrm{d}u}\\ &= -x^2\cos x + 2\int x\cos x\mathrm{d}x\end{aligned}$$

Using integration by parts again, we want to be differentiating (not antidifferentiating) x, so we choose $u = x$, $\mathrm{d}v = \cos x\mathrm{d}x$, and then $\mathrm{d}u = \mathrm{d}x$ (x went away!), $v = \sin x$.

$$\begin{aligned}&= -x^2\cos x + 2\left[\underbrace{x\sin x}_{uv} - \int \underbrace{\sin x\mathrm{d}x}_{v\mathrm{d}u}\right]\\ &= -x^2\cos x + 2x\sin x + 2\cos x + C\\ &= (2-x^2)\cos x + 2x\sin x + C\end{aligned}$$

1.7.2.13. Solution. This problem is similar to Questions 6 and 7: integrating a polynomial multiplied by a logarithm. Just as in these questions, if we use integration by parts with $u = \log t$, then $\mathrm{d}u = \dfrac{1}{t}\mathrm{d}t$, and our new integrand will consist of powers of t–which are easy to antidifferentiate.
So, we use $u = \log t$, $\mathrm{d}v = 3t^2 - 5t + 6$, $\mathrm{d}u = \frac{1}{t}\mathrm{d}t$, and $v = t^3 - \frac{5}{2}t^2 + 6t$.

$$\begin{aligned}&\int (3t^2 - 5t + 6)\log t\mathrm{d}t\\ &\qquad= \underbrace{\log t}_{u}\left(\underbrace{t^3 - \frac{5}{2}t^2 + 6t}_{v}\right) - \int \underbrace{\frac{1}{t}\left(t^3 - \frac{5}{2}t^2 + 6t\right)\mathrm{d}t}_{v\mathrm{d}u}\\ &\qquad= \left(t^3 - \frac{5}{2}t^2 + 6t\right)\log t - \int\left(t^2 - \frac{5}{2}t + 6\right)\mathrm{d}t\\ &\qquad= \left(t^3 - \frac{5}{2}t^2 + 6t\right)\log t - \frac{1}{3}t^3 + \frac{5}{4}t^2 - 6t + C\end{aligned}$$

1.7.2.14. Solution. Before we jump to integration by parts, we notice that the square roots lend themselves to substitution. Let's take $w = \sqrt{s}$. Then $\mathrm{d}w = \dfrac{1}{2\sqrt{s}}\mathrm{d}s$, so $2w\mathrm{d}w = \mathrm{d}s$.

$$\int \sqrt{s}e^{\sqrt{s}}\mathrm{d}s = \int w\cdot e^w\cdot 2w\mathrm{d}w = 2\int w^2e^w\mathrm{d}w$$

Now we have nearly the situation of Question 10. We can repeatedly use integration by parts with u as the power of w to get rid of the polynomial part. We'll start with $u = w^2$ $\mathrm{d}v = e^w\mathrm{d}w$, $\mathrm{d}u = 2w\mathrm{d}w$, and $v = e^w$.

$$\begin{aligned} &= 2\left[\underbrace{w^2e^w}_{uv} - \int \underbrace{2we^w\mathrm{d}w}_{v\mathrm{d}u}\right] \\ &= 2w^2e^w - 4\int we^w\mathrm{d}w \end{aligned}$$

We use integration by parts again, this time with $u = w$, $\mathrm{d}v = e^w\mathrm{d}w$, $\mathrm{d}u = \mathrm{d}w$, and $v = e^w$.

$$\begin{aligned} &= 2w^2e^w - 4\left[\underbrace{we^w}_{uv} - \int \underbrace{e^w\mathrm{d}w}_{v\mathrm{d}u}\right] \\ &= 2w^2e^w - 4we^w + 4e^w + C \\ &= e^w\left(2w^2 - 4w + 4\right) + C \\ &= e^{\sqrt{s}}\left(2s - 4\sqrt{s} + 4\right) + C \end{aligned}$$

1.7.2.15. Solution. Let's use integration by parts. What are our parts? We have a few options.

- Solution 1: Following Example 1.7.8, we choose $u = \log^2 x$ and $\mathrm{d}v = \mathrm{d}x$, so that $\mathrm{d}u = \frac{2}{x}\log x\mathrm{d}x$ and $v = x$.

$$\int \log^2 x\mathrm{d}x = \underbrace{x\log^2 x}_{uv} - \int \underbrace{2\log x\mathrm{d}x}_{v\mathrm{d}u}$$

  Here we can either use the antiderivative of logarithm from memory, or re-derive it. We do the latter, using integration by parts with $u = \log x$, $\mathrm{d}v = 2\mathrm{d}x$, $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$, and $v = 2x$.

$$\begin{aligned} &= x\log^2 x - \left[\underbrace{2x\log x}_{uv} - \int \underbrace{2\mathrm{d}x}_{v\mathrm{d}u}\right] \\ &= x\log^2 x - 2x\log x + 2x + C \end{aligned}$$

- Solution 2: Our integrand is two functions multiplied together: $\log x$ and $\log x$. So, we will use integration by parts with $u = \log x$, $\mathrm{d}v = \log x$, $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$, and (using the antiderivative of logarithm, found in Example 1.7.8 in the text) $v = x\log x - x$.

$$\int \log^2 x\mathrm{d}x = \underbrace{(\log x)}_{u}\underbrace{(x\log x - x)}_{v} - \int \underbrace{(x\log x - x)}_{v}\underbrace{\frac{1}{x}\mathrm{d}x}_{\mathrm{d}u}$$

$$\begin{aligned}&= x\log^2 x - x\log x - \int (\log x - 1)\,\mathrm{d}x\\ &= x\log^2 x - x\log x - [(x\log x - x) - x] + C\\ &= x\log^2 x - 2x\log x + 2x + C\end{aligned}$$

1.7.2.16. Solution. This is your friendly reminder that to a person with a hammer, everything looks like a nail. The integral in the problem is a classic example of an integral to solve using substitution. We have an "inside function," x^2+1, whose derivative shows up multiplied to the rest of the integrand. We take $u = x^2+1$, then $\mathrm{d}u = 2x\mathrm{d}x$, so

$$\int 2xe^{x^2+1}\mathrm{d}x = \int e^u \mathrm{d}u = e^u + C = e^{x^2+1} + C$$

1.7.2.17. *. Solution. In Example 1.7.9, we saw that integration by parts was useful when the integrand has a derivative that works nicely when multiplied by x. We use the same idea here. Let $u = \arccos y$ and $\mathrm{d}v = \mathrm{d}y$, so that $v = y$ and $\mathrm{d}u = -\frac{\mathrm{d}y}{\sqrt{1-y^2}}$.

$$\int \arccos y\,\mathrm{d}y = \underbrace{y\arccos y}_{uv} + \underbrace{\int \frac{y}{\sqrt{1-y^2}}\mathrm{d}y}_{-v\mathrm{d}u}$$

Using the substitution $u = 1-y^2$, $\mathrm{d}u = -2y\mathrm{d}y$,

$$\begin{aligned}&= y\arccos y - \frac{1}{2}\int u^{-1/2}\mathrm{d}u\\ &= y\arccos y - u^{1/2} + C\\ &= y\arccos y - \sqrt{1-y^2} + C\end{aligned}$$

Exercises — Stage 3

1.7.2.18. *. Solution. We integrate by parts, using $u = \arctan(2y)$, $\mathrm{d}v = 4y\,\mathrm{d}y$, so that $v = 2y^2$ and $\mathrm{d}u = \frac{2\,\mathrm{d}y}{1+(2y)^2}$:

$$\int 4y\arctan(2y)\,\mathrm{d}y = \underbrace{2y^2\arctan(2y)}_{uv} - \underbrace{\int \frac{4y^2}{(2y)^2+1}\,\mathrm{d}y}_{v\mathrm{d}u}$$

The integrand $\frac{4y^2}{(2y)^2+1}$ is a rational function. So the remaining integral can be evaluated using the method of partial fractions, starting with long division. But it is easier to just notice that $\frac{4y^2}{4y^2+1} = \frac{4y^2+1}{4y^2+1} - \frac{1}{4y^2+1}$. We therefore have:

$$\int \frac{4y^2}{4y^2+1}\,\mathrm{d}y = \int \left(1 - \frac{1}{4y^2+1}\right)\,\mathrm{d}y = y - \frac{1}{2}\arctan(2y) + C$$

The final answer is then

$$\int 4y\arctan(2y)\,\mathrm{d}y = 2y^2\arctan(2y) - y + \frac{1}{2}\arctan(2y) + C$$

1.7.2.19. Solution. We've got an integrand that consists of two functions multiplied together, and no obvious substitution. So, we think about integration by parts. Let's consider our options. Note in Example 1.7.9, we found that the antiderivative of arctangent is $x\arctan x - \frac{1}{2}\log(1+x^2) + C$.

Option 1:	$u = \arctan x$	$\mathrm{d}u = \dfrac{1}{1+x^2}\,\mathrm{d}x$
	$\mathrm{d}v = x^2\,\mathrm{d}x$	$v = \dfrac{1}{3}x^3$
Option 2:	$u = x^2$	$\mathrm{d}u = 2x\,\mathrm{d}x$
	$\mathrm{d}v = \arctan x\mathrm{d}x$	$v = x\arctan x - \dfrac{1}{2}\log(1+x^2)$

- Option 1: Option 1 seems likelier. Let's see how it plays out. We use integration by parts with $u = \arctan x$, $\mathrm{d}v = x^2\mathrm{d}x$, $\mathrm{d}u = \frac{\mathrm{d}x}{1+x^2}$, and $v = \frac{1}{3}x^3$.

$$\begin{aligned}\int x^2\arctan x\mathrm{d}x &= \underbrace{\frac{x^3}{3}\arctan x}_{uv} - \underbrace{\int \frac{x^3}{3(1+x^2)}\mathrm{d}x}_{v\mathrm{d}u}\\ &= \frac{x^3}{3}\arctan x - \frac{1}{3}\int\frac{x^3}{1+x^2}\mathrm{d}x\end{aligned}$$

This is starting to look like a candidate for a substitution! Let's try the denominator, $s = 1+x^2$. Then $\mathrm{d}s = 2x\mathrm{d}x$, and $x^2 = s-1$.

$$\begin{aligned}&= \frac{x^3}{3}\arctan x - \frac{1}{6}\int\frac{x^2}{1+x^2}\cdot 2x\mathrm{d}x\\ &= \frac{x^3}{3}\arctan x - \frac{1}{6}\int\frac{s-1}{s}\mathrm{d}s\\ &= \frac{x^3}{3}\arctan x - \frac{1}{6}\int 1 - \frac{1}{s}\mathrm{d}s\\ &= \frac{x^3}{3}\arctan x - \frac{1}{6}s + \frac{1}{6}\log|s| + C\\ &= \frac{x^3}{3}\arctan x - \frac{1}{6}(1+x^2) + \frac{1}{6}\log(1+x^2) + C\end{aligned}$$

- Option 2: What if we had tried the other option? That is, $u = x^2$, $\mathrm{d}u = 2x\mathrm{d}x$, $\mathrm{d}v = \arctan x$, and $v = x\arctan x - \frac{1}{2}\log(1+x^2)$. It's not always the case that both options work, but sometimes they do. (They are almost never of equal difficulty.) This solution takes advantage of two previously hard-won results: the antiderivatives of logarithm and arctangent.

$$\begin{aligned}\int x^2 \arctan x \mathrm{d}x &= \underbrace{x^2}_{u} \underbrace{\left(x \arctan x - \frac{1}{2}\log(1+x^2)\right)}_{v} \\ &\qquad - \int \underbrace{\left(x \arctan x - \frac{1}{2}\log(1+x^2)\right)}_{v} \cdot \underbrace{2x\mathrm{d}x}_{\mathrm{d}u} \\ &= x^3 \arctan x - \frac{x^2}{2}\log(1+x^2) - 2\int x^2 \arctan x \mathrm{d}x \\ &\qquad + \int x \log(1+x^2)\mathrm{d}x\end{aligned}$$

Adding $\quad 2\int x^2 \arctan x \mathrm{d}x \quad$ to both sides:

$$\begin{aligned}3\int x^2 \arctan x \mathrm{d}x &= x^3 \arctan x - \frac{x^2}{2}\log(1+x^2) \\ &\qquad + \int x \log(1+x^2)\mathrm{d}x \\ \int x^2 \arctan x \mathrm{d}x &= \frac{x^3}{3}\arctan x - \frac{x^2}{6}\log(1+x^2) \\ &\qquad + \frac{1}{3}\int x \log(1+x^2)\mathrm{d}x\end{aligned}$$

Using the substitution $s = 1 + x^2$, $\mathrm{d}s = 2x\mathrm{d}x$:

$$= \frac{x^3}{3}\arctan x - \frac{x^2}{6}\log(1+x^2) + \frac{1}{6}\int \log s \mathrm{d}s$$

Using the antiderivative of logarithm found in Example 1.7.8,

$$\begin{aligned}&= \frac{x^3}{3}\arctan x - \frac{x^2}{6}\log(1+x^2) + \frac{1}{6}(s\log s - s) + C \\ &= \frac{x^3}{3}\arctan x - \frac{x^2}{6}\log(1+x^2) + \frac{1}{6}\big((1+x^2)\log(1+x^2) \\ &\qquad\qquad - (1+x^2)\big) + C \\ &= \frac{x^3}{3}\arctan x + \left[-\frac{x^2}{6} + \frac{1+x^2}{6}\right]\log(1+x^2) - \frac{1}{6}(1+x^2) + C \\ &= \frac{x^3}{3}\arctan x + \frac{1}{6}\log(1+x^2) - \frac{1}{6}(1+x^2) + C\end{aligned}$$

1.7.2.20. Solution. This example is similar to Example 1.7.10 in the text. The functions $e^{x/2}$ and $\cos(2x)$ both do not substantially alter when we differentiate or antidifferentiate them. If we use integration by parts twice, we'll end up with an expression that includes our original integral. Then we can just solve for the original

integral in the equation, without actually antidifferentiating.
Let's use $u = \cos(2x)$ and $\mathrm{d}v = e^{x/2}\mathrm{d}x$, so $\mathrm{d}u = -2\sin(2x)\mathrm{d}x$ and $v = 2e^{x/2}$.

$$\begin{aligned}\int e^{x/2}\cos(2x)\mathrm{d}x &= \underbrace{2e^{x/2}\cos(2x)}_{uv} - \int \underbrace{-4e^{x/2}\sin(2x)\mathrm{d}x}_{v\mathrm{d}u} \\ &= 2e^{x/2}\cos(2x) + 4\int e^{x/2}\sin(2x)\mathrm{d}x\end{aligned}$$

Similarly to our first integration by parts, we use $u = \sin(2x)$, $\mathrm{d}v = e^{x/2}\mathrm{d}x$, $\mathrm{d}u = 2\cos(2x)\mathrm{d}x$, and $v = 2e^{x/2}$.

$$= 2e^{x/2}\cos(2x) + 4\left[\underbrace{2e^{x/2}\sin(2x)}_{uv} - \int \underbrace{4e^{x/2}\cos(2x)\mathrm{d}x}_{v\mathrm{d}u}\right]$$

So, we've found the equation

$$\begin{aligned}\int e^{x/2}\cos(2x)\mathrm{d}x = 2e^{x/2}\cos(2x) &+ 8e^{x/2}\sin(2x) \\ &- 16\int e^{x/2}\cos(2x)\mathrm{d}x + C\end{aligned}$$

We add $\quad 16\int e^{x/2}\cos(2x)\mathrm{d}x \quad$ to both sides.

$$\begin{aligned}17\int e^{x/2}\cos(2x)\mathrm{d}x &= 2e^{x/2}\cos(2x) + 8e^{x/2}\sin(2x) + C \\ \int e^{x/2}\cos(2x)\mathrm{d}x &= \frac{2}{17}e^{x/2}\cos(2x) + \frac{8}{17}e^{x/2}\sin(2x) + C\end{aligned}$$

Remark: remember that C is a stand-in for "we can add any real constant". Since C can be any number in $(-\infty, \infty)$, also $\frac{C}{17}$ can be any number in $(-\infty, \infty)$. So, rather than write $\frac{C}{17}$ in the last line, we re-named $\frac{C}{17}$ to C.

1.7.2.21. Solution.

- Solution 1: This question looks like a substitution, since we have an "inside function." So, let's see where that leads: let $u = \log x$. Then $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$. We don't see this right away in our function, but we can bring it into the function by multiplying and dividing by x, and noting from our substitution that $e^u = x$.

$$\begin{aligned}\int \sin(\log x)\mathrm{d}x &= \int \frac{x\sin(\log x)}{x}\mathrm{d}x \\ &= \int e^u \sin u\mathrm{d}u\end{aligned}$$

Using the result of Example 1.7.11:

$$\begin{aligned}
&= \frac{1}{2}e^{u}\left(\sin u - \cos u\right) + C\\
&= \frac{1}{2}e^{\log x}\left(\sin(\log x) - \cos(\log x)\right) + C\\
&= \frac{1}{2}x\left(\sin(\log x) - \cos(\log x)\right) + C
\end{aligned}$$

- Solution 2: It's not clear how to antidifferentiate the integrand, but we can certainly differentiate it. So, keeping in mind the method of Example 1.7.11 in the text, we take $u = \sin(\log x)$ and $\mathrm{d}v = \mathrm{d}x$, so $\mathrm{d}u = \frac{1}{x}\cos(\log x)\mathrm{d}x$ and $v = x$.

$$\int \sin(\log x)\mathrm{d}x = \underbrace{x\sin(\log x)}_{uv} - \int \underbrace{\cos(\log x)\mathrm{d}x}_{v\mathrm{d}u}$$

Continuing on, we again use integration by parts, with $u = \cos(\log x)$, $\mathrm{d}v = \mathrm{d}x$, $\mathrm{d}u = -\frac{1}{x}\sin(\log x)\mathrm{d}x$, and $v = x$.

$$= x\sin(\log x) - \left[\underbrace{x\cos(\log x)}_{uv} + \int \underbrace{\sin(\log x)}_{-v\mathrm{d}u}\mathrm{d}x\right]$$

That is, we have

$$\int \sin(\log x)\mathrm{d}x = x\left[\sin(\log x) - \cos(\log x)\right] - \int \sin(\log x)\mathrm{d}x + C$$

Adding $\int \sin(\log x)\mathrm{d}x$ to both sides,

$$\begin{aligned}
2\int \sin(\log x)\mathrm{d}x &= x\left[\sin(\log x) - \cos(\log x)\right] + C\\
\int \sin(\log x)\mathrm{d}x &= \frac{x}{2}\left[\sin(\log x) - \cos(\log x)\right] + C
\end{aligned}$$

Remark: remember that C is a stand-in for "we can add any real constant". Since C can be any number in $(-\infty, \infty)$, also $\frac{C}{2}$ can be any number in $(-\infty, \infty)$. So, rather than write $\frac{C}{2}$ in the last line, we re-named $\frac{C}{2}$ to C.

1.7.2.22. Solution. We begin by simplifying the integrand.

$$\int 2^{x+\log_2 x}\mathrm{d}x = \int 2^{x}\cdot 2^{\log_2 x}\mathrm{d}x = \int 2^{x}\cdot x\mathrm{d}x$$

This is similar to the integral $\displaystyle\int xe^{x}\mathrm{d}x$, which we saw in Example 1.7.1. Let's write $2 = e^{\log 2}$ to take advantage of the easy integrability of e^{x}.

$$= \int x\cdot e^{x\log 2}\mathrm{d}x$$

We use integration by parts with $u = x$, $\mathrm{d}v = e^{x\log 2}\mathrm{d}x$; $\mathrm{d}u = \mathrm{d}x$, $v = \frac{1}{\log 2}e^{x\log 2}$. (Remember $\log 2$ is a constant. If you'd prefer, you can do a substitution with $s = x\log 2$ first, to have a simpler exponent of e.)

$$\begin{aligned}
&= \underbrace{\frac{x}{\log 2}e^{x\log 2}}_{uv} - \int \underbrace{\frac{1}{\log 2}e^{x\log 2}\mathrm{d}x}_{v\mathrm{d}u} \\
&= \frac{x}{\log 2}e^{x\log 2} - \frac{1}{(\log 2)^2}e^{x\log 2} + C \\
&= \frac{x}{\log 2}2^x - \frac{1}{(\log 2)^2}2^x + C
\end{aligned}$$

1.7.2.23. Solution. It's not obvious where to start, but in general it's nice to have the arguments of our trig functions the same. So, we use the identity $\sin(2x) = 2\sin x\cos x$.

$$\int e^{\cos x}\sin(2x)\mathrm{d}x = 2\int e^{\cos x}\cos x\sin x\,\mathrm{d}x$$

Now we can use the substitution $w = \cos x$, $\mathrm{d}w = -\sin x\mathrm{d}x$.

$$= -2\int we^w\mathrm{d}w$$

From here the integral should look more familiar. We can use integration by parts with $u = w$, $\mathrm{d}v = e^w\mathrm{d}w$, $\mathrm{d}u = \mathrm{d}w$, and $v = e^w$.

$$\begin{aligned}
&= -2\left[\underbrace{we^w}_{uv} - \int\underbrace{e^w\mathrm{d}w}_{v\mathrm{d}u}\right] \\
&= 2e^w\,[1-w] + C \\
&= 2e^{\cos x}[1-\cos x] + C
\end{aligned}$$

1.7.2.24. Solution. We've got an integrand that consists of several functions multiplied together, and no obvious substitution. So, we think about integration by parts. We know an antiderivative for $\frac{1}{(1-x)^2}$, because we know $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} = \frac{1}{(1-x)^2}$. So let's try $\mathrm{d}v = \frac{\mathrm{d}x}{(1-x)^2}$ and $u = xe^{-x}$. Then $v = \frac{1}{1-x}$ and $\mathrm{d}u = (1-x)e^{-x}\,\mathrm{d}x$. So, by integration by parts,

$$\begin{aligned}
\int \underbrace{xe^{-x}}_{u}\underbrace{\frac{\mathrm{d}x}{(1-x)^2}}_{\mathrm{d}v} &= \underbrace{\frac{xe^{-x}}{1-x}}_{uv} - \int\underbrace{\frac{1}{1-x}}_{v}\underbrace{(1-x)e^{-x}\mathrm{d}x}_{\mathrm{d}u} \\
&= \frac{xe^{-x}}{1-x} - \int e^{-x}\,\mathrm{d}x \\
&= \frac{xe^{-x}}{1-x} + e^{-x} + C = \frac{e^{-x}}{1-x} + C
\end{aligned}$$

1.7.2.25. $*$. Solution. (a) The "parts" in the integrand are powers of sine. Looking at the right hand side of the reduction formula, we see that it looks a little like the derivative of $\sin^{n-1} x$, although not exactly. So, let's integrate by parts with $u = \sin^{n-1} x$ and $\mathrm{d}v = \sin x\,\mathrm{d}x$, so that $\mathrm{d}u = (n-1)\sin^{n-2} x \cos x$ and $v = -\cos x$.

$$\int \sin^n x\,\mathrm{d}x = \underbrace{-\sin^{n-1} x\ \cos x}_{uv} + \underbrace{(n-1)\int \cos^2 x\ \sin^{n-2} x\ \mathrm{d}x}_{-\int v\mathrm{d}u}$$

Using the identity $\sin^2 x + \cos^2 x = 1$,

$$\begin{aligned}
&= -\sin^{n-1} x\ \cos x + (n-1)\int (1-\sin^2 x)\sin^{n-2} x\ \mathrm{d}x \\
&= -\sin^{n-1} x\ \cos x + (n-1)\int \sin^{n-2} x\ \mathrm{d}x - (n-1)\int \sin^n x\ \mathrm{d}x
\end{aligned}$$

Moving the last term on the right hand side to the left hand side gives

$$n\int \sin^n x\,\mathrm{d}x = -\sin^{n-1} x\ \cos x + (n-1)\int \sin^{n-2} x\ \mathrm{d}x$$

Dividing across by n gives the desired reduction formula.
(b) By the reduction formula of part (a), if $n \geq 2$,

$$\int_0^{\pi/2} \sin^n(x)\,\mathrm{d}x = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}(x)\,\mathrm{d}x$$

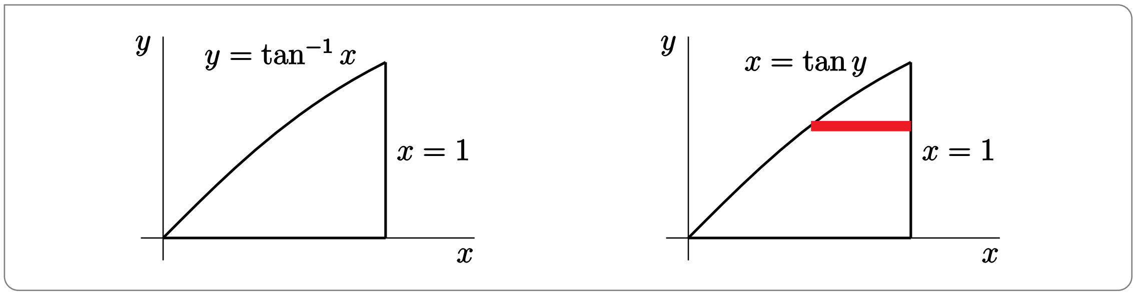
since $\sin 0 = \cos\frac{\pi}{2} = 0$. Applying this reduction formula, with $n = 8, 6, 4, 2$:

$$\begin{aligned}
\int_0^{\pi/2} \sin^8(x)\,\mathrm{d}x &= \frac{7}{8}\int_0^{\pi/2} \sin^6(x)\,\mathrm{d}x = \frac{7}{8}\cdot\frac{5}{6}\int_0^{\pi/2} \sin^4(x)\,\mathrm{d}x \\
&= \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\int_0^{\pi/2} \sin^2(x)\,\mathrm{d}x \\
&= \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\int_0^{\pi/2} \mathrm{d}x \\
&= \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \\
&= \frac{35}{256}\pi
\end{aligned}$$

Using a calculator, we see this is approximately 0.4295.

1.7.2.26. $*$. Solution. (a) The sketch is the figure on the left below. By integration by parts with $u = \arctan x$, $\mathrm{d}v = \mathrm{d}x$, $v = x$ and $\mathrm{d}u = \frac{1}{1+x^2}\,\mathrm{d}x$, and then the substitution $s = 1 + x^2$,

$$\begin{aligned} A &= \int_0^1 \arctan x \mathrm{d}x = \underbrace{x \arctan x}_{uv}\Big|_0^1 - \int_0^1 \underbrace{\frac{x}{1+x^2}\mathrm{d}x}_{v\mathrm{d}u} \\ &= \arctan 1 - \frac{1}{2}\log(1+x^2)\Big|_0^1 \\ &= \frac{\pi}{4} - \frac{\log 2}{2} \end{aligned}$$



(b) We'll use horizontal washers as in Example 1.6.5.

- We cut R into thin horizontal strips of width $\mathrm{d}y$ as in the figure on the right above.
- When we rotate R about the y–axis, each strip sweeps out a thin washer
  - whose inner radius is $r_{in} = \tan y$ and outer radius is $r_{out} = 1$, and
  - whose thickness is $\mathrm{d}y$ and hence
  - whose volume $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}y = \pi(1 - \tan^2 y)\mathrm{d}y$.
- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = \frac{\pi}{4}$ (since at the top $x = 1$ and $x = \tan y$), the total

$$\begin{aligned} \text{Volume} &= \int_0^{\frac{\pi}{4}} \pi(1 - \tan^2 y)\mathrm{d}y = \int_0^{\frac{\pi}{4}} \pi(2 - \sec^2 y)\mathrm{d}y \\ &= \pi\big[2y - \tan y\big]_0^{\frac{\pi}{4}} \\ &= \frac{\pi^2}{2} - \pi \end{aligned}$$

1.7.2.27. $*$. **Solution**. For a fixed value of x, if we rotate about the x-axis, we form a washer of inner radius $B(x)$ and outer radius $T(x)$ and hence of area $\pi[T(x)^2 - B(x)^2]$. We integrate this function from $x = 0$ to $x = 3$ to find the total volume V:

$$\begin{aligned} V &= \int_0^3 \pi[T(x)^2 - B(x)^2]\,\mathrm{d}x \\ &= \pi\int_0^3 (\sqrt{x}e^{3x})^2 - (\sqrt{x}(1+2x))^2\,\mathrm{d}x \end{aligned}$$

$$\begin{aligned}
&= \pi \int_0^3 \big(xe^{6x} - (x + 4x^2 + 4x^3)\big)\,\mathrm{d}x \\
&= \pi \int_0^3 xe^{6x}\,\mathrm{d}x - \pi\left[\frac{x^2}{2} + \frac{4x^3}{3} + x^4\right]_0^3 \\
&= \pi \int_0^3 xe^{6x}\,\mathrm{d}x - \pi\left[\frac{3^2}{2} + \frac{4\cdot 3^3}{3} + 3^4\right]
\end{aligned}$$

For the first integral, we use integration by parts with $u(x) = x$, $\mathrm{d}v = e^{6x}\mathrm{d}x$, so that $\mathrm{d}u = \mathrm{d}x$ and $v(x) = \frac{1}{6}e^{6x}$:

$$\begin{aligned}
\int_0^3 xe^{6x}\,\mathrm{d}x &= \underbrace{\frac{xe^{6x}}{6}}_{uv}\bigg|_0^3 - \int_0^3 \underbrace{\frac{1}{6}e^{6x}\,\mathrm{d}x}_{v\mathrm{d}u} \\
&= \frac{3e^{18}}{6} - 0 - \frac{1}{36}e^{6x}\bigg|_0^3 = \frac{e^{18}}{2} - \left(\frac{e^{18}}{36} - \frac{1}{36}\right).
\end{aligned}$$

Therefore, the total volume is

$$\begin{aligned}
V &= \pi\left[\frac{e^{18}}{2} - \left(\frac{e^{18}}{36} - \frac{1}{36}\right)\right] - \pi\left[\frac{3^2}{2} + \frac{4\cdot 3^3}{3} + 3^4\right] \\
&= \pi\left(\frac{17e^{18} - 4373}{36}\right).
\end{aligned}$$

1.7.2.28. $\ast$. Solution. To get rid of the square root in the argument of f'', we make the change of variables (also called "substitution") $x = t^2, \mathrm{d}x = 2t\,\mathrm{d}t$.

$$\int_0^4 f''\big(\sqrt{x}\big)\,\mathrm{d}x = 2\int_0^2 t f''(t)\,\mathrm{d}t$$

Then, to convert f'' into f', we integrate by parts with $u = t$, $\mathrm{d}v = f''(t)\,\mathrm{d}t$, $v = f'(t)$.

$$\begin{aligned}
\int_0^4 f''\big(\sqrt{x}\big)\,\mathrm{d}x &= 2\left\{\Big[\underbrace{tf'(t)}_{uv}\Big]_0^2 - \int_0^2 \underbrace{f'(t)\,\mathrm{d}t}_{v\mathrm{d}u}\right\} \\
&= 2\Big[tf'(t) - f(t)\Big]_0^2 \\
&= 2\big[2f'(2) - f(2) + f(0)\big] = 2\big[2\times 4 - 3 + 1\big] \\
&= 12
\end{aligned}$$

1.7.2.29. Solution. As we saw in Section 1.1, there are many different ways to interpret a limit as a Riemann sum. In the absence of instructions that restrain our choices, we go with the most convenient interpretations.
With that in mind, we choose:

- that our Riemann sum is a right Riemann sum (because we see i, not $i-1$ or $i-\frac{1}{2}$)

- $\Delta x = \frac{2}{n}$ (because it is multiplied by the rest of the integrand, and also shows up multiplied by i),
- then $x_i = a + i\Delta x = \frac{2}{n}i - 1$, which leads us to $a = -1$ and
- $f(x) = xe^x$.
- Finally, since $\Delta x = \frac{b-a}{n} = \frac{2}{n}$ and $a = -1$, we have $b = 1$.

So, the limit is equal to the definite integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(\frac{2}{n}i - 1\right) e^{\frac{2}{n}i-1} = \int_{-1}^{1} xe^x \mathrm{d}x$$

which we evaluate using integration by parts with $u = x$, $\mathrm{d}v = e^x\mathrm{d}x$, $\mathrm{d}u = \mathrm{d}x$, and $v = e^x$.

$$\begin{aligned} &= \Big[\underbrace{xe^x}_{uv}\Big]_{-1}^{1} - \int_{-1}^{1} \underbrace{e^x\mathrm{d}x}_{v\mathrm{d}u} \\ &= \left(e + \frac{1}{e}\right) - \left(e - \frac{1}{e}\right) = \frac{2}{e} \end{aligned}$$

1.8 · Trigonometric Integrals

1.8.4 · Exercises

Exercises — Stage 1

1.8.4.1. Solution. If $u = \cos x$, then $\mathrm{d}u = -\sin x\,\mathrm{d}x$. If $n \neq -1$, then

$$\begin{aligned} \int_0^{\pi/4} \sin x \cos^n x \mathrm{d}x &= -\int_1^{1/\sqrt{2}} u^n \mathrm{d}u = \left[-\frac{1}{n+1}u^{n+1}\right]_1^{1/\sqrt{2}} \\ &= \frac{1}{n+1}\left(1 - \frac{1}{\sqrt{2}^{n+1}}\right) \end{aligned}$$

If $n = -1$, then

$$\begin{aligned} \int_0^{\pi/4} \sin x \cos^n x \mathrm{d}x &= -\int_1^{1/\sqrt{2}} u^n \mathrm{d}u = -\int_1^{1/\sqrt{2}} \frac{1}{u}\mathrm{d}u \\ &= \Big[-\log|u|\Big]_1^{1/\sqrt{2}} \\ &= -\log\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}\log 2 \end{aligned}$$

So, (e) n can be any real number.

1.8.4.2. Solution. We use the substitution $u = \sec x$, $\mathrm{d}u = \sec x \tan x \mathrm{d}x$.

$$\int \sec^n x \tan x \mathrm{d}x = \int \sec^{n-1} x \cdot \sec x \tan x \mathrm{d}x = \int u^{n-1} \mathrm{d}u$$

Since n is positive, $n-1 \neq -1$, so we antidifferentiate using the power rule.

$$= \frac{u^n}{n} + C = \frac{1}{n} \sec^n x + C$$

1.8.4.3. Solution. We divide both sides by $\cos^2 x$, and simplify.

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
\frac{\sin^2 x + \cos^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x} \\
\frac{\sin^2 x}{\cos^2 x} + 1 &= \sec^2 x \\
\tan^2 x + 1 &= \sec^2 x
\end{aligned}$$

Exercises — Stage 2

1.8.4.4. *. Solution. The power of cosine is odd, and the power of sine is even (zero). Following the strategy in the text, we make the substitution $u = \sin x$, so that $\mathrm{d}u = \cos x \,\mathrm{d}x$ and $\cos^2 x = 1 - \sin^2 x = 1 - u^2$:

$$\begin{aligned}
\int \cos^3 x \,\mathrm{d}x &= \int (1 - \sin^2 x) \cos x \,\mathrm{d}x = \int (1 - u^2) \,\mathrm{d}u \\
&= u - \frac{u^3}{3} + C = \sin x - \frac{\sin^3 x}{3} + C
\end{aligned}$$

1.8.4.5. *. Solution. Using the trig identity $\cos^2 x = \dfrac{1 + \cos(2x)}{2}$, we have

$$\int \cos^2 x \mathrm{d}x = \frac{1}{2} \int_0^\pi \big[1 + \cos(2x)\big] \mathrm{d}x = \frac{1}{2} \Big[x + \frac{1}{2} \sin(2x)\Big]_0^\pi = \frac{\pi}{2}$$

1.8.4.6. *. Solution. Since the power of cosine is odd, following the strategies in the text, we make the substitution $u = \sin t$, so that $\mathrm{d}u = \cos t \,\mathrm{d}t$ and $\cos^2 t = 1 - \sin^2 t = 1 - u^2$.

$$\begin{aligned}
\int \sin^{36} t \cos^3 t \,\mathrm{d}t &= \int \sin^{36} t \,(1 - \sin^2 t) \cos t \,\mathrm{d}t = \int u^{36} (1 - u^2) \,\mathrm{d}u \\
&= \frac{u^{37}}{37} - \frac{u^{39}}{39} + C = \frac{\sin^{37} t}{37} - \frac{\sin^{39} t}{39} + C
\end{aligned}$$

1.8.4.7. Solution. Since the power of sine is odd (and positive), we can reserve one sine for $\mathrm{d}u$, and turn the rest into cosines using the identity $\sin^2 + \cos^2 x = 1$. This allows us to use the substitution $u = \cos x$, $\mathrm{d}u = -\sin x \mathrm{d}x$, and $\sin^2 x =$

$1 - \cos^2 x = 1 - u^2$.

$$\begin{aligned}
\int \frac{\sin^3 x}{\cos^4 x}\mathrm{d}x &= \int \frac{\sin^2 x}{\cos^4 x}\sin x\mathrm{d}x = \int -\frac{1-u^2}{u^4}\mathrm{d}u \\
&= \int \left(-\frac{1}{u^4} + \frac{1}{u^2}\right)\mathrm{d}u = \frac{1}{3u^3} - \frac{1}{u} + C \\
&= \frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C
\end{aligned}$$

1.8.4.8. Solution. Both sine and cosine have even powers (four and zero, respectively), so we don't have the option of using a substitution like $u = \sin x$ or $u = \cos x$. Instead, we use the identity $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$.

$$\begin{aligned}
\int_0^{\pi/3} \sin^4 x\mathrm{d}x &= \int_0^{\pi/3} \left(\sin^2 x\right)^2\mathrm{d}x = \int_0^{\pi/3}\left(\frac{1-\cos(2x)}{2}\right)^2\mathrm{d}x \\
&= \frac{1}{4}\int_0^{\pi/3}\left(1 - 2\cos(2x) + \cos^2(2x)\right)\mathrm{d}x \\
&= \frac{1}{4}\int_0^{\pi/3}\left(1-2\cos(2x)\right)\mathrm{d}x + \frac{1}{4}\int_0^{\pi/3}\cos^2(2x)\mathrm{d}x
\end{aligned}$$

We can antidifferentiate the first integral right away. For the second integral, we use the identity $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$, with $\theta = 2x$.

$$\begin{aligned}
&= \frac{1}{4}\Big[x - \sin(2x)\Big]_0^{\pi/3} + \frac{1}{8}\int_0^{\pi/3}(1+\cos(4x))\mathrm{d}x \\
&= \frac{1}{4}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right] + \frac{1}{8}\left[x + \frac{1}{4}\sin(4x)\right]_0^{\pi/3} \\
&= \frac{1}{4}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right] + \frac{1}{8}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{8}\right] \\
&= \frac{\pi}{8} - \frac{9\sqrt{3}}{64}
\end{aligned}$$

1.8.4.9. Solution. Since the power of sine is odd, we can reserve one sine for $\mathrm{d}u$, and change the remaining four into cosines. This sets us up to use the substitution $u = \cos x$, $\mathrm{d}u = -\sin x\mathrm{d}x$.

$$\begin{aligned}
\int \sin^5 x\mathrm{d}x &= \int \sin^4 x \cdot \sin x\mathrm{d}x = \int (1-\cos^2 x)^2\sin x\mathrm{d}x \\
&= -\int (1-u^2)^2\mathrm{d}u = -\int(1-2u^2+u^4)\mathrm{d}u \\
&= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C \\
&= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C
\end{aligned}$$

1.8.4.10. Solution. If we use the substitution $u = \sin x$, then $\mathrm{d}u = \cos x\mathrm{d}x$, which very conveniently shows up in the integrand.

$$\int \sin^{1.2} x \cos x\mathrm{d}x = \int u^{1.2}\mathrm{d}u = \frac{u^{2.2}}{2.2} + C = \frac{1}{2.2}\sin^{2.2} x + C$$

Note this is exactly the strategy described in the text when the power of cosine is odd. The non-integer power of sine doesn't cause a problem.

1.8.4.11. Solution.

- Solution 1: Let's use the substitution $u = \tan x$, $\mathrm{d}u = \sec^2 x\mathrm{d}x$:

$$\int \tan x \sec^2 x\mathrm{d}x = \int u\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2 x + C$$

- Solution 2: We can also use the substitution $u = \sec x$, $\mathrm{d}u = \sec x \tan x\mathrm{d}x$:

$$\int \tan x \sec^2 x\mathrm{d}x = \int u\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}\sec^2 x + C$$

We note that because $\tan^2 x$ and $\sec^2 x$ only differ by a constant, the two answers are equivalent.

1.8.4.12. *. Solution.

- Solution 1: Substituting $u = \cos x$, $\mathrm{d}u = -\sin x\,\mathrm{d}x$, $\sin^2 x = 1 - \cos^2 x = 1 - u^2$, gives

$$\begin{aligned}\int \tan^3 x \sec^5 x\,\mathrm{d}x &= \int \frac{\sin^3 x}{\cos^8 x}\,\mathrm{d}x = \int \frac{(1-\cos^2 x)\sin x}{\cos^8 x}\,\mathrm{d}x\\ &= -\int \frac{1-u^2}{u^8}\,\mathrm{d}u = -\Big[\frac{u^{-7}}{-7} - \frac{u^{-5}}{-5}\Big] + C\\ &= \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C\end{aligned}$$

- Solution 2: Alternatively, substituting $u = \sec x$, $\mathrm{d}u = \sec x \tan x\,\mathrm{d}x$, $\tan^2 x = \sec^2 x - 1 = u^2 - 1$, gives

$$\begin{aligned}\int \tan^3 x \sec^5 x\,\mathrm{d}x &= \int \tan^2 x \sec^4 x\ (\tan x \sec x)\,\mathrm{d}x\\ &= \int (u^2-1)u^4\,\mathrm{d}u = \Big[\frac{u^7}{7} - \frac{u^5}{5}\Big] + C\\ &= \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + C\end{aligned}$$

1.8.4.13. *. Solution. Use the substitution $u = \tan x$, so that $\mathrm{d}u = \sec^2 x\,\mathrm{d}x$:

$$\int \sec^4 x\ \tan^{46} x\,\mathrm{d}x = \int (\tan^2 x + 1)\tan^{46} x\ \sec^2 x\,\mathrm{d}x$$

$$= \int (u^2+1)u^{46}\,\mathrm{d}u = \frac{u^{49}}{49} + \frac{u^{47}}{47} + C$$
$$= \frac{\tan^{49} x}{49} + \frac{\tan^{47} x}{47} + C$$

1.8.4.14. Solution. We use the substitution $u = \sec x$, $\mathrm{d}u = \sec x \tan x \mathrm{d}x$. Then $\tan^2 x = \sec^2 x - 1 = u^2 - 1$.

$$\begin{aligned}\int \tan^3 x \sec^{1.5} x \mathrm{d}x &= \int \tan^2 x \cdot \sec^{0.5} x \cdot \sec x \tan x \mathrm{d}x \\ &= \int (u^2-1)u^{0.5}\mathrm{d}u = \int \left(u^{2.5} - u^{0.5}\right)\mathrm{d}u \\ &= \frac{u^{3.5}}{3.5} - \frac{u^{1.5}}{1.5} + C \\ &= \frac{1}{3.5}\sec^{3.5} x - \frac{1}{1.5}\sec^{1.5} x + C\end{aligned}$$

Note this solution used the same method as Example 1.8.13 for the case that the power of tangent is odd and there is at least one secant.

1.8.4.15. Solution. We'll give two solutions.

- Solution 1: As in Question 14, we have an odd power of tangent and at least one secant. So, as in strategy (2) of Section 1.8.2, we can use the substitution $u = \sec x$, $\mathrm{d}u = \sec x \tan x \mathrm{d}x$, and $\tan^2 x = \sec^2 x - 1 = u^2 - 1$.

$$\begin{aligned}\int \tan^3 x \sec^2 x \mathrm{d}x &= \int \tan^2 x \sec x \cdot \sec x \tan x \mathrm{d}x \\ &= \int (u^2-1)u\mathrm{d}u = \int \left(u^3 - u\right)\mathrm{d}u \\ &= \frac{1}{4}u^4 - \frac{1}{2}u^2 + C \\ &= \frac{1}{4}\sec^4 x - \frac{1}{2}\sec^2 x + C\end{aligned}$$

- Solution 2: We have an even, strictly positve, power of $\sec x$. So, as in strategy (3) of Section 1.8.2, we can use the substitution $u = \tan x$, $\mathrm{d}u = \sec^2 x\ \mathrm{d}x$.

$$\begin{aligned}\int \tan^3 x \sec^2 x\ \mathrm{d}x &= \int \tan^3 x \cdot \sec^2 x\ \mathrm{d}x \\ &= \int u^3\ \mathrm{d}u \\ &= \frac{1}{4}u^4 + C \\ &= \frac{1}{4}\tan^4 x + C\end{aligned}$$

It looks like we have two different answers. But, because $\tan^2 x = \sec^2 x - 1$,

$$\frac{1}{4}\tan^4 = \frac{1}{4}(\sec^2 x - 1)^2 = \frac{1}{4}\sec^4 x - \frac{1}{2}\sec^2 x + \frac{1}{4}$$

and the two answers are really the same, except that the arbitrary constant C of Solution 1 is $\frac{1}{4}$ plus the arbitrary constant C of Solution 2.

1.8.4.16. Solution. In contrast to Questions 14 and 15, we do not have an odd power of tangent, so we should consider a different substitution. Luckily, if we choose $u = \tan x$, then $\mathrm{d}u = \sec^2 x\mathrm{d}x$, and this fits our integrand nicely.

$$\int \tan^4 x \sec^2 x \mathrm{d}x = \int u^4 \mathrm{d}u = \frac{1}{5}u^5 + C = \frac{1}{5}\tan^5 x + C$$

1.8.4.17. Solution.

- Solution 1: Since the power of tangent is odd, let's try to use the substitution $u = \sec x$, $\mathrm{d}u = \sec x \tan x\mathrm{d}x$, and $\tan^2 x = \sec^2 x - 1 = u^2 - 1$, as in Questions 14 and 15. In order to make this work, we need to see $\sec x \tan x\mathrm{d}x$ in the integrand, so we do a little algebraic manipulation.

$$\begin{aligned}\int \tan^3 x \sec^{-0.7} x\mathrm{d}x &= \int \frac{\tan^3 x}{\sec^{0.7x}}\mathrm{d}x = \int \frac{\tan^3 x}{\sec^{1.7x}}\sec x\mathrm{d}x\\ &= \int \frac{\tan^2 x}{\sec^{1.7} x}\cdot \sec x \tan x\mathrm{d}x\\ &= \int \frac{u^2-1}{u^{1.7}}\mathrm{d}u = \int \left(u^{0.3} - u^{-1.7}\right)\mathrm{d}u\\ &= \frac{u^{1.3}}{1.3} + \frac{1}{0.7u^{0.7}} + C\\ &= \frac{1}{1.3}\sec^{1.3} x + \frac{1}{0.7\sec^{0.7} x} + C\\ &= \frac{1}{1.3}\sec^{1.3} x + \frac{1}{0.7}\cos^{0.7} x + C\end{aligned}$$

- Solution 2: Let's convert the secants and tangents to sines and cosines.

$$\begin{aligned}\int \tan^3 x \sec^{-0.7} x\mathrm{d}x &= \int \frac{\sin^3 x}{\cos^3 x}\cdot \cos^{0.7} x\mathrm{d}x\\ &= \int \frac{\sin^3 x}{\cos^{2.3} x}\mathrm{d}x = \int \frac{\sin^2 x}{\cos^{2.3} x}\cdot \sin x\mathrm{d}x\end{aligned}$$

Using the substitution $u = \cos x$, $\mathrm{d}u = -\sin \mathrm{d}x$, and $\sin^2 x = 1 - \cos^2 x = 1 - u^2$:

$$\begin{aligned}&= -\int \frac{1-u^2}{u^{2.3}}\mathrm{d}u = \int \left(-u^{-2.3} + u^{-0.3}\right)\mathrm{d}u\\ &= \frac{1}{1.3}u^{-1.3} + \frac{1}{0.7}u^{0.7} + C\\ &= \frac{1}{1.3}\sec^{1.3} x + \frac{1}{0.7}\cos^{0.7} x + C\end{aligned}$$

1.8.4.18. Solution. We replace $\tan x$ with $\dfrac{\sin x}{\cos x}$.

$$\int \tan^5 x \mathrm{d}x = \int \left(\frac{\sin x}{\cos x}\right)^5 \mathrm{d}x = \int \frac{\sin^4 x}{\cos^5 x} \cdot \sin x \mathrm{d}x$$

Now we use the substitution $u = \cos x$, $\mathrm{d}u = -\sin x \mathrm{d}x$, and $\sin^2 x = 1 - \cos^2 x = 1 - u^2$.

$$\begin{aligned}
&= -\int \frac{(1-u^2)^2}{u^5}\mathrm{d}u = \int \left(-u^{-5} + 2u^{-3} - u^{-1}\right)\mathrm{d}u \\
&= \frac{1}{4}u^{-4} - u^{-2} - \log|u| + C \\
&= \frac{1}{4}\sec^4 x - \sec^2 x - \log|\cos x| + C \\
&= \frac{1}{4}\sec^4 x - \sec^2 x + \log|\sec x| + C
\end{aligned}$$

where in the last line, we used the logarithm rule $\log(b^a) = a\log b$, with $b^a = \cos x = (\sec x)^{-1}$.

1.8.4.19. Solution. Integrating even powers of tangent is surprisingly different from integrating odd powers of tangent. For even powers, we use the identity $\tan^2 x = \sec^2 x - 1$, then use the substitution $u = \tan x$, $\mathrm{d}u = \sec^2 x \mathrm{d}x$ on (perhaps only a part of) the resulting integral.

$$\begin{aligned}
\int_0^{\pi/6} \tan^6 x \mathrm{d}x &= \int_0^{\pi/6} \tan^4 x(\sec^2 x - 1)\mathrm{d}x \\
&= \int_0^{\pi/6} \Big(\underbrace{\tan^4 x \sec^2 x}_{u^4\mathrm{d}u} - \tan^4 x\Big)\mathrm{d}x \\
&= \int_0^{\pi/6} \Big(\tan^4 x \sec^2 x - \tan^2 x(\sec^2 x - 1)\Big)\mathrm{d}x \\
&= \int_0^{\pi/6} \Big(\tan^4 x \sec^2 x - \underbrace{\tan^2 x \sec^2 x}_{u^2\mathrm{d}u} + \tan^2 x\Big)\mathrm{d}x \\
&= \int_0^{\pi/6} \Big(\tan^4 x \sec^2 x - \tan^2 x \sec^2 x + (\underbrace{\sec^2 x}_{\mathrm{d}u} - 1)\Big)\mathrm{d}x \\
&= \int_0^{\pi/6} \left(\tan^4 x - \tan^2 x + 1\right)\sec^2 x \mathrm{d}x - \int_0^{\pi/6} 1\mathrm{d}x
\end{aligned}$$

Note $\tan(0) = 0$, and $\tan(\pi/6) = 1/\sqrt{3}$.

$$\begin{aligned}
&= \int_0^{1/\sqrt{3}} (u^4 - u^2 + 1)\mathrm{d}u - \big[x\big]_0^{\pi/6} \\
&= \left[\frac{1}{5}u^5 - \frac{1}{3}u^3 + u\right]_0^{1/\sqrt{3}} - \frac{\pi}{6}
\end{aligned}$$

$$= \frac{1}{5\sqrt{3}^5} - \frac{1}{3\sqrt{3}^3} + \frac{1}{\sqrt{3}} - \frac{\pi}{6}$$
$$= \frac{41}{45\sqrt{3}} - \frac{\pi}{6}$$

1.8.4.20. Solution. Since there is an even power of secant in the integrand, we can reserve two secants for $\mathrm{d}u$ and change the rest to tangents. That sets us up nicely to use the substitution $u = \tan x$, $\mathrm{d}u = \sec^2 x\mathrm{d}x$. Note $\tan(0) = 0$ and $\tan(\pi/4) = 1$.

$$\begin{aligned}
\int_0^{\pi/4} \tan^8 x \sec^4 x\mathrm{d}x &= \int_0^{\pi/4} \tan^8 x(\tan^2 x + 1)\sec^2 x\mathrm{d}x \\
&= \int_0^1 u^8(u^2+1)\mathrm{d}u \\
&= \int_0^1 u^{10} + u^8\mathrm{d}u \\
&= \frac{1}{11} + \frac{1}{9}
\end{aligned}$$

1.8.4.21. Solution.

- Solution 1: Let's use the substitution $u = \sec x$, $\mathrm{d}u = \sec x\tan x\mathrm{d}x$. In order to make this work, we need to see $\sec x\tan x$ in the integrand, so we start with some algebraic manipulation.

$$\begin{aligned}
\int \tan x\sqrt{\sec x}\left(\frac{\sqrt{\sec x}}{\sqrt{\sec x}}\right)\mathrm{d}x &= \int \frac{1}{\sqrt{\sec x}}\sec x\tan x\mathrm{d}x \\
&= \int \frac{1}{\sqrt{u}}\mathrm{d}u = 2\sqrt{u} + C \\
&= 2\sqrt{\sec x} + C
\end{aligned}$$

- Solution 2: Let's turn our secants and tangents into sines and cosines.

$$\int \tan x\sqrt{\sec x}\mathrm{d}x = \int \frac{\sin x}{\cos x\cdot\sqrt{\cos x}}\mathrm{d}x = \int \frac{\sin x}{\cos^{1.5} x}\mathrm{d}x$$

We use the substitution $u = \cos x$, $\mathrm{d}u = -\sin x\mathrm{d}x$.

$$\begin{aligned}
&= \int -u^{-1.5}\mathrm{d}u = \frac{2}{\sqrt{u}} + C \\
&= 2\sqrt{\sec x} + C
\end{aligned}$$

1.8.4.22. Solution. Since the power of secant is even and positive, we can reserve two secants for $\mathrm{d}u$, and change the rest into tangents, setting the stage for the substitution $u = \tan\theta$, $\mathrm{d}u = \sec^2\theta\mathrm{d}\theta$.

$$\int \sec^8\theta\tan^e\theta\mathrm{d}\theta = \int \sec^6\theta\tan^e\theta\sec^2\theta\mathrm{d}\theta$$

$$
\begin{aligned}
&= \int (\tan^2\theta + 1)^3 \tan^e\theta \sec^2\theta \mathrm{d}\theta \\
&= \int (u^2+1)^3 \cdot u^e \mathrm{d}u \\
&= \int (u^6 + 3u^4 + 3u^2 + 1) \cdot u^e \mathrm{d}u \\
&= \int (u^{6+e} + 3u^{4+e} + 3u^{2+e} + u^e)\mathrm{d}u \\
&= \frac{1}{7+e}u^{7+e} + \frac{3}{5+e}u^{5+e} + \frac{3}{3+e}u^{3+e} + \frac{1}{1+e}u^{1+e} + C \\
&= \frac{1}{7+e}\tan^{7+e}\theta + \frac{3}{5+e}\tan^{5+e}\theta + \frac{3}{3+e}\tan^{3+e}\theta \\
&\qquad + \frac{1}{1+e}\tan^{1+e}\theta + C \\
&= \tan^{1+e}\theta\left(\frac{\tan^6\theta}{7+e} + \frac{3\tan^4\theta}{5+e} + \frac{3\tan^2\theta}{3+e} + \frac{1}{1+e}\right) + C
\end{aligned}
$$

Exercises — Stage 3

1.8.4.23. *. Solution. (a) Using the trig identity $\tan^2 x = \sec^2 x - 1$ and the substitution $y = \tan x$, $\mathrm{d}y = \sec^2 x\ \mathrm{d}x$,

$$
\begin{aligned}
\int \tan^n x\mathrm{d}x &= \int \tan^{n-2} x\ \tan^2 x\mathrm{d}x \\
&= \int \tan^{n-2} x\ \sec^2 x\mathrm{d}x - \int \tan^{n-2} x\mathrm{d}x \\
&= \int y^{n-2}\,\mathrm{d}y - \int \tan^{n-2} x\mathrm{d}x \\
&= \frac{y^{n-1}}{n-1} - \int \tan^{n-2} x\mathrm{d}x \\
&= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\mathrm{d}x
\end{aligned}
$$

(b) By the reduction formula of part (a),

$$
\begin{aligned}
\int_0^{\pi/4} \tan^n(x)\,\mathrm{d}x &= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2}(x)\,\mathrm{d}x \\
&= \frac{1}{n-1} - \int_0^{\pi/4} \tan^{n-2}(x)\,\mathrm{d}x
\end{aligned}
$$

for all integers $n \ge 2$, since $\tan 0 = 0$ and $\tan\frac{\pi}{4} = 1$. We apply this reduction formula, with $n = 6, 4, 2$.

$$
\begin{aligned}
\int_0^{\pi/4} \tan^6(x)\,\mathrm{d}x &= \frac{1}{5} - \int_0^{\pi/4} \tan^4(x)\,\mathrm{d}x \\
&= \frac{1}{5} - \frac{1}{3} + \int_0^{\pi/4} \tan^2(x)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{5} - \frac{1}{3} + 1 - \int_0^{\pi/4} \mathrm{d}x \\
&= \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4} = \frac{13}{15} - \frac{\pi}{4}
\end{aligned}
$$

Using a calculator, we see this is approximately 0.0813.
Notice how much faster this was than the method of Question 19.

1.8.4.24. Solution. Recall $\tan x = \dfrac{\sin x}{\cos x}$.

$$\int \tan^5 x \cos^2 x \mathrm{d}x = \int \frac{\sin^5 x}{\cos^5 x} \cos^2 x \mathrm{d}x = \int \frac{\sin^5 x}{\cos^3 x} \mathrm{d}x$$

Substitute $u = \cos x$, so $\mathrm{d}u = -\sin x \mathrm{d}x$ and $\sin^2 x = 1 - \cos^2 x = 1 - u^2$.

$$
\begin{aligned}
&= \int \frac{\sin^4 x}{\cos^3 x} \sin x \mathrm{d}x = -\int \frac{(1-u^2)^2}{u^3} \mathrm{d}u \\
&= -\int \frac{1 - 2u^2 + u^4}{u^3} \mathrm{d}u = \int \left(-\frac{1}{u^3} + \frac{2}{u} - u \right) \mathrm{d}u \\
&= \frac{1}{2u^2} + 2 \log |u| - \frac{1}{2} u^2 + C \\
&= \frac{1}{2\cos^2 x} + 2 \log |\cos x| - \frac{1}{2} \cos^2 x + C
\end{aligned}
$$

1.8.4.25. Solution. We can use the definition of secant to make this integral look more familiar.

$$\int \frac{1}{\cos^2 \theta} \mathrm{d}\theta = \int \sec^2 \theta \mathrm{d}\theta = \tan \theta + C$$

1.8.4.26. Solution. We re-write $\cot x = \dfrac{\cos x}{\sin x}$, and use the substitution $u = \sin x$, $\mathrm{d}u = \cos x \mathrm{d}x$.

$$
\begin{aligned}
\int \cot x \mathrm{d}x &= \int \frac{\cos x}{\sin x} \mathrm{d}x = \int \frac{1}{u} \mathrm{d}u \\
&= \log |u| + C = \log |\sin x| + C
\end{aligned}
$$

1.8.4.27. Solution.

- Solution 1: We begin with the obvious substitution, $w = e^x$, $\mathrm{d}w = e^x \mathrm{d}w$.

$$\int e^x \sin(e^x) \cos(e^x) \mathrm{d}x = \int \sin w \cos w \mathrm{d}w$$

Now we see another substitution, $u = \sin w$, $\mathrm{d}u = \cos w \mathrm{d}w$.

$$
\begin{aligned}
&= \int u \mathrm{d}u = \frac{1}{2} u^2 + C = \frac{1}{2} \sin^2 w + C \\
&= \frac{1}{2} \sin^2(e^x) + C
\end{aligned}
$$

- Solution 2: Notice that $\frac{\mathrm{d}}{\mathrm{d}x}\{\sin(e^x)\} = e^x \cos(e^x)$. This suggests to us the substitution $u = \sin(e^x)$, $\mathrm{d}u = e^x \cos(e^x)\mathrm{d}x$.

$$\int e^x \sin(e^x)\cos(e^x)\mathrm{d}x = \int u\mathrm{d}u = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2(e^x) + C$$

1.8.4.28. Solution. Since we have an "inside function," we start with the substitution $s = \cos x$, so $-\mathrm{d}s = \sin x\mathrm{d}x$ and $\sin^2 x = 1 - \cos^2 x = 1 - s^2$.

$$\begin{aligned}\int \sin(\cos x)\sin^3 x\mathrm{d}x &= \int \sin(\cos x)\cdot\sin^2 x\cdot\sin x\mathrm{d}x\\ &= -\int \sin(s)\cdot(1-s^2)\mathrm{d}s\end{aligned}$$

We use integration by parts with $u = (1-s^2)$, $\mathrm{d}v = \sin s\mathrm{d}s$; $\mathrm{d}u = -2s\mathrm{d}s$, and $v = -\cos s$.

$$\begin{aligned}&= -\left[-(1-s^2)\cos s - \int 2s\cos s\mathrm{d}s\right]\\ &= (1-s^2)\cos s + \int 2s\cos s\mathrm{d}s\end{aligned}$$

We integrate by parts again, with $u = 2s$, $\mathrm{d}v = \cos s\mathrm{d}s$; $\mathrm{d}u = 2\mathrm{d}s$, and $v = \sin s$.

$$\begin{aligned}&= (1-s^2)\cos s + 2s\sin s - \int 2\sin s\mathrm{d}s\\ &= (1-s^2)\cos s + 2s\sin s + 2\cos s + C\\ &= \sin^2 x\cdot\cos(\cos x) + 2\cos x\cdot\sin(\cos x) + 2\cos(\cos x) + C\\ &= (\sin^2 x + 2)\cos(\cos x) + 2\cos x\cdot\sin(\cos x) + C\end{aligned}$$

1.8.4.29. Solution. Since the integrand is the product of polynomial and trigonometric functions, we suspect it might yield to integration by parts. There are a number of ways this can be accomplished.

- Solution 1: Before we choose parts, let's use the identity $\sin(2x) = 2\sin x\cos x$.

  $$\int x\sin x\cos x\mathrm{d}x = \frac{1}{2}\int x\sin(2x)\mathrm{d}x$$

  Now let $u = x$, $\mathrm{d}v = \sin(2x)\mathrm{d}x$; $\mathrm{d}u = \mathrm{d}x$, and $v = -\frac{1}{2}\cos(2x)$. Using integration by parts:

  $$\begin{aligned}&= \frac{1}{2}\left[-\frac{x}{2}\cos(2x) + \frac{1}{2}\int\cos(2x)\mathrm{d}x\right]\\ &= -\frac{x}{4}\cos(2x) + \frac{1}{8}\sin(2x) + C\\ &= -\frac{x}{4}(1 - 2\sin^2 x) + \frac{1}{4}\sin x\cos x + C\\ &= -\frac{x}{4} + \frac{x}{2}\sin^2 x + \frac{1}{4}\sin x\cos x + C\end{aligned}$$

- Solution 2: If we let $u = x$, then $du = dx$, and this seems desirable for integration by parts. If $u = x$, then $dv = \sin x \cos x dx$. To find v we can use the substitution $u = \sin x$, $du = \cos x dx$.

$$v = \int \sin x \cos x dx = \int u du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$$

So, we take $v = \frac{1}{2}\sin^2 x$. Now we can apply integration by parts to our original integral.

$$\int x \sin x \cos x dx = \frac{x}{2}\sin^2 x - \int \frac{1}{2}\sin^2 x dx$$

Apply the identity $\sin^2 x = \dfrac{1 - \cos(2x)}{2}$.

$$\begin{aligned} &= \frac{x}{2}\sin^2 x - \frac{1}{4}\int 1 - \cos(2x) dx \\ &= \frac{x}{2}\sin^2 x - \frac{x}{4} + \frac{1}{8}\sin(2x) + C \\ &= \frac{x}{2}\sin^2 x - \frac{x}{4} + \frac{1}{4}\sin x \cos x + C \end{aligned}$$

- Solution 3: Let $u = x \sin x$ and $dv = \cos x dx$; then $du = (x \cos x + \sin x)dx$ and $v = \sin x$.

$$\begin{aligned} \int x \sin x \cos x dx &= x \sin^2 x - \int \sin x (x \cos x + \sin x) dx \\ &= x \sin^2 x - \int x \sin x \cos x dx - \int \sin^2 x dx \end{aligned}$$

Apply the identity $\sin^2 x = \dfrac{1 - \cos(2x)}{2}$ to the second integral.

$$\begin{aligned} &= x \sin^2 x - \int x \sin x \cos x dx - \int \frac{1 - \cos(2x)}{2} dx \\ &= x \sin^2 x - \int x \sin x \cos x dx - \frac{x}{2} + \frac{1}{4}\sin(2x) + C \end{aligned}$$

So, we have the equation

$$\begin{aligned} \int x \sin x \cos x dx &= x \sin^2 x - \int x \sin x \cos x dx \\ &\qquad - \frac{x}{2} + \frac{1}{4}\sin(2x) + C \\ 2\int x \sin x \cos x dx &= x \sin^2 x - \frac{x}{2} + \frac{1}{4}\sin(2x) + C \\ \int x \sin x \cos x dx &= \frac{x}{2}\sin^2 x - \frac{x}{4} + \frac{1}{8}\sin(2x) + \frac{C}{2} \\ &= \frac{x}{2}\sin^2 x - \frac{x}{4} + \frac{1}{4}\sin x \cos x + \frac{C}{2} \end{aligned}$$

Since C is an arbitrary constant that can take any number in $(-\infty, \infty)$, also $\frac{C}{2}$ is an arbitrary constant that can take any number in $(-\infty, \infty)$, so we're free to rename $\frac{C}{2}$ to C.

1.9 · Trigonometric Substitution

1.9.2 · Exercises

Exercises — Stage 1

1.9.2.1. $*$. Solution. In the text, there is a template for choosing an appropriate substitution, but for this problem we will explain the logic of the choices.
The trig identities that we can use are:

$$1-\sin^2\theta=\cos^2\theta \qquad \tan^2\theta+1=\sec^2\theta \qquad \sec^2\theta-1=\tan^2\theta$$

They have the following forms:

$$\text{constant} - \text{function} \qquad \text{function} + \text{constant} \qquad \text{function} - \text{constant}$$

In order to cancel out the square root, we should choose a substitution that will match the argument under the square root with the trig identity of the corresponding form.
(a) There's not an obvious non-trig substitution for evaluating this problem, so we want a trigonometric substitution to get rid of the square root in the denominator. Under the square root is the function $9x^2-16$, which has the form (function) $-$ (constant). This form matches the trig identity $\sec^2\theta-1=\tan^2\theta$. We can set x to be whatever we need it to be, but we don't have the same control over the constant, 16. So, to make the substitution work, we use a different form of the trig identity: multiplying both sides by 16, we get

$$16\sec^2\theta-16=16\tan^2\theta$$

What we want is a substitution that gives us

$$\begin{aligned} 9x^2-16&=16\sec^2\theta-16\\ \text{So,}\qquad 9x^2&=16\sec^2\theta\\ x&=\frac{4}{3}\sec\theta \end{aligned}$$

Using this substitution,

$$\begin{aligned} \sqrt{9x^2-16}&=\sqrt{16\sec^2\theta-16}\\ &=\sqrt{16\tan^2\theta}\\ &=4|\tan\theta| \end{aligned}$$

So, we eliminated the square root.
(b) There's not an obvious non-trig substitution for evaluating this problem, so we want a trigonometric substitution to get rid of the square root in the denominator. Under the square root is the function $1-4x^2$, which has the form (constant) $-$ (function). This form matches the trig identity $1-\sin^2\theta=\cos^2\theta$. What we want is a substitution that gives us

$$\begin{aligned} 1-4x^2&=1-\sin^2\theta\\ \text{So,}\qquad 4x^2&=\sin^2\theta\\ x&=\frac{1}{2}\sin\theta \end{aligned}$$

Using this substitution,

$$\begin{aligned}\sqrt{1-4x^2} &= \sqrt{1-\sin^2\theta}\\ &= \sqrt{\cos^2\theta}\\ &= |\cos\theta|\end{aligned}$$

So, we eliminated the square root. We remark that the absolute value signs are not needed in $|\cos\theta|$, because, for $-\frac{1}{2} \le x \le \frac{1}{2}$, we have $\theta = \arcsin(2x)$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and $\cos(\theta) \ge 0$ for those θ's.
(c) There's not an obvious non-trig substitution for evaluating this problem, so we want a trigonometric substitution to get rid of the fractional power. (That is, we want to eliminate the square root.) The function under the power is $25 + x^2$, which has the form (constant) + (function). This form matches the trig identity $\tan^2\theta + 1 = \sec^2\theta$. We can set x to be whatever we need it to be, but we don't have the same control over the constant, 25. So, to make the substitution work, we use a different form of the trig identity: multiplying both sides by 25, we get

$$25\tan^2\theta + 25 = 25\sec^2\theta$$

What we want is a substitution that gives us

$$\begin{aligned}25 + x^2 &= 25\tan^2\theta + 25\\ \text{So,}\qquad x^2 &= 25\tan^2\theta\\ x &= 5\tan\theta\end{aligned}$$

Using this substitution,

$$\begin{aligned}(25+x^2)^{-5/2} &= (25+25\tan^2\theta)^{-5/2}\\ &= (25\sec^2\theta)^{-5/2}\\ &= (5|\sec\theta|)^{-5}\end{aligned}$$

So, we eliminated the square root. We remark that the absolute value signs are not needed in $|\sec\theta|$, because, for $-\infty < x < \infty$, we have $\theta = \arctan(x/5)$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and $\sec(\theta) \ge 0$ for those θ's.

1.9.2.2. Solution. Just as in Question 1, we want to choose a trigonometric substitution that will allow us to eliminate the square roots. Before we can make that choice, though, we need to complete the square. In subsequent problems, we won't show the algebra behind completing the square, but for this problem we'll work it out explicitly. After some practice, you'll be able to do this step in your head for many cases.
After the squares are completed, the choice of trig substitution follows the logic outlined in the solutions to Question 1, or (equivalently) the template in the text.

a The quadratic function under the square root is $x^2 - 4x + 1$. To complete the square, we match the non-constant terms to those of a perfect square.

$$\begin{aligned} (ax+b)^2 &= a^2x^2 + 2abx + b^2 \\ x^2 - 4x + 1 &= a^2x^2 + 2abx + b^2 + c \quad \text{for some constant } c \end{aligned}$$

- Looking at the leading term tells us $a = 1$.
- Then the second term tells us $-4 = 2ab = 2b$, so $b = -2$.
- Finally, the constant terms give us $1 = b^2 + c = 4 + c$, so $c = -3$.

$$\begin{aligned} \int \frac{1}{\sqrt{x^2-4x+1}}\mathrm{d}x &= \int \frac{1}{\sqrt{(x-2)^2-3}}\mathrm{d}x \\ &= \int \frac{1}{\sqrt{(x-2)^2-\sqrt{3}^2}}\mathrm{d}x \end{aligned}$$

So we use the substitution $(x-2) = \sqrt{3}\sec u$, which eliminates the square root:

$$\sqrt{(x-2)^2-3} = \sqrt{3\sec^2 u - 3} = \sqrt{3\tan^2 u} = \sqrt{3}|\tan u|$$

b The quadratic function under the square root is $-x^2+2x+4 = -[x^2-2x-4]$. To complete the square, we match the non-constant terms to those of a perfect square. We factored out the negative to make things a little easier — don't forget to put it back in before choosing a substitution!

$$\begin{aligned} (ax+b)^2 &= a^2x^2 + 2abx + b^2 \\ x^2 - 2x - 4 &= a^2x^2 + 2abx + b^2 + c \quad \text{for some constant } c \end{aligned}$$

- Looking at the leading term tells us $a = 1$.
- Then the second term tells us $-2 = 2ab = 2b$, so $b = -1$.
- Finally, the constant terms give us $-4 = b^2 + c = 1 + c$, so $c = -5$.
- Then

$$\begin{aligned} -x^2+2x+4 &= -[x^2-2x-4] = -[(x-1)^2-5] \\ &= 5-(x-1)^2 \end{aligned}$$

$$\begin{aligned} \int \frac{(x-1)^6}{(-x^2+2x+4)^{3/2}}\mathrm{d}x &= \int \frac{(x-1)^6}{(5-(x-1)^2)^{3/2}}\mathrm{d}x \\ &= \int \frac{(x-1)^6}{\left(\sqrt{5}^2-(x-1)^2\right)^{3/2}}\mathrm{d}x \end{aligned}$$

So we use the substitution $(x-1) = \sqrt{5}\sin u$, which eliminates the square root (fractional power):

$$\begin{aligned} \left(5-(x-1)^2\right)^{3/2} &= \left(5-5\sin^2 u\right)^{3/2} = \left(5\cos^2 u\right)^{3/2} \\ &= 5\sqrt{5}|\cos^3 u| \end{aligned}$$

c The quadratic function under the square root is $4x^2 + 6x + 10$. To complete the square, we match the non-constant terms to those of a perfect square.

$$\begin{aligned} (ax+b)^2 &= a^2x^2 + 2abx + b^2 \\ 4x^2 + 6x + 10 &= a^2x^2 + 2abx + b^2 + c \quad \text{for some constant } c \end{aligned}$$

- Looking at the leading term tells us $a = 2$.
- Then the second term tells us $6 = 2ab = 4b$, so $b = \frac{3}{2}$.
- Finally, the constant terms give us $10 = b^2 + c = \frac{9}{4} + c$, so $c = \frac{31}{4}$.

$$\begin{aligned} \int \frac{1}{\sqrt{4x^2+6x+10}} \mathrm{d}x &= \int \frac{1}{\sqrt{\left(2x+\frac{3}{2}\right)^2 + \frac{31}{4}}} \mathrm{d}x \\ &= \int \frac{1}{\sqrt{\left(2x+\frac{3}{2}\right)^2 + \left(\frac{\sqrt{31}}{2}\right)^2}} \mathrm{d}x \end{aligned}$$

So we use the substitution $\left(2x + \frac{3}{2}\right) = \frac{\sqrt{31}}{2} \tan u$, which eliminates the square root:

$$\begin{aligned} \sqrt{\left(2x+\frac{3}{2}\right)^2 + \frac{31}{4}} &= \sqrt{\frac{31}{4}\tan^2 u + \frac{31}{4}} = \sqrt{\frac{31}{4}\sec^2 u} \\ &= \frac{\sqrt{31}}{2}|\sec u| \end{aligned}$$

d The quadratic function under the square root is $x^2 - x$. To complete the square, we match the non-constant terms to those of a perfect square.

$$\begin{aligned} (ax+b)^2 &= a^2x^2 + 2abx + b^2 \\ x^2 - x &= a^2x^2 + 2abx + b^2 + c \quad \text{for some constant } c \end{aligned}$$

- Looking at the leading term tells us $a = 1$.
- Then the second term tells us $-1 = 2ab = 2b$, so $b = -\frac{1}{2}$.
- Finally, the constant terms give us $0 = b^2 + c = \frac{1}{4} + c$, so $c = -\frac{1}{4}$.

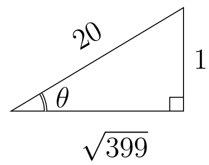
$$\begin{aligned} \int \sqrt{x^2 - x}\mathrm{d}x &= \int \sqrt{\left(x - \frac{1}{2}\right)^2 - \frac{1}{4}} \mathrm{d}x \\ &= \int \sqrt{\left(x - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \mathrm{d}x \end{aligned}$$

So we use the substitution $(x - 1/2) = \frac{1}{2}\sec u$, which eliminates the square root:

1.9.2.3. Solution.

$$\sqrt{\left(x - \frac{1}{2}\right)^2 - \frac{1}{4}} = \sqrt{\frac{1}{4}\sec^2 u - \frac{1}{4}} = \sqrt{\frac{1}{4}\tan^2 u} = \frac{1}{2}|\tan u|$$

a If $\sin\theta = \dfrac{1}{20}$ and θ is between 0 and $\pi/2$, then we can draw a right triangle with angle θ that has opposite side length 1, and hypotenuse length 20. By the Pythagorean Theorem, the adjacent side has length $\sqrt{20^2 - 1^2} = \sqrt{399}$. So, $\cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{\sqrt{399}}{20}$.



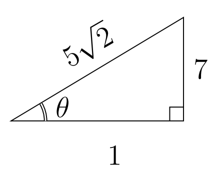
We can do a quick "reasonableness" check here: $\frac{1}{20}$ is pretty close to 0, so we might expect θ to be pretty close to 0, and so $\cos\theta$ should be pretty close to 1. Indeed it is: $\dfrac{\sqrt{399}}{20} \approx \dfrac{\sqrt{400}}{20} = \dfrac{20}{20} = 1$.
Alternatively, we can solve this problem using identities.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\left(\frac{1}{20}\right)^2 + \cos^2\theta = 1$$
$$\cos\theta = \pm\sqrt{1 - \frac{1}{400}} = \pm\frac{\sqrt{399}}{20}$$

Since $0 \le \theta \le \frac{\pi}{2}$, $\cos\theta \ge 0$, so

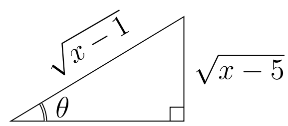
$$\cos\theta = \frac{\sqrt{399}}{20}$$

b If $\tan\theta = 7$ and θ is between 0 and $\pi/2$, then we can draw a right triangle with angle θ that has opposite side length 7 and adjacent side length 1. By the Pythagorean Theorem, the hypotenuse has length $\sqrt{7^2 + 1^2} = \sqrt{50} = 5\sqrt{2}$. So, $\csc\theta = \dfrac{\text{hyp}}{\text{opp}} = \dfrac{5\sqrt{2}}{7}$.



Again, we can do a quick reasonableness check. Since 7 is much larger than 1, the triangle we're thinking of doesn't look much like the triangle in our standardized picture above: it's really quite tall, with a small base. So, the opposite side and hypotenuse are pretty close in length. Indeed, $\dfrac{5\sqrt{2}}{7} \approx 7.071$, so this dimension seems reasonable.

c If $\sec\theta = \dfrac{\sqrt{x-1}}{2}$ and θ is between 0 and $\pi/2$, then we can draw a right triangle with angle θ that has hypotenuse length $\sqrt{x-1}$ and adjacent side length 2. By the Pythagorean Theorem, the opposite side has length $\sqrt{\sqrt{x-1}^2 - 2^2} = \sqrt{x-1-4} = \sqrt{x-5}$. So, $\tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{\sqrt{x-5}}{2}$.



We can also solve this using identities. Note that since $\sec\theta$ exists, $\theta \neq \frac{\pi}{2}$.

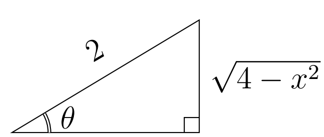
$$\begin{aligned}
\tan^2\theta + 1 &= \sec^2\theta \\
\tan^2\theta + 1 &= \left(\frac{\sqrt{x-1}}{2}\right)^2 = \frac{x-1}{4} \\
\tan\theta &= \pm\sqrt{\frac{x-1}{4} - 1} = \pm\frac{\sqrt{x-5}}{2}
\end{aligned}$$

Since $0 \leq \theta < \frac{\pi}{2}$, $\tan\theta \geq 0$, so

$$\tan\theta = \frac{\sqrt{x-5}}{2}$$

1.9.2.4. Solution.

a Let $\theta = \arccos\left(\frac{x}{2}\right)$. That is, $\cos(\theta) = \frac{x}{2}$, and $0 \leq \theta \leq \pi$. Then we can draw the corresponding right triangle with angle θ with adjacent side of signed length x (we note that if $\theta > \frac{\pi}{2}$, then x is negative) and hypotenuse of length 2. By the Pythagorean Theorem, the opposite side of the triangle has length $\sqrt{4-x^2}$.



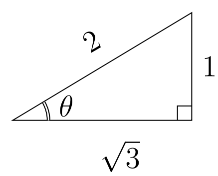
So,

$$\sin\left(\arccos\left(\frac{x}{2}\right)\right) = \sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{4-x^2}}{2}$$

b Let $\theta = \arctan\left(\frac{1}{\sqrt{3}}\right)$. That is, $\tan(\theta) = \frac{1}{\sqrt{3}}$, and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

- Solution 1: Then $\theta = \dfrac{\pi}{6}$, so $\sin\theta = \dfrac{1}{2}$.

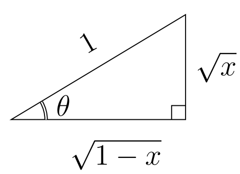
- Solution 2: Then we can draw the corresponding right triangle with angle θ with opposite side of length 1 and adjacent side of length $\sqrt{3}$. By the Pythagorean Theorem, the hypotenuse of the triangle has length $\sqrt{\sqrt{3}^2 + 1^2} = 2$.



So,

$$\sin\left(\arctan\left(\frac{1}{\sqrt{3}}\right)\right) = \sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2}$$

c Let $\theta = \arcsin\left(\sqrt{x}\right)$. That is, $\sin(\theta) = \sqrt{x}$, and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then we can draw the corresponding right triangle with angle θ with opposite side of length $\sqrt{x}$ and hypotenuse of length 1. By the Pythagorean Theorem, the adjacent side of the triangle has length $\sqrt{1-x}$.



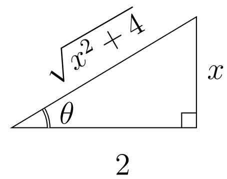
So,

$$\sec\left(\arcsin\left(\sqrt{x}\right)\right) = \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{1}{\sqrt{1-x}}$$

Exercises — Stage 2

1.9.2.5. *. Solution. Let $x = 2\tan\theta$, so that $x^2+4 = 4\tan^2\theta + 4 = 4\sec^2\theta$ and $\mathrm{d}x = 2\sec^2\theta\,\mathrm{d}\theta$. Then

$$\begin{aligned}
\int \frac{1}{(x^2+4)^{3/2}}\,\mathrm{d}x &= \int \frac{1}{(4\sec^2\theta)^{3/2}} \cdot 2\sec^2\theta\,\mathrm{d}\theta \\
&= \int \frac{2\sec^2\theta}{8\sec^3\theta}\,\mathrm{d}\theta \\
&= \frac{1}{4}\int \cos\theta\,\mathrm{d}\theta \\
&= \frac{1}{4}\sin\theta + C = \frac{1}{4}\frac{x}{\sqrt{x^2+4}} + C
\end{aligned}$$



To find $\sin\theta$ in terms of x, we construct the right triangle above. Since $\tan\theta = \frac{x}{2} = \frac{\text{opp}}{\text{adj}}$, we label the opposite side x and the adjacent side 2. By the Pythagorean Theorem, the hypotenuse has length $\sqrt{x^2+4}$. Then $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{x^2+4}}$.

To see why we could write $(\sec^2\theta)^{3/2} = \sec^3\theta$, as opposed to $(\sec^2\theta)^{3/2} = \big|\sec^3\theta\big|$, in the second line above, see Example 1.9.5.

As a check, we observe that the derivative of the answer

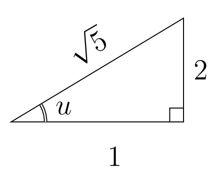
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{4}\frac{x}{\sqrt{x^2+4}}+C\right) &= \frac{1}{4}\frac{1}{\sqrt{x^2+4}} - \frac{1}{2\times 4}\frac{x(2x)}{\big(x^2+4\big)^{3/2}} \\ &= \frac{\frac{x^2}{4}+1-\frac{x^2}{4}}{\big(x^2+4\big)^{3/2}} = \frac{1}{\big(x^2+4\big)^{3/2}}\end{aligned}$$

is exactly the integrand.

1.9.2.6. $*$. Solution.

- Solution 1: As in Question 5, substitute $x = 2\tan u$, $\mathrm{d}x = 2\sec^2 u\,\mathrm{d}u$. Note that when $x = 4$ we have $4 = 2\tan u$, so that $\tan u = 2$.

$$\begin{aligned}\int_0^4 \frac{1}{(4+x^2)^{3/2}}\,\mathrm{d}x &= \int_0^{\arctan 2}\frac{1}{(4+4\tan^2 u)^{3/2}}\,2\sec^2 u\,\mathrm{d}u \\ &= \int_0^{\arctan 2}\frac{2\sec^2 u}{(2\sec u)^3}\,\mathrm{d}u \\ &= \frac{1}{4}\int_0^{\arctan 2}\frac{\sec^2 u}{\sec^3 u}\,\mathrm{d}u \\ &= \frac{1}{4}\int_0^{\arctan 2}\cos u\,\mathrm{d}u \\ &= \left[\frac{1}{4}\sin u\right]_0^{\arctan 2} \\ &= \frac{1}{4}\big(\sin(\arctan 2) - 0\big) = \frac{1}{2\sqrt{5}}\end{aligned}$$



To find $\sin(\arctan 2)$, we use the right triangle above, with angle $u = \arctan 2$. Since $\tan u = 2 = \dfrac{\text{opp}}{\text{adj}}$, we label the opposite side as 2, and the adjacent side as 1. The Pythagorean Theorem tells us the hypotenuse has length $\sqrt{5}$, so $\sin u = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{2}{\sqrt{5}}$.

- Solution 2: Using our result from Question 5,

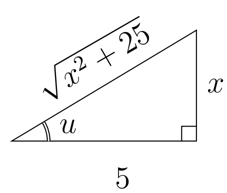
$$\begin{aligned}\int_0^4 \frac{1}{(4+x^2)^{3/2}}\,\mathrm{d}x &= \frac{1}{4}\left[\frac{x}{\sqrt{x^2+4}}\right]_0^4\\ &= \frac{1}{4}\cdot\frac{4}{\sqrt{4^2+4}} = \frac{1}{2\sqrt{5}}\end{aligned}$$

1.9.2.7. *. Solution. Make the change of variables $x = 5\sin\theta$, $\mathrm{d}x = 5\cos\theta\,\mathrm{d}\theta$. Since $x = 0$ corresponds to $\theta = 0$ and $x = \frac{5}{2}$ correponds to $\sin\theta = \frac{1}{2}$ or $\theta = \frac{\pi}{6}$,

$$\int_0^{5/2} \frac{\mathrm{d}x}{\sqrt{25-x^2}} = \int_0^{\pi/6} \frac{5\cos\theta\,\mathrm{d}\theta}{\sqrt{25-25\sin^2\theta}} = \int_0^{\pi/6} \mathrm{d}\theta = \frac{\pi}{6}$$

1.9.2.8. *. Solution. Substitute $x = 5\tan u$, so that $\mathrm{d}x = 5\sec^2 u\,\mathrm{d}u$.

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+25}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{25\tan^2 u+25}}\,5\sec^2 u\,\mathrm{d}u\\ &= \int \frac{5\sec^2 u}{5\sec u}\,\mathrm{d}u = \int \sec u\,\mathrm{d}u\\ &= \log\big|\sec u + \tan u\big| + C\\ &= \log\left|\sqrt{1+\frac{x^2}{25}} + \frac{x}{5}\right| + C\end{aligned}$$



To find $\sec u$ and $\tan u$, we have two options. One is to set up a right triangle with angle u and $\tan u = \frac{x}{5}$. Then we can label the opposite side x and the adjacent side 5, and use Pythagoras to find that the hypotenuse is $\sqrt{x^2+25}$.
Another option is to look back at our work a little more closely — in fact, we've already found what we're looking for. Since we used the substitution $x = 5\tan u$, this gives us $\tan u = \frac{x}{5}$. In the denominator of the integrand, we simplified $\sqrt{x^2+25} = 5\sec u$, so $\sec u = \frac{1}{5}\sqrt{x^2+25} = \sqrt{1+\frac{x^2}{25}}$.
To see why we could write $\sqrt{x^2+25} = 5\sec u$, as opposed to $\sqrt{x^2+25} = 5|\sec u|$, see Example 1.9.5.

1.9.2.9. Solution. The quadratic formula underneath the square root makes us think of a trig substitution, but in the interest of developing good habits, let's check for an easier way first. If we let $u = 2x^2 + 4x$, then $\mathrm{d}u = (4x + 4)\mathrm{d}x$, so $\frac{1}{4}\,\mathrm{d}u = (x + 1)\,\mathrm{d}x$. This substitution looks easier than a trig substitution (which would start with completing the square).

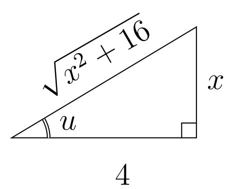
$$\int \frac{x+1}{\sqrt{2x^2+4x}}\,\mathrm{d}x = \frac{1}{4}\int \frac{1}{\sqrt{u}}\,\mathrm{d}u = \frac{1}{2}\sqrt{u} + C = \frac{1}{2}\sqrt{2x^2+4x} + C$$

1.9.2.10. *. Solution. Substitute $x = 4\tan u$, $\mathrm{d}x = 4\sec^2 u\,\mathrm{d}u$.

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2+16}}\,\mathrm{d}x &= \int \frac{1}{16\tan^2 u\sqrt{16\tan^2 u+16}}\,4\sec^2 u\,\mathrm{d}u\\ &= \int \frac{\sec^2 u}{16\tan^2 u\sec u}\,\mathrm{d}u = \frac{1}{16}\int \frac{\sec u}{\tan^2 u}\,\mathrm{d}u\\ &= \frac{1}{16}\int \frac{\cos u}{\sin^2 u}\,\mathrm{d}u\end{aligned}$$

To finish off the integral, we'll substitute $v = \sin u$, $\mathrm{d}v = \cos u\,\mathrm{d}u$.

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2+16}}\,\mathrm{d}x &= \frac{1}{16}\int \frac{\cos u}{\sin^2 u}\,\mathrm{d}u = \frac{1}{16}\int \frac{\mathrm{d}v}{v^2} = -\frac{1}{16v} + C\\ &= -\frac{1}{16\sin u} + C = -\frac{1}{16}\frac{\sqrt{x^2+16}}{x} + C\end{aligned}$$



To find $\sin u$, we draw a right triangle with angle u and $\tan u = \frac{x}{4}$. We label the opposite side x and the adjacent side 4, and then from Pythagoras we find that the hypotenuse has length $\sqrt{x^2+16}$. So, $\sin u = \dfrac{\sqrt{x^2+16}}{x}$.

As a check, we observe that the derivative of the answer

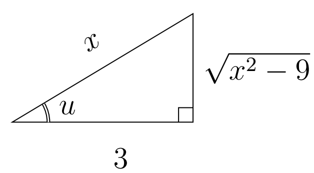
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{16}\frac{\sqrt{x^2+16}}{x} + C\right) &= \frac{1}{16}\frac{\sqrt{x^2+16}}{x^2} - \frac{1}{16}\frac{x}{x\sqrt{x^2+16}}\\ &= \frac{1}{16}\frac{(x^2+16)-x^2}{x^2\sqrt{x^2+16}}\\ &= \frac{1}{x^2\sqrt{x^2+16}}\end{aligned}$$

is exactly the integrand.

1.9.2.11. $*$. **Solution**. Substitute $x = 3\sec u$ with $0 \le u < \frac{\pi}{2}$. Then $\mathrm{d}x = 3\sec u\tan u\,\mathrm{d}u$ and $\sqrt{x^2-9} = \sqrt{9\sec^2 u - 9} = \sqrt{9\tan^2 u} = 3\tan u$, so that

$$\begin{aligned}\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-9}} &= \int \frac{3\sec u\tan u\mathrm{d}u}{9\sec^2 u\sqrt{9\tan^2 u}}\\ &= \frac{1}{9}\int \frac{\mathrm{d}u}{\sec u}\\ &= \frac{1}{9}\int \cos u\mathrm{d}u = \frac{1}{9}\sin u + C.\end{aligned}$$

To evaluate $\sin u$, we make a right triangle with angle u. Since $\sec u = \frac{x}{3} = \frac{\text{hyp}}{\text{adj}}$, we label the hypotenuse x and the adjacent side 3.



Using the Pythagorean Theorem, the opposite side has length $\sqrt{x^2-9}$. So, $\sin u = \dfrac{\sqrt{x^2-9}}{x}$ and

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-9}} = \frac{\sqrt{x^2-9}}{9x} + C.$$

As a check, we observe that the derivative of the answer

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sqrt{x^2-9}}{9x} + C\right) &= -\frac{\sqrt{x^2-9}}{9x^2} + \frac{x}{9x\sqrt{x^2-9}}\\ &= \frac{1}{9}\,\frac{-(x^2-9)+x^2}{x^2\sqrt{x^2-9}} = \frac{1}{x^2\sqrt{x^2-9}}\end{aligned}$$

is exactly the integrand. (We remark that this is the case even for $x \le -3$.)

1.9.2.12. $*$. **Solution**. (a) We'll use the trig identity $\cos 2\theta = 2\cos^2\theta - 1$. It implies that

$$\begin{aligned}\cos^2\theta = \frac{\cos 2\theta + 1}{2} \implies \cos^4\theta &= \frac{1}{4}\big[\cos^2 2\theta + 2\cos 2\theta + 1\big]\\ &= \frac{1}{4}\Big[\frac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1\Big]\\ &= \frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8}\end{aligned}$$

So,

$$\int_0^{\pi/4} \cos^4\theta\mathrm{d}\theta = \int_0^{\pi/4}\Big(\frac{\cos 4\theta}{8} + \frac{\cos 2\theta}{2} + \frac{3}{8}\Big)\mathrm{d}\theta$$

$$\begin{aligned}
&= \left[\frac{\sin 4\theta}{32} + \frac{\sin 2\theta}{4} + \frac{3}{8}\theta\right]_0^{\pi/4} \\
&= \frac{1}{4} + \frac{3}{8}\cdot\frac{\pi}{4} \\
&= \frac{8+3\pi}{32}
\end{aligned}$$

as required.
(b) We'll use the trig substitution $x = \tan\theta$, $\mathrm{d}x = \sec^2\theta\mathrm{d}\theta$. Note that when $\theta = \pm\frac{\pi}{4}$, we have $x = \pm 1$. Also note that dividing the trig identity $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$ gives the trig identity $\tan^2\theta + 1 = \sec^2\theta$. So

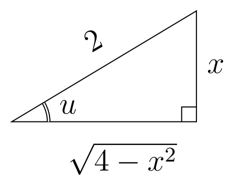
$$\begin{aligned}
\int_{-1}^{1} \frac{\mathrm{d}x}{(x^2+1)^3} &= 2\int_0^1 \frac{\mathrm{d}x}{(x^2+1)^3} && \text{(even integrand)} \\
&= 2\int_0^{\pi/4} \frac{\sec^2\theta\mathrm{d}\theta}{(\tan^2\theta+1)^3} \\
&= 2\int_0^{\pi/4} \frac{\sec^2\theta\mathrm{d}\theta}{(\sec^2\theta)^3} \\
&= 2\int_0^{\pi/4} \cos^4\theta\mathrm{d}\theta \\
&= \frac{8+3\pi}{16}
\end{aligned}$$

by part (a).

1.9.2.13. Solution. The integrand is an odd function, and the limits of integration are symmetric, so $\displaystyle\int_{-\pi/12}^{\pi/12} \frac{15x^3}{(x^2+1)\sqrt{9-x^2}^5}\mathrm{d}x = 0$.

1.9.2.14. *. Solution. Substitute $x = 2\sin u$, so that $\mathrm{d}x = 2\cos u\,\mathrm{d}u$.

$$\begin{aligned}
\int \sqrt{4-x^2}\,\mathrm{d}x &= \int \sqrt{4-4\sin^2 u}\;2\cos u\,\mathrm{d}u \\
&= \int \sqrt{4\cos^2 u}\;2\cos u\,\mathrm{d}u \\
&= \int 4\cos^2 u\,\mathrm{d}u = 2\int \big(1+\cos(2u)\big)\,\mathrm{d}u \\
&= 2u + \sin(2u) + C \\
&= 2u + 2\sin u\cos u + C \\
&= 2\arcsin\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2} + C
\end{aligned}$$



To see why we could write $\sqrt{4\cos^2 u} = 2\cos u$, as opposed to $\sqrt{4\cos^2 u} = 2|\cos u|$, in the third line above, see Example 1.9.2.

We used the substitution $x = 2\sin u$, so we know $\sin u = \frac{x}{2}$ and $u = \arcsin(\frac{x}{2})$. We have three options for finding $\cos u$.

First, we can draw a right triangle with angle u. Since $\sin u = \frac{x}{2}$, we label the opposite side x and the hypotenuse 2, then by the Pythagorean Theorem the adjacent side has length $\sqrt{4-x^2}$. So, $\cos u = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{\sqrt{4-x^2}}{2}$.

Second, we can look back carefully at our work. We simplified $\sqrt{4-x^2} = 2\cos u$, so $\cos u = \dfrac{\sqrt{4-x^2}}{2}$.

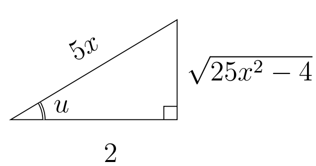
Third, we could use the identity $\sin^2 u + \cos^2 u = 1$. Then $\cos u = \pm\sqrt{1-\sin^2 u} = \pm\sqrt{1-\frac{x^2}{4}}$. Since $u = \arcsin(x/2)$, u is in the range of arcsine, which means $-\frac{\pi}{2} \le u \le \frac{\pi}{2}$. Therefore, $\cos u \ge 0$, so $\cos u = \sqrt{1-\frac{x^2}{4}} = \frac{\sqrt{4-x^2}}{2}$.

So,

$$\begin{aligned}\int \sqrt{4-x^2}\,\mathrm{d}x &= 2u + 2\sin u\cos u + C\\ &= 2\arcsin\frac{x}{2} + x\cdot\frac{\sqrt{4-x^2}}{2} + C\end{aligned}$$

1.9.2.15. *. Solution. Substitute $x = \frac{2}{5}\sec u$ with $0 < u < \frac{\pi}{2}$, so that $\mathrm{d}x = \frac{2}{5}\sec u\,\tan u\,\mathrm{d}u$ and $\sqrt{25x^2-4} = \sqrt{4(\sec^2 u - 1)} = \sqrt{4\tan^2 u} = 2\tan u$. Then

$$\begin{aligned}\int \frac{\sqrt{25x^2-4}}{x}\,\mathrm{d}x &= \int \frac{2\tan u}{\frac{2}{5}\sec u}\cdot\frac{2}{5}\sec u\tan u\,\mathrm{d}u\\ &= 2\int \tan^2 u\,\mathrm{d}u = 2\int \big(\sec^2 u - 1\big)\,\mathrm{d}u\\ &= 2\tan u - 2u + C\\ &= \sqrt{25x^2-4} - 2\,\mathrm{arcsec}\,\tfrac{5x}{2} + C\end{aligned}$$



To find $\tan u$, we draw a right triangle with angle u. Since $\sec u = \dfrac{5x}{2}$, we label the hypotenuse $5x$ and the adjacent side 2. Then the Pythagorean Theorem gives us

the opposite side as length $\sqrt{25x^2-4}$. Then $\tan u = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{25x^2-4}}{2}$.
Alternately, we can notice that in our work, we already showed $2\tan u = \sqrt{25x^2-4}$, so $\tan u = \frac{1}{2}\sqrt{25x^2-4}$.
As a check, we observe that the derivative of the answer

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{25x^2-4} - 2\,\mathrm{arcsec}\,\frac{5x}{2} + C\right) \\
&= \frac{25x}{\sqrt{25x^2-4}} - 2\frac{\frac{5}{2}}{\left|\frac{5x}{2}\right|\sqrt{\frac{25x^2}{4}-1}} \\
&= \frac{25x}{\sqrt{25x^2-4}} - \frac{4}{x\sqrt{25x^2-4}} \\
&\qquad \text{since } x>0 \\
&= \frac{25x^2-4}{x\sqrt{25x^2-4}} \\
&= \frac{\sqrt{25x^2-4}}{x}
\end{aligned}$$

is exactly the integrand (provided $x > \frac{2}{5}$).

1.9.2.16. Solution. The integrand has a quadratic polynomial under a square root, which makes us think of trig substitutions. However, it's good practice to look for simpler methods before we jump into more complicated ones, and in this case we find something nicer than a trig substitution: the substitution $u = x^2-1$, $\mathrm{d}u = 2x\,\mathrm{d}x$. Then $x\mathrm{d}x = \frac{1}{2}\mathrm{d}u$, and $x^2 = u+1$. When $x=\sqrt{10}$, $u=9$, and when $x=\sqrt{17}$, $u=16$.

$$\begin{aligned}
\int_{\sqrt{10}}^{\sqrt{17}} \frac{x^3}{\sqrt{x^2-1}}\,\mathrm{d}x &= \int_{\sqrt{10}}^{\sqrt{17}} \frac{x^2}{\sqrt{x^2-1}} \cdot x\mathrm{d}x \\
&= \frac{1}{2}\int_9^{16} \frac{u+1}{\sqrt{u}}\,\mathrm{d}u \\
&= \frac{1}{2}\int_9^{16} \left(u^{1/2}+u^{-1/2}\right)\,\mathrm{d}u \\
&= \frac{1}{2}\left[\frac{2}{3}u^{3/2} + 2u^{1/2}\right]_9^{16} \\
&= \frac{1}{2}\left[\frac{2}{3}\cdot 4^3 + 2\cdot 4 - \frac{2}{3}\cdot 3^3 - 2\cdot 3\right] \\
&= \frac{40}{3}
\end{aligned}$$

1.9.2.17. *. Solution. This integrand looks very different from those above.

But it is only slightly disguised. If we complete the square

$$\int \frac{\mathrm{d}x}{\sqrt{3-2x-x^2}} = \int \frac{\mathrm{d}x}{\sqrt{4-(x+1)^2}}$$

and make the substitution $y = x+1$, $\mathrm{d}y = \mathrm{d}x$

$$\int \frac{\mathrm{d}x}{\sqrt{3-2x-x^2}} = \int \frac{\mathrm{d}x}{\sqrt{4-(x+1)^2}} = \int \frac{\mathrm{d}y}{\sqrt{4-y^2}}$$

we get a typical trig substitution integral. So, we substitute $y = 2\sin\theta$, $\mathrm{d}y = 2\cos\theta\,\mathrm{d}\theta$ to get

$$\begin{aligned}\int \frac{\mathrm{d}x}{\sqrt{3-2x-x^2}} = \int \frac{\mathrm{d}y}{\sqrt{4-y^2}} &= \int \frac{2\cos\theta\,\mathrm{d}\theta}{\sqrt{4-4\sin^2\theta}} = \int \frac{2\cos\theta\,\mathrm{d}\theta}{\sqrt{4\cos^2\theta}} \\ &= \int \mathrm{d}\theta = \theta + C = \arcsin\frac{y}{2} + C \\ &= \arcsin\frac{x+1}{2} + C\end{aligned}$$

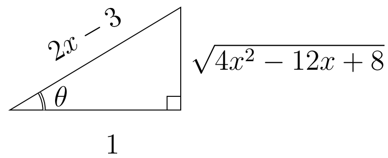
An experienced integrator would probably substitute $x+1 = 2\sin\theta$ directly, without going through y.

1.9.2.18. Solution. Completing the square, we see $4x^2-12x+8 = (2x-3)^2-1$.

$$\int \frac{1}{(2x-3)^3\sqrt{4x^2-12x+8}}\mathrm{d}x = \int \frac{1}{(2x-3)^3\sqrt{(2x-3)^2-1}}\mathrm{d}x$$

As $x > 2$, we have $2x-3 > 1$. We use the substitution $2x-3 = \sec\theta$ with $0 \le \theta < \frac{\pi}{2}$. So $2\,\mathrm{d}x = \sec\theta\tan\theta\ \mathrm{d}\theta$ and $\sqrt{(2x-3)^2-1} = \sqrt{\sec^2\theta-1} = \sqrt{\tan^2\theta} = \tan\theta$.

$$\begin{aligned}&= \frac{1}{2}\int \frac{1}{\sec^3\theta\sqrt{\sec^2\theta-1}}\sec\theta\tan\theta\mathrm{d}\theta \\ &= \frac{1}{2}\int \frac{1}{\sec^3\theta\tan\theta}\sec\theta\tan\theta\mathrm{d}\theta \\ &= \frac{1}{2}\int \frac{1}{\sec^2\theta}\mathrm{d}\theta \\ &= \frac{1}{2}\int \cos^2\theta\mathrm{d}\theta \\ &= \frac{1}{4}\int (1+\cos(2\theta))\mathrm{d}\theta \\ &= \frac{1}{4}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C \\ &= \frac{1}{4}(\theta + \sin\theta\cos\theta) + C \\ &= \frac{1}{4}\left(\arccos\left(\frac{1}{2x-3}\right) + \frac{\sqrt{4x^2-12x+8}}{(2x-3)^2}\right) + C\end{aligned}$$



Since $2x-3=\sec\theta$, we know $\cos\theta=\frac{1}{2x-3}$ and $\theta=\arccos\left(\frac{1}{2x-3}\right)$. (Equivalently, $\theta=\text{arcsec}(2x-3)$.) To find $\sin\theta$, we draw a right triangle with adjacent side of length 1, and hypotenuse of length $2x-3$. By the Pythagorean Theorem, the opposite side has length $\sqrt{4x^2-12x+8}$.

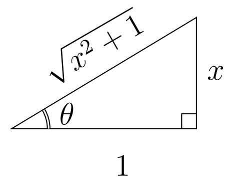
1.9.2.19. Solution. We use the substitution $x=\tan u$, $\mathrm{d}x=\sec^2 u\mathrm{d}u$. Note $\tan 0=0$ and $\tan\frac{\pi}{4}=1$.

$$\begin{aligned}
\int_0^1 \frac{x^2}{\sqrt{x^2+1}^3}\mathrm{d}x &= \int_0^{\pi/4} \frac{\tan^2 u}{\sqrt{\tan^2 u+1}^3}\sec^2 u\mathrm{d}u\\
&= \int_0^{\pi/4} \frac{\tan^2 u}{\sqrt{\sec^2 u}^3}\sec^2 u\mathrm{d}u\\
&= \int_0^{\pi/4} \frac{\tan^2 u}{\sec u}\mathrm{d}u\\
&= \int_0^{\pi/4} \frac{\sec^2 u-1}{\sec u}\mathrm{d}u\\
&= \int_0^{\pi/4} \big(\sec u-\cos u\big)\mathrm{d}u\\
&= \Big[\log|\sec u+\tan u|-\sin u\Big]_0^{\pi/4}\\
&= \left(\log\left|\sqrt{2}+1\right|-\frac{1}{\sqrt{2}}\right)-(\log|1+0|-0)\\
&= \log(1+\sqrt{2})-\frac{1}{\sqrt{2}}
\end{aligned}$$

1.9.2.20. Solution. There's no square root, but we can still make use of the substitution $x=\tan\theta$, $\mathrm{d}x=\sec^2\theta\mathrm{d}\theta$.

$$\begin{aligned}
\int \frac{1}{(x^2+1)^2}\mathrm{d}x &= \int \frac{1}{(\tan^2\theta+1)^2}\sec^2\theta\mathrm{d}\theta\\
&= \int \frac{1}{\sec^4\theta}\sec^2\theta\mathrm{d}\theta = \int \cos^2\theta\mathrm{d}\theta\\
&= \frac{1}{2}\int \big(1+\cos(2\theta)\big)\mathrm{d}\theta\\
&= \frac{1}{2}\left(\theta+\frac{1}{2}\sin(2\theta)\right)+C\\
&= \frac{1}{2}\left(\theta+\sin\theta\cos\theta\right)+C
\end{aligned}$$

$$= \frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right) + C$$



Since $x = \tan\theta$, we can draw a right triangle with angle θ, opposite side x, and adjacent side 1. Then by the Pythagorean Theorem, its hypotenuse has length $\sqrt{x^2+1}$, which allows us to find $\sin\theta$ and $\cos\theta$.

Exercises — Stage 3

1.9.2.21. Solution. We complete the square to find $x^2 - 2x + 2 = (x-1)^2 + 1$.

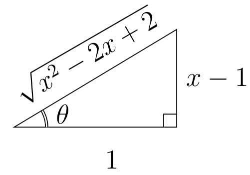
$$\int \frac{x^2}{\sqrt{x^2-2x+2}}\mathrm{d}x = \int \frac{x^2}{\sqrt{(x-1)^2+1}}\mathrm{d}x$$

We use the substitution $x - 1 = \tan\theta$, which implies $\mathrm{d}x = \sec^2\theta \mathrm{d}\theta$ and $x = \tan\theta + 1$

$$\begin{aligned}
&= \int \frac{(\tan\theta+1)^2}{\sqrt{(\tan\theta)^2+1}}\sec^2\theta \mathrm{d}\theta \\
&= \int \frac{\tan^2\theta + 2\tan\theta + 1}{\sec\theta}\sec^2\theta \mathrm{d}\theta \\
&= \int (\sec^2\theta + 2\tan\theta)\sec\theta \mathrm{d}\theta \\
&= \int \left(\sec^3\theta + 2\tan\theta\sec\theta\right)\mathrm{d}\theta \\
&= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\log|\sec\theta+\tan\theta| + 2\sec\theta + C \\
&= \frac{1}{2}\sqrt{x^2-2x+2}(x-1) + \frac{1}{2}\log\left|\sqrt{x^2-2x+2}+x-1\right| \\
&\qquad + 2\sqrt{x^2-2x+2} + C \\
&= \frac{3+x}{2}\sqrt{x^2-2x+2} + \frac{1}{2}\log\left|\sqrt{x^2-2x+2}+x-1\right| + C
\end{aligned}$$

To see why we could write $\sqrt{(\tan\theta)^2+25} = \sec\theta$, as opposed to $\sqrt{(\tan\theta)^2+25} = |\sec\theta|$, see Example 1.9.5.

From our substitution, we know $\tan\theta = x - 1$. To find $\sec\theta$, we can notice that in our work we already simplified $\sqrt{x^2-2x+1} = \sec\theta$.



Alternately, we can draw a right triangle with angle θ, opposite side $x-1$, adjacent side 1, and use the Pythagorean Theorem to find the hypotenuse.

1.9.2.22. Solution. First, we complete the square. The constants aren't integers, but we can still use the same method as in Question 2. The quadratic function under the square root is $3x^2+5x$. We match the non-constant terms to those of a perfect square.

$$\begin{aligned}&(ax+b)^2 = a^2x^2+2abx+b^2\\ &3x^2+5x = a^2x^2+2abx+b^2+c \quad \text{for some constant } c\end{aligned}$$

- Looking at the leading term tells us $a=\sqrt{3}$.
- Then the second term tells us $5=2ab=2\sqrt{3}b$, so $b=\frac{5}{2\sqrt{3}}$.
- Finally, the constant terms give us $0=b^2+c=\frac{25}{12}+c$, so $c=-\frac{25}{12}$.

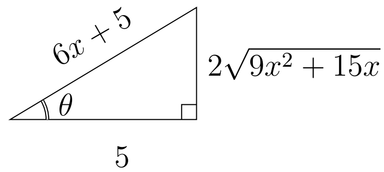
So, $3x^2+5x=\left(\sqrt{3}x+\frac{5}{2\sqrt{3}}\right)^2-\frac{25}{12}$.

$$\int\frac{1}{\sqrt{3x^2+5x}}\mathrm{d}x=\int\frac{1}{\sqrt{\left(\sqrt{3}x+\frac{5}{2\sqrt{3}}\right)^2-\frac{25}{12}}}\mathrm{d}x$$

We use the substitution $\sqrt{3}x+\frac{5}{2\sqrt{3}}=\frac{5}{2\sqrt{3}}\sec\theta$, which leads to $\sqrt{3}\mathrm{d}x=\frac{5}{2\sqrt{3}}\sec\theta\tan\theta\mathrm{d}\theta$, i.e. $\mathrm{d}x=\frac{5}{6}\sec\theta\tan\theta\mathrm{d}\theta$.

$$\begin{aligned}&=\int\frac{1}{\sqrt{\left(\frac{5}{2\sqrt{3}}\sec\theta\right)^2-\frac{25}{12}}}\cdot\frac{5}{6}\sec\theta\tan\theta\mathrm{d}\theta\\
&=\int\frac{1}{\sqrt{\frac{25}{12}\sec^2\theta-\frac{25}{12}}}\cdot\frac{5}{6}\sec\theta\tan\theta\mathrm{d}\theta\\
&=\int\frac{1}{\sqrt{\frac{25}{12}\tan^2\theta}}\cdot\frac{5}{6}\sec\theta\tan\theta\mathrm{d}\theta\\
&=\int\frac{1}{\frac{5}{2\sqrt{3}}\tan\theta}\cdot\frac{5}{6}\sec\theta\tan\theta\mathrm{d}\theta\\
&=\frac{1}{\sqrt{3}}\int\sec\theta\mathrm{d}\theta\\
&=\frac{1}{\sqrt{3}}\log|\sec\theta+\tan\theta|+C\end{aligned}$$

$$= \frac{1}{\sqrt{3}} \log\left|\left(\frac{6}{5}x+1\right)+\frac{2}{5}\sqrt{9x^2+15x}\right|+C$$



Since we used the substitution $\sqrt{3}x+\frac{5}{2\sqrt{3}} = \frac{5}{2\sqrt{3}}\sec\theta$, we have $\sec\theta = \frac{6}{5}x+1 = \frac{6x+5}{5}$. To find $\tan\theta$ in terms of x, we have two options. We can make a right triangle with angle θ, hypotenuse $6x+5$, and adjacent side 5, then use the Pythagorean Theorem to find the opposite side. Or, we can look through our work and see that $\sqrt{3x^2+5} = \frac{5}{2\sqrt{3}}\tan\theta$, so $\tan\theta = \frac{2\sqrt{3}}{5}\sqrt{3x^2+5} = \frac{2}{5}\sqrt{9x^2+15}$.
As a check, we observe that the derivative of the answer

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sqrt{3}}\log\left|\left(\frac{6}{5}x+1\right)+\frac{2}{5}\sqrt{9x^2+15x}\right|+C\right)\\
&= \frac{1}{\sqrt{3}}\,\frac{\frac{6}{5}+\frac{1}{5}\frac{18x+15}{\sqrt{9x^2+15x}}}{\left(\frac{6}{5}x+1\right)+\frac{2}{5}\sqrt{9x^2+15x}}\\
&= \frac{1}{\sqrt{3}}\,\frac{6+3\frac{6x+5}{\sqrt{9x^2+15x}}}{(6x+5)+2\sqrt{9x^2+15x}} = \sqrt{3}\,\frac{2+\frac{6x+5}{\sqrt{9x^2+15x}}}{(6x+5)+2\sqrt{9x^2+15x}}\\
&= \frac{\sqrt{3}}{\sqrt{9x^2+15x}} = \frac{1}{\sqrt{3x^2+5x}}
\end{aligned}$$

is exactly the integrand.
Remark: in applications, often the numbers involved are messier than they are in textbooks. The ideas of this problem are similar to other problems in this section, but it's good practice to apply them in a slightly messy context.

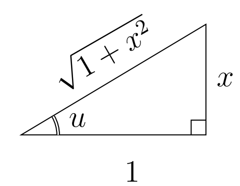
1.9.2.23. Solution. We use the substitution $x=\tan u$, $\mathrm{d}x = \sec^2 u\mathrm{d}u$.

$$\begin{aligned}
\int \frac{\sqrt{1+x^2}^3}{x}\mathrm{d}x &= \int \frac{\sqrt{1+\tan^2 u}^3}{\tan u}\sec^2 u\mathrm{d}u\\
&= \int \frac{\sec^3 u}{\tan u}\sec^2 u\mathrm{d}u\\
&= \int \frac{(\sec^2 u)^2}{\tan u}\sec u\mathrm{d}u\\
&= \int \frac{(\tan^2 u+1)^2}{\tan u}\sec u\mathrm{d}u\\
&= \int \frac{\tan^4 u+2\tan^2 u+1}{\tan u}\sec u\mathrm{d}u\\
&= \int \tan^3 u\sec u\mathrm{d}u + \int 2\sec u\tan u\mathrm{d}u + \int \frac{\sec u}{\tan u}\mathrm{d}u
\end{aligned}$$

For the first integral, we use the substitution $w = \sec u$. The second is the antiderivative of $2\sec u$. The third we simplify as $\frac{\sec u}{\tan u} = \frac{1}{\cos u} \cdot \frac{\cos u}{\sin u} = \csc u$. This brings us to

$$\begin{aligned}
&\int \frac{\sqrt{1+x^2}^3}{x}\mathrm{d}x \\
&= \int \Big((\sec^2 u - 1)\sec u \tan u \Big)\mathrm{d}u + 2\sec u + \log|\cot u - \csc u| + C \\
&= \int (w^2-1)\mathrm{d}w + 2\sec u + \log|\cot u - \csc u| + C \\
&= \frac{1}{3}w^3 - w + 2\sec u + \log|\cot u - \csc u| + C \\
&= \frac{1}{3}\sec^3 u - \sec u + 2\sec u + \log|\cot u - \csc u| + C \\
&= \frac{1}{3}\sec^3 u + \sec u + \log|\cot u - \csc u| + C
\end{aligned}$$

We began with the substitution $x = \tan u$. Then $\cot u = \frac{1}{x}$. To find $\csc u$ and $\sec u$, we draw a right triangle with angle u, opposite side x, and adjacent side 1. The Pythagorean Theorem gives us the hypotenuse.



So

$$\begin{aligned}
&\int \frac{\sqrt{1+x^2}^3}{x}\mathrm{d}x \\
&= \frac{1}{3}\sqrt{1+x^2}^3 + \sqrt{1+x^2} + \log\left|\frac{1}{x} - \frac{\sqrt{1+x^2}}{x}\right| + C \\
&= \frac{1}{3}\sqrt{1+x^2}(4+x^2) + \log\left|\frac{1-\sqrt{1+x^2}}{x}\right| + C
\end{aligned}$$

1.9.2.24. Solution. The half of the ellipse to the right of the y-axis is given by the equation

$$x = f(y) = 4\sqrt{1-\left(\frac{y}{2}\right)^2}$$

The area we want is twice the area between the right-hand side of the curve and the y-axis, from $y=-1$ to $y=1$. In other words,

$$\text{Area} = 2\int_{-1}^{1} 4\sqrt{1-\left(\frac{y}{2}\right)^2}\mathrm{d}y = 8\int_{-1}^{1}\sqrt{1-\left(\frac{y}{2}\right)^2}\mathrm{d}y$$

Since the integrand $\sqrt{1-\left(\frac{y}{2}\right)^2}$ is an even function of y,

$$\int_{-1}^{1}\sqrt{1-\left(\frac{y}{2}\right)^2}\,\mathrm{d}y = 2\int_{0}^{1}\sqrt{1-\left(\frac{y}{2}\right)^2}\,\mathrm{d}y$$

So

$$\text{Area} = 16\int_{0}^{1}\sqrt{1-\left(\frac{y}{2}\right)^2}\,\mathrm{d}y$$

We use the substitution $\frac{y}{2}=\sin\theta$, $\frac{1}{2}\mathrm{d}y=\cos\theta\mathrm{d}\theta$. When $y=0$, $\sin\theta=0$ so that $\theta=0$, and when $y=1$, $\sin\theta=\frac{1}{2}$ so that $\theta=\frac{\pi}{6}$. Hence

$$\begin{aligned}
\text{Area} &= 16\int_{0}^{\pi/6}\sqrt{1-(\sin\theta)^2}\cdot 2\cos\theta\,\mathrm{d}\theta\\
&= 32\int_{0}^{\pi/6}\sqrt{\cos^2\theta}\cos\theta\,\mathrm{d}\theta\\
&= 32\int_{0}^{\pi/6}\cos^2\theta\,\mathrm{d}\theta\\
&= 16\int_{0}^{\pi/6}\big(1+\cos(2\theta)\big)\,\mathrm{d}\theta\\
&= 16\left[\theta+\frac{1}{2}\sin(2\theta)\right]_{0}^{\pi/6}\\
&= 16\left(\frac{\pi}{6}+\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)\\
&= \frac{8\pi}{3}+4\sqrt{3}
\end{aligned}$$

Remark: we also investigated areas of ellipses in Question 1.2.3.16, Section 1.2.

1.9.2.25. Solution. Note that $f(x)$ is an even function, nonnegative over its entire domain.
(a) To find the area of R, we evaluate

$$\text{Area} = \int_{-1/2}^{1/2}\frac{|x|}{\sqrt[4]{1-x^2}}\mathrm{d}x = 2\int_{0}^{1/2}\frac{x}{\sqrt[4]{1-x^2}}\mathrm{d}x$$

We use the substitution $u=1-x^2$, $\mathrm{d}u=-2x\mathrm{d}x$.

$$\begin{aligned}
&= -\int_{1}^{3/4}\frac{1}{u^{1/4}}\mathrm{d}u\\
&= -\left[\frac{4}{3}u^{3/4}\right]_{1}^{3/4} = -\frac{4}{3}\left(\left(\frac{3}{4}\right)^{3/4}-1\right)\\
&= \frac{4}{3}-\sqrt[4]{\frac{4}{3}}
\end{aligned}$$

(b) We slice the solid of rotation into circular disks of width $\mathrm{d}x$ and radius $\dfrac{|x|}{\sqrt[4]{1-x^2}}$.

$$\begin{aligned}\text{Volume} &= \int_{-1/2}^{1/2} \pi\left(\frac{|x|}{\sqrt[4]{1-x^2}}\right)^2 \mathrm{d}x\\ &= 2\pi\int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x\end{aligned}$$

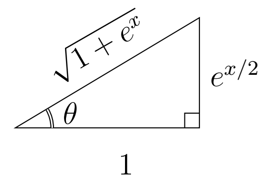
We use the substitution $x = \sin\theta$, $\mathrm{d}x = \cos\theta\mathrm{d}\theta$, so $\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$. Note $\sin 0 = 0$ and $\sin\frac{\pi}{6} = \frac{1}{2}$.

$$\begin{aligned} &= 2\pi\int_0^{\pi/6} \frac{\sin^2\theta}{\cos\theta}\cos\theta\mathrm{d}\theta\\ &= 2\pi\int_0^{\pi/6}\sin^2\theta\mathrm{d}\theta\\ &= \pi\int_0^{\pi/6}\big(1-\cos(2\theta)\big)\mathrm{d}\theta\\ &= \pi\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\pi/6}\\ &= \pi\left(\frac{\pi}{6} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)\\ &= \frac{\pi^2}{6} - \frac{\sqrt{3}\pi}{4}\end{aligned}$$

1.9.2.26. Solution. If we think of e^x as $\left(e^{x/2}\right)^2$, the function under the square root suggests the substitution $e^{x/2} = \tan\theta$. Then $\frac{1}{2}e^{x/2}\mathrm{d}x = \sec^2\theta\mathrm{d}\theta$, so $\mathrm{d}x = \frac{2}{e^{x/2}}\sec^2\theta\mathrm{d}\theta = \frac{2}{\tan\theta}\sec\theta\mathrm{d}\theta$.

$$\begin{aligned}\int\sqrt{1+e^x}\mathrm{d}x &= \int\frac{2\sqrt{1+\tan^2\theta}}{\tan\theta}\sec^2\theta\mathrm{d}\theta\\ &= 2\int\frac{\sec^3\theta}{\tan\theta}\mathrm{d}\theta\\ &= 2\int\frac{\sec\theta(\tan^2\theta+1)}{\tan\theta}\mathrm{d}\theta\\ &= 2\int\left(\sec\theta\tan\theta + \frac{\sec\theta}{\tan\theta}\right)\mathrm{d}\theta\\ &= 2\int\big(\sec\theta\tan\theta + \csc\theta\big)\mathrm{d}\theta\\ &= 2\sec\theta + 2\log|\cot\theta - \csc\theta| + C\\ &= 2\sqrt{1+e^x} + 2\log\left|\frac{1}{e^{x/2}} - \frac{\sqrt{1+e^x}}{e^{x/2}}\right| + C\end{aligned}$$

$$= 2\sqrt{1+e^x} + 2\log\left|1-\sqrt{1+e^x}\right| - 2\log(e^{x/2}) + C$$
$$= 2\sqrt{1+e^x} + 2\log\left|1-\sqrt{1+e^x}\right| - x + C$$



We used the substitution $e^{x/2} = \tan\theta$, so $\cot\theta = \frac{1}{e^{x/2}}$. To find $\sec\theta$ and $\csc\theta$, we draw a right triangle with opposite side $e^{x/2}$ and adjacent side 1. They by the Pythagorean Theorem, the hypotenuse has length $\sqrt{1+e^x}$.

Remark: if we use the substitution $u = \sqrt{1+e^x}$, then we can change the integral to $\displaystyle\int \frac{2u^2}{u^2-1}\mathrm{d}u$. We can integrate this using the method of partial fractions, which we'll learn in the next section. You can explore this option in Question 1.10.4.26, Section 1.10.

1.9.2.27. Solution.

a We can save ourselves some trouble by applying logarithm rules before we differentiate.

$$\begin{aligned}
\log\left|\frac{1+x}{\sqrt{1-x^2}}\right| &= \log|1+x| - \log|\sqrt{1-x^2}|\\
&= \log|1+x| - \frac{1}{2}\log|1-x^2|\\
&= \log|1+x| - \frac{1}{2}\log|(1+x)(1-x)|\\
&= \log|1+x| - \frac{1}{2}\log|1+x| - \frac{1}{2}\log|1-x|
\end{aligned}$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left\{\log\left|\frac{1+x}{\sqrt{1-x^2}}\right|\right\}\\
&\quad= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\log|1+x| - \frac{1}{2}\log|1+x| - \frac{1}{2}\log|1-x|\right\}\\
&\quad= \frac{1}{1+x} - \frac{1/2}{1+x} + \frac{1/2}{1-x}\\
&\quad= \frac{1/2}{1+x} + \frac{1/2}{1-x}\\
&\quad= \frac{1}{1-x^2}
\end{aligned}$$

Notice this is the integrand from our work in blue.

b False: $\displaystyle\int_2^3 \frac{1}{1-x^2}\mathrm{d}x$ is a number, because it is the area under a finite portion of a continuous curve. (We note that the integrand is continuous over the interval $[2,3]$, although it is not continuous everywhere.) However, $\displaystyle\left[\log\left|\frac{1+x}{\sqrt{1-x^2}}\right|\right]_{x=2}^{x=3}$ is not defined, since the denominator takes the square root of a negative number. So, these two expressions are not the same.

c The work in the question is not correct. The most salient problem is that when we make the substitution $x = \sin\theta$, we restrict the possible values of x to $[-1, 1]$, since this is the range of the sine function. However, the original integral had no such restriction.

How can we be sure we avoid this problem in the future? In the introductory text to Section 1.9 (before Example 1.9.1), the notes tell us that we are allowed to write our old variable as a function of a new variable (say $x = s(u)$) *as long as that function is invertible to recover our original variable* x. There is one very obvious reason why invertibility is necessary: after we antidifferentiate using our new variable u, we need to get it back in terms of our original variable, so we need to be able to recover x. Moreover, invertibility reconciles potential problems with domains: if an inverse function $u = s^{-1}(x)$ exists, then for any x, there exists a u with $s(u) = x$. (This was not the case in the work for the question, because we chose $x = \sin\theta$, but if $x = 2$, there is no corresponding θ. Note, however, that $x = \sin\theta$ is invertible over $[-1, 1]$, so the work is correct if we restrict x to those values.)

Remark: in the next section, you will learn to use partial fractions to find $\displaystyle\int \frac{1}{1-x^2}\mathrm{d}x = \log|1+x| - \frac{1}{2}\log|1-x|$. When $-1 < x < 1$, this is equivalent to $\displaystyle\log\left|\frac{1+x}{\sqrt{1-x^2}}\right|$.

1.9.2.28. Solution. Remember that for any value X,

$$|X| = \begin{cases} X & \text{if } X \geq 0 \\ -X & \text{if } X \leq 0 \end{cases}$$

So, $|X| \neq X$ precisely when $X < 0$.
(a) The range of arcsine is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. So, since $u = \arcsin(x/a)$, u is in the range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore $\cos u \geq 0$. Since a is positive, $a\cos u \geq 0$, so $a\cos u = |a\cos u|$. That is,

$$\sqrt{a^2 - x^2} = |a\cos u| = a\cos u$$

all the time.
(b) The range of arctangent is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. So, since $u = \arctan(x/a)$, u is in the range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Therefore $\sec u = \frac{1}{\cos u} > 0$. Since a is positive, $a\sec u > 0$, so $a\sec u = |a\sec u|$.That is,

$$\sqrt{a^2 + x^2} = |a\sec u| = a\sec u$$

all the time.
(c) The range of arccosine is $\big[0,\pi\big]$. So, since $u = \operatorname{arcsec}(x/a) = \arccos(a/x)$, u is in the range $\big[0,\pi\big]$. (Actually, it's in the range $[0,\frac{\pi}{2}) \cup (\frac{\pi}{2},\pi]$, since secant is undefined at $\pi/2$.) If $|a\tan u| \neq a\tan u$, then $\tan u < 0$, which happens when u is in the range $\big(\frac{\pi}{2},\pi\big)$. This is the same range over which $-1 < \cos u < 0$, and so $-1 < \frac{a}{x} < 0$. Since $\frac{a}{x} < 0$, a and x have different signs, so $x < 0$. Then since $-1 < \frac{a}{x}$, also $x < -a$.
So,

$$\sqrt{x^2 - a^2} = |a\tan u| = -a\tan u \neq a\tan u$$

happens precisely when when $x < -a$.

1.10 · Partial Fractions
1.10.4 · Exercises

Exercises — Stage 1

1.10.4.1. Solution. If a quadratic function can be factored as $(ax+b)(cx+d)$ for some constants a,b,c,d, then it has roots $-\frac{b}{a}$ and $-\frac{d}{c}$. So, if a quadratic function has no roots, it is irreducible: this is the case for the function in graph (d).
If a quadratic function has two different roots, then $(ax+b) \neq \alpha(cx+d)$ for any constant α. That is, the quadratic function is the product of distinct linear factors. This is the case for the functions graphed in (b) and (c), since these each have two distinct places where they cross the x-axis.
Finally, if a quadratic function has precisely one root, then $\frac{b}{a} = \frac{d}{c}$, so:

$$\begin{aligned}(ax+b)(cx+d) &= a(x+\tfrac{b}{a})(cx+d) = a(x+\tfrac{d}{c})(cx+d)\\ &= \tfrac{a}{c}(cx+d)(cx+d)\end{aligned}$$

That is, the quadratic function is the product of a repeated linear factor, and a constant $\frac{a}{c}$ (which might simply be $\frac{a}{c} = 1$).

1.10.4.2. *. Solution. Our first step is to fully factor the denominator:

$$(x^2-1)^2(x^2+1) = (x-1)^2(x+1)^2(x^2+1)$$

Once a term is linear, it can't be factored further; for quadratic terms, we should check that they are irreducible. Since x^2+1 has no real roots (we are familiar with its graph, which is entirely above the x-axis), it is irreducible, so now our denominator is fully factored.

$$\begin{aligned}\frac{x^3+3}{(x^2-1)^2(x^2+1)} &= \frac{x^3+3}{(x-1)^2(x+1)^2(x^2+1)}\\ &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{Ex+F}{x^2+1}\end{aligned}$$

Notice $(x-1)$ and $(x+1)$ are (repeated) linear factors, while (x^2+1) is an irreducible quadratic factor. This accounts for the difference in the numerators of their corresponding terms.

1.10.4.3. *. Solution. The partial fraction decomposition has the form

$$\frac{3x^3 - 2x^2 + 11}{x^2(x-1)(x^2+3)} = \frac{A}{x-1} + \text{various terms}$$

When we multiply through by the original denominator, this becomes

$$3x^3 - 2x^2 + 11 = x^2(x^2+3)A + (x-1)(\text{other terms}).$$

Evaluating both sides at $x = 1$ yields $3 \cdot 1^3 - 2 \cdot 1^2 + 11 = 1^2(1^2+3)A + 0$, or $A = 3$.

1.10.4.4. Solution.

a We start by dividing. The leading term of the numerator is x times the leading term of the denominator. The remainder is $x + 2$.

$$\begin{array}{r|l} & \quad\quad x \\ \hline x^2+1 & \;\; x^3 + 2x + 2 \\ & -x^3 \;\; -x \\ \hline & \quad\quad\;\; x + 2 \end{array}$$

That is, $x^3 + 2x + 2 = x(x^2+1) + (x+2)$. So,

$$\frac{x^3 + 2x + 2}{x^2 + 1} = x + \frac{x+2}{x^2+1}$$

b We start by dividing. The leading term of the numerator is $3x^2$ times the leading term of the denominator.

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad 3x^2 \\ \hline 5x^2+2x+8 & \;\; 15x^4 + 6x^3 + 34x^2 + 4x + 20 \\ & -15x^4 - 6x^3 - 24x^2 \\ \hline & \quad\quad\quad\quad\quad 10x^2 + 4x + 20 \end{array}$$

Then $5x^2$ goes into $10x^2$ twice, so

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad 3x^2 \quad\quad\quad + 2 \\ \hline 5x^2+2x+8 & \;\; 15x^4 + 6x^3 + 34x^2 + 4x + 20 \\ & -15x^4 - 6x^3 - 24x^2 \\ \hline & \quad\quad\quad\quad\quad 10x^2 + 4x + 20 \\ & \quad\quad\quad\quad -10x^2 - 4x - 16 \\ \hline & \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 4 \end{array}$$

Our remainder is 4. That is,

$$\frac{15x^4 + 6x^3 + 34x^2 + 4x + 20}{5x^2 + 2x + 8} = 3x^2 + 2 + \frac{4}{5x^2 + 2x + 8}.$$

c We start by dividing. The leading term of the numerator is x^3 times the leading term of the denominator.

$$\begin{array}{r|l} & x^3 \\ \hline 2x^2+5 & 2x^5+9x^3+12x^2+10x+30 \\ & -2x^5-5x^3 \\ \cline{2-2} & \quad 4x^3+12x^2+10x \end{array}$$

Then $2x^2(2x)$ gives us $4x^3$.

$$\begin{array}{r|l} & x^3+2x \\ \hline 2x^2+5 & 2x^5+9x^3+12x^2+10x+30 \\ & -2x^5-5x^3 \\ \cline{2-2} & \quad 4x^3+12x^2+10x \\ & \quad -4x^3 \qquad -10x \\ \cline{2-2} & \qquad 12x^2 \qquad +30 \end{array}$$

Finally, $2x^2$ goes into $12x^2$ six times.

$$\begin{array}{r|l} & x^3+2x+6 \\ \hline 2x^2+5 & 2x^5+9x^3+12x^2+10x+30 \\ & -2x^5-5x^3 \\ \cline{2-2} & \quad 4x^3+12x^2+10x \\ & \quad -4x^3 \qquad -10x \\ \cline{2-2} & \qquad 12x^2 \qquad +30 \\ & \qquad -12x^2 \qquad -30 \\ \cline{2-2} & \qquad\qquad\qquad 0 \end{array}$$

Since there is no remainder,

$$\frac{2x^5+9x^3+12x^2+10x+30}{2x^2+5} = x^3+2x+6$$

Remark: if we wanted to be pedantic about the question statement, we could write our final answer as $x^3+2x+6+\frac{0}{x}$, so that we are indeed adding a polynomial to a rational function whose numerator has degree strictly smaller than its denominator.

1.10.4.5. Solution.

a The polynomial $5x^3-3x^2-10x+6$ has a repeated pattern: the ratio of the first two coefficients is the same as the ratio of the last two coefficients. We can use this to factor.

$$\begin{aligned} 5x^3-3x^2-10x+6 &= x^2(5x-3)-2(5x-3) \\ &= (x^2-2)(5x-3) \\ &= (x+\sqrt{2})(x-\sqrt{2})(5x-3) \end{aligned}$$

b The polynomial x^4-3x^2-5 has only even powers of x, so we can (temporarily) replace them with $x^2=y$ to turn our quartic polynomial into a quadratic.

$$x^4-3x^2-5=y^2-3y-5$$

There's no obvious factoring here, but we can find its roots, if any, using the quadratic equation.

$$\begin{aligned} y&=\frac{3\pm\sqrt{3^2-4(1)(-5)}}{2}\\ &=\frac{3\pm\sqrt{29}}{2}\\ \text{So, } y^2-3y-5&=\left(y-\frac{3+\sqrt{29}}{2}\right)\left(y-\frac{3-\sqrt{29}}{2}\right)\\ \text{Therefore, } x^4-3x^2-5&=\left(x^2-\frac{3+\sqrt{29}}{2}\right)\left(x^2-\frac{3-\sqrt{29}}{2}\right)\end{aligned}$$

We'd like to use the difference of two squares to factor these quadratic expressions. For this to work, the constants must be positive (so their square roots are real). Since $\sqrt{29}>3$, only the first quadratic is factorable. The other is irreducible — it's always positive, so it had no roots.

$$x^4-3x^2-5=\left(x+\sqrt{\frac{3+\sqrt{29}}{2}}\right)\left(x-\sqrt{\frac{3+\sqrt{29}}{2}}\right)\left(x^2+\frac{\sqrt{29}-3}{2}\right)$$

c Without seeing any obvious patterns, we start hunting for roots. Since we have all integer coefficients, if there are any integer roots, they will divide our constant term, -6. So, our candidates for roots are ±1, ±2, ±3, and ±6. To save time, we don't need to know exactly the value of our polynomial at these points: only whether or not it is 0. Write $f(x)=x^4-4x^3-10x^2-11x-6$.

$$\begin{array}{cccc} f(-1)=0 & f(-2)\neq0 & f(-3)\neq0 & f(-6)\neq0\\ f(1)\neq0 & f(2)\neq0 & f(3)\neq0 & f(6)=0 \end{array}$$

Since $x=-1$ and $x=6$ are roots of our polynomial, it has factors $(x+1)$ and $(x-6)$. Note $(x+1)(x-6)=x^2-5x-6$. We use long division to figure out what else is lurking in our polynomial.

$$\begin{array}{r}
x^2 \quad + x + 1 \\
x^2 - 5x - 6 \overline{) \; x^4 - 4x^3 - 10x^2 - 11x - 6} \\
\underline{- x^4 + 5x^3 \; + 6x^2 \qquad\qquad} \\
x^3 \; - 4x^2 - 11x \quad \\
\underline{- x^3 \; + 5x^2 \; + 6x \quad} \\
x^2 \; - 5x - 6 \\
\underline{- x^2 \; + 5x + 6} \\
0
\end{array}$$

So, $x^4 - 4x^3 - 10x^2 - 11x - 6 = (x+1)(x-6)(x^2+x+1)$.

We should check whether $x^2 + x + 1$ is reducible or not. If we try to find its roots with the quadratic equation, we get $\dfrac{-1 \pm \sqrt{-3}}{2}$, which are not real numbers. So, we're at the end of our factoring.

d Without seeing any obvious patterns, we start hunting for roots. Since we have all integer coefficients, if there are any integer roots, they will divide our constant term, -15. So, our candidates for roots are ± 1, ± 3, ± 5, and ± 15. Write $f(x) = 2x^4 + 12x^3 - x^2 - 52x + 15$.

$$\begin{array}{cccc}
f(-1) \neq 0 & f(-3) = 0 & f(-5) = 0 & f(-5) \neq 0 \\
f(1) \neq 0 & f(3) \neq 0 & f(5) \neq 0 & f(15) \neq 0
\end{array}$$

Since $x = -3$ and $x = -5$ are roots of our polynomial, it has factors $(x+3)$ and $(x+5)$. Note $(x+3)(x+5) = x^2 + 8x + 15$. We use long division to move forward.

$$\begin{array}{r}
2x^2 \; - 4x \; + 1 \\
x^2 + 8x + 15 \overline{) \; 2x^4 + 12x^3 \quad - x^2 - 52x + 15} \\
\underline{- 2x^4 - 16x^3 - 30x^2 \qquad\qquad} \\
- 4x^3 - 31x^2 - 52x \quad \\
\underline{4x^3 + 32x^2 + 60x \quad} \\
x^2 \; + 8x + 15 \\
\underline{- x^2 \; - 8x - 15} \\
0
\end{array}$$

So, $2x^4 + 12x^3 - x^2 - 52x + 15 = (x+3)(x+5)(2x^2 - 4x + 1)$.

We should check whether $2x^2 - 4x + 1$ is reducible or not. There's not an obvious way to factor it, but we can use the quadratic equation. This gives us roots $\dfrac{4 \pm \sqrt{16-8}}{4} = 1 \pm \dfrac{\sqrt{2}}{2}$. So, we have two more linear factors.

Specifically:

$$2x^4 + 12x^3 - x^2 - 52x + 15 = (x+3)(x+5)\left(x - \left(1 + \frac{\sqrt{2}}{2}\right)\right)$$

$$\left(x - \left(1 - \frac{\sqrt{2}}{2}\right)\right)$$

1.10.4.6. Solution. The goal of partial fraction decomposition is to write our integrand in a form that is easy to integrate. The antiderivative of (1) can be easily determined with the substitution $u = (ax + b)$. It's less clear how to find the antiderivative of (2).

Exercises — Stage 2

1.10.4.7. *. Solution. The integrand is a rational function, so it's a candidate for partial fraction. We quickly rule out any obvious substitution or integration by parts, so we go ahead with the decomposition.

We start by expressing the integrand, i.e. the fraction $\frac{1}{x+x^2} = \frac{1}{x(1+x)}$, as a linear combination of the simpler fractions $\frac{1}{x}$ and $\frac{1}{x+1}$ (which we already know how to integrate). We will have

$$\frac{1}{x + x^2} = \frac{1}{x(1 + x)} = \frac{a}{x} + \frac{b}{x + 1} = \frac{a(x + 1) + bx}{x(1 + x)}$$

The fraction on the left hand side is the same as the fraction on the right hand side if and only if the numerator on the left hand side, which is $1 = 0x + 1$, is equal to the numerator on the right hand side, which is $a(x + 1) + bx = (a + b)x + a$. This in turn is the case if and only of $a = 1$ (i.e. the constant terms are the same in the two numerators) and $a + b = 0$ (i.e. the coefficients of x are the same in the two numerators). So $a = 1$ and $b = -1$. Now we can easily evaluate the integral

$$\begin{aligned}\int_1^2 \frac{\mathrm{d}x}{x + x^2} &= \int_1^2 \frac{\mathrm{d}x}{x(x + 1)} = \int_1^2 \left(\frac{1}{x} - \frac{1}{x + 1}\right) \mathrm{d}x \\ &= \Big[\log x - \log(x + 1)\Big]_1^2 = \log 2 - \log\frac{3}{2} = \log\frac{4}{3}\end{aligned}$$

1.10.4.8. *. Solution. We'll first do a partial fraction decomposition. The sneaky way is to temporarily rename x^2 to y. Then $x^4 + x^2 = y^2 + y$ and

$$\frac{1}{x^4 + x^2} = \frac{1}{y(y + 1)} = \frac{1}{y} - \frac{1}{y + 1}$$

as we found in Question 7. Now we restore y to x^2.

$$\int \frac{1}{x^4 + x^2}\,\mathrm{d}x = \int \left(\frac{1}{x^2} - \frac{1}{x^2 + 1}\right) \mathrm{d}x = -\frac{1}{x} - \arctan x + C$$

1.10.4.9. *. Solution. The integrand is of the form $N(x)/D(x)$ with $D(x)$ already factored and $N(x)$ of lower degree. We immediately look for a partial fraction decomposition:

$$\frac{12x + 4}{(x - 3)(x^2 + 1)} = \frac{A}{x - 3} + \frac{Bx + C}{x^2 + 1}.$$

Multiplying through by the denominator yields

$$12x + 4 = A(x^2 + 1) + (Bx + C)(x - 3) \tag{$*$}$$

Setting $x = 3$ we find:

$$36 + 4 = A(9 + 1) + 0 \implies 40 = 10A \implies A = 4$$

Substituting $A = 4$ in $(*)$ gives

$$\begin{aligned} 12x + 4 &= 4(x^2 + 1) + (Bx + C)(x - 3) \\ \implies -4x^2 + 12x &= (x - 3)(Bx + C) \\ \implies (-4x)(x - 3) &= (Bx + C)(x - 3) \\ \implies B = -4, C = 0 \end{aligned}$$

So we have found that $A = 4$, $B = -4$, and $C = 0$. Therefore

$$\begin{aligned} \int \frac{12x + 4}{(x - 3)(x^2 + 1)}\,\mathrm{d}x &= \int \left(\frac{4}{x - 3} - \frac{4x}{x^2 + 1}\right)\mathrm{d}x \\ &= 4\log|x - 3| - 2\log(x^2 + 1) + C \end{aligned}$$

The second integral was found just by guessing an antiderivative. Alternatively, one could use the substitution $u = x^2 + 1$, $\mathrm{d}u = 2x\,\mathrm{d}x$.

1.10.4.10. $*$. **Solution**. The integrand is of the form $N(x)/D(x)$ with $D(x)$ already factored and $N(x)$ of lower degree. With no obvious substitution available, we look for a partial fraction decomposition.

$$\frac{3x^2 - 4}{(x - 2)(x^2 + 4)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 4}$$

Multiplying through by the denominator gives

$$3x^2 - 4 = A(x^2 + 4) + (Bx + C)(x - 2) \tag{$*$}$$

Setting $x = 2$ we find:

$$12 - 4 = A(4 + 4) + 0 \implies 8 = 8A \implies A = 1$$

Substituting $A = 1$ in $(*)$ gives

$$\begin{aligned} 3x^2 - 4 &= (x^2 + 4) + (x - 2)(Bx + C) \\ \implies 2x^2 - 8 &= (x - 2)(Bx + C) \\ \implies (x - 2)(2x + 4) &= (x - 2)(Bx + C) \\ \implies B = 2, C = 4 \end{aligned}$$

Thus, we have:

$$\frac{3x^2 - 4}{(x - 2)(x^2 + 4)} = \frac{1}{x - 2} + \frac{2x + 4}{x^2 + 4} = \frac{1}{x - 2} + \frac{2x}{x^2 + 4} + \frac{4}{x^2 + 4}$$

The first two of these are directly integrable:

$$F(x) = \log|x-2| + \log|x^2+4| + \int \frac{4}{x^2+4}\,\mathrm{d}x$$

(The second integral was found just by guessing an antiderivative. Alternatively, one could use the substitution $u = x^2+4$, $\mathrm{d}u = 2x\,\mathrm{d}x$.) For the final integral, we substitute: $x = 2y$, $\mathrm{d}x = 2\mathrm{d}y$, and see that:

$$\int \frac{4}{x^2+4}\,\mathrm{d}x = 2\int \frac{1}{y^2+1}\,\mathrm{d}y = 2\arctan y + D = 2\arctan(x/2) + D$$

for any constant D. All together we have:

$$F(x) = \log|x-2| + \log|x^2+4| + 2\arctan(x/2) + D$$

1.10.4.11. *. Solution. This integrand is a rational function, with no obvious substitution. This sure looks like a partial fraction problem. Let's go through our protocol.

- The degree of the numerator $x-13$ is one, which is strictly smaller than the dergee of the denominator x^2-x-6, which is two. So we don't need long division to pull out a polynomial.

- Next we factor the denominator.

$$x^2 - x - 6 = (x-3)(x+2)$$

- Next we find the partial fraction decomposition of the integrand. It is of the form

$$\frac{x-13}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

To find A and B, using the sneaky method, we cross multiply by the denominator.

$$x - 13 = A(x+2) + B(x-3)$$

Now we can find A by evaluating at $x = 3$

$$3 - 13 = A(3+2) + B(3-3) \implies A = -2$$

and find B by evaluating at $x = -2$.

$$-2 - 13 = A(-2+2) + B(-2-3) \implies B = 3$$

(Hmmm. A and B are nice round numbers. Sure looks like a rigged exam or homework question.) Our partial fraction decomposition is

$$\frac{x-13}{(x-3)(x+2)} = \frac{-2}{x-3} + \frac{3}{x+2}$$

As a check, we recombine the right hand side and make sure that it matches the left hand side.

$$\frac{-2}{x-3} + \frac{3}{x+2} = \frac{-2(x+2)+3(x-3)}{(x-3)(x+2)} = \frac{x-13}{(x-3)(x+2)}$$

- Finally, we evaluate the integral.

$$\begin{aligned}\int \frac{x-13}{x^2-x-6}\mathrm{d}x &= \int \left(\frac{-2}{x-3} + \frac{3}{x+2}\right)\mathrm{d}x \\ &= -2\log|x-3| + 3\log|x+2| + C\end{aligned}$$

1.10.4.12. $*$. Solution. Again, this sure looks like a partial fraction problem. So let's go through our protocol.

- The degree of the numerator $5x+1$ is one, which is strictly smaller than the dergee of the denominator x^2+5x+6, which is two. So we do not long divide to pull out a polynomial.
- Next we factor the denominator.

$$x^2+5x+6 = (x+2)(x+3)$$

- Next we find the partial fraction decomposition of the integrand. It is of the form

$$\frac{5x+1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

To find A and B, using the sneaky method, we cross multiply by the denominator.

$$5x+1 = A(x+3) + B(x+2)$$

Now we can find A by evaluating at $x=-2$

$$-10+1 = A(-2+3) + B(-2+2) \implies A = -9$$

and find B by evaluating at $x=-3$.

$$-15+1 = A(-3+3) + B(-3+2) \implies B = 14$$

So our partial fraction decomposition is

$$\frac{5x+1}{(x+2)(x+3)} = \frac{-9}{x+2} + \frac{14}{x+3}$$

As a check, we recombine the right hand side and make sure that it matches the left hand side

$$\frac{-9}{x+2} + \frac{14}{x+3} = \frac{-9(x+3)+14(x+2)}{(x+2)(x+3)} = \frac{5x+1}{(x+2)(x+3)}$$

- Finally, we evaluate the integral

$$\begin{aligned}\int \frac{5x+1}{x^2+5x+6}\mathrm{d}x &= \int \left(\frac{-9}{x+2} + \frac{14}{x+3}\right)\mathrm{d}x \\ &= -9\log|x+2| + 14\log|x+3| + C\end{aligned}$$

1.10.4.13. Solution. We have a rational function with no obvious substitution, so let's use partial fraction decomposition.

- Since the degree of the numerator is the same as the degree of the denominator, we need to pull out a polynomial.

$$\begin{array}{r|l} & 5 \\ \hline x^2-1 & 5x^2-3x-1 \\ & -5x^2 \qquad +5 \\ \hline & -3x+4 \end{array}$$

That is,

$$\begin{aligned} \int \frac{5x^2-3x-1}{x^2-1}\mathrm{d}x &= \int \left(5+\frac{-3x+4}{x^2-1}\right)\mathrm{d}x \\ &= 5x + \int \frac{-3x+4}{x^2-1}\mathrm{d}x \end{aligned}$$

- Again, there's no obvious substitution for the new integrand, so we want to use partial fraction. The denominator factors as $(x-1)(x+1)$, so our decomposition has this form:

$$\begin{aligned} \frac{-3x+4}{x^2-1} &= \frac{-3x+4}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \\ &= \frac{(A+B)x+(A-B)}{(x-1)(x+1)} \end{aligned}$$

So, (1) $A+B=-3$ and (2) $A-B=4$.

- We solve (2) for A in terms of B, namely $A=4+B$. Plugging this into (1), we see $(4+B)+B=-3$. So, $B=-\frac{7}{2}$, and $A=\frac{1}{2}$.

- Now we can write our integral in a friendlier form and evaluate.

$$\begin{aligned} \int \frac{5x^2-3x-1}{x^2-1}\mathrm{d}x = &= 5x + \int \frac{-3x+4}{x^2-1}\mathrm{d}x \\ &= 5x + \int \frac{1/2}{x-1} - \frac{7/2}{x+1}\mathrm{d}x \\ &= 5x + \frac{1}{2}\log|x-1| - \frac{7}{2}\log|x+1| + C \end{aligned}$$

1.10.4.14. Solution. The integrand is a rational function with no obvious substitution, so we use partial fraction decomposition.

- The degree of the numerator is the same as the degree of the denominator. Since it's not smaller, we need to re-write our integrand. We could do this using long division, but this case is simple enough to do more informally.

$$\begin{aligned}\frac{4x^4+14x^2+2}{4x^4+x^2} &= \frac{4x^4+x^2+13x^2+2}{4x^4+x^2} \\ &= \frac{4x^4+x^2}{4x^4+x^2}+\frac{13x^2+2}{4x^4+x^2} \\ &= 1+\frac{13x^2+2}{4x^4+x^2}\end{aligned}$$

- The denominator factors as $x^2(4x^2+1)$.
- We want to find the partial fraction decomposition of the fractional part of our simplified integrand.

$$\frac{13x^2+2}{4x^4+x^2} = \frac{13x^2+2}{x^2(4x^2+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{4x^2+1}$$

Multiply through by the original denominator.

$$13x^2+2 = Ax(4x^2+1)+B(4x^2+1)+(Cx+D)x^2 \tag{1}$$

Setting $x=0$ gives us:

$$2 = B$$

We use $B=2$ to simplify Equation (1).

$$\begin{aligned}13x^2+2 &= Ax(4x^2+1)+2(4x^2+1)+(Cx+D)x^2 \\ 5x^2 &= Ax(4x^2+1)+(Cx+D)x^2 \\ 5x &= A(4x^2+1)+(Cx+D)x \qquad (2)\end{aligned}$$

Again, let $x=0$.

$$0 = A$$

Using $A=0$, simplify Equation (2).

$$\begin{aligned}5x &= (Cx+D)x \\ 5 &= Cx+D \\ C &= 0, \qquad D=5\end{aligned}$$

- Now we can write our integral in pieces.

$$\begin{aligned}\int \frac{4x^4+14x^2+2}{4x^4+x^2}\mathrm{d}x &= \int\left(1+\frac{13x^2+2}{4x^4+x^2}\right)\mathrm{d}x \\ &= \int\left(1+\frac{2}{x^2}+\frac{5}{4x^2+1}\right)\mathrm{d}x\end{aligned}$$

$$= x - \frac{2}{x} + \int \frac{5}{(2x)^2 + 1}\mathrm{d}x$$

Substitute $u = 2x$, $\mathrm{d}u = 2\mathrm{d}x$.

$$\begin{aligned}
&= x - \frac{2}{x} + \int \frac{5/2}{u^2 + 1}\mathrm{d}u \\
&= x - \frac{2}{x} + \frac{5}{2}\arctan u + C \\
&= x - \frac{2}{x} + \frac{5}{2}\arctan(2x) + C
\end{aligned}$$

1.10.4.15. Solution. The integrand is a rational function with no obvious substitution, so we'll use a partial fraction decomposition.

- Since the numerator has strictly smaller degree than the denominator, we don't need to start off with a long division.
- We do, however, need to factor the denominator. We can immediately pull out x^2; the remaining part is $x^2 - 2x + 1 = (x-1)^2$.
- Now we can perform our partial fraction decomposition.

$$\frac{x^2 + 2x - 1}{x^4 - 2x^3 + x^2} = \frac{x^2 + 2x - 1}{x^2(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

Multiply both sides by the original denominator.

$$x^2 + 2x - 1 = Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2 \tag{1}$$

To be sneaky, we set $x = 0$, and find:

$$-1 = B$$

We also set $x = 1$, and find:

$$2 = D$$

We use B and D to simplify Equation (1).

$$\begin{aligned}
x^2 + 2x - 1 &= Ax(x-1)^2 - 1(x-1)^2 + Cx^2(x-1) + 2x^2 \\
0 &= Ax(x-1)^2 + Cx^2(x-1) \\
&= x(x-1)[(A+C)x - A] \\
\text{So,}\qquad 0 &= (A+C)x - A
\end{aligned}$$

That is, $A = C = 0$.

- Now we can evaluate our integral.

$$\begin{aligned}
\int \frac{x^2 + 2x - 1}{x^4 - 2x^3 + x^2}\mathrm{d}x &= \int \left(\frac{-1}{x^2} + \frac{2}{(x-1)^2}\right)\mathrm{d}x \\
&= \frac{1}{x} - \frac{2}{x-1} + C
\end{aligned}$$

1.10.4.16. Solution. Our integrand is a rational function with no obvious substitution, so we'll use the method of partial fractions.

- The degree of the numerator is less than the degree of the denominator.
- We need to factor the denominator. The first two terms have the same ratio as the last two terms.

$$\begin{aligned} 2x^3 - x^2 - 8x + 4 &= x^2(2x-1) - 4(2x-1) \\ &= (x^2-4)(2x-1) \\ &= (x-2)(x+2)(2x-1) \end{aligned}$$

- Now we find our partial fraction decomposition.

$$\begin{aligned} \frac{3x^2-4x-10}{2x^3-x^2-8x+4} &= \frac{3x^2-4x-10}{(x-2)(x+2)(2x-1)} \\ &= \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{2x-1} \end{aligned}$$

Multiply both sides by the original denominator.

$$\begin{aligned} 3x^2 - 4x - 10 = A(x+2)(2x-1) + B(x-2)(2x-1) \\ + C(x-2)(x+2) \end{aligned}$$

Distinct linear factors is the best possible scenario for the sneaky method. First, let's set $x = 2$.

$$\begin{aligned} 3(4) - 4(2) - 10 &= A(4)(3) + B(0) + C(0) \\ A &= -\frac{1}{2} \end{aligned}$$

Now, let $x = -2$.

$$\begin{aligned} 3(4) - 4(-2) - 10 &= A(0) + B(-4)(-5) + C(0) \\ B &= \frac{1}{2} \end{aligned}$$

Finally, let $x = \frac{1}{2}$.

$$\begin{aligned} \frac{3}{4} - 2 - 10 &= A(0) + B(0) + C\left(-\frac{3}{2}\right)\left(\frac{5}{2}\right) \\ C &= 3 \end{aligned}$$

- Now we can evaluate our integral in its new form.

$$\begin{aligned} \int \frac{3x^2-4x-10}{2x^3-x^2-8x+4}\mathrm{d}x &= \int \left(\frac{-1/2}{x-2} + \frac{1/2}{x+2} + \frac{3}{2x-1}\right)\mathrm{d}x \\ &= -\frac{1}{2}\log|x-2| + \frac{1}{2}\log|x+2| + \frac{3}{2}\log|2x-1| + C \\ &= \frac{1}{2}\log\left|\frac{x+2}{x-2}\right| + \frac{3}{2}\log|2x-1| + C \end{aligned}$$

1.10.4.17. Solution. The integrand is a rational function with no obvious substitution, so we use the method of partial fractions.

- The numerator has smaller degree than the denominator.
- We need to factor the denominator. In the absence of any clues, we look for an integer root. The constant term is 5, so the possible integer roots are ± 1 and ± 5. Name $f(x) = 2x^3 + 11x^2 + 6x + 5$.

  $$f(-1) \neq 0 \qquad f(-5) = 0 \qquad f(1) \neq 0 \qquad f(5) \neq 0$$

  So, $(x+5)$ is a factor of the denominator.
- We use long division to pull out the factor of $(x+5)$.

  $$\begin{array}{r|l} & 2x^2 + x + 1 \\ \hline x+5 & 2x^3 + 11x^2 + 6x + 5 \\ & -2x^3 - 10x^2 \\ \hline & x^2 + 6x \\ & -x^2 - 5x \\ \hline & x + 5 \\ & -x - 5 \\ \hline & 0 \end{array}$$

  That is, our denominator is $(x+5)(2x^2 + x + 1)$.
- The quadratic function $2x^2 + x + 1$ is irreducible: we can see this by using the quadratic equation, and finding no real roots. So, we are ready to find our partial fraction decomposition.

  $$\frac{10x^2 + 24x + 8}{2x^3 + 11x^2 + 6x + 5} = \frac{10x^2 + 24x + 8}{(x+5)(2x^2 + x + 1)} = \frac{A}{x+5} + \frac{Bx + C}{2x^2 + x + 1}$$

  Multiply through by the original denominator.

  $$10x^2 + 24x + 8 = A(2x^2 + x + 1) + (Bx + C)(x+5) \qquad (1)$$

  Set $x = -5$.

  $$10(25) - 24(5) + 8 = A(2(25) - 5 + 1) + (B(-5) + C)(0)$$
  $$A = 3$$

  Using our value of A, we simplify Equation (1).

  $$10x^2 + 24x + 8 = 3(2x^2 + x + 1) + (Bx + C)(x+5)$$
  $$4x^2 + 21x + 5 = (Bx + C)(x+5)$$

  We factor the left side. We know $(x+5)$ must be one of its factors.

  $$(4x + 1)(x+5) = (Bx + C)(x+5)$$
  $$4x + 1 = Bx + C$$

  So, $B = 4$ and $C = 1$.

- Now we can write our integral in smaller pieces.

$$\int_0^1 \frac{10x^2+24x+8}{2x^3+11x^2+6x+5}\mathrm{d}x = \int_0^1 \left(\frac{3}{x+5}+\frac{4x+1}{2x^2+x+1}\right)\mathrm{d}x$$

The antiderivative of the left fraction is $3\log|x+5|$. For the right fraction, we use the substitution $u = 2x^2+x+1$, $\mathrm{d}u = (4x+1)\mathrm{d}x$ to antidifferentiate.

$$\begin{aligned}
&= \left[3\log|x+5| + \log|2x^2+x+1|\right]_0^1 \\
&= 3\log 6 + \log 4 - 3\log 5 - \log 1 \\
&= \log\left(\frac{4\cdot 6^3}{5^3}\right)
\end{aligned}$$

Exercises — Stage 3

1.10.4.18. Solution. We follow the example in the text.

$$\int \csc x\mathrm{d}x = \int \frac{1}{\sin x}\mathrm{d}x = \int \frac{\sin x}{\sin^2 x}\mathrm{d}x = \int \frac{\sin x}{1-\cos^2 x}\mathrm{d}x$$

Let $u = \cos x$, $\mathrm{d}u = -\sin x\mathrm{d}x$.

$$= \int \frac{-1}{1-u^2}\mathrm{d}u = \int \frac{-1}{(1+u)(1-u)}\mathrm{d}u$$

We see an opportunity for partial fraction.

$$\frac{-1}{(1+u)(1-u)} = \frac{A}{1+u}+\frac{B}{1-u}$$

Multiply both sides by the original denominator.

$$-1 = A(1-u) + B(1+u)$$

Let $u = 1$.

$$-1 = 2B \qquad \Rightarrow B = -\frac{1}{2}$$

Let $u = -1$.

$$-1 = 2A \qquad \Rightarrow A = -\frac{1}{2}$$

We can now re-write our integral.

$$\begin{aligned}
\int \csc x\mathrm{d}x = \int \frac{-1}{(1+u)(1-u)}\mathrm{d}u &= \int \left(\frac{-1/2}{1+u}+\frac{-1/2}{1-u}\right)\mathrm{d}u \\
&= -\frac{1}{2}\log|1+u| + \frac{1}{2}\log|1-u| + C \\
&= \frac{1}{2}\log\left|\frac{1-u}{1+u}\right| + C
\end{aligned}$$

$$= \frac{1}{2}\log\left|\frac{1-\cos x}{1+\cos x}\right| + C$$

Remark: Elsewhere in the text, and in many tables of integrals, the antiderivative of cosecant is given as $\log|\csc x - \cot x|$. We show that this is equivalent to our result.

$$\begin{aligned}
\log|\csc x - \cot x| &= \frac{1}{2}\log\left|(\csc x - \cot x)^2\right| \\
&= \frac{1}{2}\log\left|\csc^2 x - 2\csc x\cot x + \cot^2 x\right| \\
&= \frac{1}{2}\log\left|\frac{1}{\sin^2 x} - \frac{2\cos x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}\right| \\
&= \frac{1}{2}\log\left|\frac{1-2\cos x+\cos^2 x}{\sin^2 x}\right| = \frac{1}{2}\log\left|\frac{(1-\cos x)^2}{1-\cos^2 x}\right| \\
&= \frac{1}{2}\log\left|\frac{(1-\cos x)^2}{(1-\cos x)(1+\cos x)}\right| = \frac{1}{2}\log\left|\frac{1-\cos x}{1+\cos x}\right|
\end{aligned}$$

1.10.4.19. Solution. We follow the example in the text.

$$\int \csc^3 x\,\mathrm{d}x = \int \frac{1}{\sin^3 x}\mathrm{d}x = \int \frac{\sin x}{\sin^4 x}\mathrm{d}x = \int \frac{\sin x}{(1-\cos^2 x)^2}\mathrm{d}x$$

Let $u = \cos x$, $\mathrm{d}u = -\sin x\mathrm{d}x$.

$$= \int \frac{-1}{(1-u^2)^2}\mathrm{d}u$$

In Question 18 we saw $\frac{1}{1-u^2} = \frac{1/2}{1+u} + \frac{1/2}{1-u}$, so

$$\begin{aligned}
&\int \frac{-1}{(1-u^2)^2}\mathrm{d}u = -\int\left(\frac{1}{1-u^2}\right)^2\mathrm{d}u = -\int\left(\frac{1/2}{1+u} + \frac{1/2}{1-u}\right)^2\mathrm{d}u \\
&= -\frac{1}{4}\int\left(\frac{1}{(1+u)^2} + \frac{2}{1-u^2} + \frac{1}{(1-u)^2}\right)\mathrm{d}u \\
&= -\frac{1}{4}\int\left(\frac{1}{(1+u)^2} + \frac{1}{1+u} + \frac{1}{1-u} + \frac{1}{(1-u)^2}\right)\mathrm{d}u \\
&= -\frac{1}{4}\left(-\frac{1}{1+u} + \log|1+u| - \log|1-u| + \frac{1}{1-u}\right) + C \\
&= -\frac{1}{4}\left(\frac{2u}{1-u^2} + \log\left|\frac{1+u}{1-u}\right|\right) + C \\
&= \frac{-u}{2(1-u^2)} + \frac{1}{4}\log\left|\frac{1-u}{1+u}\right| + C \\
&= \frac{-\cos x}{2\sin^2 x} + \frac{1}{4}\log\left|\frac{1-\cos x}{1+\cos x}\right| + C
\end{aligned}$$

Remark: In Example 1.8.23, and in many tables of integrals, the antiderivative of $\csc^3 x$ is given as $-\frac{1}{2}\cot x\csc x + \frac{1}{2}\log|\csc x - \cot x| + C$. This is equivalent to our result. Recall in the remark after the solution to Question 18, we saw $\frac{1}{2}\log\left|\frac{1-\cos x}{1+\cos x}\right| = \log|\csc x - \cot x|$.

$$\begin{aligned}
&-\frac{1}{2}\cot x\csc x + \frac{1}{2}\log|\csc x - \cot x| \\
&\qquad = -\frac{1}{2}\cot x\csc x + \frac{1}{4}\log\left|\frac{1-\cos x}{1+\cos x}\right| \\
&\qquad = -\frac{1}{2}\left(\frac{\cos x}{\sin x}\right)\left(\frac{1}{\sin x}\right) + \frac{1}{4}\log\left|\frac{1-\cos x}{1+\cos x}\right| \\
&\qquad = \frac{-\cos x}{2\sin^2 x} + \frac{1}{4}\log\left|\frac{1-\cos x}{1+\cos x}\right|
\end{aligned}$$

1.10.4.20. Solution. This is a rational function, and there's no obvious substitution, so we'll use partial fraction decomposition.

- First, we check that the numerator has strictly smaller degree than the denominator, so we don't have to use long division.
- Second, we factor the denominator. We can immediately pull out a factor of x^2; then we're left with the quadratic polynomial $x^2 + 5x + 10$. Using the quadratic equation, we check that this has no real roots, so it is irreducible.
- Once we know the factorization of the denominator, we can set up our decomposition.

$$\begin{aligned}
\frac{3x^3+15x^2+35x+10}{x^4+5x^3+10x^2} &= \frac{3x^3+15x^2+35x+10}{x^2(x^2+5x+10)} \\
&= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+5x+10}
\end{aligned}$$

We multiply both sides by the original denominator.

$$\begin{aligned}
&3x^3+15x^2+35x+10 \\
&\quad = Ax(x^2+5x+10) + B(x^2+5x+10) + (Cx+D)x^2 \qquad (1)
\end{aligned}$$

Following the "Sneaky Method," we plug in $x = 0$.

$$\begin{aligned}
0+10 &= A(0) + B(10) + (C(0)+D)(0) \\
B &= 1
\end{aligned}$$

- Knowing B allows us to simplify our Equation (1).

$$\begin{aligned}
3x^3+15x^2+35x+10 &= Ax(x^2+5x+10) + 1(x^2+5x+10) \\
&\qquad + (Cx+D)x^2 \\
3x^3+14x^2+30x &= Ax(x^2+5x+10) + (Cx+D)x^2
\end{aligned}$$

We can factor x out of both sides of the equation.

$$3x^2+14x+30 = A(x^2+5x+10) + (Cx+D)x \qquad (2)$$

- Again, we set $x = 0$.

$$\begin{aligned} 0 + 30 &= A(10) + (C(0 + D)(0) \\ A &= 3 \end{aligned}$$

- We simplify Equation (2), using $A = 3$.

$$\begin{aligned} 3x^2 + 14x + 30 &= 3(x^2 + 5x + 10) + (Cx + D)x \\ -x &= Cx^2 + Dx \\ C &= 0, \quad D = -1 \end{aligned}$$

- Now that we have our coefficients, we can re-write our integral in a friendlier form.

$$\begin{aligned} &\int_1^2 \frac{3x^3 + 15x^2 + 35x + 10}{x^4 + 5x + 10x^2} \mathrm{d}x \\ &\quad = \int_1^2 \left(\frac{3}{x} + \frac{1}{x^2} - \frac{1}{x^2 + 5x + 10} \right) \mathrm{d}x \\ &\quad = \left[3\log|x| - \frac{1}{x} \right]_1^2 - \int_1^2 \frac{1}{x^2 + 5x + 10} \mathrm{d}x \\ &\quad = 3\log 2 + \frac{1}{2} - \int_1^2 \frac{1}{x^2 + 5x + 10} \mathrm{d}x \end{aligned}$$

The remaining integral is the reciprocal of a quadratic polynomial, much like $\frac{1}{1+x^2}$, whose antiderivative is arctangent. We complete the square and use the substitution $u = \left(\frac{2x+5}{\sqrt{15}}\right)$, $\mathrm{d}u = \frac{2}{\sqrt{15}}\mathrm{d}x$.

$$\begin{aligned} \int_1^2 \frac{1}{x^2 + 5x + 10} \mathrm{d}x &= \int_1^2 \frac{1}{\left(x + \frac{5}{2}\right)^2 + \frac{15}{4}} \mathrm{d}x \\ &= \frac{4}{15} \int_1^2 \frac{1}{\left(\frac{2x+5}{\sqrt{15}}\right)^2 + 1} \mathrm{d}x \\ &= \frac{2}{\sqrt{15}} \int_{7/\sqrt{15}}^{9/\sqrt{15}} \frac{1}{u^2 + 1} \mathrm{d}u \\ &= \frac{2}{\sqrt{15}} \Big[\arctan u \Big]_{7/\sqrt{15}}^{9/\sqrt{15}} \\ &= \frac{2}{\sqrt{15}} \left(\arctan\left(\frac{9}{\sqrt{15}}\right) - \arctan\left(\frac{7}{\sqrt{15}}\right) \right) \end{aligned}$$

So, all together,

$$\int_1^2 \frac{3x^3 + 15x^2 + 35x + 10}{x^4 + 5x^3 + 10x^2} \mathrm{d}x$$

$$= 3\log 2 + \frac{1}{2} - \frac{2}{\sqrt{15}}\left(\arctan\left(\frac{9}{\sqrt{15}}\right) - \arctan\left(\frac{7}{\sqrt{15}}\right)\right)$$

1.10.4.21. Solution. Our integrand is already in the nice form that would come out of a partial fractions decomposition. Let's consider its different pieces.

- First piece: $\int \frac{3}{x^2+2}\mathrm{d}x$. The fraction looks somewhat like the derivative of arctangent, so we can massage it to find an appropriate substitution.

$$\int \frac{3}{x^2+2}\mathrm{d}x = \frac{3}{2}\int \frac{1}{\left(\frac{x}{\sqrt{2}}\right)^2+1}\mathrm{d}x$$

  Use the substitution $u = \frac{x}{\sqrt{2}}$, $\mathrm{d}u = \frac{1}{\sqrt{2}}\mathrm{d}x$.

$$\begin{aligned} &= \frac{3}{\sqrt{2}}\int \frac{1}{u^2+1}\mathrm{d}u \\ &= \frac{3}{\sqrt{2}}\arctan u + C \\ &= \frac{3}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C \end{aligned}$$

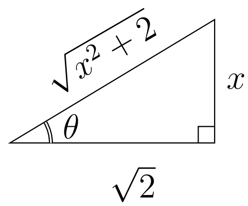
- The next piece is $\int \frac{x-3}{(x^2+2)^2}\mathrm{d}x$. If the numerator were only x (and no constant), we could use the substitution $u = x^2+2$, $\mathrm{d}u = 2x\mathrm{d}x$. So, to that end, we can break up that fraction into $\frac{x}{(x^2+2)^2} - \frac{3}{(x^2+2)^2}$. For now, we only evaluate the first half.

$$\begin{aligned} \int \frac{x}{(x^2+2)^2}\mathrm{d}x &= \frac{1}{2}\int \frac{1}{u^2}\mathrm{d}u = -\frac{1}{2u} + C \\ &= -\frac{1}{2x^2+4} + C \end{aligned}$$

- That leaves us with the final piece, $\frac{3}{(x^2+2)^2}$, which is the hardest. We saw something similar in Question 1.9.2.20 in Section 1.9: we can use the substitution $x = \sqrt{2}\tan\theta$, $\mathrm{d}x = \sqrt{2}\sec^2\theta\mathrm{d}\theta$.

$$\begin{aligned} \int \frac{3}{(x^2+2)^2}\mathrm{d}x &= \int \frac{3}{(2\tan^2\theta+2)^2}\sqrt{2}\sec^2\theta\mathrm{d}\theta \\ &= \int \frac{3}{4\sec^4\theta}\sqrt{2}\sec^2\theta\mathrm{d}\theta \\ &= \frac{3}{2\sqrt{2}}\int \cos^2\theta\mathrm{d}\theta \end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{4\sqrt{2}} \int \big(1+\cos(2\theta)\big) \mathrm{d}\theta \\
&= \frac{3}{4\sqrt{2}} \left(\theta + \frac{1}{2}\sin(2\theta)\right) + C \\
&= \frac{3}{4\sqrt{2}} (\theta + \sin\theta\cos\theta) + C \\
&= \frac{3}{4\sqrt{2}} \left(\arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{x\sqrt{2}}{x^2+2}\right) + C
\end{aligned}
$$



From our substitution, $\tan\theta = \frac{x}{\sqrt{2}}$. So, we can draw a right triangle with angle θ, opposite side x, and adjacent side $\sqrt{2}$. Then by the Pythagorean Theorem, the hypotenuse has length $\sqrt{x^2+2}$, and this gives us $\sin\theta$ and $\cos\theta$.

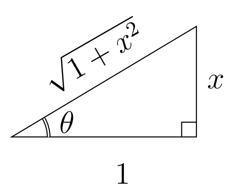
Now we have our integral.

$$
\begin{aligned}
&\int \left(\frac{3}{x^2+2} + \frac{x-3}{(x^2+2)^2}\right) \mathrm{d}x \\
&= \int \frac{3}{x^2+2} \mathrm{d}x + \int \frac{x}{(x^2+2)^2} \mathrm{d}x - \int \frac{3}{(x^2+2)^2} \mathrm{d}x \\
&= \frac{3}{\sqrt{2}} \arctan\left(\frac{x}{\sqrt{2}}\right) - \frac{1}{2x^2+4} \\
&\qquad - \frac{3}{4\sqrt{2}} \left(\arctan\left(\frac{x}{\sqrt{2}}\right) + \frac{x\sqrt{2}}{x^2+2}\right) + C \\
&= \frac{9}{4\sqrt{2}} \arctan\left(\frac{x}{\sqrt{2}}\right) - \frac{1}{2(x^2+2)} - \frac{3x}{4(x^2+2)} + C \\
&= \frac{9}{4\sqrt{2}} \arctan\left(\frac{x}{\sqrt{2}}\right) - \frac{2+3x}{4(x^2+2)} + C
\end{aligned}
$$

1.10.4.22. Solution. This is already as simplified as we can make it using partial fraction. Indeed, this is the kind of term that could likely come out of the partial fraction decomposition of a scarier rational function. So, we need to know how to integrate it. Similar to the last piece we integrated in Question 21, we can use the substitution $x = \tan\theta$, $\mathrm{d}x = \sec^2\theta \mathrm{d}\theta$.

$$
\int \frac{1}{(1+x^2)^3} \mathrm{d}x = \int \frac{\sec^2\theta}{(1+\tan^2\theta)^3} \mathrm{d}\theta = \int \frac{\sec^2\theta}{(\sec^2\theta)^3} \mathrm{d}\theta
$$

$$
\begin{aligned}
&= \int \cos^4 \theta \mathrm{d}\theta = \int \left[\frac{1+\cos(2\theta)}{2}\right]^2 \mathrm{d}\theta \\
&= \frac{1}{4}\int (1+\cos(2\theta))^2 \mathrm{d}\theta \\
&= \frac{1}{4}\int \big(1+2\cos(2\theta)+\cos^2(2\theta)\big)\, \mathrm{d}\theta \\
&= \frac{1}{4}\int \left(1+2\cos(2\theta)+\frac{1}{2}(1+\cos(4\theta))\right) \mathrm{d}\theta \\
&= \frac{1}{4}\int \left(\frac{3}{2}+2\cos(2\theta)+\frac{1}{2}\cos(4\theta))\right) \mathrm{d}\theta \\
&= \frac{1}{4}\left(\frac{3}{2}\theta+\sin(2\theta)+\frac{1}{8}\sin(4\theta)\right)+C \\
&= \frac{3}{8}\theta+\frac{1}{4}\sin(2\theta)+\frac{1}{32}\sin(4\theta)+C \\
&= \frac{3}{8}\theta+\frac{1}{2}\sin\theta\cos\theta+\frac{1}{16}\sin(2\theta)\cos(2\theta)+C \\
&= \frac{3}{8}\theta+\frac{1}{2}\sin\theta\cos\theta+\frac{1}{8}\sin\theta\cos\theta(\cos^2\theta-\sin^2\theta)+C \\
&= \frac{3}{8}\arctan x+\frac{x}{2(1+x^2)}+\frac{1}{8}\left(\frac{x}{1+x^2}\right)\left(\frac{1-x^2}{1+x^2}\right)+C \\
&= \frac{3}{8}\arctan x+\frac{3x^3+5x}{8(1+x^2)^2}+C
\end{aligned}
$$



To change our variables from θ to x, recall we used the substitution $x = \tan\theta$. So, we draw a right triangle with angle θ, opposite side length x, and adjacent side length 1. By the Pythagorean Theorem, the hypotenuse has length $\sqrt{1+x^2}$. This allows us to find $\sin\theta$ and $\cos\theta$.

1.10.4.23. Solution. Our integrand is already as simplified as the method of partial fractions can make it. The first term is easy to antidifferentiate. The second term would be easier if it were broken into two pieces: one where the numerator is a constant, and one where the numerator is a multiple of x.

$$
\begin{aligned}
&\int \left(3x+\frac{3x+1}{x^2+5}+\frac{3x}{(x^2+5)^2}\right) \mathrm{d}x \\
&\quad = \frac{3}{2}x^2+\int \left(\frac{1}{x^2+5}+\frac{3x}{x^2+5}+\frac{3x}{(x^2+5)^2}\right) \mathrm{d}x
\end{aligned}
$$

$$= \frac{3}{2}x^2 + \int \frac{1}{x^2+5}\mathrm{d}x + \int \left(\frac{3x}{x^2+5} + \frac{3x}{(x^2+5)^2}\right)\mathrm{d}x$$

The first integral looks similar to the derivative of arctangent. For the second integral, we use the substitution $u = x^2+5$, $\mathrm{d}u = 2x\mathrm{d}x$.

$$= \frac{3}{2}x^2 + \frac{1}{5}\int \frac{1}{\left(\frac{x}{\sqrt{5}}\right)^2+1}\mathrm{d}x + \int \left(\frac{3/2}{u} + \frac{3/2}{u^2}\right)\mathrm{d}u$$

For the first integral, use the substitution $w = \frac{x}{\sqrt{5}}$, $\mathrm{d}w = \frac{1}{\sqrt{5}}\mathrm{d}x$.

$$\begin{aligned}
&= \frac{3}{2}x^2 + \frac{1}{\sqrt{5}}\int \frac{1}{w^2+1}\mathrm{d}w + \frac{3}{2}\log|u| - \frac{3}{2u} \\
&= \frac{3}{2}x^2 + \frac{1}{\sqrt{5}}\arctan w + \frac{3}{2}\log|x^2+5| - \frac{3}{2x^2+10} + C \\
&= \frac{3}{2}x^2 + \frac{1}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right) + \frac{3}{2}\log|x^2+5| - \frac{3}{2x^2+10} + C
\end{aligned}$$

1.10.4.24. Solution. If our denominator were all sines, we could use the substitution $x = \sin\theta$. To that end, we apply the identity $\cos^2\theta = 1 - \sin^2\theta$.

$$\begin{aligned}
\int \frac{\cos\theta}{3\sin\theta + \cos^2\theta - 3}\mathrm{d}\theta &= \int \frac{\cos\theta}{3\sin\theta + 1 - \sin^2\theta - 3}\mathrm{d}\theta \\
&= \int \frac{\cos\theta}{3\sin\theta - \sin^2\theta - 2}\mathrm{d}\theta
\end{aligned}$$

We use the substitution $x = \sin\theta$, $\mathrm{d}x = \cos\theta\mathrm{d}\theta$.

$$= \int \frac{1}{3x - x^2 - 2}\mathrm{d}x = \int \frac{-1}{x^2 - 3x + 2}\mathrm{d}x = \int \frac{-1}{(x-1)(x-2)}\mathrm{d}x$$

Now we can find a partial fraction decomposition.

$$\begin{aligned}
\frac{-1}{(x-1)(x-2)} &= \frac{A}{x-1} + \frac{B}{x-2} \\
-1 &= A(x-2) + B(x-1)
\end{aligned}$$

Setting $x = 1$ and $x = 2$, we see

$$A = 1, \quad B = -1$$

Now, we can evaluate our integral.

$$\begin{aligned}
\int \frac{\cos\theta}{3\sin\theta + \cos^2\theta - 3}\mathrm{d}\theta &= \int \frac{-1}{(x-1)(x-2)}\mathrm{d}x \\
&= \int \left(\frac{1}{x-1} - \frac{1}{x-2}\right)\mathrm{d}x \\
&= \log|x-1| - \log|x-2| + C = \log\left|\frac{x-1}{x-2}\right| + C \\
&= \log\left|\frac{\sin\theta - 1}{\sin\theta - 2}\right| + C
\end{aligned}$$

1.10.4.25. Solution. This looks a lot like a rational function, but with the function e^t in place of the variable. So, we would like to make the substitution $x = e^t$, $\mathrm{d}x = e^t \mathrm{d}t$. Then $\mathrm{d}t = \frac{1}{e^t}\,\mathrm{d}x = \frac{1}{x}\,\mathrm{d}x$.

$$\int \frac{1}{e^{2t}+e^t+1}\mathrm{d}t = \int \frac{1}{x\left(x^2+x+1\right)}\mathrm{d}x$$

The factor x^2+x+1 is an irreducible quadratic, so the denominator is completely factored. Now we can use partial fraction decomposition.

$$\begin{aligned}\frac{1}{x\left(x^2+x+1\right)} &= \frac{A}{x} + \frac{Bx+C}{x^2+x+1}\\ 1 &= A(x^2+x+1) + (Bx+C)x\\ 1 &= (A+B)x^2 + (A+C)x + A\end{aligned}$$

The constant terms tell us $A = 1$; then the coefficient of x tells us $C = -A = -1$. Finally, the coefficient of x^2 tells us $B = -A = -1$. Now we can evaluate our integral.

$$\begin{aligned}\int \frac{1}{e^{2t}+e^t+1}\mathrm{d}t &= \int \frac{1}{x\left(x^2+x+1\right)}\mathrm{d}x\\ &= \int \left(\frac{1}{x} - \frac{x+1}{x^2+x+1}\right)\mathrm{d}x\\ &= \int \left(\frac{1}{x} - \frac{x+1/2+1/2}{x^2+x+1}\right)\mathrm{d}x \qquad (*)\\ &= \int \frac{1}{x}\mathrm{d}x - \int \frac{x+1/2}{x^2+x+1}\mathrm{d}x - \int \frac{1/2}{x^2+x+1}\mathrm{d}x\\ &= \log|x| - \frac{1}{2}\log|x^2+x+1| - \int \frac{1/2}{x^2+x+1}\mathrm{d}x\end{aligned}$$

In step $(*)$, we set ourselves up so that we could evaluate the second integral with the substitution $u = x^2+x+1$. For the remaining integral, we complete the square, so that the integrand looks something like the derivative of arctangent.

$$\begin{aligned}&= \log|x| - \frac{1}{2}\log|x^2+x+1| - \int \frac{1/2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}x\\ &= \log|x| - \frac{1}{2}\log|x^2+x+1| - \frac{2}{3}\int \frac{1}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}\mathrm{d}x\end{aligned}$$

We use the substitution $u = \frac{2x+1}{\sqrt{3}}$, $\mathrm{d}u = \frac{2}{\sqrt{3}}$.

$$\begin{aligned}&= \log|x| - \frac{1}{2}\log|x^2+x+1| - \frac{1}{\sqrt{3}}\int \frac{1}{u^2+1}\mathrm{d}u\\ &= \log|x| - \frac{1}{2}\log|x^2+x+1| - \frac{1}{\sqrt{3}}\arctan u + C\end{aligned}$$

$$\begin{aligned}
&= \log|x| - \frac{1}{2}\log|x^2+x+1| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C \\
&= \log|e^t| - \frac{1}{2}\log|e^{2t}+e^t+1| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2e^t+1}{\sqrt{3}}\right) + C \\
&= t - \frac{1}{2}\log|e^{2t}+e^t+1| - \frac{1}{\sqrt{3}}\arctan\left(\frac{2e^t+1}{\sqrt{3}}\right) + C
\end{aligned}$$

1.10.4.26. Solution.

- Solution 1: We use the substitution $u = \sqrt{1+e^x}$.

  Then $\mathrm{d}u = \dfrac{e^x}{2\sqrt{1+e^x}}\mathrm{d}x$, so $\mathrm{d}x = \dfrac{2u}{u^2-1}\mathrm{d}u$.

$$\begin{aligned}
\int \sqrt{1+e^x}\mathrm{d}x &= \int u\cdot\frac{2u}{u^2-1}\mathrm{d}u = \int \frac{2u^2}{u^2-1}\mathrm{d}u \\
&= \int \frac{2(u^2-1)+2}{u^2-1}\mathrm{d}u = \int \left(2+\frac{2}{u^2-1}\right)\mathrm{d}u
\end{aligned}$$

  We use a partial fraction decomposition on the fractional part of the integrand.

$$\begin{aligned}
\frac{2}{u^2-1} = \frac{2}{(u-1)(u+1)} &= \frac{A}{u-1} + \frac{B}{u+1} \\
&= \frac{(A+B)u+(A-B)}{(u-1)(u+1)}
\end{aligned}$$

  For the right hand side to match the left hand side, we need

$$A+B=0, \quad A-B=2 \implies A=1, \quad B=-1$$

  So the integral

$$\begin{aligned}
&\int \sqrt{1+e^x}\mathrm{d}x = \int\left(2+\frac{2}{u^2-1}\right)\mathrm{d}u \\
&= \int\left(2+\frac{1}{u-1}-\frac{1}{u+1}\right)\mathrm{d}u \\
&= 2u + \log|u-1| - \log|u+1| + C = 2u + \log\left|\frac{u-1}{u+1}\right| + C \\
&= 2\sqrt{1+e^x} + \log\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C
\end{aligned}$$

- Solution 2: It might not occur to us right away to use the fruitful substitution in Solution 1. More realistically, we might start with the "inside function," $u = 1+e^x$. Then $\mathrm{d}u = e^x\,\mathrm{d}x$, so $\mathrm{d}x = \frac{1}{u-1}\mathrm{d}u$.

$$\int \sqrt{1+e^x}\mathrm{d}x = \int \frac{\sqrt{u}}{u-1}\mathrm{d}u$$

This isn't quite a rational function, because we have a square root on top. If we could turn it into a rational function, we could use partial fraction. To that end, let $w = \sqrt{u}$, $\mathrm{d}w = \frac{1}{2\sqrt{u}}\mathrm{d}u$, so $\mathrm{d}u = 2w\mathrm{d}w$.

$$\begin{aligned} &= \int \frac{w}{w^2-1} 2w\mathrm{d}w = \int \frac{2w^2}{w^2-1}\mathrm{d}w \\ &= \int \frac{2(w^2-1)+2}{w^2-1}\mathrm{d}w = \int 2 + \frac{2}{w^2-1}\mathrm{d}w \end{aligned}$$

Now we can use partial fraction decomposition.

$$\begin{aligned} \frac{2}{w^2-1} &= \frac{2}{(w-1)(w+1)} = \frac{A}{w-1} + \frac{B}{w+1} \\ &= \frac{(A+B)w + (A-B)}{(w-1)(w+1)} \end{aligned}$$

For the left and right hand sides to match, we need

$$A + B = 0, \quad A - B = 2 \implies A = 1, \quad B = -1$$

This allows us to antidifferentiate.

$$\begin{aligned} \int \sqrt{1+e^x}\mathrm{d}x &= \int \left(2 + \frac{2}{w^2-1}\right)\mathrm{d}w \\ &= \int \left(2 + \frac{1}{w-1} - \frac{1}{w+1}\right)\mathrm{d}w \\ &= 2w + \log|w-1| - \log|w+1| + C \\ &= 2w + \log\left|\frac{w-1}{w+1}\right| + C \\ &= 2\sqrt{u} + \log\left|\frac{\sqrt{u}-1}{\sqrt{u}+1}\right| + C \\ &= 2\sqrt{1+e^x} + \log\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C \end{aligned}$$

Remark: we also evaluated this integral using trigonometric substitution in Section 1.9, Question 1.9.2.26. In that question, we found the antiderivative to be $2\sqrt{1+e^x} + 2\log\left|1-\sqrt{1+e^x}\right| - x + C$. These expressions are equivalent:

$$\begin{aligned} &\log\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| = \log\left|\sqrt{1+e^x}-1\right| + \log\left|\frac{1}{\sqrt{1+e^x}+1}\right| \\ &= \log\left|\sqrt{1+e^x}-1\right| + \log\left|\left(\frac{1}{\sqrt{1+e^x}+1}\right)\left(\frac{1-\sqrt{1+e^x}}{1-\sqrt{1+e^x}}\right)\right| \\ &= \log\left|\sqrt{1+e^x}-1\right| + \log\left|\frac{1-\sqrt{1+e^x}}{1-(1+e^x)}\right| \\ &= \log\left|\sqrt{1+e^x}-1\right| + \log\left|\frac{1-\sqrt{1+e^x}}{-e^x}\right| \end{aligned}$$

$$= \log\left|\sqrt{1+e^x}-1\right| + \log\left|1-\sqrt{1+e^x}\right| - \log|-e^x|$$
$$= 2\log\left|\sqrt{1+e^x}-1\right| - x$$

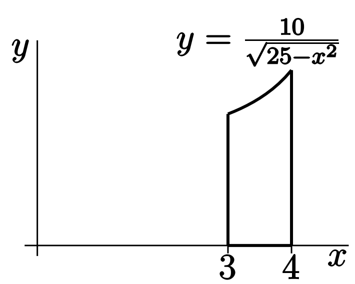
1.10.4.27. ***.** **Solution**. (a) Let's graph $y = \dfrac{10}{\sqrt{25-x^2}}$. We start with the endpoints: $(3, \frac{5}{2})$ and $(4, \frac{10}{3})$. Then we consider the first derivative:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{10}{\sqrt{25-x^2}}\right\} = \frac{10x}{\sqrt{25-x^2}^3}$$

Over the interval $[3, 4]$, this is always positive, so our function is increasing over the entire interval. The second derivative,

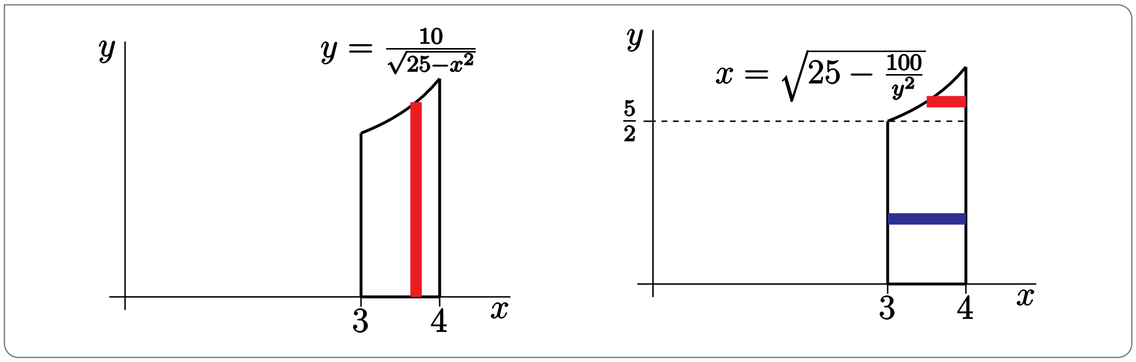
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{\frac{10}{\sqrt{25-x^2}}\right\} = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{10x}{\sqrt{25-x^2}^3}\right\} = \frac{10(2x^2+25)}{\sqrt{25-x^2}^5},$$

is always positive, so our function is concave up over the entire interval. So, the region R is:



(b) Let $\mathcal{V}_1$ be the solid obtained by revolving R about the x-axis. The portion of $\mathcal{V}_1$ with x-coordinate between x and $x+\mathrm{d}x$ is obtained by rotating the red vertical strip in the figure on the left below about the x-axis. That portion is a disk of radius $\frac{10}{\sqrt{25-x^2}}$ and thickness $\mathrm{d}x$. The volume of this disk is $\pi\left(\frac{10}{\sqrt{25-x^2}}\right)^2\mathrm{d}x$. So the total volume of $\mathcal{V}_1$ is

$$\begin{aligned}
\int_3^4 \pi\Big(\frac{10}{\sqrt{25-x^2}}\Big)^2\mathrm{d}x &= 100\pi\int_3^4 \frac{1}{25-x^2}\mathrm{d}x \\
&= 100\pi\int_3^4 \frac{1}{(5-x)(5+x)}\mathrm{d}x \\
&= 10\pi\int_3^4 \Big(\frac{1}{5-x}+\frac{1}{5+x}\Big)\mathrm{d}x \\
&= 10\pi\Big[-\log(5-x)+\log(5+x)\Big]_3^4 \\
&= 10\pi\Big[-\log 1+\log 9+\log 2-\log 8\Big] \\
&= 10\pi\log\frac{9}{4} = 20\pi\log\frac{3}{2}
\end{aligned}$$



(c) We'll use horizontal washers as in Example 1.6.5.

- We cut $\mathcal{R}$ into thin horizontal strips of width $\mathrm{d}y$ as in the figure on the right above.
- When we rotate $\mathcal{R}$ about the y-axis, each strip sweeps out a thin washer
  - whose outer radius is $r_{out} = 4$, and
  - whose inner radius is $r_{in} = \sqrt{25 - \frac{100}{y^2}}$ when $y \ge \frac{10}{\sqrt{25-3^2}} = \frac{10}{4} = \frac{5}{2}$ (see the red strip in the figure on the right above), and whose inner radius is $r_{in} = 3$ when $y \le \frac{5}{2}$ (see the blue strip in the figure on the right above) and
  - whose thickness is $\mathrm{d}y$ and hence
  - whose volume is $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}y = \pi\big(\frac{100}{y^2} - 9\big)\mathrm{d}y$ when $y \ge \frac{5}{2}$ and whose volume is $\pi(r_{out}^2 - r_{in}^2)\mathrm{d}y = 7\pi\,\mathrm{d}y$ when $y \le \frac{5}{2}$ and
- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = \frac{10}{3}$ (since at the top $x = 4$ and $y = \frac{10}{\sqrt{25-x^2}} = \frac{10}{\sqrt{25-4^2}} = \frac{10}{3}$), the volume is

$$\begin{aligned}
&\int_{5/2}^{10/3} \pi\Big(\frac{100}{y^2} - 9\Big)\mathrm{d}y + \int_0^{5/2} 7\pi\,\mathrm{d}y \\
&= \pi\Big[-\frac{100}{y} - 9y\Big]_{5/2}^{10/3} + \frac{35}{2}\pi \\
&= \pi\Big[-30 + 40 - 30 + \frac{45}{2}\Big] + \frac{35}{2}\pi \\
&= 20\pi
\end{aligned}$$

1.10.4.28. Solution. In order to find the area between the curves, we need to know which one is on top, and which on the bottom. Let's start by finding where they meet.

$$\begin{aligned}
\frac{4}{3+x^2} &= \frac{2}{x(x+1)} \\
2x^2 + 2x &= 3 + x^2 \\
x^2 + 2x - 3 &= 0
\end{aligned}$$

$$(x-1)(x+3)=0$$

In the interval $[\frac{1}{4},3]$, the curves only meet at $x=1$. So, to find which is on top and on bottom in the intervals $[\frac{1}{4},1)$ and $(1,3]$, it suffices to check some point in each interval.

x	$\frac{4}{3+x^2}$	$\frac{2}{x(x+1)}$	Top:
1/2	16/13	8/3	$\frac{2}{x(x+1)}$
2	4/7	1/3	$\frac{4}{3+x^2}$

So, $\frac{2}{x(x+1)}$ is the top function when $\frac{1}{4}\le x<1$, and $\frac{4}{3+x^2}$ is the top function when $1<x\le 3$. Then the area we want to find is:

$$\text{Area}=\int_{\frac{1}{4}}^{1}\left(\frac{2}{x(x+1)}-\frac{4}{3+x^2}\right)\mathrm{d}x+\int_{1}^{3}\left(\frac{4}{3+x^2}-\frac{2}{x(x+1)}\right)\mathrm{d}x$$

We'll need to antidifferentiate both these functions. We can antidifferentiate $\dfrac{2}{x(x+1)}$ using partial fraction decomposition.

$$\frac{2}{x(x+1)}=\frac{A}{x}+\frac{B}{x+1}=\frac{(A+B)x+A}{x(x+1)}$$

Matching coefficients

$$A+B=0,\ A=2\implies A=2,\quad B=-2$$

and the integral

$$\begin{aligned}\int\frac{2}{x(x+1)}\mathrm{d}x&=\int\left(\frac{2}{x}-\frac{2}{x+1}\right)\mathrm{d}x=2\log|x|-2\log|x+1|+C\\&=2\log\left|\frac{x}{x+1}\right|+C\end{aligned}$$

We can antidifferentiate $\dfrac{4}{3+x^2}$ using the substitution $u=\frac{x}{\sqrt{3}}$, $\mathrm{d}u=\frac{1}{\sqrt{3}}\mathrm{d}x$.

$$\begin{aligned}\int\frac{4}{3+x^2}\mathrm{d}x&=\int\frac{4}{3\left(1+\left(\frac{x}{\sqrt{3}}\right)^2\right)}\mathrm{d}x=\int\frac{4\sqrt{3}}{3\left(1+u^2\right)}\mathrm{d}u\\&=\frac{4}{\sqrt{3}}\arctan u+C=\frac{4}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right)+C\end{aligned}$$

Now, we can find our area.

$$\text{Area}=\int_{\frac{1}{4}}^{1}\left(\frac{2}{x(x+1)}-\frac{4}{3+x^2}\right)\mathrm{d}x+\int_{1}^{3}\left(\frac{4}{3+x^2}-\frac{2}{x(x+1)}\right)\mathrm{d}x$$

$$
\begin{aligned}
&= \left[2\log\left|\frac{x}{x+1}\right| - \frac{4}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right)\right]_{1/4}^{1} \\
&\qquad + \left[\frac{4}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) - 2\log\left|\frac{x}{x+1}\right|\right]_{1}^{3} \\
&= \left(2\log\frac{1}{2} - \frac{4}{\sqrt{3}}\cdot\frac{\pi}{6} - 2\log\frac{1}{5} + \frac{4}{\sqrt{3}}\arctan\frac{1}{4\sqrt{3}}\right) + \\
&\quad \left(\frac{4}{\sqrt{3}}\cdot\frac{\pi}{3} - 2\log\frac{3}{4} - \frac{4}{\sqrt{3}}\cdot\frac{\pi}{6} + 2\log\frac{1}{2}\right) \\
&= 2\log\frac{5}{3} + \frac{4}{\sqrt{3}}\arctan\frac{1}{4\sqrt{3}}
\end{aligned}
$$

1.10.4.29. Solution. (a) To antidifferentiate $\frac{1}{t^2-9}$, we use a partial fraction decomposition.

$$
\begin{aligned}
&\frac{1}{t^2-9} = \frac{1}{(t-3)(t+3)} = \frac{A}{t-3} + \frac{B}{t+3} = \frac{(A+B)t + 3(A-B)}{(t-3)(t+3)} \\
&A+B = 0, \quad A-B = \frac{1}{3} \\
&\quad A = \frac{1}{6}, \quad B = -\frac{1}{6} \\
&F(x) = \int_1^x \frac{1}{t^2-9}\mathrm{d}x = \int_1^x \left(\frac{1/6}{t-3} - \frac{1/6}{t+3}\right)\mathrm{d}x \\
&\qquad = \left[\frac{1}{6}\log|t-3| - \frac{1}{6}\log|t+3|\right]_1^x \\
&\qquad = \left(\frac{1}{6}\log|x-3| - \frac{1}{6}\log|x+3| - \frac{1}{6}\log 2 + \frac{1}{6}\log 4\right) \\
&\qquad = \frac{1}{6}\left(\log\left|2\cdot\frac{x-3}{x+3}\right|\right)
\end{aligned}
$$

(b) Rather than differentiate our answer from (a), we use the Fundamental Theorem of Calculus Part 1 to conclude

$$
F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_1^x \frac{1}{t^2-9}\mathrm{d}t\right\} = \frac{1}{x^2-9}
$$

1.11 · Numerical Integration

1.11.6 · Exercises

Exercises — Stage 1

1.11.6.1. Solution. The absolute error is the difference between the two values:

$$
|1.387 - 1.5| = 0.113
$$

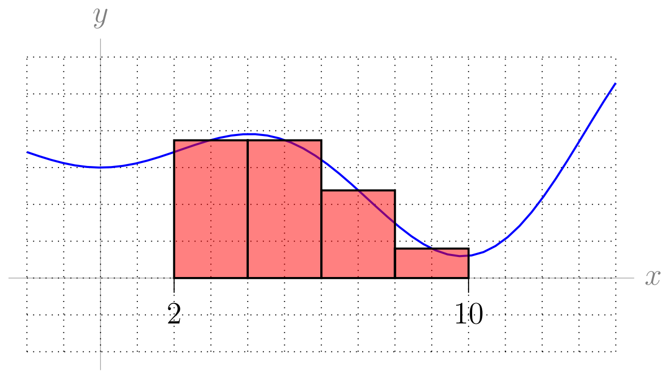
The relative error is the absolute error divided by the exact value:

$$\frac{0.113}{1.387} \approx 0.08147$$

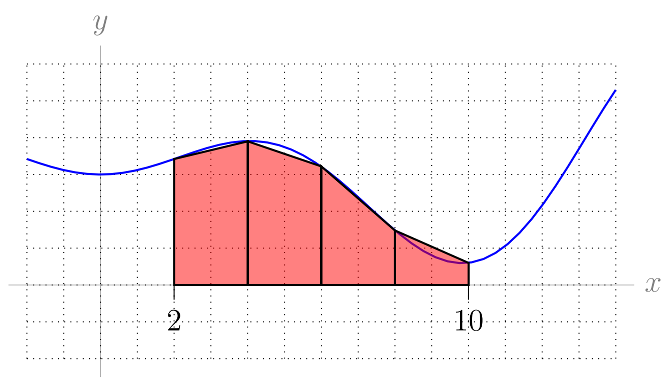
The percent error is 100 times the relative error:

$$\approx 8.147\%$$

1.11.6.2. Solution. Midpoint rule:



Trapezoidal rule:

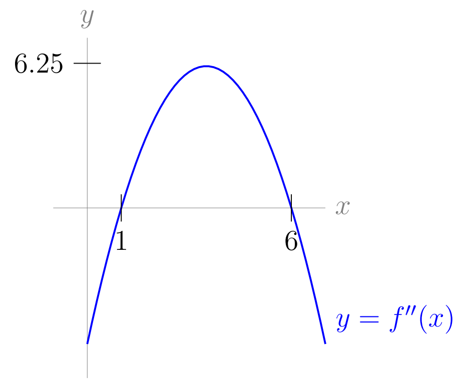


1.11.6.3. Solution.

a Differentiating, we find $f''(x) = -x^2 + 7x - 6$. Since $f''(x)$ is quadratic, we have a pretty good idea of what it looks like.

- It factors as $f(x) = -(x-6)(x-1)$, so its two roots are at $x = 6$ and $x = 1$.
- The "flat part" of the parabola is at $x = 3.5$ (since this is exactly half way between $x = 1$ and $x = 6$; alternately, we can check that $f'''(3.5) = 0$).
- Since the coefficient of x^2 is negative, $f(x)$ is increasing from $-\infty$ to 3.5, then decreasing from 3.5 to ∞.

Therefore, over the interval $[1, 6]$, the largest positive value of $f''(x)$ occurs when $x = 3.5$, and this is $f''(3.5) = -(3.5-6)(3.5-1) = 6.25$.



So, we take $M = 6.25$.

b We differentiate further to find $f^{(4)}(x) = -2$. This is constant everywhere, so we take $L = |-2| = 2$.

1.11.6.4. Solution. Let's start by differentiating.

$$\begin{aligned} f(x) &= x \sin x + 2\cos x \\ f'(x) &= x\cos x + \sin x - 2\sin x = x\cos x - \sin x \\ f''(x) &= -x\sin x + \cos x - \cos x = -x\sin x \end{aligned}$$

For any value of x, $|\sin x| \le 1$. When $-3 \le x \le 2$, then $|x| \le 3$. So, it is true (and not unreasonably sloppy) that

$$f''(x) \le 3$$

whenever x is in the interval $[-3, 2]$. So, we can take $M = 3$.
Note that $|f''(x)|$ is actually *smaller* than 3 whenever x is in the interval $[-3, 2]$, because when $x = -3$, $\sin x \ne 1$. In fact, since 3 is pretty close to π, $\sin 3$ is pretty small. (The actual maximum value of $|f''(x)|$ when $-3 \le x \le 2$ is about 1.8.) However, we find parameters like M for the purpose of computing error bounds. There is often not much to be gained from taking the time to find the actual maximum of a function, so we content ourselves with reasonable upper bounds. Question 31 has a further investigation of "sloppy" bounds like this.

1.11.6.5. Solution.

a Let $f(x) = \cos x$. Then $f^{(4)}(x) = \cos x$, so $|f^{(4)}(x)| \le 1$ when $-\pi \le x \le \pi$. So, using $L = 1$, we find the upper bound of the error using Simpson's rule with $n = 4$ is:

$$\frac{L(b-a)^5}{180n^4} = \frac{(2\pi)^5}{180 \cdot 4^4} = \frac{\pi^5}{180 \cdot 8} \approx 0.2$$

The error bound comes from Theorem 1.11.13 in the text. We used a calculator to find the approximate decimal value.

b We use the general form of Simpson's rule (Equation 1.11.9 in the text) with

$\Delta x = \frac{b-a}{n} = \frac{2\pi}{4} = \frac{\pi}{2}$.

$$\begin{aligned} A &\approx \frac{\Delta x}{3}\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)\right) \\ &= \frac{\pi/2}{3}\left(f(-\pi)+4f(\tfrac{-\pi}{2})+2f(0)+4f(\tfrac{\pi}{2})+f(\pi)\right) \\ &= \frac{\pi}{6}\left(-1+4(0)+2(1)+4(0)-1\right)=0 \end{aligned}$$

c To find the actual error in our approximation, we compare the approximation from (b) to the exact value of A. In fact, $A = 0$: this is a fact you've probably seen before by considering the symmetry of cosine, but it's easy enough to calculate:

$$A = \int_{-\pi}^{\pi} \cos x \mathrm{d}x = \sin\pi - \sin(-\pi) = 0$$

So, our approximation was exactly the same as our exact value. The absolute error is 0.

Remark: the purpose of this question was to remind you that the error bounds we calculate are not (usually) the same as the actual error. Often our approximations are better than we give them credit for. In normal circumstances, we would be approximating an integral precisely to avoid evaluating it exactly, so we wouldn't find our exact error. The bound is a quick way of ensuring that our approximation is not *too* far off.

1.11.6.6. Solution. Using Theorem 1.11.13 in the text, the error using the trapezoidal rule as described is *at most*

$$\frac{M(b-a)^3}{12\cdot n^2} = \frac{M}{48} \le \frac{3}{48} = \frac{1}{16}.$$

So, we're really being asked to find a function with the maximum possible error using the trapezoidal rule, given its second derivative.

With that in mind, our function should have the largest second derivative possible: let's set $f''(x) = 3$ for every x. Then:

$$\begin{aligned} f''(x) &= 3 \\ f'(x) &= 3x + C \\ f(x) &= \frac{3}{2}x^2 + Cx + D \end{aligned}$$

for some constants C and D. Now we can find the exact and approximate values of $\displaystyle\int_0^1 f(x)\mathrm{d}x$.

Exact:
$$\begin{aligned} \int_0^1 f(x)\mathrm{d}x &= \int_0^1 \left(\frac{3}{2}x^2 + Cx + D\right)\mathrm{d}x \\ &= \left[\frac{1}{2}x^3 + \frac{C}{2}x^2 + Dx\right]_0^1 \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}+\frac{C}{2}+D \\
\text{Approximate:}\qquad \int_0^1 f(x)\mathrm{d}x &\approx \Delta x\left[\frac{1}{2}f(0)+f(\tfrac{1}{2})+\frac{1}{2}f(1)\right] \\
&= \frac{1}{2}\left[\frac{1}{2}(D)+\left(\frac{3}{8}+\frac{C}{2}+D\right)\right. \\
&\qquad\qquad \left.+\frac{1}{2}\left(\frac{3}{2}+C+D\right)\right] \\
&= \frac{1}{2}\left[\frac{9}{8}+C+2D\right] \\
&= \frac{9}{16}+\frac{C}{2}+D
\end{aligned}
$$

So, the absolute error associated with the trapezoidal approximation is:

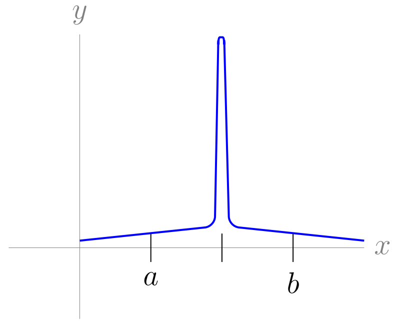
$$\left|\left(\frac{1}{2}+\frac{C}{2}+D\right)-\left(\frac{9}{16}+\frac{C}{2}+D\right)\right| = \frac{1}{16}$$

So, for any constants C and D, $f(x)=\frac{3}{2}x^2+Cx+D$ has the desired error.
Remark: contrast this question with Question 5. In this problem, our absolute error was exactly as bad as the bound predicted, but sometimes it is much better. The thing to remember is that, in general, we *don't know* our absolute error. We only guarantee that it's not any worse than some worst-case-scenario bound.

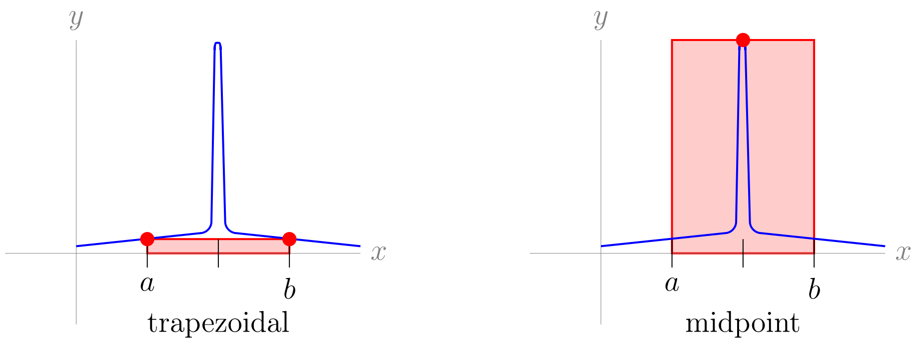
1.11.6.7. Solution. Under any reasonable assumptions [a], my mother is older than I am.

[a] Anyone caught trying to come up with a scenario in which I am older than my mother will be sent to maximum security grad school.

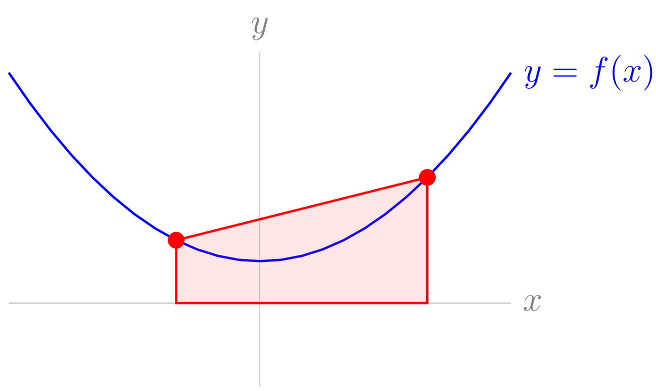
1.11.6.8. Solution. (a) Since both expressions are positive, and $\frac{1}{24}\le\frac{1}{12}$, the inequality is true.
(b) False. The reasoning is the same as in Question 7. The *error bound* given by Theorem 1.11.13 is always better for the trapezoid rule, but this doesn't necessarily mean the *error* is better.
To see how the trapezoid approximation could be better than the corresponding midpoint approximation in some cases, consider the function $f(x)$ sketched below.



The trapezoidal approximation of $\int_a^b f(x)\mathrm{d}x$ with $n = 1$ misses the thin spike, and gives a mild underapproximation. By contrast, the midpoint approximation with $n = 1$ takes the spike as the height of the entire region, giving a vast overapproximation.



1.11.6.9. *. Solution. True. Because $f(x)$ is positive and concave up, the graph of $f(x)$ is always below the top edges of the trapezoids used in the trapezoidal rule.



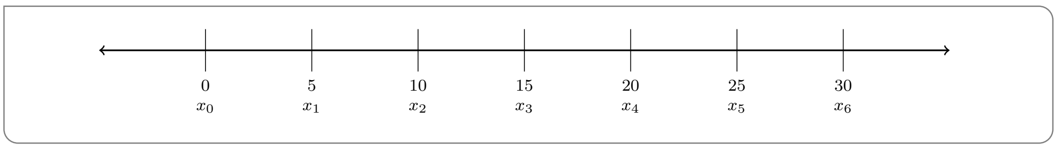
1.11.6.10. Solution. According to Theorem 1.11.13 in the text, the error associated with the Simpson's rule approximation is no more than $\frac{L}{180}\frac{(b-a)^5}{n^4}$, where L is a constant such that $|f^{(4)}(x)| \le L$ for all x in $[a, b]$. If $L = 0$, then the error is no more than 0 regardless of a, b, or n–that is, the approximation is exact.
Any polynomial $f(x)$ of degree at most 3 has $f^{(4)}(x) = 0$ for all x. So, any polyno-

mial of degree at most 3 is an acceptable answer. For example, $f(x) = 5x^3 - 27$, or $f(x) = x^2$.

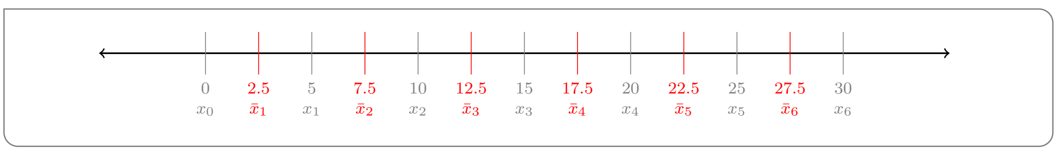
Exercises — Stage 2

1.11.6.11. Solution.

- For all three approximations, $\Delta x = \dfrac{b-a}{n} = \dfrac{30-0}{6} = 5$.
- For the trapezoidal rule and Simpson's rule, the x-values where we evaluate $\dfrac{1}{x^3+1}$ start at $x = a = 0$ and move up by $\Delta x = 5$: $x_0 = 0$, $x_1 = 5$, $x_2 = 10$, $x_3 = 15$, $x_4 = 20$, $x_5 = 25$, and $x_6 = 30$.



- For the midpoint rule, the x-values where we evaluate $\dfrac{1}{x^3+1}$ start at $x = 2.5 = \frac{x_0+x_1}{2}$ and move up by $\Delta x = 5$: $\bar{x}_1 = 2.5$, $\bar{x}_2 = 7.5$, $\bar{x}_3 = 12.5$, $\bar{x}_4 = 17.5$, $\bar{x}_5 = 22.5$, and $\bar{x}_6 = 27.5$.



- Following Equation 1.11.2, the midpoint rule approximation is:

$$\int_0^{30} \frac{1}{x^3+1}\,\mathrm{d}x \approx \Big[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)\Big]\Delta x$$
$$= \Bigg[\tfrac{1}{(2.5)^3+1} + \frac{1}{(7.5)^3+1} + \frac{1}{(12.5)^3+1} + \frac{1}{(17.5)^3+1} + \frac{1}{(22.5)^3+1} + \frac{1}{(27.5)^3+1}\Bigg]5$$

- Following Equation 1.11.6, the trapezoidal rule approximation is:

$$\int_0^{30} \frac{1}{x^3+1}\,\mathrm{d}x$$
$$\approx \Big[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\Big]\Delta x$$
$$= \Bigg[\frac{1/2}{0^3+1} + \frac{1}{5^3+1} + \frac{1}{10^3+1} + \frac{1}{15^3+1} + \frac{1}{20^3+1} + \frac{1}{25^3+1} + \frac{1/2}{30^3+1}\Bigg]5$$

- Following Equation 1.11.9, the Simpson's rule approximation is:

$$\begin{aligned}
&\int_0^{30} \frac{1}{x^3+1}\,\mathrm{d}x \\
&\approx \Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+4f(x_5)+f(x_6)\Big]\frac{\Delta x}{3} \\
&= \bigg[\frac{1}{0^3+1}+\frac{4}{5^3+1}+\frac{2}{10^3+1}+\frac{4}{15^3+1}+\frac{2}{20^3+1} \\
&\qquad\qquad\qquad\qquad\qquad +\frac{4}{25^3+1}+\frac{1}{30^3+1}\bigg]\frac{5}{3}
\end{aligned}$$

1.11.6.12. *. Solution. By Equation 1.11.2, the midpoint rule approximation to $\int_a^b f(x)\mathrm{d}x$ with $n=3$ is

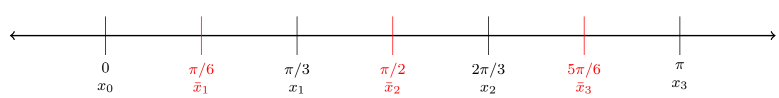
$$\int_a^b f(x)\,\mathrm{d}x \approx \big[f(\bar{x}_1)+f(\bar{x}_2)+f(\bar{x}_3)\big]\Delta x$$

where $\Delta x = \frac{b-a}{3}$ and

$$\begin{array}{llll}
x_0 = a & x_1 = a+\Delta x & x_2 = a+2\Delta x & x_3 = b \\
 & \bar{x}_1 = \frac{x_0+x_1}{2} & \bar{x}_2 = \frac{x_1+x_2}{2} & \bar{x}_3 = \frac{x_2+x_3}{2}
\end{array}$$

For this problem, $a=0$, $b=\pi$ and $f(x)=\sin x$, so that $\Delta x = \frac{\pi}{3}$ and

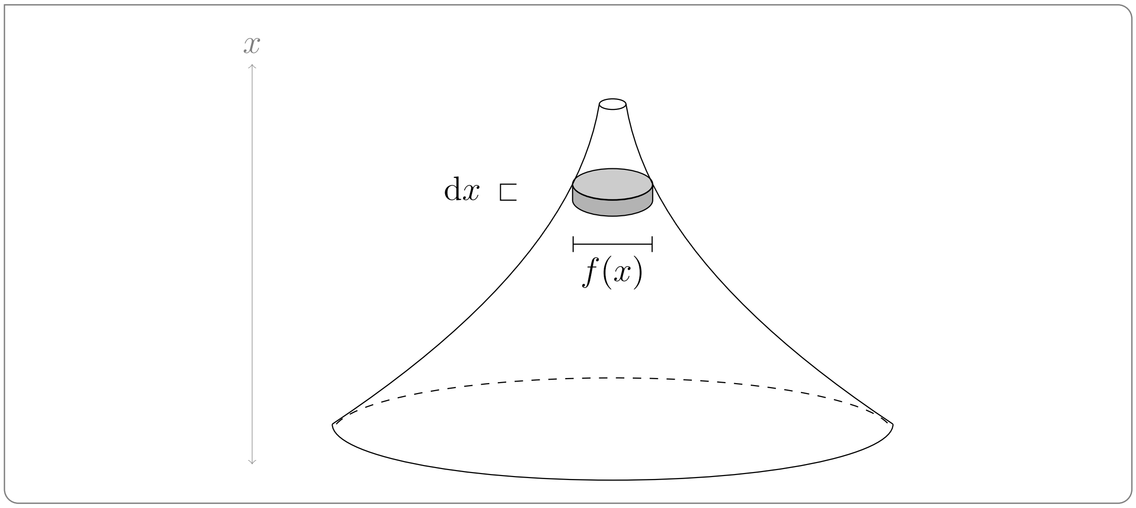
$$\begin{array}{llll}
x_0 = 0 & x_1 = \frac{\pi}{3} & x_2 = \frac{2\pi}{3} & x_3 = \pi \\
 & \bar{x}_1 = \frac{\pi}{6} & \bar{x}_2 = \frac{\pi}{2} & \bar{x}_3 = \frac{5\pi}{6}
\end{array}$$



Therefore,

$$\int_0^{\pi} \sin x\mathrm{d}x \approx \left[\sin\frac{\pi}{6}+\sin\frac{\pi}{2}+\sin\frac{5\pi}{6}\right]\frac{\pi}{3} = \left[\frac{1}{2}+1+\frac{1}{2}\right]\frac{\pi}{3} = \frac{2\pi}{3}$$

1.11.6.13. *. Solution. Let $f(x)$ denote the diameter at height x. As in Example 1.6.6, we slice V into thin horizontal "pancakes", which in this case are circular.



- We are told that the pancake at height x is a circular disk of diameter $f(x)$ and so
- has cross-sectional area $\pi\left(\frac{f(x)}{2}\right)^2$ and thickness dx and hence
- has volume $\pi\left(\frac{f(x)}{2}\right)^2 \mathrm{d}x$.

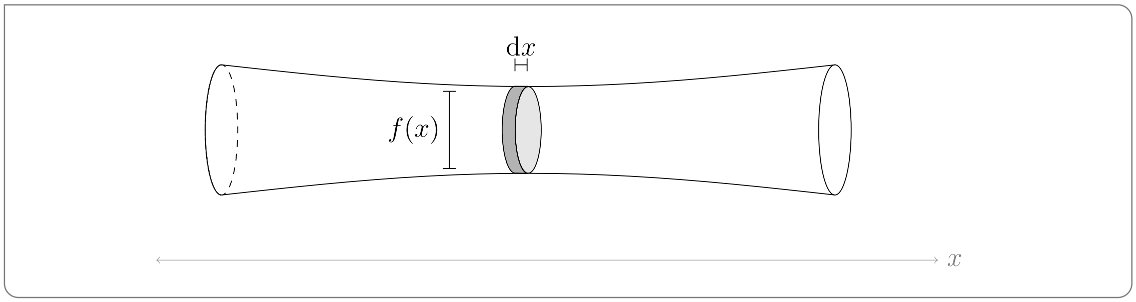
Hence the volume of V is

$$\begin{aligned}\int_0^{40} \pi\left[\frac{f(x)}{2}\right]^2 \mathrm{d}x \\ &\approx \frac{\pi}{4}10\left[\frac{1}{2}f(0)^2 + f(10)^2 + f(20)^2 + f(30)^2 + \frac{1}{2}f(40)^2\right] \\ &= \frac{\pi}{4}10\left[\frac{1}{2}24^2 + 16^2 + 10^2 + 6^2 + \frac{1}{2}4^2\right] \\ &= 688 \times 2.5\pi = 1720\pi \approx 5403.5\end{aligned}$$

where we have approximated the integral using the trapezoidal rule with $\Delta x = 10$, and used a calculator to get a decimal approximation.

1.11.6.14. ∗. Solution. Let $f(x)$ be the diameter a distance x from the left end of the log. If we slice our log into thin disks, the disks x metres from the left end of the log has

- radius $\frac{f(x)}{2}$,
- width dx, and so
- volume $\pi\left(\frac{f(x)}{2}\right)^2 \mathrm{d}x = \frac{\pi}{4}f(x)^2 \mathrm{d}x$.

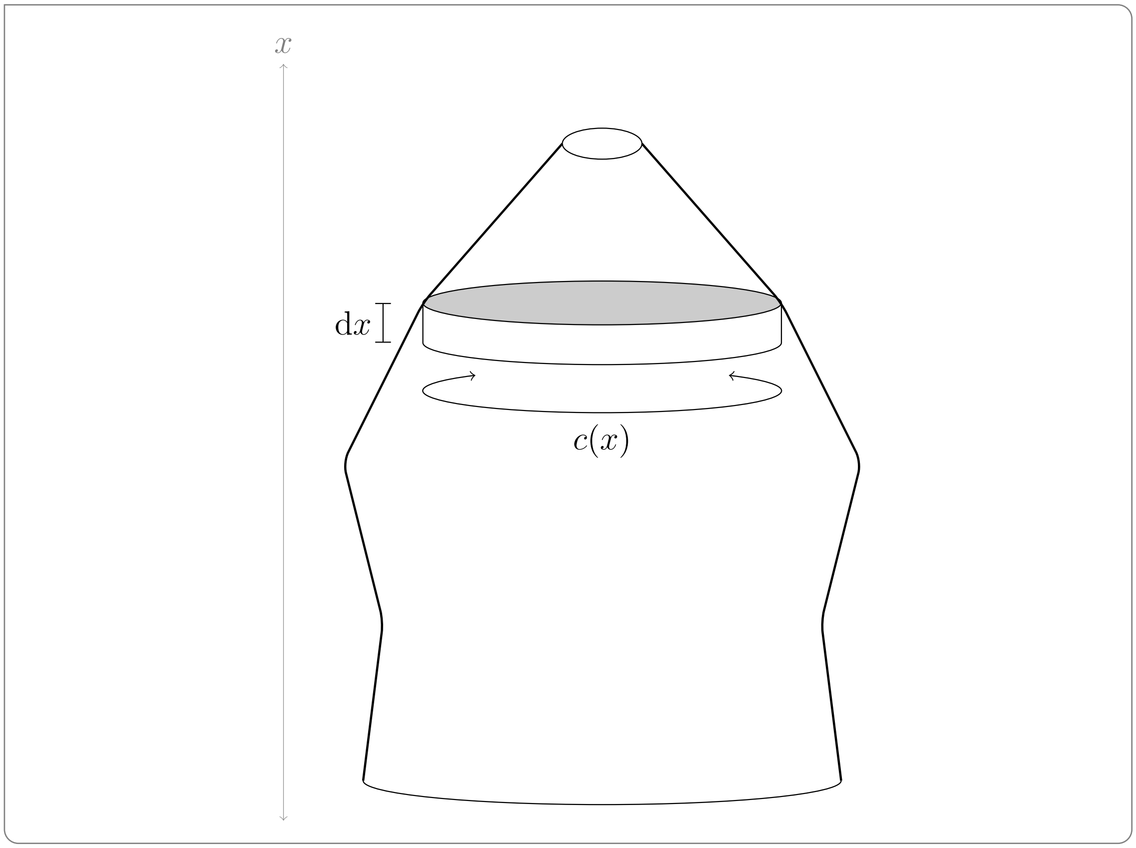


Using Simpson's Rule with $\Delta x = 1$, the volume of the log is:

$$\begin{aligned}
V &= \int_0^6 \frac{\pi}{4} f(x)^2\,\mathrm{d}x \\
&\approx \frac{\pi}{4}\frac{1}{3}\Big[f(0)^2 + 4f(1)^2 + 2f(2)^2 + 4f(3)^2 + 2f(4)^2 + 4f(5)^2 + f(6)^2\Big] \\
&= \frac{\pi}{12}\Big[1.2^2 + 4(1)^2 + 2(0.8)^2 + 4(0.8)^2 + 2(1)^2 + 4(1)^2 + 1.2^2\Big] \\
&= \frac{\pi}{12}(16.72) \\
&\approx 4.377 \text{ m}^3
\end{aligned}$$

where we used a calculator to approximate the decimal value.

1.11.6.15. $*$. Solution. At height x metres, let the circumference of the tree be $c(x)$. The corresponding radius is $\frac{c(x)}{2\pi}$, so the corresponding cross–sectional area is $\pi\left(\frac{c(x)}{2\pi}\right)^2 = \frac{c(x)^2}{4\pi}$.



The height of a very thin cross–sectional disk is $\mathrm{d}x$, so the volume of a cross-sectional disk is $\dfrac{c(x)^2}{4\pi}\mathrm{d}x$. Therefore, total volume of the tree is:

$$\begin{aligned}\int_0^8 \frac{c(x)^2}{4\pi}\,\mathrm{d}x &\approx \frac{1}{4\pi}\frac{2}{3}\Big[c(0)^2+4c(2)^2+2c(4)^2+4c(6)^2+c(8)^2\Big]\\ &= \frac{1}{6\pi}\Big[1.2^2+4(1.1)^2+2(1.3)^2+4(0.9)^2+0.2^2\Big]\\ &= \frac{12.94}{6\pi}\approx 0.6865\end{aligned}$$

where we used Simpson's rule with $\Delta x = 2$ and $n = 4$ to approximate the value of the integral based on the values of $c(x)$ given in the table.

1.11.6.16. $*$. Solution. For both approximations, $\Delta x = 10$ and $n = 6$.
(a) The Trapezoidal Rule gives

$$\begin{aligned}V &= \int_0^{60} A(h)\,\mathrm{d}h\\ &\approx 10\Big[\frac{1}{2}A(0)+A(10)+A(20)+A(30)+A(40)+A(50)+\frac{1}{2}A(60)\Big]\\ &= 363{,}500\end{aligned}$$

(b) Simpson's Rule gives

$$\begin{aligned}V &= \int_0^{60} A(h)\,\mathrm{d}h \\ &\approx \frac{10}{3}\Big[A(0)+4A(10)+2A(20)+4A(30)+2A(40)+4A(50)+A(60)\Big] \\ &= 367{,}000\end{aligned}$$

1.11.6.17. *. Solution. Call the curve in the graph $y = f(x)$. It looks like

$$f(2) = 3 \qquad f(3) = 8 \qquad f(4) = 7 \qquad f(5) = 6 \qquad f(6) = 4$$

We're estimating $\int_2^6 f(x)\mathrm{d}x$ with $n = 4$, so $\Delta x = \frac{6-2}{4} = 1$.
(a) The trapezoidal rule gives

$$T_4 = \left[\frac{3}{2} + 8 + 7 + 6 + \frac{4}{2}\right] \times 1 = \frac{49}{2}$$

(b) Simpson's rule gives

$$S_4 = \frac{1}{3}\big[3 + 4 \times 8 + 2 \times 7 + 4 \times 6 + 4\big] \times 1 = \frac{77}{3}$$

1.11.6.18. *. Solution. Let $f(x) = \sin(x^2)$. Then $f'(x) = 2x\cos(x^2)$ and

$$f''(x) = 2\cos(x^2) - 4x^2\sin(x^2).$$

Since $|x^2| \le 1$ when $|x| \le 1$, and $|\sin\theta| \le 1$ and $|\cos\theta| \le 1$ for all θ, we have

$$\begin{aligned}\big|2\cos(x^2) - 4x^2\sin(x^2)\big| &\le 2|\cos(x^2)| + 4x^2|\sin(x^2)| \\ &\le 2 \times 1 + 4 \times 1 \times 1 = 2 + 4 = 6\end{aligned}$$

We can therefore choose $M = 6$, and it follows that the error is at most

$$\frac{M[b-a]^3}{24n^2} \le \frac{6 \cdot [1-(-1)]^3}{24 \cdot 1000^2} = \frac{2}{10^6} = 2 \cdot 10^{-6}$$

1.11.6.19. *. Solution. Setting $f(x) = 2x^4$ and $b - a = 1 - (-2) = 3$, we compute $f''(x) = 24x^2$. The largest value of $24x^2$ on the interval $[-2, 1]$ occurs at $x = -2$, so we can take $M = 24 \cdot (-2)^2 = 96$. Thus the total error for the midpoint rule with $n = 60$ points is bounded by

$$\frac{M(b-a)^3}{24n^2} = \frac{96 \times 3^3}{24 \times 60 \times 60} = \frac{3}{100}$$

That is: we are guaranteed our absolute error is certainly no more [a] than $\frac{3}{100}$, and using the bound stated in the problem we cannot give a better guarantee. (The second part of the previous sentence comes from the fact that we used the smallest possible M: if we had used a larger value of M, we would still have some true statement about the error, for example "the error is no more than $\frac{5}{100}$," but it would not be the *best* true statement we could make.)

a This is what the error bound always tells us.

1.11.6.20. *. Solution. (a) Since $a = 0$, $b = 2$ and $n = 6$, we have $\Delta x = \frac{b-a}{n} = \frac{2-0}{6} = \frac{1}{3}$, and so $x_0 = 0$, $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$, $x_3 = 1$, $x_4 = \frac{4}{3}$, $x_5 = \frac{5}{3}$, and $x_6 = 2$. Since Simpson's Rule with $n = 6$ in general is

$$\frac{\Delta x}{3}\big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\big],$$

the desired approximation is

$$\frac{1/3}{3}\bigg((-3)^5 + 4\Big(\frac{1}{3} - 3\Big)^5 + 2\Big(\frac{2}{3} - 3\Big)^5 + 4(-2)^5 + 2\Big(\frac{4}{3} - 3\Big)^5 + 4\Big(\frac{5}{3} - 3\Big)^5 + (-1)^5\bigg)$$

(b) Here $f(x) = (x-3)^5$, which has derivatives

$$f'(x) = 5(x-3)^4 \qquad f''(x) = 20(x-3)^3$$
$$f^{(3)}(x) = 60(x-3)^2 \qquad f^{(4)}(x) = 120(x-3).$$

For $0 \le x \le 2$, $(x-3)$ runs from -3 to -1, so the maximum absolute values are found at $x = 0$, giving $M = 20 \cdot |0-3|^3 = 540$ and $L = 120 \cdot |0-3| = 360$. Consequently, for the Midpoint Rule with $n = 100$,

$$|E_M| \le \frac{M(b-a)^3}{24n^2} = \frac{540 \times 2^3}{24 \times 10^4} = \frac{180}{10^4};$$

whereas for Simpson's Rule with $n = 10$,

$$|E_S| \le \frac{360 \times 2^5}{180 \times 10^4} = \frac{64}{10^4}.$$

Since $64 < 180$, Simpson's Rule results in a smaller error bound.

1.11.6.21. *. Solution. In general the error in approximating $\int_a^b f(x)\mathrm{d}x$ using Simpson's rule with n steps is bounded by $\frac{L(b-a)}{180}(\Delta x)^4$ where $\Delta x = \frac{b-a}{n}$ and $L \ge |f^{(4)}(x)|$ for all $a \le x \le b$. In this case, $a = 1$, $b = 5$, $n = 4$ and $f(x) = \frac{1}{x}$. We need to find L, so we differentiate.

$$f'(x) = -\frac{1}{x^2} \qquad f''(x) = \frac{2}{x^3} \qquad f^{(3)}(x) = -\frac{6}{x^4} \qquad f^{(4)}(x) = \frac{24}{x^5}$$

and

$$\big|f^{(4)}(x)\big| \le 24 \text{ for all } x \ge 1$$

So we may take $L = 24$ and $\Delta x = \frac{5-1}{4} = 1$, which leads to

$$|\text{Error }| \le \frac{24(5-1)}{180}(1)^4 = \frac{24}{45} = \frac{8}{15}$$

1.11.6.22. *. Solution. In general, the error in approximating $\int_a^b f(x)\mathrm{d}x$ using Simpson's rule with n steps is bounded by $\dfrac{L(b-a)}{180}(\Delta x)^4$ where $\Delta x = \dfrac{b-a}{n}$ and $L \ge |f^{(4)}(x)|$ for all $a \le x \le b$. In this case, $a = 0$, $b = 1$, $n = 6$ and $f(x) = e^{-2x} + 3x^3$. We need to find L, so we differentiate.

$$f'(x) = -2e^{-2x} + 9x^2 \qquad f''(x) = 4e^{-2x} + 18x$$
$$f^{(3)}(x) = -8e^{-2x} + 18 \qquad f^{(4)}(x) = 16e^{-2x}$$

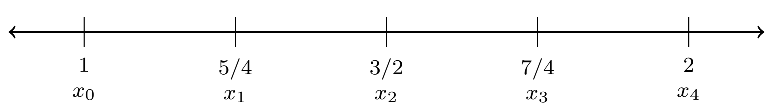
Since $e^{-2x} = \dfrac{1}{e^{2x}}$, we see $f^{(4)}(x)$ is a positive, decreasing function. So, its maximum occurs when x is as small as possible. In the interval $[0,1]$, that means $x = 0$.

$$\left|f^{(4)}(x)\right| \le f(0) = 16 \text{ for all } x \ge 0$$

So, we take $L = 16$ and $\Delta x = \frac{1-0}{6} = \frac{1}{6}$.

$$\begin{aligned}|\text{Error}| &\le \frac{L(b-a)}{180}(\Delta x)^4 = \frac{16(1-0)}{180}(1/6)^4 = \frac{16}{180 \times 6^4}\\ &= \frac{1}{180 \times 3^4} = \frac{1}{14580}\end{aligned}$$

1.11.6.23. *. Solution. For both approximations, $a = 1$, $b = 2$, $n = 4$, $f(x) = \frac{1}{x}$ and $\Delta x = \frac{b-a}{n} = \frac{1}{4}$.
Then $x_0 = 1$, $x_1 = \frac{5}{4}$, $x_2 = \frac{3}{2}$, $x_3 = \frac{7}{4}$, and $x_4 = 2$.



(a)

$$\begin{aligned}T_4 &= \Delta x\left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + f(x_3) + \frac{1}{2}f(x_4)\right]\\ &= \Delta x\left[\frac{1}{2}f(1) + f(5/4) + f(3/2) + f(7/4) + \frac{1}{2}f(2)\right]\\ &= \frac{1}{4}\left[\left(\frac{1}{2}\times 1\right) + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \left(\frac{1}{2}\times\frac{1}{2}\right)\right]\end{aligned}$$

(b)

$$\begin{aligned}S_4 &= \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]\\ &= \frac{\Delta x}{3}\left[f(1) + 4f(5/4) + 2f(3/2) + 4f(7/4) + f(2)\right]\end{aligned}$$

$$= \frac{1}{12}\left[1 + \left(4 \times \frac{4}{5}\right) + \left(2 \times \frac{2}{3}\right) + \left(4 \times \frac{4}{7}\right) + \frac{1}{2}\right]$$

(c) In this case, $a = 1$, $b = 2$, $n = 4$ and $f(x) = \frac{1}{x}$. We need to find L, so we differentiate.

$$f'(x) = -\frac{1}{x^2} \qquad f''(x) = \frac{2}{x^3} \qquad f^{(3)}(x) = -\frac{6}{x^4} \qquad f^{(4)}(x) = \frac{24}{x^5}$$

So,

$$\left|f^{(4)}(x)\right| \le 24 \text{ for all } x \text{ in the interval } [1,2]$$

We take $L = 24$.

$$|\text{Error }| \le \frac{L(b-a)^5}{180 \times n^4} \le \frac{24(2-1)^5}{180 \times 4^4} = \frac{24}{180 \times 4^4} = \frac{3}{5760}$$

1.11.6.24. *. Solution. Set $a = 0$ and $b = 8$. Since we have information about $s(x)$ when x is 0, 2, 4, 6, and 8, we set $\Delta x = \frac{b-a}{n} = 2$, so $n = 4$. (Recall with the trapezoid rule and Simpson's rule, $n = 4$ intervals actually uses the value of the function at 5 points.)
We could perform the trapezoidal approximations with fewer intervals, for example $n = 2$, but this would involve ignoring some of the points we're given. Since the question asks for the best estimation we can give, we use $n = 4$ intervals and no fewer.

a

$$\begin{aligned}
T_4 &= \Delta x\left[\frac{1}{2}s(0) + s(2) + s(4) + s(6) + \frac{1}{2}s(8)\right] \\
&= 2\left[\frac{1.00664}{2} + 1.00543 + 1.00435 + 1.00331 + \frac{1.00233}{2}\right] \\
&= 8.03515 \\
S_4 &= \frac{\Delta x}{3}\left[s(0) + 4s(2) + 2s(4) + 4s(6) + s(8)\right] \\
&= \frac{2}{3}\big[1.00664 + 4 \times 1.00543 + 2 \times 1.00435 + 4 \times 1.00331 \\
&\qquad + 1.00233\big] \\
&\approx 8.03509
\end{aligned}$$

b The information $\left|s^{(k)}(x)\right| \le \dfrac{k}{1000}$, with $k = 2$, tells us $|s''(x)| \le \frac{2}{1000}$ for all x in the interval $[0,8]$. So, we take K_2 (also called M in your text) to be $\frac{2}{1000}$. Then the absolute error associated with our trapezoid rule approximation is at most

$$\left|\int_a^b f(x)\,\mathrm{d}x - T_n\right| \le \frac{K_2(b-a)^3}{12n^2} \le \frac{2}{1000} \cdot \frac{8^3}{12(4)^2} \le 0.00533$$

For $k = 4$, we see $|s^{(4)}(x)| \le \frac{4}{1000}$ for all x in the interval $[0, 8]$. So, we take K_4 (also called L in your text) to be $\frac{4}{1000}$.

Then the absolute error associated with our Simpson's rule approximation is at most

$$\left| \int_a^b f(x)\,\mathrm{d}x - S_n \right| \le \frac{K_4(b-a)^5}{180n^4} \le \frac{4}{1000} \cdot \frac{8^5}{180(4)^4} \le 0.00284$$

1.11.6.25. $*$. Solution. In this case, $a = 1$, $b = 4$. Since $-2 \le f''(x) \le 0$ over the relevant interval, we take $M = 2$. (Remember M is an upper bound on $|f''(x)|$, not $f''(x)$.) So we need n to obey

$$\frac{2(4-1)^3}{12n^2} \le 0.001 \iff n^2 \ge \frac{2(3)^3}{12}1000 = \frac{27000}{6} = \frac{9000}{2} = 4500$$

One obvious allowed n is 100. Since $\sqrt{4500} \approx 67.01$, and n has to be a whole number, any $n \ge 68$ works.

Exercises — Stage 3

1.11.6.26. $*$. Solution. Denote by $f(x)$ the width of the pool x feet from the left-hand end. From the sketch, $f(0) = 0$, $f(2) = 10$, $f(4) = 12$, $f(6) = 10$, $f(8) = 8$, $f(10) = 6$, $f(12) = 8$, $f(14) = 10$ and $f(16) = 0$.

A cross-section of the pool x feet from the left end is half of a circular disk with diameter $f(x)$ (so, radius $\frac{f(x)}{2}$) and thickness $\mathrm{d}x$. So, the volume of the part of the pool with x–coordinate running from x to $x + \mathrm{d}x$ is $\frac{1}{2}\pi\left(\frac{f(x)}{2}\right)^2 \mathrm{d}x = \frac{\pi}{8}[f(x)]^2\,\mathrm{d}x$. The total volume is given by the following integral.

$$\begin{aligned}
V &= \frac{\pi}{8}\int_0^{16} f(x)^2\,\mathrm{d}x \\
&\approx \frac{\pi}{8} \cdot \frac{\Delta x}{3}\Big[f(0)^2 + 4f(2)^2 + 2f(4)^2 + 4f(6)^2 + 2f(8)^2 + 4f(10)^2 \\
&\qquad\qquad + 2f(12)^2 + 4f(14)^2 + f(16)^2\Big] \\
&= \frac{\pi}{8} \cdot \frac{2}{3}\Big[0 + 4(10)^2 + 2(12)^2 + 4(10)^2 + 2(8)^2 + 4(6)^2 + 2(8)^2 \\
&\qquad\qquad + 4(10)^2 + 0\Big] \\
&= \frac{472}{3}\pi \approx 494 \text{ ft}^3
\end{aligned}$$

1.11.6.27. $*$. Solution. (a) The Trapezoidal Rule with $n = 4$, $a = 0$, $b = 1$, and $\Delta x = \frac{1}{4}$ gives:

$$\begin{aligned}
W &= 2\pi 10^{-6}\int_0^1 rg(r)\,\mathrm{d}r \\
&\approx 2\pi 10^{-6}\Delta x\left[\frac{1}{2}x_0 g(x_0) + x_1 g(x_1) + x_2 g(x_2) + x_3 g(x_3) + \frac{1}{2}x_4 g(x_4)\right]
\end{aligned}$$

$$\begin{aligned}
&= 2\pi 10^{-6}\ \frac{1}{4}\Big[\frac{1}{2}0g(0) + \frac{1}{4}g\Big(\frac{1}{4}\Big) + \frac{1}{2}g\Big(\frac{1}{2}\Big) + \frac{3}{4}g\Big(\frac{3}{4}\Big) + \frac{1}{2}g(1)\Big] \\
&= \pi 10^{-6}\ \frac{1}{2}\Big[\frac{8100}{4} + \frac{8144}{2} + \frac{3\cdot 8170}{4} + \frac{8190}{2}\Big] \\
&= \frac{32639\pi}{4\cdot 10^6} \approx 0.025635
\end{aligned}$$

(b) Using the product rule, the integrand $f(r) = 2\pi 10^{-6} r g(r)$ obeys

$$f''(r) = 2\pi 10^{-6}\frac{\mathrm{d}}{\mathrm{d}r}\big[g(r) + rg'(r)\big] = 2\pi 10^{-6}\big[2g'(r) + rg''(r)\big]$$

and hence, for $0 \le r \le 1$,

$$\big|f''(r)\big| \le 2\pi 10^{-6}\big[2\times 200 + 1\times 150\big] = 1.1\pi 10^{-3}$$

So,

$$|\text{Error}| \le \frac{1.1\pi 10^{-3}(1-0)^3}{12(4)^2} \le 1.8\times 10^{-5}$$

1.11.6.28. $*$. **Solution**. (a) Let $f(x) = \frac{1}{x}$, $a = 1$, $b = 2$ and $\Delta x = \frac{b-a}{6} = \frac{1}{6}$. Using Simpson's rule:

$$\begin{aligned}
\int_1^2 \frac{1}{x}\mathrm{d}x \approx \frac{\Delta x}{3}\Big[f(1) &+ 4f\Big(\frac{7}{6}\Big) + 2f\Big(\frac{8}{6}\Big) + 4f\Big(\frac{9}{6}\Big) + 2f\Big(\frac{10}{6}\Big) \\
&+ 4f\Big(\frac{11}{6}\Big) + f(2)\Big] \\
&= \frac{1}{18}\Big[1 + \frac{24}{7} + \frac{12}{8} + \frac{24}{9} + \frac{12}{10} + \frac{24}{11} + \frac{1}{2}\Big] \approx 0.6931698
\end{aligned}$$

(b) The integrand is $f(x) = \frac{1}{x}$. The first four derivatives of $f(x)$ are:

$$f'(x) = -\frac{1}{x^2}, \qquad f''(x) = \frac{2}{x^3}, \qquad f^{(3)}(x) = -\frac{6}{x^4}, \qquad f^{(4)}(x) = \frac{24}{x^5}$$

On the interval $1 \le x \le 2$, the fourth derivative is never bigger in magnitude than $L = 24$.

$$|E_n| \le \frac{L(b-a)^5}{180n^4} = \frac{24(2-1)^5}{180n^4} = \frac{4}{30n^4}$$

So, we want an even number n such that

$$\begin{aligned}
\frac{4}{30n^4} &\le 0.00001 = \frac{1}{10^5} \\
n^4 &\ge \frac{40000}{3} \\
n &\ge \sqrt[4]{\frac{40000}{3}} \approx 10.7
\end{aligned}$$

So, any even number greater than or equal to 12 will do.

1.11.6.29. *. Solution. (a) From the figure, we see that the magnitude of $|f''''(x)|$ never exceeds 310 for $0 \le x \le 2$. So, the absolute error is bounded by

$$\frac{310(2-0)^5}{180 \times 8^4} \le 0.01345$$

(b) We want to choose n such that:

$$\begin{aligned} \frac{310(2-0)^5}{180 \times n^4} &\le 10^{-4} \\ n^4 &\ge \frac{310 \times 2^5}{180} 10^4 \\ n &\ge 10 \sqrt[4]{\frac{310 \times 32}{180}} \approx 27.2 \end{aligned}$$

For Simpson's rule, n must be even, so any even integer obeying $n \ge 28$ will guarantee us the requisite accuracy.

1.11.6.30. *. Solution. Let $g(x) = \displaystyle\int_0^x \sin(\sqrt{t})\, dt$. By the Fundamental Theorem of Calculus Part 1, $g'(x) = \sin(\sqrt{x})$. By its definition, $f(x) = g(x^2)$, so we use the chain rule to differentiate $f(x)$.

$$f'(x) = 2xg'(x^2) = 2x \sin x \qquad\qquad f''(x) = 2\sin x + 2x\cos x$$

Since $|\sin x|, |\cos x| \le 1$, we have $|f''(x)| \le 2 + 2|x|$ and, for $0 \le t \le 1$, $|f''(t)| \le 4$. When the trapezoidal rule with n subintervals is applied, the resulting error E_n obeys

$$E_n \le \frac{4(1-0)^3}{12n^2} = \frac{1}{3n^2}$$

We want an integer n such that

$$\begin{aligned} \frac{1}{3n^2} &\le 0.000005 \\ n^2 &\ge \frac{4}{12 \times 0.000005} \\ n &\ge \sqrt{\frac{1}{3 \times 0.000005}} \approx 258.2 \end{aligned}$$

Any integer $n \ge 259$ will do.

1.11.6.31. Solution.

a When $0 \le x \le 1$, then $x^2 \le 1$ and $x + 1 \ge 1$, so $|f''(x)| = \dfrac{x^2}{|x+1|} \le \dfrac{1}{1} = 1$.

b To find the maximum value of a function over a closed interval, we test the function's values at the endpoints of the interval and at its critical points inside the interval. The critical points are where the function's derivative is zero or does not exist.

The function we're trying to maximize is $|f''(x)| = \frac{x^2}{|x+1|} = \frac{x^2}{x+1} = f''(x)$ (since our interval only contains nonnegative numbers). So, the critical points occur when $f'''(x) = 0$ or does not exist. We find $f'''(x)$ Using the quotient rule.

$$\begin{aligned}
f'''(x) &= \frac{(x+1)(2x) - x^2}{(x+1)^2} = \frac{x^2+2x}{(x+1)^2} \\
0 &= \frac{x(x+2)}{x+1} \\
0 &= x \quad \text{or} \quad x = -1 \quad \text{or} \quad x = -2
\end{aligned}$$

The only critical point in $[0, 1]$ is $x = 0$. So, the extrema of $f''(x)$ over $[0, 1]$ will occur at its endpoints. Indeed, since $f'''(x) \geq 0$ for all x in $[0, 1]$, $f''(x)$ is increasing over this interval, so its maximum occurs at $x = 1$. That is,

$$|f''(x)| \leq f''(1) = \frac{1}{2}$$

c The absolute error using the midpoint rule is at most $\dfrac{M(b-a)^3}{24n^2}$. Using $M = 1$, if we want this to be no more than 10^{-5}, we find an acceptable value of n with the following calculation:

$$\begin{aligned}
\frac{M(b-a)^3}{24n^2} &\leq 10^{-5} \\
\frac{1}{24n^2} &\leq 10^{-5} && (b - a = 1,\ M = 1) \\
\frac{10^5}{24} &\leq n^2 \\
n &\geq 65
\end{aligned}$$

d The absolute error using the midpoint rule is at most $\dfrac{M(b-a)^3}{24n^2}$. Using $M = \frac{1}{2}$, if we want this to be no more than 10^{-5}, we find an acceptable value of n with the following calculation:

$$\begin{aligned}
\frac{M(b-a)^3}{24n^2} &\leq 10^{-5} \\
\frac{1}{48n^2} &\leq 10^{-5} && (b - a = 1,\ M = \frac{1}{2}) \\
\frac{10^5}{48} &\leq n^2 \\
n &\geq 46
\end{aligned}$$

Remark: how accurate you want to be in these calculations depends a lot on your circumstances. Imagine, for instance, that you were finding M by hand, using this to find n by hand, then programming a computer to evaluate the approximation. For a simple integral like this, the difference between computing time for 65 intervals versus 46 is likely to be miniscule. So, there's not much to be gained by the extra work in (b). However, if your original sloppy M gave you something like $n = 1000000$, you might want to put some time into improving it, to shorten computation time. Moreover, if you were finding the approximation by hand, the difference between adding 46 terms and adding 65 terms would be considerable, and you would probably want to put in the effort up front to find the most accurate M possible.

1.11.6.32. Solution. Before we can take our Simpson's rule approximation of $\displaystyle\int_1^x \frac{1}{t}\mathrm{d}t$, we need to know how many intervals to use. That means we need to bound our error, which means we need to bound $\frac{\mathrm{d}^4}{\mathrm{d}t^4}\left\{\frac{1}{t}\right\}$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{t}\right\} = -\frac{1}{t^2} \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left\{\frac{1}{t}\right\} = \frac{2}{t^3}$$

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}\left\{\frac{1}{t}\right\} = -\frac{6}{t^4} \qquad \frac{\mathrm{d}^4}{\mathrm{d}t^4}\left\{\frac{1}{t}\right\} = \frac{24}{t^5}$$

So, over the interval $[1,3]$, $\left|\dfrac{\mathrm{d}^4}{\mathrm{d}t^4}\left\{\dfrac{1}{t}\right\}\right| \le 24$.

Now, we can find an appropriate n to ensure our error will be be less than 0.1 for any x in $[1,3]$:

$$\begin{aligned}\frac{L(b-a)^5}{180n^4} &< 0.1\\ \frac{24(x-1)^5}{180n^4} &< \frac{1}{10}\\ n^4 &> \frac{24\cdot(x-1)^5}{18}\end{aligned}$$

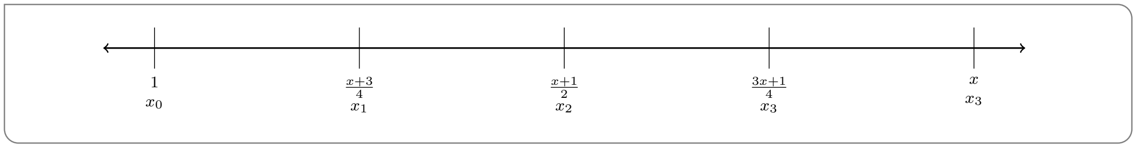
Because $x-1 \le 2$ for every x in $[1,3]$, if $n^4 > \dfrac{24\cdot 2^5}{18}$, then $n^4 > \dfrac{24\cdot(x-1)^5}{18}$ for every allowed x.

$$\begin{aligned}n^4 &> \frac{24\cdot 2^5}{18} = \frac{128}{3}\\ n &> \sqrt[4]{\frac{128}{3}} \approx 2.6\end{aligned}$$

Since n must be even, $n = 4$ is enough intervals to guarantee our error is not too high for any x in $[1,3]$. Now we find our Simpson's rule approximation with $n = 4$, $a = 1$, $b = x$, and $\Delta x = \dfrac{x-1}{4}$. The points where we evaluate $\frac{1}{t}$ are:

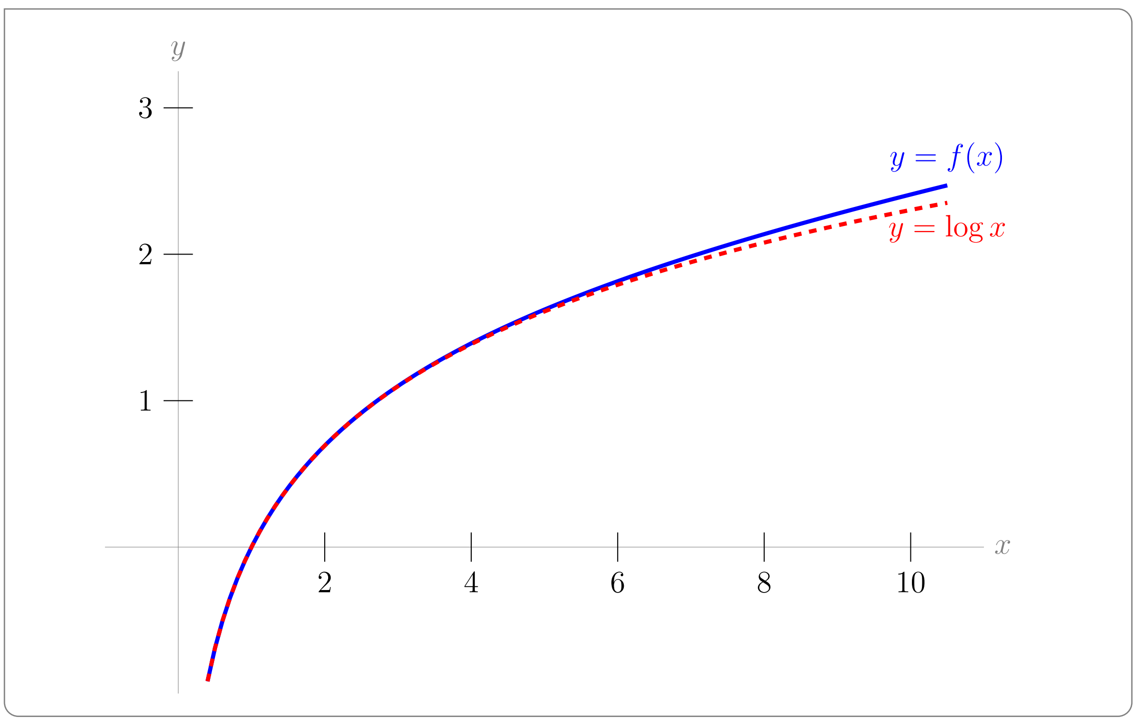
$$x_0 = 1 \qquad \begin{aligned}x_1 &= 1 + \frac{x-1}{4}\\ &= \frac{x+3}{4}\end{aligned} \qquad \begin{aligned}x_2 &= 1 + 2\cdot\frac{x-1}{4}\\ &= \frac{x+1}{2}\end{aligned}$$

$$x_3 = 1 + 3 \cdot \frac{x-1}{4} \qquad\qquad x_4 = 1 + 4 \cdot \frac{x-1}{4}$$
$$= \frac{3x+1}{4} \qquad\qquad\qquad = x$$



$$\begin{aligned}
\log x = \int_1^x \frac{1}{t}\mathrm{d}t &\approx \frac{\Delta x}{3}\left[\frac{1}{x_0} + \frac{4}{x_1} + \frac{2}{x_2} + \frac{4}{x_3} + \frac{1}{x_4}\right] \\
&= \frac{x-1}{12}\left[1 + \frac{16}{x+3} + \frac{4}{x+1} + \frac{16}{3x+1} + \frac{1}{x}\right] \\
&= f(x)
\end{aligned}$$

Below is a graph of our approximation $f(x)$ and natural logarithm on the same axes. The natural logarithm function is shown red and dashed, while our approximating function is solid blue. Our approximation appears to be quite accurate for small, positive values of x.



1.11.6.33. Solution. First, we want a strategy for approximating $\arctan 2$. Our hints are that involves integrating $\frac{1}{1+x^2}$, which is the antiderivative of arctangent, and the number $\frac{\pi}{4}$, which is the same as arctan(1). With that in mind:

$$\int_1^2 \frac{1}{1+x^2}\mathrm{d}x = \arctan(2) - \arctan(1) = \arctan(2) - \frac{\pi}{4}$$

$$\text{So,}\qquad \arctan(2) = \frac{\pi}{4} + \int_1^2 \frac{1}{1+x^2}\mathrm{d}x \qquad (*)$$

We won't know the value of the integral exactly, but we'll have an approximation A bounded by some positive error bound ε. Then,

$$\begin{aligned} -\varepsilon \le \left(\int_1^2 \frac{1}{1+x^2}\mathrm{d}x - A\right) &\le \varepsilon \\ A - \varepsilon \le \left(\int_1^2 \frac{1}{1+x^2}\mathrm{d}x\right) &\le A + \varepsilon \\ \text{So, from } (*), \qquad \frac{\pi}{4} + A - \varepsilon \le \arctan(2) &\le \frac{\pi}{4} + A + \varepsilon \end{aligned}$$

Which approximation should we use? We're given the fourth derivative of $\frac{1}{1+x^2}$, which is the derivative we need for Simpson's rule. Simpson's rule is also usually quite efficient, and we're very interested in not adding up dozens of terms, so we choose Simpson's rule.

Now that we've chosen Simpson's rule, we should decide how many intervals to use. In order to bound our error, we need to find a bound for the fourth derivative. To that end, define $N(x) = 24(5x^4 - 10x^2 + 1)$. Then $N'(x) = 24(20x^3 - 20x) = 480x(x^2-1)$, which is positive over the interval $[1, 2]$. So, $N(x) \le N(2) = 24(5 \cdot 2^4 - 10 \cdot 2^2 + 1) = 984$ when $1 \le x \le 2$. Furthermore, let $D(x) = (x^2+1)^5$. If $1 \le x \le 2$, then $D(x) \ge 2^5$. Now we can find a reasonable value of L:

$$|f^{(4)}(x)| = \left|\frac{25(5x^4 - 10x^2 + 1)}{(x^2+1)^5}\right| = \left|\frac{N(x)}{D(x)}\right| \le \frac{984}{2^5} = \frac{123}{4} = 30.75$$

So, we take $L = 30.75$.

We want $\left[\frac{\pi}{4} + A - \varepsilon, \frac{\pi}{4} + A + \varepsilon\right]$ to look something like $\left[\frac{\pi}{4} + 0.321, \frac{\pi}{4} + 0.323\right]$. Note ε is half the length of the first interval. Half the length of the second interval is $0.001 = \frac{1}{1000}$. So, we want a value of ε that is no larger than this. Now we can find our n:

$$\begin{aligned} \frac{L(b-a)^5}{180 \cdot n^4} &\le \frac{1}{1000} \\ \frac{30.75}{180 \cdot n^4} &\le \frac{1}{1000} \\ n^4 &\ge \frac{30.75 \times 1000}{180} \\ n &\ge \sqrt[4]{\frac{30750}{180}} \approx 3.62 \end{aligned}$$

So, we choose $n = 4$), and are guaranteed that the absolute error in our approximation will be no more than $\frac{30.75}{180 \cdot 4^4} < 0.00067$.

Since $n = 4$, then $\Delta x = \frac{b-a}{n} = \frac{1}{4}$, so:

$$x_0 = 1 \qquad x_1 = \frac{5}{4} \qquad x_2 = \frac{3}{2} \qquad x_3 = \frac{7}{4} \qquad x_4 = 2$$

Now we can find our Simpson's rule approximation A:

$$\begin{aligned}
\int_0^1 \frac{1}{1+x^2}\mathrm{d}x &\approx \frac{\Delta x}{3}\big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\big] \\
&= \frac{1/4}{3}\big[f(1) + 4f(5/4) + 2f(3/2) + 4f(7/4) + f(2)\big] \\
&= \frac{1}{12}\left[\frac{1}{1+1} + \frac{4}{25/16+1} + \frac{2}{9/4+1} + \frac{4}{49/16+1} + \frac{1}{4+1}\right] \\
&= \frac{1}{12}\left[\frac{1}{2} + \frac{4 \cdot 16}{25+16} + \frac{2 \cdot 4}{9+4} + \frac{4 \cdot 16}{49+16} + \frac{1}{5}\right] \\
&= \frac{1}{12}\left[\frac{1}{2} + \frac{64}{41} + \frac{8}{13} + \frac{64}{65} + \frac{1}{5}\right] \\
&\approx 0.321748 = A
\end{aligned}$$

As we saw before, the error associated with this approximation is at most $\frac{30.75}{180 \cdot 4^4} < 0.00067 = \varepsilon$. So,

$$\begin{aligned}
& A - \varepsilon \le \int_1^2 \frac{1}{1+x^2}\mathrm{d}x \le A + \varepsilon \\
\Rightarrow \quad & 0.321748 - 0.00067 \le \int_1^2 \frac{1}{1+x^2}\mathrm{d}x \le 0.321748 + 0.00067 \\
\Rightarrow \quad & 0.321078 \le \int_1^2 \frac{1}{1+x^2}\mathrm{d}x \le 0.322418 \\
\Rightarrow \quad & 0.321 \le \int_1^2 \frac{1}{1+x^2}\mathrm{d}x \le 0.323 \\
\Rightarrow \quad & \frac{\pi}{4} + 0.321 \le \int_1^2 \frac{1}{1+x^2}\mathrm{d}x + \frac{\pi}{4} \le \frac{\pi}{4} + 0.323 \\
\Rightarrow \quad & \frac{\pi}{4} + 0.321 \le \arctan(2) \le \frac{\pi}{4} + 0.323;
\end{aligned}$$

This is precisely what we wanted to show.

1.12 · Improper Integrals

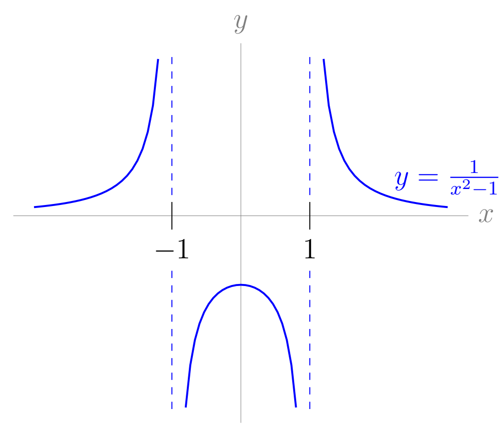
1.12.4 · Exercises

Exercises — Stage 1

1.12.4.1. Solution. If $b = \pm\infty$, then our integral is improper because one limit is not a real number.

Furthermore, our integral will be improper if its domain of integration contains either of its infinite discontinuities, $x = 1$ and $x = -1$. Since one limit of integration is 0, the integral is improper if $b \geq 1$ or if $b \leq -1$.

Below, we've graphed $\frac{1}{x^2-1}$ to make it clearer why values of b in $(-1, 1)$ are the only values that *don't* result in an improper integral when the other limit of integration is $a = 0$.



1.12.4.2. Solution. Since the integrand is continuous for all real x, the only kind of impropriety available to us is to set $b = \pm\infty$.

1.12.4.3. Solution. For large values of x, $|\text{red function}| \leq (\text{blue function})$ and $0 \leq (\text{blue function})$. If the blue function's integral converged, then the red function's integral would as well (by the comparison test, Theorem 1.12.17 in the text). Since one integral converges and the other diverges, the blue function is $g(x)$ and the red function is $f(x)$.

1.12.4.4. *. Solution. False. The inequality goes the "wrong way" for Theorem 1.12.17: the area under the curve $f(x)$ is finite, but the area under $g(x)$ could be much larger, even infinitely larger.

For example, if $f(x) = e^{-x}$ and $g(x) = 1$, then $0 \leq f(x) \leq g(x)$ and $\displaystyle\int_1^\infty f(x)\,\mathrm{d}x$ converges, but $\displaystyle\int_1^\infty g(x)\,\mathrm{d}x$ diverges.

1.12.4.5. Solution.

a Not enough information to decide. For example, consider $h(x) = 0$ versus $h(x) = -1$. In both cases, $h(x) \leq f(x)$. However, $\displaystyle\int_0^\infty 0\mathrm{d}x$ converges to 0, while $\displaystyle\int_0^\infty -1\mathrm{d}x$ diverges.

Note: if we had also specified $0 \leq h(x)$, then we would be able to conclude that $\int_0^\infty h(x)\mathrm{d}x$ converges by the comparison test.

b Not enough information to decide. For example, consider $h(x) = f(x)$ versus

$h(x) = g(x)$. In both cases, $f(x) \le h(x) \le g(x)$.

c $\displaystyle\int_0^\infty h(x)\mathrm{d}x$ converges.

- From the given information, $|h(x)| \le 2f(x)$.
- We claim $\displaystyle\int_0^\infty 2f(x)\mathrm{d}x$ converges.
    - We can see this by writing $\displaystyle\int_0^\infty 2f(x)\mathrm{d}x = 2\int_0^\infty f(x)\mathrm{d}x$ and noting that the second integral converges.
    - Alternately, we can use the limiting comparison test, Theorem 1.12.22. Since $f(x) \ge 0$, $\displaystyle\int_0^\infty f(x)\mathrm{d}x$ converges, and $\displaystyle\lim_{x\to\infty} \frac{2f(x)}{f(x)} = 2$ (the limit exists), we conclude $\displaystyle\int_0^\infty 2f(x)\mathrm{d}x$ converges.
- So, comparing $h(x)$ to $2f(x)$, by the comparison test (Theorem 1.12.17) $\displaystyle\int_0^\infty h(x)\mathrm{d}x$ converges.

Exercises — Stage 2

1.12.4.6. *. Solution. The denominator is zero when $x = 1$, but the numerator is not, so the integrand has a singularity (infinite discontinuity) at $x = 1$. Let's replace the limit $x = 1$ with a variable that creeps toward 1.

$$\int_0^1 \frac{x^4}{x^5-1}\,\mathrm{d}x = \lim_{t\to 1^-}\int_0^t \frac{x^4}{x^5-1}\,\mathrm{d}x$$

To evaluate this integral we use the substitution $u = x^5$, $\mathrm{d}u = 5x^4\mathrm{d}x$. When $x = 0$ we have $u = 0$, and when $x = t$ we have $u = t^5$, so

$$\begin{aligned}\int_0^1 \frac{x^4}{x^5-1}\,\mathrm{d}x &= \lim_{t\to 1^-}\int_0^t \frac{x^4}{x^5-1}\,\mathrm{d}x = \lim_{t\to 1^-}\int_{u=0}^{u=t^5} \frac{1}{5(u-1)}\,\mathrm{d}u \\ &= \lim_{t\to 1^-}\left(\left[\frac{1}{5}\log|u-1|\right]_0^{t^5}\right) = \lim_{t\to 1^-}\frac{1}{5}\log|t^5-1| = -\infty\end{aligned}$$

The limit diverges, so the integral diverges as well.

1.12.4.7. *. Solution. The denominator of the integrand is zero when $x = -1$, but the numerator is not. So, the integrand has a singularity (infinite discontinuity) at $x = -1$. This is the only "source of impropriety" in this integral, so we only need to make one break in the domain of integration.

$$\int_{-2}^2 \frac{1}{(x+1)^{4/3}}\,\mathrm{d}x = \lim_{t\to -1^-}\int_{-2}^t \frac{1}{(x+1)^{4/3}}\,\mathrm{d}x + \lim_{t\to -1^+}\int_t^2 \frac{1}{(x+1)^{4/3}}\,\mathrm{d}x$$

Let's start by considering the left limit.

$$\begin{aligned}\lim_{t\to-1^-}\int_{-2}^{t}\frac{1}{(x+1)^{4/3}}\,\mathrm{d}x &= \lim_{t\to-1^-}\left(\left[-\frac{3}{(x+1)^{1/3}}\right|_{-2}^{t}\right)\\ &= \lim_{t\to-1^-}\left(-\frac{3}{(t+1)^{1/3}}+\frac{3}{(-1)^{1/3}}\right)=\infty\end{aligned}$$

Since this limit diverges, the integral diverges. (A similar argument shows that the second integral diverges. Either one of them diverging is enough to conclude that the original integral diverges.)

1.12.4.8. *. Solution. First, let's identify all "sources of impropriety." The integrand has a singularity when $4x^2-x=0$, that is, when $x(4x-1)=0$, so at $x=0$ and $x=\frac{1}{4}$. Neither of these are in our domain of integration, so the only "source of impropriety" is the unbounded domain of integration.
We could antidifferentiate this function (it looks like a nice candidate for a trig substitution), but is seems easier to use a comparison. For large values of x, the term x^2 will be much larger than x, so we might guess that our integral behaves similarly to $\int_1^\infty \frac{1}{\sqrt{4x^2}}\mathrm{d}x=\int_1^\infty\frac{1}{2x}\mathrm{d}x$.
For all $x\ge1$, $\sqrt{4x^2-x}\le\sqrt{4x^2}=2x$. So, $\frac{1}{\sqrt{4x^2-x}}\ge\frac{1}{2x}$. Note $\int_1^\infty\frac{1}{2x}\mathrm{d}x$ diverges:

$$\lim_{t\to\infty}\int_1^t\frac{1}{2x}\,\mathrm{d}x=\lim_{t\to\infty}\left(\frac{1}{2}\big[\log x\big]_1^t\right)=\lim_{t\to\infty}\frac{1}{2}\log t=\infty$$

So:

- $\frac{1}{2x}$ and $\frac{1}{\sqrt{4x^2-x}}$ are defined and continuous for all $x\ge1$,
- $\frac{1}{2x}\ge0$ for all $x\ge1$,
- $\frac{1}{\sqrt{4x^2-x}}\ge\frac{1}{\sqrt{4x^2}}=\frac{1}{2x}$ for all $x\ge1$, and
- $\int_1^\infty\frac{1}{2x}\mathrm{d}x$ diverges.

By the comparison test, Theorem 1.12.17 in the text, the integral does not converge.

1.12.4.9. *. Solution. The integrand is positive everywhere. So, either the integral converges to some finite number, or it is infinite. We want to generate a guess as to which it is.
When x is small, $\sqrt{x}>x^2$, so we might guess that our integral behaves like the integral of $\frac{1}{\sqrt{x}}$ when x is near to 0. On the other hand, when x is large, $\sqrt{x}<x^2$, so we might guess that our integral behaves like the integral of $\frac{1}{x^2}$ as x goes to infinity. This is the hunch that drives the following work:

$$0\le\frac{1}{x^2+\sqrt{x}}\le\frac{1}{\sqrt{x}}$$

and the integral $\displaystyle\int_0^1\frac{\mathrm{d}x}{\sqrt{x}}$ converges by Example 1.12.9, and

$$0 \le \frac{1}{x^2+\sqrt{x}} \le \frac{1}{x^2}$$

and the integral $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{x^2}$ converges by Example 1.12.8

Note that $\frac{1}{x^2+\sqrt{x}}$ is defined and continuous for all $x > 0$, $\frac{1}{\sqrt{x}}$ is defined and continuous for $x > 0$, and $\frac{1}{x^2}$ is defined and continuous for $x \ge 1$. So, the integral converges by the comparison test, Theorem 1.12.17 in the text, together with Remark 1.12.16.

1.12.4.10. Solution. There are two "sources of impropriety:" the two (infinite) limits of integration. So, we break our integral into two pieces.

$$\begin{aligned}\int_{-\infty}^{\infty} \cos x\mathrm{d}x &= \int_{-\infty}^{0} \cos x\mathrm{d}x + \int_0^\infty \cos x\mathrm{d}x\\ &= \lim_{a\to\infty}\left[\int_{-a}^{0} \cos x\mathrm{d}x\right] + \lim_{b\to\infty}\left[\int_0^b \cos x\mathrm{d}x\right]\end{aligned}$$

These are easy enough to antdifferentiate.

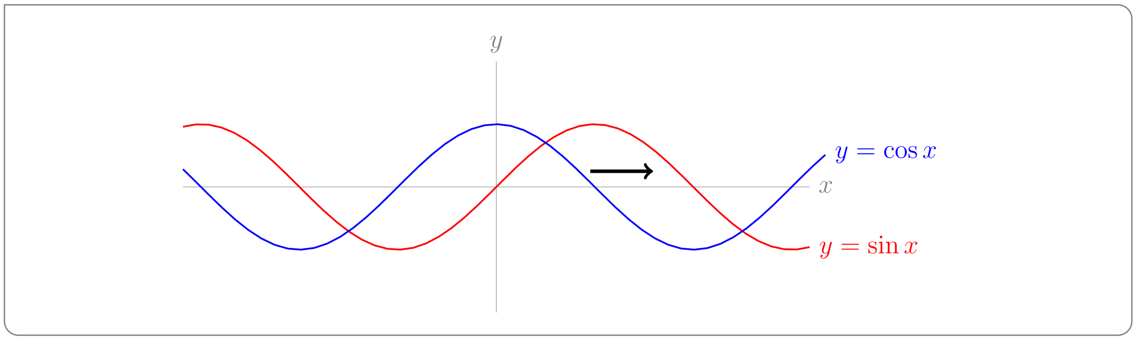
$$\begin{aligned}&= \lim_{a\to\infty}[\sin 0 - \sin(-a)] + \lim_{b\to\infty}[\sin b - \sin 0]\\ &= \text{DNE}\end{aligned}$$

Since the limits don't exist, the integral diverges. (It happens that both limits don't exist; even if only one failed to exist, the integral would still diverge.)

1.12.4.11. Solution. There are two "sources of impropriety:" the two bounds. So, we break our integral into two pieces.

$$\begin{aligned}\int_{-\infty}^{\infty} \sin x\mathrm{d}x &= \int_{-\infty}^{0} \sin x\mathrm{d}x + \int_0^\infty \sin x\mathrm{d}x\\ &= \lim_{a\to\infty}\left[\int_{-a}^{0} \sin x\mathrm{d}x\right] + \lim_{b\to\infty}\left[\int_0^b \sin x\mathrm{d}x\right]\\ &= \lim_{a\to\infty}[-\cos 0 + \cos(-a)] + \lim_{b\to\infty}[-\cos b + \cos 0]\\ &= \text{DNE}\end{aligned}$$

Since the limits don't exist, the integral diverges. (It happens that both limits don't exist; even if only one failed to exist, the integral would diverge.)
Remark: it's very tempting to think that this integral should converge, because as an odd function the area to the right of the x-axis "cancels out" the area to the left when the limits of integration are symmetric. One justification for not using this intuition is given in Example 1.12.11 in the text. Here's another: In Question 1.12.4.10 we saw that $\int_{-\infty}^{\infty} \cos x\mathrm{d}x$ diverges. Since $\sin x = \cos(x - \pi/2)$, the area bounded by sine and the area bounded by cosine over an infinite region seem to be the same–only shifted by $\pi/2$. So if $\int_{-\infty}^{\infty} \sin x\mathrm{d}x = 0$, then we ought to also have $\int_{-\infty}^{\infty} \cos x\mathrm{d}x = 0$, but we saw in Question 1.12.4.10 this is not the case.



1.12.4.12. Solution. First, we check that the integrand has no singularities. The denominator is always positive when $x \geq 10$, so our only "source of impropriety" is the infinite limit of integration.

We further note that, for large values of x, the integrand resembles $\dfrac{x^4}{x^5} = \dfrac{1}{x}$. So, we have a two-part hunch: that the integral diverges, and that we can show it diverges by comparing it to $\displaystyle\int_{10}^{\infty} \frac{1}{x} \mathrm{d}x$.

In order to use the comparison test, we'd need to show that $\dfrac{x^4 - 5x^3 + 2x - 7}{x^5 + 3x + 8} \geq \dfrac{1}{x}$. If this is true, it will be difficult to prove–and it's not at all clear that it's true. So, we will use the limiting comparison test instead, Theorem 1.12.22, with $g(x) = \dfrac{1}{x}$, $f(x) = \dfrac{x^4 - 5x^3 + 2x - 7}{x^5 + 3x + 8}$, and $a = 10$.

- Both $f(x)$ and $g(x)$ are defined and continuous for all $x > 0$, so in particular they are defined and continuous for $x \geq 10$.
- $g(x) \geq 0$ for all $x \geq 10$
- $\displaystyle\int_{10}^{\infty} g(x) \mathrm{d}x$ diverges.
- Using l'Hôpital's rule (5 times!), or simply dividing both the numerator and denominator by x^5 (the common leading term), tells us:

$$\begin{aligned}\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{\frac{x^4-5x^3+2x-7}{x^5+3x+8}}{\frac{1}{x}} = \lim_{x\to\infty} x \cdot \frac{x^4 - 5x^3 + 2x - 7}{x^5 + 3x + 8} \\ &= \lim_{x\to\infty} \frac{x^5 - 5x^4 + 2x^2 - 7x}{x^5 + 3x + 8} = 1\end{aligned}$$

  That is, the limit exists and is nonzero.

By the limiting comparison test, we conclude $\displaystyle\int_{10}^{\infty} f(x) \mathrm{d}x$ diverges.

1.12.4.13. Solution. Our domain of integration is finite, so the only potential "sources of impropriety" are infinite discontinuities in the integrand. To find these, we factor.

$$\int_0^{10} \frac{x-1}{x^2-11x+10}\mathrm{d}x = \int_0^{10} \frac{x-1}{(x-1)(x-10)}\mathrm{d}x$$

A removable discontinuity doesn't affect the integral.

$$= \int_0^{10} \frac{1}{x-10}\mathrm{d}x$$

Use the substitution $u = x - 10$, $\mathrm{d}u = \mathrm{d}x$. When $x = 0$, $u = -10$, and when $x = 10$, $u = 0$.

$$= \int_{-10}^{0} \frac{1}{u}\mathrm{d}u$$

This is a p-integral with $p = 1$. From Example 1.12.9 and Theorem 1.12.20, we know it diverges.

1.12.4.14. *. Solution. You might think that, because the integrand is odd, the integral converges to 0. This is a common mistake– see Example 1.12.11 in the text, or Question 11 in this section. In the absence of such a shortcut, we use our standard procedure: identifying problem spots over the domain of integration, and replacing them with limits.

There are two "sources of impropriety," namely $x \to +\infty$ and $x \to -\infty$. So, we split the integral in two, and treat the two halves separately. The integrals below can be evaluated with the substitution $u = x^2 + 1$, $\frac{1}{2}\mathrm{d}u = x\mathrm{d}x$.

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{x}{x^2+1}\mathrm{d}x &= \int_{-\infty}^{0} \frac{x}{x^2+1}\mathrm{d}x + \int_{0}^{+\infty} \frac{x}{x^2+1}\mathrm{d}x \\
\int_{-\infty}^{0} \frac{x}{x^2+1}\mathrm{d}x &= \lim_{R\to\infty} \int_{-R}^{0} \frac{x}{x^2+1}\mathrm{d}x = \lim_{R\to\infty} \frac{1}{2}\log(x^2+1)\Big|_{-R}^{0} \\
&= \lim_{R\to\infty} \frac{1}{2}\left[\log 1 - \log(R^2+1)\right] = \lim_{R\to\infty} -\frac{1}{2}\log(R^2+1) \\
&= -\infty \\
\int_{0}^{+\infty} \frac{x}{x^2+1}\mathrm{d}x &= \lim_{R\to\infty} \int_{0}^{R} \frac{x}{x^2+1}\mathrm{d}x = \lim_{R\to\infty} \frac{1}{2}\log(x^2+1)\Big|_{0}^{R} \\
&= \lim_{R\to\infty} \frac{1}{2}\left[\log(R^2+1) - \log 1\right] = \lim_{R\to\infty} \frac{1}{2}\log(R^2+1) \\
&= +\infty
\end{aligned}$$

Both halves diverge, so the whole integral diverges.

Once again: after we found that one of the limits diverged, we could have stopped and concluded that the original integrand diverges. Don't make the mistake of thinking that $\infty - \infty = 0$. That can get you into *big* trouble. ∞ is not a normal number. For example $2\infty = \infty$. So if ∞ were a normal number we would have both $\infty - \infty = 0$ and $\infty - \infty = 2\infty - \infty = \infty$.

1.12.4.15. $*$. **Solution**. We don't want to antidifferentiate this integrand, so let's use a comparison. Note the integrand is positive when $x > 0$.

For any x, $|\sin x| \le 1$, so $\dfrac{|\sin x|}{x^{3/2}+x^{1/2}} \le \dfrac{1}{x^{3/2}+x^{1/2}}$.

Since $x = 0$ and $x \to \infty$ both cause the integral to be improper, we need to break it into two pieces. Since both terms in the denominator give positive numbers when x is positive, $\dfrac{1}{x^{3/2}+x^{1/2}} \le \dfrac{1}{x^{3/2}}$ and $\dfrac{1}{x^{3/2}+x^{1/2}} \le \dfrac{1}{x^{1/2}}$. That gives us two options for comparison.

When x is positive and close to zero, $x^{1/2} \ge x^{3/2}$, so we guess that we should compare our integrand to $\frac{1}{x^{1/2}}$ near the limit $x = 0$. In contrast, when x is very large, $x^{1/2} \le x^{3/2}$, so we guess that we should compare our integrand to $\frac{1}{x^{3/2}}$ as x goes to infinity.

$$\frac{|\sin x|}{x^{3/2}+x^{1/2}} \le \frac{1}{x^{1/2}}$$
$$\text{and the integral } \int_0^1 \frac{\mathrm{d}x}{x^{1/2}} \text{ converges by the } p\text{-test, Example 1.12.9}$$
$$\frac{|\sin x|}{x^{3/2}+x^{1/2}} \le \frac{1}{x^{3/2}}$$
$$\text{and the integral } \int_1^\infty \frac{\mathrm{d}x}{x^{3/2}} \text{ converges by the } p\text{-test, Example 1.12.8}$$

Now we have all the data we need to apply the comparison test, Theorem 1.12.17.

- $\dfrac{|\sin x|}{x^{3/2}+x^{1/2}}$, $\dfrac{1}{x^{1/2}}$, and $\dfrac{1}{x^{3/2}}$ are defined and continuous for $x > 0$
- $\dfrac{1}{x^{1/2}}$ and $\dfrac{1}{x^{3/2}}$ are nonnegative for $x \ge 0$
- $\dfrac{|\sin x|}{x^{3/2}+x^{1/2}} \le \dfrac{1}{x^{1/2}}$ for all $x > 0$ and $\displaystyle\int_0^1 \frac{1}{x^{1/2}}\mathrm{d}x$ converges, so (using Remark 1.12.16) $\displaystyle\int_0^1 \frac{|\sin x|}{x^{3/2}+x^{1/2}}\mathrm{d}x$ converges.
- $\dfrac{|\sin x|}{x^{3/2}+x^{1/2}} \le \dfrac{1}{x^{3/2}}$ for all $x \ge 1$ and $\displaystyle\int_1^\infty \frac{1}{x^{3/2}}\mathrm{d}x$ converges, so $\displaystyle\int_1^\infty \frac{|\sin x|}{x^{3/2}+x^{1/2}}\mathrm{d}x$ converges.

Therefore, our integral $\displaystyle\int_0^\infty \frac{|\sin x|}{x^{3/2}+x^{1/2}}\mathrm{d}x$ converges.

1.12.4.16. $*$. **Solution**. The integrand is positive everywhere, so either the integral converges to some finite number or it is infinite. There are two potential "sources of impropriety" — a possible singularity at $x = 0$ and the fact that the domain of integration extends to ∞. So we split up the integral.

$$\int_0^\infty \frac{x+1}{x^{1/3}(x^2+x+1)}\,\mathrm{d}x = \int_0^1 \frac{x+1}{x^{1/3}(x^2+x+1)}\,\mathrm{d}x + \int_1^\infty \frac{x+1}{x^{1/3}(x^2+x+1)}\,\mathrm{d}x$$

Let's develop a hunch about whether the integral converges or diverges. When $x \approx 0$, x^2 and x are both a lot smaller than 1, so we guess we should compare the integrand to $\frac{1}{x^{1/3}}$.

$$\frac{x+1}{x^{1/3}(x^2+x+1)} \approx \frac{1}{x^{1/3}(1)} = \frac{1}{x^{1/3}}$$

Note $\int_0^1 \frac{1}{x^{1/3}}\mathrm{d}x$ converges by Example 1.12.9 (it's a p-type integral), so we guess $\int_0^1 \frac{x+1}{x^{1/3}(x^2+x+1)}\mathrm{d}x$ converges as well.
When x is very large, x^2 is much bigger than x, which is much bigger than 1, so we guess we should compare the integrand to $\frac{1}{x^{4/3}}$.

$$\frac{x+1}{x^{1/3}(x^2+x+1)} \approx \frac{x}{x^{1/3}(x^2)} = \frac{1}{x^{4/3}}$$

Note $\int_1^\infty \frac{1}{x^{4/3}}\mathrm{d}x$ converges by Example 1.12.8 (it's a p-type integral), so we guess $\int_1^\infty \frac{x+1}{x^{1/3}(x^2+x+1)}\mathrm{d}x$ converges as well.
Now it's time to verify our guesses with the limiting comparison test, Theorem 1.12.22. Be careful: our "$\approx$" signs are not strong enough to use either the limiting comparison test or the comparison test, they are only enough to suggest a reasonable function to compare *to*.

- $\frac{x+1}{x^{1/3}(x^2+x+1)}$, $\frac{1}{x^{1/3}}$, and $\frac{1}{x^{4/3}}$ are defined and continuous for all $x > 0$
- $\frac{1}{x^{1/3}}$ and $\frac{1}{x^{4/3}}$ are positive for all $x > 0$
- $\int_0^1 \frac{1}{x^{1/3}}\mathrm{d}x$ and $\int_1^\infty \frac{1}{x^{4/3}}\mathrm{d}x$ both converge
- $\lim\limits_{x\to 0} \dfrac{\frac{x+1}{x^{1/3}(x^2+x+1)}}{\frac{1}{x^{1/3}}} = \lim\limits_{x\to 0} \dfrac{x+1}{x^2+x+1} = \dfrac{0+1}{0+0+1} = 1$; in particular, this limit exists.
- Using the limiting comparison test (Theorem 1.12.22, together with Remark 1.12.16 because our impropriety is due to a singularity), $\int_0^1 \frac{x+1}{x^{1/3}(x^2+x+1)}\mathrm{d}x$ converges.
- $\lim\limits_{x\to \infty} \dfrac{\frac{x+1}{x^{1/3}(x^2+x+1)}}{\frac{1}{x^{4/3}}} = \lim\limits_{x\to 0} \dfrac{x(x+1)}{x^2+x+1} = 1$; in particular, this limit exists.
- Using the limiting comparison test (Theorem 1.12.22), $\int_1^\infty \frac{x+1}{x^{1/3}(x^2+x+1)}\mathrm{d}x$ converges.

We conclude $\displaystyle\int_0^\infty \frac{x+1}{x^{1/3}(x^2+x+1)}\mathrm{d}x$ converges.

Exercises — Stage 3

1.12.4.17. Solution. To find the volume of the solid, we cut it into horizontal slices, which are thin circular disks. At height y, the disk has radius $x = \frac{1}{y}$ and thickness $\mathrm{d}y$, so its volume is $\frac{\pi}{y^2}\mathrm{d}y$. The base of the solid is at height $y = 1$, and its top is at height $y = \frac{1}{a}$. So, the volume of the entire solid is:

$$\int_1^{1/a} \frac{\pi}{y^2}\mathrm{d}y = \left[-\frac{\pi}{y}\right]_1^{1/a} = \pi(1-a)$$

If we imagine sliding a closer and closer to 0, the volume increases, getting closer and closer to π units, but never quite reaching it.
So, the statement is false. For example, if we set $M = 4$, no matter which a we choose our solid has volume strictly less than M.
Remark: we've seen before that $\int_0^1 \frac{1}{x}\,\mathrm{d}x$ diverges. If we imagine the solid that would result from choosing $a = 0$, it would have a scant volume of π cubic units, but a silhouette (side view) of infinite area.

1.12.4.18. *. Solution. Our goal is to decide when this integral diverges, and where it converges. We will leave q as a variable, and antidifferentiate. In order to antidifferentiate without knowing q, we'll need different cases. The integrand is x^{-5q}, so when $-5q \neq -1$, we use the power rule (that is, $\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1}$) to antidifferentiate. Note $x^{(-5q)+1} = x^{1-5q} = \dfrac{1}{x^{5q-1}}$.

$$\begin{aligned}\int_1^t \frac{1}{x^{5q}}\,\mathrm{d}x &= \begin{cases} \left[\frac{x^{1-5q}}{1-5q}\right]_1^t \text{ with } 1-5q>0 & \text{if } q<\frac{1}{5} \\ \left[\log x\right]_1^t & \text{if } q=\frac{1}{5} \\ \left[\frac{1}{(1-5q)x^{5q-1}}\right]_1^t \text{ with } 5q-1>0 & \text{if } q>\frac{1}{5} \end{cases} \\ &= \begin{cases} \frac{1}{1-5q}(t^{1-5q}-1) \text{ with } 1-5q>0 & \text{if } q<\frac{1}{5} \\ \log t & \text{if } q=\frac{1}{5} \\ \frac{1}{5q-1}(1-\frac{1}{t^{5q-1}}) \text{ with } 5q-1>0 & \text{if } q>\frac{1}{5}. \end{cases}\end{aligned}$$

Therefore,

$$\begin{aligned}\int_1^\infty \frac{1}{x^{5q}}\,\mathrm{d}x &= \lim_{t\to\infty}\left(\int_1^t \frac{1}{x^{5q}}\,\mathrm{d}x\right) \\ &= \begin{cases} \frac{1}{1-5q}\left(\lim\limits_{t\to\infty} t^{1-5q}-1\right) = \infty & \text{if } q<\frac{1}{5} \\ \lim\limits_{t\to\infty}\log t = \infty & \text{if } q=\frac{1}{5} \\ \frac{1}{5q-1}\left(1-\lim\limits_{t\to\infty}\dfrac{1}{t^{5q-1}}\right) = \frac{1}{5q-1} & \text{if } q>\frac{1}{5}. \end{cases}\end{aligned}$$

The first two cases are divergent, and so the largest such value is $q = \frac{1}{5}$. (Alternatively, we might recognize this as a "p-integral" with $p = 5q$, and recall that the p-integral diverges precisely when $p \leq 1$.)

1.12.4.19. Solution. This integrand is a nice candidate for the substitution $u = x^2 + 1$, $\frac{1}{2}\mathrm{d}u = x\mathrm{d}x$. Remember when we use substitution on a definite integral, we also need to adjust the limits of integration.

$$\begin{aligned}
\int_0^\infty \frac{x}{(x^2+1)^p}\mathrm{d}x &= \lim_{t\to\infty}\int_0^t \frac{x}{(x^2+1)^p}\mathrm{d}x \\
&= \lim_{t\to\infty}\frac{1}{2}\int_1^{t^2+1}\frac{1}{u^p}\mathrm{d}u \\
&= \lim_{t\to\infty}\frac{1}{2}\int_1^{t^2+1} u^{-p}\mathrm{d}u \\
&= \begin{cases} \frac{1}{2}\lim\limits_{t\to\infty}\left[\dfrac{u^{1-p}}{1-p}\right]_1^{t^2+1} & \text{if } p \neq 1 \\ \frac{1}{2}\lim\limits_{t\to\infty}\Big[\log|u|\Big]_1^{t^2+1} & \text{if } p = 1 \end{cases} \\
&= \begin{cases} \frac{1}{2}\lim\limits_{t\to\infty}\dfrac{1}{1-p}\left[(t^2+1)^{1-p}-1\right] & \text{if } p \neq 1 \\ \frac{1}{2}\lim\limits_{t\to\infty}\Big[\log(t^2+1)\Big] = \infty & \text{if } p = 1 \end{cases}
\end{aligned}$$

At this point, we can see that the integral diverges when $p = 1$. When $p \neq 1$, we have the limit

$$\lim_{t\to\infty}\frac{1/2}{1-p}\left[(t^2+1)^{1-p}-1\right] = \frac{1/2}{1-p}\left[\lim_{t\to\infty}(t^2+1)^{1-p}\right] - \frac{1/2}{1-p}$$

Since $t^2 + 1 \to \infty$, this limit converges exactly when the exponent $1 - p$ is negative; that is, it converges when $p > 1$, and diverges when $p < 1$.
So, the integral in the question converges when $p > 1$.

1.12.4.20. Solution.

- First, we notice there is only one "source of impropriety:" the domain of integration is infinite. (The integrand has a singularity at $t = 1$, but this is not in the domain of integration, so it's not a problem for us.)

- We should try to get some intuition about whether the integral converges or diverges. When $t \to \infty$, notice the integrand "looks like" the function $\frac{1}{t^4}$. We know $\int_1^\infty \frac{1}{t^4}\mathrm{d}t$ converges, because it's a p-integral with $p = 4 > 1$ (see Example 1.12.8). So, our integral probably converges as well. If we were only asked show it converges, we could use a comparison test, but we're asked more than that.

- Since we guess the integral converges, we'll need to evaluate it. The integrand is a rational function, and there's no obvious substitution, so we use partial fractions.

$$\frac{1}{t^4-1} = \frac{1}{(t^2+1)(t^2-1)} = \frac{1}{(t^2+1)(t+1)(t-1)}$$

$$= \frac{At+B}{t^2+1} + \frac{C}{t+1} + \frac{D}{t-1}$$

Multiply by the original denominator.

$$1 = (At+B)(t+1)(t-1) + C(t^2+1)(t-1) + D(t^2+1)(t+1) \qquad (*)$$

Set $t = 1$.

$$1 = 0 + 0 + D(2)(2) \qquad \Rightarrow \qquad D = \frac{1}{4}$$

Set $t = -1$.

$$1 = 0 + C(2)(-2) + 0 \qquad \Rightarrow \qquad C = -\frac{1}{4}$$

Simplify $(*)$ using $D = \frac{1}{4}$ and $C = -\frac{1}{4}$.

$$\begin{aligned}
1 &= (At+B)(t+1)(t-1) - \frac{1}{4}(t^2+1)(t-1) + \frac{1}{4}(t^2+1)(t+1) \\
&= (At+B)(t+1)(t-1) + \frac{1}{2}(t^2+1) \\
&= At^3 + \left(B + \frac{1}{2}\right)t^2 - At + \left(\frac{1}{2} - B\right)
\end{aligned}$$

By matching up coefficients of corresponding powers of t, we find $A = 0$ and $B = -\frac{1}{2}$.

$$\begin{aligned}
\int_2^\infty \frac{1}{t^4-1}\mathrm{d}t &= \int_2^\infty \left(\frac{-1/2}{t^2+1} - \frac{1/4}{t+1} + \frac{1/4}{t-1}\right)\mathrm{d}t \\
&= \lim_{R\to\infty} \int_2^R \left(\frac{-1/2}{t^2+1} - \frac{1/4}{t+1} + \frac{1/4}{t-1}\right)\mathrm{d}t \\
&= \lim_{R\to\infty} \left[-\frac{1}{2}\arctan t - \frac{1}{4}\log|t+1| + \frac{1}{4}\log|t-1|\right]_2^R \\
&= \lim_{R\to\infty} \left[-\frac{1}{2}\arctan t + \frac{1}{4}\log\left|\frac{t-1}{t+1}\right|\right]_2^R \\
&= \lim_{R\to\infty} \Bigg(-\frac{1}{2}\arctan R + \frac{1}{2}\arctan 2 + \frac{1}{4}\log\left|\frac{R-1}{R+1}\right| \\
&\qquad - \frac{1}{4}\log\left|\frac{2-1}{2+1}\right|\Bigg)
\end{aligned}$$

We can use l'Hôpital's rule to see $\lim\limits_{R\to\infty}\dfrac{R-1}{R+1} = 1$. Also note $-\log(1/3) = \log 3$.

$$\begin{aligned}
&= -\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{2}\arctan 2 + \frac{1}{4}\log 1 + \frac{1}{4}\log 3 \\
&= \frac{\log 3 - \pi}{4} + \frac{1}{2}\arctan 2
\end{aligned}$$

1.12.4.21. Solution. There are three singularities in the integrand: $x = 0$, $x = 1$, and $x = 2$. We'll need to break up the integral at each of these places.

$$\begin{aligned}
&\int_{-5}^{5}\left(\frac{1}{\sqrt{|x|}}+\frac{1}{\sqrt{|x-1|}}+\frac{1}{\sqrt{|x-2|}}\right)\mathrm{d}x\\
&=\int_{-5}^{0}\left(\frac{1}{\sqrt{|x|}}+\frac{1}{\sqrt{|x-1|}}+\frac{1}{\sqrt{|x-2|}}\right)\mathrm{d}x\\
&\qquad+\int_{0}^{1}\left(\frac{1}{\sqrt{|x|}}+\frac{1}{\sqrt{|x-1|}}+\frac{1}{\sqrt{|x-2|}}\right)\mathrm{d}x\\
&+\int_{1}^{2}\left(\frac{1}{\sqrt{|x|}}+\frac{1}{\sqrt{|x-1|}}+\frac{1}{\sqrt{|x-2|}}\right)\mathrm{d}x\\
&\qquad+\int_{2}^{5}\left(\frac{1}{\sqrt{|x|}}+\frac{1}{\sqrt{|x-1|}}+\frac{1}{\sqrt{|x-2|}}\right)\mathrm{d}x
\end{aligned}$$

This looks rather unfortunate. Let's think again. If all of the integrals below converge, then we can write:

$$\begin{aligned}
&\int_{-5}^{5}\left(\frac{1}{\sqrt{|x|}}+\frac{1}{\sqrt{|x-1|}}+\frac{1}{\sqrt{|x-2|}}\right)\mathrm{d}x\\
&\qquad=\int_{-5}^{5}\frac{1}{\sqrt{|x|}}\mathrm{d}x+\int_{-5}^{5}\frac{1}{\sqrt{|x-1|}}\mathrm{d}x+\int_{-5}^{5}\frac{1}{\sqrt{|x-2|}}\mathrm{d}x
\end{aligned}$$

That looks a lot better. Also, we have a good reason to guess these integrals converge–they look like p-integrals with $p = \frac{1}{2}$. Let's take a closer look at each one.

$$\begin{aligned}
\int_{-5}^{5}\frac{1}{\sqrt{|x|}}\mathrm{d}x &= \int_{-5}^{0}\frac{1}{\sqrt{|x|}}\mathrm{d}x+\int_{0}^{5}\frac{1}{\sqrt{|x|}}\mathrm{d}x\\
&=2\int_{0}^{5}\frac{1}{\sqrt{|x|}}\mathrm{d}x \qquad \text{(even function)}\\
&=2\int_{0}^{5}\frac{1}{\sqrt{x}}\mathrm{d}x
\end{aligned}$$

This is a p-integral, with $p = \frac{1}{2}$. By Example 1.12.9 (and Theorem 1.12.20, since the upper limit of integration is not 1), it converges. The other two pieces behave similarly.

$$\int_{-5}^{5}\frac{1}{\sqrt{|x-1|}}\mathrm{d}x=\int_{-5}^{1}\frac{1}{\sqrt{|x-1|}}\mathrm{d}x+\int_{1}^{5}\frac{1}{\sqrt{|x-1|}}\mathrm{d}x$$

Use $u = x - 1$, $\mathrm{d}u = \mathrm{d}x$

$$=\int_{-6}^{0}\frac{1}{\sqrt{|u|}}\mathrm{d}u+\int_{0}^{4}\frac{1}{\sqrt{|u|}}\mathrm{d}x$$

$$= \int_0^6 \frac{1}{\sqrt{u}} \mathrm{d}u + \int_0^4 \frac{1}{\sqrt{u}} \mathrm{d}x$$

Since our function is even, we use the reasoning of Example 1.2.10 in the text to consider the area under the curve when $x \geq 0$, rather than when $x \leq 0$. Again, these are p-integrals with $p = \frac{1}{2}$, so they both converge. Finally:

$$\int_{-5}^5 \frac{1}{\sqrt{|x-2|}} \mathrm{d}x = \int_{-5}^2 \frac{1}{\sqrt{|x-2|}} \mathrm{d}x + \int_2^5 \frac{1}{\sqrt{|x-2|}} \mathrm{d}x$$

Use $u = x - 2$, $\mathrm{d}u = \mathrm{d}x$.

$$\begin{aligned}
&= \int_{-7}^0 \frac{1}{\sqrt{|u|}} \mathrm{d}u + \int_0^3 \frac{1}{\sqrt{|u|}} \mathrm{d}u \\
&= \int_0^7 \frac{1}{\sqrt{u}} \mathrm{d}u + \int_0^3 \frac{1}{\sqrt{u}} \mathrm{d}u
\end{aligned}$$

Since $p = \frac{1}{2}$, so they both converge. We conclude our original integral, as the sum of convergent integrals, converges.

1.12.4.22. Solution. We can use integration by parts twice to find the antiderivative of $e^{-x} \sin x$, as in Example 1.7.10. To keep our work a little simpler, we'll find the antiderivative first, then take the limit.
Let $u = e^{-x}$, $\mathrm{d}v = \sin x \mathrm{d}x$, so $\mathrm{d}u = -e^{-x}\mathrm{d}x$ and $v = -\cos x$.

$$\int e^{-x} \sin x \mathrm{d}x = -e^{-x} \cos x - \int e^{-x} \cos x \mathrm{d}x$$

Now let $u = e^{-x}$, $\mathrm{d}v = \cos x \mathrm{d}x$, so $\mathrm{d}u = -e^{-x}\mathrm{d}x$ and $v = \sin x$.

$$\begin{aligned}
&= -e^{-x} \cos x - \left[e^{-x} \sin x + \int e^{-x} \sin x \mathrm{d}x\right] \\
&= -e^{-x} \cos x - e^{-x} \sin x - \int e^{-x} \sin x \mathrm{d}x
\end{aligned}$$

All together, we found

$$\begin{aligned}
\int e^{-x} \sin x \mathrm{d}x &= -e^{-x} \cos x - e^{-x} \sin x - \int e^{-x} \sin x \mathrm{d}x + C \\
2\int e^{-x} \sin x \mathrm{d}x &= -e^{-x} \cos x - e^{-x} \sin x + C \\
\int e^{-x} \sin x \mathrm{d}x &= -\frac{1}{2e^x}(\cos x + \sin x) + C
\end{aligned}$$

(Remember, since C is an arbitrary constant, we can rename $\frac{C}{2}$ to simply C.) Now we can evaluate our improper integral.

$$\int_0^\infty e^{-x} \sin x \mathrm{d}x = \lim_{b \to \infty} \int_0^b e^{-x} \sin x \mathrm{d}x$$

$$\begin{aligned}
&= \lim_{b\to\infty}\left[-\frac{1}{2e^x}(\cos x+\sin x)\right]_0^b \\
&= \lim_{b\to\infty}\left(\frac{1}{2}-\frac{1}{2e^b}(\cos b+\sin b)\right)
\end{aligned}$$

To find the limit, we use the Squeeze Theorem (see the CLP-1 text). Since $|\sin b|, |\cos b| \le 1$ for any b, we can use the fact that $-2 \le \cos b + \sin b \le 2$ for any b.

$$\begin{aligned}
&\frac{-2}{2e^b} \le \frac{1}{2e^b}(\cos b+\sin b) \le \frac{2}{2e^b} \\
&\lim_{b\to\infty}\frac{-2}{2e^b} = 0 = \frac{2}{2e^b} \\
&\text{So,} \qquad \lim_{b\to\infty}\left[\frac{1}{2e^b}(\cos b+\sin b)\right] = 0 \\
\text{Therefore,} \qquad &\frac{1}{2} = \lim_{b\to\infty}\left(\frac{1}{2}-\frac{1}{2e^b}(\cos b+\sin b)\right)
\end{aligned}$$

That is, $\displaystyle\int_0^\infty e^{-x}\sin x\mathrm{d}x = \frac{1}{2}$.

1.12.4.23. $*$. **Solution**. The integrand is positive everywhere. So either the integral converges to some finite number or it is infinite. There are two potential "sources of impropriety" — a possible singularity at $x = 0$ and the fact that the domain of integration extends to ∞. So, we split up the integral.

$$\int_0^\infty \frac{\sin^4 x}{x^2}\,\mathrm{d}x = \int_0^1 \frac{\sin^4 x}{x^2}\,\mathrm{d}x + \int_1^\infty \frac{\sin^4 x}{x^2}\,\mathrm{d}x$$

Let's consider the first integral. By l'Hôpital's rule (see the CLP-1 text),

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\cos x}{1} = \cos 0 = 1$$

Consequently,

$$\lim_{x\to 0}\frac{\sin^4 x}{x^2} = \Big(\lim_{x\to 0}\sin^2 x\Big)\Big(\lim_{x\to 0}\frac{\sin x}{x}\Big)\Big(\lim_{x\to 0}\frac{\sin x}{x}\Big) = 0\times 1\times 1 = 0$$

and the first integral is not even improper.
Now for the second integral. Since $|\sin x| \le 1$, we'll compare it to $\int_1^\infty \frac{1}{x^2}$.

- $\frac{\sin^4 x}{x^2}$ and $\frac{1}{x^2}$ are defined and continuous for every $x \ge 1$
- $0 \le \frac{\sin^4 x}{x^2} \le \frac{1^4}{x^2} = \frac{1}{x^2}$ for every $x \ge 1$
- $\int_1^\infty \frac{1}{x^2}\mathrm{d}x$ converges by Example 1.12.8 (it's a p-type integral with $p > 1$)

By the comparison test, Theorem 1.12.17, $\displaystyle\int_1^\infty \frac{\sin^4 x}{x^2}\,\mathrm{d}x$ converges.

Since $\displaystyle\int_0^1 \frac{\sin^4 x}{x^2}\,\mathrm{d}x$ and $\displaystyle\int_1^\infty \frac{\sin^4 x}{x^2}\,\mathrm{d}x$ both converge, we conclude $\displaystyle\int_0^\infty \frac{\sin^4 x}{x^2}\,\mathrm{d}x$ converges as well.

1.12.4.24. Solution. Since the denominator is positive for all $x \geq 0$, the integrand is continuous over $[0, \infty)$. So, the only "source of impropriety" is the infinite domain of integration.

- Solution 1: Let's try to use a direct comparison. Note $\dfrac{x}{e^x + \sqrt{x}} \geq 0$ whenever $x \geq 0$. Also note that, for large values of x, e^x is much larger than $\sqrt{x}$. That leads us to consider the following inequalty:

$$0 \leq \frac{x}{e^x + \sqrt{x}} \leq \frac{x}{e^x}$$

If $\int_0^\infty \frac{x}{e^x} \mathrm{d}x$ converges, we're in business. Let's figure it out. The integrand looks like a candidate for integration by parts: take $u = x$, $\mathrm{d}v = e^{-x}\mathrm{d}x$, so $\mathrm{d}u = \mathrm{d}x$ and $v = -e^{-x}$.

$$\begin{aligned}
\int_0^\infty \frac{x}{e^x}\mathrm{d}x &= \lim_{b\to\infty}\int_0^b \frac{x}{e^x}\mathrm{d}x = \lim_{b\to\infty}\left(\left[-\frac{x}{e^x}\right]_0^b + \int_0^b e^{-x}\mathrm{d}x\right) \\
&= \lim_{b\to\infty}\left(-\frac{b}{e^b} + \left[-e^{-x}\right]_0^b\right) = \lim_{b\to\infty}\left(-\frac{b}{e^b} - \frac{1}{e^b} + 1\right) \\
&= \lim_{b\to\infty}\left(1 - \underbrace{\frac{b+1}{e^b}}_{\substack{\text{num}\to\infty \\ \text{den}\to\infty}}\right) = \lim_{b\to\infty}\left(1 - \frac{1}{e^b}\right) = 1
\end{aligned}$$

Using l'Hôpital's rule, we see $\int_0^\infty \frac{x}{e^x}\mathrm{d}x$ converges. All together:

  - $\frac{x}{e^x}$ and $\frac{x}{e^x+\sqrt{x}}$ are defined and continuous for all $x \geq 0$,
  - $\left|\frac{x}{e^x+\sqrt{x}}\right| \leq \frac{x}{e^x}$, and
  - $\int_0^\infty \frac{x}{e^x}\mathrm{d}x$ converges.

So, by Theorem 1.12.17, our integral $\displaystyle\int_0^\infty \frac{x}{e^x + \sqrt{x}}\mathrm{d}x$ converges.

- Solution 2: Let's try to use a different direct comparison from Solution 1, and avoid integration by parts. We'd like to compare to something like $\dfrac{1}{e^x}$, but the inequality goes the wrong way. So, we make a slight modification: we consider $2e^{-x/2}$. To that end, we claim $x < 2e^{x/2}$ for all $x \geq 0$. We can prove this by noting the following two facts:

  - $0 < 2 = 2e^{0/2}$, and
  - $\frac{\mathrm{d}}{\mathrm{d}x}\{x\} = 1 \leq e^{x/2} = \frac{\mathrm{d}}{\mathrm{d}x}\{2e^{x/2}\}$.

So, when $x = 0$, $x < 2e^{x/2}$, and then as x increases, $2e^{x/2}$ grows faster than x. Now we can make the following comparison:

$$0 \leq \frac{x}{e^x + \sqrt{x}} \leq \frac{x}{e^x} < \frac{2e^{x/2}}{e^x} = \frac{2}{e^{x/2}}$$

We have a hunch that $\int_0^\infty \frac{2}{e^{x/2}}\mathrm{d}x$ converges, just like $\int_0^\infty \frac{1}{e^x}\mathrm{d}x$. This is easy enough to prove. We can guess an antiderivative, or use the substitution $u = x/2$.

$$\begin{aligned}\int_0^\infty \frac{2}{e^{x/2}}\mathrm{d}x &= \lim_{R\to\infty}\int_0^R \frac{2}{e^{x/2}}\mathrm{d}x = \lim_{R\to\infty}\left[-\frac{4}{e^{x/2}}\right]_0^R \\ &= \lim_{R\to\infty}\left[\frac{4}{e^0} - \frac{4}{e^{R/2}}\right]_0^R = 4\end{aligned}$$

Now we know:

- $0 \le \frac{x}{e^x+\sqrt{x}} \le \frac{2}{e^{x/2}}$, and
- $\int_0^\infty \frac{2}{e^{x/2}}\mathrm{d}x$ converges.
- Furthermore, $\frac{x}{e^x+\sqrt{x}}$ and $\frac{2}{e^{x/2}}$ are defined and continuous for all $x \ge 0$.

By the comparison test (Theorem 1.12.17), we conclude the integral converges.

- Solution 3: Let's use the limiting comparison test (Theorem 1.12.22). We have a hunch that our integral behaves similarly to $\int_0^\infty \frac{1}{e^x}\,\mathrm{d}x$, which converges (see Example 1.12.18). Unfortunately, if we choose $g(x) = \frac{1}{e^x}$ (and, of course, $f(x) = \frac{x}{e^x+\sqrt{x}}$), then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{x}{e^x+\sqrt{x}}\cdot e^x = \lim_{x\to\infty}\frac{x}{1+\underbrace{\frac{\sqrt{x}}{e^x}}_{\to 0}} = \infty$$

That is, the limit does not exist, so the limiting comparison test does not apply. (To find $\lim\limits_{x\to\infty}\frac{\sqrt{x}}{e^x}$, you can use l'Hôpital's rule.)

This setback encourages us to try a slightly different angle. If $g(x)$ gave larger values, then we could decrease $\frac{f(x)}{g(x)}$. So, let's try $g(x) = \frac{1}{e^{x/2}} = e^{-x/2}$. Now,

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{x}{e^x+\sqrt{x}} \div \frac{1}{e^{x/2}} = \lim_{x\to\infty}\frac{x}{e^{x/2}+\frac{\sqrt{x}}{e^{x/2}}}$$

Hmm... this looks hard. Instead of dealing with it directly, let's use the squeeze theorem (see CLP-1 notes).

$$0 \le \frac{x}{e^{x/2}+\frac{\sqrt{x}}{e^{x/2}}} \le \frac{x}{e^{x/2}}$$

Using l'Hôpital's rule,

$$\lim_{x\to\infty}\underbrace{\frac{x}{e^{x/2}}}_{\substack{\text{num}\to\infty\\ \text{den}\to\infty}} = \lim_{x\to\infty}\frac{1}{\frac{1}{2}e^{x/2}} = 0 = \lim_{x\to\infty} 0$$

So, by the squeeze theorem $\lim_{x\to 0} \frac{\frac{x}{e^x+\sqrt{x}}}{\frac{1}{e^{x/2}}} = 0$. Since this limit exists, $\frac{1}{e^{x/2}}$ is a reasonable function to use in the limiting comparison test (provided its integral converges). So, we need to show that $\int_0^\infty \frac{1}{e^{x/2}}\,\mathrm{d}x$ converges. This can be done by simply evaluating it:

$$\begin{aligned}\int_0^\infty \frac{1}{e^{x/2}}\,\mathrm{d}x &= \lim_{b\to\infty}\int_0^b e^{-x/2}\,\mathrm{d}x = \lim_{b\to\infty} -\frac{1}{2}\left[e^{-x/2}\right]_0^b \\ &= \lim_{b\to\infty} -\frac{1}{2}\left[\frac{1}{e^{b/2}}-1\right] = \frac{1}{2}\end{aligned}$$

So, all together:

- The functions $\frac{x}{e^x+\sqrt{x}}$ and $\frac{1}{e^{x/2}}$ are defined and continuous for all $x \geq 0$, and $\frac{1}{e^{x/2}} \geq 0$ for all $x \geq 0$.
- $\int_0^\infty \frac{1}{e^{x/2}}\,\mathrm{d}x$ converges.
- The limit $\lim_{x\to\infty} \frac{\frac{x}{e^x}+\sqrt{x}}{\frac{1}{e^{x/2}}}$ exists (it's equal to 0).
- So, the limiting comparison test (Theorem 1.12.17) tells us that $\int_0^\infty \frac{x}{e^x+\sqrt{x}}\,\mathrm{d}x$ converges as well.

1.12.4.25. $*$. Solution. There are two sources of error: the upper bound is t, rather than infinity, and we're using an approximation with some finite number of intervals, n. Our plan is to first find a value of t that introduces an error of no more than $\frac{1}{2}10^{-4}$. That is, we'll find a value of t such that $\int_t^\infty \frac{e^{-x}}{1+x}\mathrm{d}x \leq \frac{1}{2}10^{-4}$. After that, we'll find a value of n that approximates $\int_0^t \frac{e^{-x}}{x+1}\mathrm{d}x$ to within $\frac{1}{2}10^{-4}$. Then, all together, our error will be at most $\frac{1}{2}10^{-4} + \frac{1}{2}10^{-4} = 10^{-4}$, as desired. (Note we could have broken up the error in another way—it didn't have to be $\frac{1}{2}10^{-4}$ and $\frac{1}{2}10^{-4}$. This will give us one of many possible answers.)
Let's find a t such that $\int_t^\infty \frac{e^{-x}}{1+x}\mathrm{d}x \leq \frac{1}{2}10^{-4}$. For all $x \geq 0$, $0 < \frac{e^{-x}}{1+x} \leq e^{-x}$, so

$$\int_t^\infty \frac{e^{-x}}{1+x}\mathrm{d}x \leq \int_t^\infty e^{-x}\mathrm{d}x = e^{-t} \stackrel{(*)}{\leq} \frac{1}{2}10^{-4}$$
$$\text{where } (*) \text{ is true if} \quad t \geq -\log\left(\frac{1}{2}10^{-4}\right) \approx 9.90$$

Choose, for example, $t = 10$.
Now it's time to decide how many intervals we're going to use to approximate $\displaystyle\int_0^t \frac{e^{-x}}{x+1}\mathrm{d}x$. Again, we want our error to be less than $\frac{1}{2}10^{-4}$. To bound our error, we need to know the second derivative of $\frac{e^{-x}}{x+1}$.

$$f(x) = \frac{e^{-x}}{1+x} \implies f'(x) = -\frac{e^{-x}}{1+x} - \frac{e^{-x}}{(1+x)^2}$$

$$\implies f''(x) = \frac{e^{-x}}{1+x} + 2\frac{e^{-x}}{(1+x)^2} + 2\frac{e^{-x}}{(1+x)^3}$$

Since $f''(x)$ is positive, and decreases as x increases,

$$|f''(x)| \le f''(0) = 5 \implies |E_n| \le \frac{5(10-0)^3}{24n^2} = \frac{5000}{24n^2} = \frac{625}{3n^2}$$

and $|E_n| \le \frac{1}{2}10^{-4}$ if

$$\begin{aligned} && \frac{625}{3n^2} &\le \frac{1}{2}10^{-4} \\ \iff && n^2 &\ge \frac{1250 \times 10^4}{3} \\ \iff && n &\ge \sqrt{\frac{1.25 \times 10^7}{3}} \approx 2041.2 \end{aligned}$$

So $t = 10$ and $n = 2042$ will do the job. There are many other correct answers.

1.12.4.26. Solution.

a Since $f(x)$ is odd, using the reasoning of Example 1.2.11,

$$\begin{aligned} \int_{-\infty}^{-1} f(x)\mathrm{d}x &= \lim_{t\to\infty} \int_{-t}^{-1} f(x)\mathrm{d}x = \lim_{t\to\infty} -\int_1^t f(x)\mathrm{d}x \\ &= -\lim_{t\to\infty} \int_1^t f(x)\mathrm{d}x \end{aligned}$$

Since $\displaystyle\int_1^\infty f(x)\mathrm{d}x$ converges, the last limit above converges. Therefore, $\displaystyle\int_{-\infty}^{-1} f(x)\mathrm{d}x$ converges.

b Since $f(x)$ is even, using the reasoning of Example 1.2.10,

$$\begin{aligned} \int_{-\infty}^{-1} f(x)\mathrm{d}x &= \lim_{t\to\infty} \int_{-t}^{-1} f(x)\mathrm{d}x = \lim_{t\to\infty} \int_1^t f(x)\mathrm{d}x \\ &= \lim_{t\to\infty} \int_1^t f(x)\mathrm{d}x \end{aligned}$$

Since $\displaystyle\int_1^\infty f(x)\mathrm{d}x$ converges, the last limit above converges. Since $f(x)$ is continuous everywhere, by Theorem 1.12.20, $\displaystyle\int_{-1}^\infty f(x)\mathrm{d}x$ converges (note the adjusted lower limit). Then, since

$$\int_{-\infty}^\infty f(x)\mathrm{d}x = \int_{-\infty}^{-1} f(x)\mathrm{d}x + \int_{-1}^\infty f(x)\mathrm{d}x$$

and both terms converge, our original integral converges as well.

1.12.4.27. Solution. Define $F(x) = \int_0^x \frac{1}{e^t} \mathrm{d}t$.

$$F(x) = \int_0^x \frac{1}{e^t} \mathrm{d}t = \left[-\frac{1}{e^t}\right]_0^x = \frac{1}{e^0} - \frac{1}{e^x} < \frac{1}{e^0} = 1$$

So, the statement is false: there is no x such that $F(x) = 1$. For every real x, $F(x) < \frac{1}{e^0} = 1$.

We note here that $\lim\limits_{x\to\infty} \displaystyle\int_0^x \frac{1}{e^t}\mathrm{d}t = 1$. So, as x grows larger, the gap between $F(x)$ and 1 grows infintesimally small. But there is no real value of x where $F(x)$ is exactly equal to 1.

1.13 · More Integration Examples · Exercises

Exercises — Stage 1

1.13.1. Solution. (A) Note $\int f'(x)f(x)\,\mathrm{d}x = \int u\,\mathrm{d}u$ if we substitute $u = f(x)$. This is the kind of integrand described in (I). It's quite possible that a $u = f(x)$ substitution would work on the others, as well, but (I) is the most reliable kind of integrand for a $u = f(x)$ substitution.

(B) A trigonometric substitution usually allows us to cancel out a square root containing a quadratic function, as in (IV).

(C) We can often antidifferentiate the product of a polynomial with an exponential function using integration by parts: see Examples 1.7.1, 1.7.6. If we let u be the polynomial function and $\mathrm{d}v$ be the exponential, as long as we can antidifferentiate $\mathrm{d}v$, we can repeatedly apply integration by parts until the polynomial function goes away. So, we go with (II)

(D) We apply partial fractions to rational functions, (III).

Note: without knowing more about the functions, there's no guarantee that the methods we chose will be the best methods, or even that they will work (with the exception of (I)). With practice, you gain intuition about likely methods for different integrals. Luckily for you, there's lots of practice below.

Exercises — Stage 2

1.13.2. Solution. The integrand is a product of powers of sine and cosine. Since cosine has an odd power, we want to substitute $u = \sin x$, $\mathrm{d}u = \cos x\mathrm{d}x$. Therefore, we should:

- reserve one cosine for the derivative of sine in our substitution, and
- change the rest of the cosines to sines using the identity $\sin^2 x + \cos^2 x = 1$.

$$\begin{aligned}\int_0^{\pi/2} \sin^4 x \cos^5 x\mathrm{d}x &= \int_0^{\pi/2} \sin^4 x(\cos^2 x)^2 \cos x\mathrm{d}x \\ &= \int_0^{\pi/2} \sin^4 x(1-\sin^2 x)^2 \underbrace{\cos x\mathrm{d}x}_{\mathrm{d}u}\end{aligned}$$

$$\begin{aligned}
&= \int_{\sin(0)}^{\sin(\pi/2)} u^4(1-u^2)^2 \mathrm{d}u \\
&= \int_0^1 u^4(1-2u^2+u^4)\mathrm{d}u \\
&= \int_0^1 (u^4-2u^6+u^8)\mathrm{d}u \\
&= \left[\frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9\right]_{u=0}^{u=1} \\
&= \left(\frac{1}{5}-\frac{2}{7}+\frac{1}{9}\right) - 0 \\
&= \frac{8}{315}
\end{aligned}$$

1.13.3. Solution. We notice that there is a quadratic equation under the square root. If that equation were a perfect square, we could get rid of the square root: so we'll mould it into a perfect square using a trig substitution.
Our candidates will use one of the following identities:

$$1-\sin^2\theta = \cos^2\theta \qquad \tan^2\theta + 1 = \sec^2\theta \qquad \sec^2\theta - 1 = \tan^2\theta$$

We'll be substituting $x =$(something), so we notice that $3-5x^2$ has the general form of (constant)−(function), as does $1-\sin^2\theta$. In order to get the constant right, we multiply through by three:

$$3 - 3\sin^2\theta = 3\cos^2\theta$$

Our goal is to get $3-5x^2 = 3-3\sin^2\theta$; so we solve this equation for x and decide on the substitution

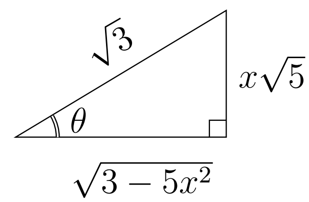
$$x = \sqrt{\frac{3}{5}}\sin\theta, \qquad \mathrm{d}x = \sqrt{\frac{3}{5}}\cos\theta\mathrm{d}\theta$$

Now we evaluate our integral.

$$\begin{aligned}
\int \sqrt{3-5x^2}\mathrm{d}x &= \int \sqrt{3-5\left(\sqrt{\frac{3}{5}}\sin\theta\right)^2}\sqrt{\frac{3}{5}}\cos\theta d\theta \\
&= \int \sqrt{3-3\sin^2\theta}\sqrt{3/5}\cos\theta\mathrm{d}\theta \\
&= \int \sqrt{3\cos^2\theta}\sqrt{3/5}\cos\theta\mathrm{d}\theta \\
&= \int \sqrt{3}\cos\theta\sqrt{3/5}\cos\theta\mathrm{d}\theta \\
&= \frac{3}{\sqrt{5}}\int \cos^2\theta\mathrm{d}\theta \\
&= \frac{3}{\sqrt{5}}\int \frac{1+\cos 2\theta}{2}\mathrm{d}\theta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{2\sqrt{5}} \int (1+\cos 2\theta)\mathrm{d}\theta \\
&= \frac{3}{2\sqrt{5}} \left[\theta + \frac{1}{2}\sin(2\theta)\right] + C \\
&= \frac{3}{2\sqrt{5}} \left[\theta + \sin\theta\cos\theta\right] + C
\end{aligned}
$$

From our substitution $x = \sqrt{3/5}\sin\theta$, we glean $\sin\theta = x\sqrt{5/3}$, and $\theta = \arcsin\left(x\sqrt{5/3}\right)$. To figure out $\cos\theta$, we draw a right triangle. Let θ be one angle, and since $\sin\theta = \dfrac{x\sqrt{5}}{\sqrt{3}}$, we let the hypotenuse be $\sqrt{3}$ and the side opposite θ be $x\sqrt{5}$. By Pythagoras, the missing side (adjacent to θ) has length $\sqrt{3-5x^2}$.



Therefore, $\cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{\sqrt{3-5x^2}}{\sqrt{3}}$. So our integral evaluates to:

$$
\begin{aligned}
&\frac{3}{2\sqrt{5}} \left[\theta + \sin\theta\cos\theta\right] + C \\
&\qquad = \frac{3}{2\sqrt{5}} \left[\arcsin(x\sqrt{5/3}) + x\sqrt{5/3}\cdot\frac{\sqrt{3-5x^2}}{\sqrt{3}}\right] + C \\
&\qquad = \frac{3}{2\sqrt{5}} \arcsin(x\sqrt{5/3}) + \frac{x}{2}\cdot\sqrt{3-5x^2} + C
\end{aligned}
$$

1.13.4. Solution. First, we note the integral is improper. So, we'll need to replace the top bound with a variable, and take a limit. Second, we're going to have to antidifferentiate. The integrand is the product of an exponential function, e^{-x}, with a polynomial function, $x-1$, so we use integration by parts with $u = x-1$, $\mathrm{d}v = e^{-x}\mathrm{d}u$, $\mathrm{d}u = \mathrm{d}x$, and $v = -e^{-x}$.

$$
\begin{aligned}
\int \frac{x-1}{e^x}\mathrm{d}x &= -(x-1)e^{-x} + \int e^{-x}\mathrm{d}x \\
&= -(x-1)e^{-x} - e^{-x} + C = -xe^{-x} + C
\end{aligned}
$$

$$
\begin{aligned}
\text{So,} \qquad \int_0^\infty \frac{x-1}{e^x}\mathrm{d}x &= \lim_{b\to\infty} \int_0^b \frac{x-1}{e^x}\mathrm{d}x \\
&= \lim_{b\to\infty}\left[-\frac{x}{e^x}\right]_0^b = \lim_{b\to\infty}\left[-\underbrace{\frac{b}{e^b}}_{\substack{\text{num}\to\infty\\ \text{den}\to\infty}}\right]
\end{aligned}
$$

$$\stackrel{(*)}{=} \lim_{b\to\infty} -\frac{1}{e^b} = 0$$

(In the equality marked $(*)$, we used l'Hôpital's rule.)
So, $\displaystyle\int_0^\infty \frac{x-1}{e^x}\mathrm{d}x = 0$.
Remark: this shows that, interestingly, $\displaystyle\int_0^\infty \frac{x}{e^x}\mathrm{d}x = \int_0^\infty \frac{1}{e^x}\mathrm{d}x$.

1.13.5. Solution.

- Solution 1: Notice the denominator factors as $(x+1)(3x+1)$. Since the integrand is a rational function (the quotient of two polynomials), we can use partial fraction decomposition.

$$\begin{aligned}\frac{-2}{3x^2+4x+1} &= \frac{-2}{(x+1)(3x+1)}\\ &= \frac{A}{x+1}+\frac{B}{3x+1}\\ &= \frac{A(3x+1)+B(x+1)}{(x+1)(3x+1)}\\ &= \frac{(3A+B)x+(A+B)}{(x+1)(3x+1)}\end{aligned}$$

So:

$$\begin{aligned}-2 &= (3A+B)x+(A+B)\\ 0 &= 3A+B \text{ and } -2 = A+B\\ B &= -3A \text{ and hence } -2 = A+(-3A)\\ A&= 1 \text{ so then } B = -3\end{aligned}$$

So now:

$$\begin{aligned}\frac{-2}{3x^2+4x+1} &= \frac{1}{x+1}-\frac{3}{3x+1}\\ \int \frac{-2}{3x^2+4x+1}\mathrm{d}x &= \int\left(\frac{1}{x+1}-\frac{3}{3x+1}\right)\mathrm{d}x\\ &= \log|x+1| - \log|3x+1| + C\\ &= \log\left|\frac{x+1}{3x+1}\right| + C\end{aligned}$$

- Solution 2: The previous solution is probably the nicest. However, for the foolhardy or the brave, this integral can also be evaluated using trigonometric substitution.

  We start by completing the square on the denominator.

$$3x^2+4x+1 = 3\left(x^2+\frac{4}{3}x+\frac{1}{3}\right)$$

$$
\begin{aligned}
&= 3\left(x^2 + 2 \cdot \frac{2}{3}x + \frac{4}{9} - \frac{4}{9} + \frac{1}{3}\right) \\
&= 3\left(\left(x + \frac{2}{3}\right)^2 - \frac{4}{9} + \frac{3}{9}\right) \\
&= 3\left(\left(x + \frac{2}{3}\right)^2 - \frac{1}{9}\right) \\
&= 3\left(x + \frac{2}{3}\right)^2 - \frac{1}{3}
\end{aligned}
$$

This has the form of a function minus a constant, which matches the trigonometric identity $\sec^2\theta - 1 = \tan^2\theta$. Multiplying through by $\frac{1}{3}$, we see we can use the identity $\frac{1}{3}\sec^2\theta - \frac{1}{3} = \frac{1}{3}\tan^2\theta$. So, to get the substitution right, we want to choose a substitution that makes the following true:

$$
\begin{aligned}
3\left(x + \frac{2}{3}\right)^2 - \frac{1}{3} &= \frac{1}{3}\sec^2\theta - \frac{1}{3} \\
3\left(x + \frac{2}{3}\right)^2 &= \frac{1}{3}\sec^2\theta \\
9\left(x + \frac{2}{3}\right)^2 &= \sec^2\theta \\
{\color{red}3x + 2} &= {\color{blue}\sec\theta}
\end{aligned}
$$

And, accordingly:

$$
{\color{red}3\mathrm{d}x} = {\color{blue}\sec\theta\tan\theta\mathrm{d}\theta}
$$

Now, let's simplify a little and use this substitution on our integral:

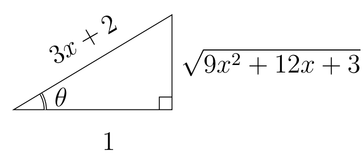
$$
\begin{aligned}
\int \frac{-2}{3x^2 + 4x + 1}\mathrm{d}x &= \int \frac{-2}{3\left(x + \frac{2}{3}\right)^2 - \frac{1}{3}}\mathrm{d}x \\
&= \int \frac{-2}{9\left(x + \frac{2}{3}\right)^2 - 1} 3\mathrm{d}x \\
&= \int \frac{-2}{\left({\color{red}3x + 2}\right)^2 - 1} {\color{red}3\mathrm{d}x} \\
&= \int \frac{-2}{\left({\color{blue}\sec\theta}\right)^2 - 1} {\color{blue}\sec\theta\tan\theta\mathrm{d}\theta} \\
&= \int \frac{-2}{\tan^2\theta}\sec\theta\tan\theta\mathrm{d}\theta \\
= \qquad & \int -2\frac{\sec\theta}{\tan\theta}\mathrm{d}\theta \\
&= \int -2\frac{1}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta}\mathrm{d}\theta \\
&= \int -2\frac{1}{\sin\theta}\mathrm{d}\theta
\end{aligned}
$$

$$= \int -2 \csc\theta \mathrm{d}\theta$$

Using the result of Example 1.8.21, or a table of integrals:

$$= 2\log|\csc\theta + \cot\theta| + C$$

Our final task is to translate this back from θ to x. Recall we used the substitution $3x+2 = \sec\theta$. Using this information, and $\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent}}$, we can fill in two sides of a right triangle with angle θ. The Pythagorean theorem tells us the third side (opposite to θ) has measure $\sqrt{(3x+2)^2-1} = \sqrt{9x^2-12x+3}$.



$$\begin{aligned}
&2\log|\csc\theta + \cot\theta| + C\\
&= 2\log\left|\frac{3x+2}{\sqrt{9x^2+12x+3}} + \frac{1}{\sqrt{9x^2+12x+3}}\right| + C\\
&= 2\log\left|\frac{3x+3}{\sqrt{9x^2+12x+3}}\right| + C\\
&= \log\left|\frac{(3x+3)^2}{\sqrt{9x^2+12x+3}^2}\right| + C\\
&= \log\left|\frac{(3x+3)^2}{9x^2+12x+3}\right| + C\\
&= \log\left|\frac{9(x+1)^2}{3(3x+1)(x+1)}\right| + C\\
&= \log\left|\frac{3(x+1)^2}{(3x+1)(x+1)}\right| + C\\
&= \log\left|\frac{3(x+1)}{3x+1}\right| + C\\
&= \log\left|\frac{x+1}{3x+1}\right| + \log 3 + C
\end{aligned}$$

Since C is an arbitrary constant, we can write our final answer as

$$\log\left|\frac{x+1}{3x+1}\right| + C$$

1.13.6. Solution. We see that we have two functions multiplied, but they don't simplify nicely with each other. However, if we differentiate logarithm, and integrate x^2, we'll get a polynomial. So, let's use integration by parts.

$$u = \log x \qquad \mathrm{d}v = x^2 \mathrm{d}x$$

$$\mathrm{d}u = (1/x)\mathrm{d}x \qquad v = x^3/3$$

First, let's antidifferentiate. We'll deal with the limits of integration later.

$$\begin{aligned} \int x^2 \log x \mathrm{d}x &= \underbrace{(\log x)}_{u}\underbrace{(x^3/3)}_{v}) - \int \underbrace{(x^3/3)}_{v}\underbrace{(1/x)\mathrm{d}x}_{\mathrm{d}u} \\ &= \frac{1}{3}x^3 \log x - \frac{1}{3}\int x^2 \mathrm{d}x \\ &= \frac{1}{3}x^3 \log x - \frac{1}{3}\cdot\frac{1}{3}x^3 + C \\ &= \frac{1}{3}x^3 \log x - \frac{1}{9}x^3 + C \end{aligned}$$

We use the Fundamental Theorem of Calculus Part 2 to evaluate the definite integral.

$$\begin{aligned} \int_1^2 x^3 \log x \mathrm{d}x &= \left[\frac{1}{3}x^3 \log x - \frac{1}{9}x^3\right]_1^2 \\ &= \left[\frac{1}{3}2^3 \log 2 - \frac{1}{9}2^3\right] - \left[\frac{1}{3}1^3 \log 1 - \frac{1}{9}1^3\right] \\ &= \frac{8\log 2}{3} - \frac{8}{9} - 0 + \frac{1}{9} \\ &= \frac{8}{3}\log 2 - \frac{7}{9} \end{aligned}$$

1.13.7. *. Solution. The derivative of the denominator shows up in the numerator, only differing by a constant, so we perform a substitution. Specifically, substitute $u = x^2 - 3$, $\mathrm{d}u = 2x\,\mathrm{d}x$. This gives

$$\int \frac{x}{x^2-3}\mathrm{d}x = \int \frac{\mathrm{d}u/2}{u} = \frac{1}{2}\log|u| + C = \frac{1}{2}\log\left|x^2-3\right| + C$$

1.13.8. *. Solution. (a) Although a quadratic under a square root often suggests trigonometric substitution, in this case we have an easier substitution. Specifically, let $y = 9 + x^2$. Then $\mathrm{d}y = 2x\mathrm{d}x$, $x\mathrm{d}x = \frac{\mathrm{d}y}{2}$, $y(0) = 9$, and $y(4) = 25$.

$$\int_0^4 \frac{x}{\sqrt{9+x^2}}\,\mathrm{d}x = \int_9^{25} \frac{1}{\sqrt{y}}\,\frac{\mathrm{d}y}{2} = \frac{1}{2}\cdot\frac{\sqrt{y}}{1/2}\bigg|_9^{25} = 5 - 3 = 2$$

(b) The power of cosine is odd, so we can reserve one cosine for the differential and change the rest to sines. Substituting $y = \sin x$, $\mathrm{d}y = \cos x$, $\mathrm{d}x$, $y(0) = 0$,

$y(\pi/2) = 1$, $\cos^2 x = 1 - y^2$:

$$\begin{aligned}
\int_0^{\pi/2} \cos^3 x\ \sin^2 x \,\mathrm{d}x &= \int_0^{\pi/2} \cos^2 x\ \sin^2 x\ \cos x \,\mathrm{d}x \\
&= \int_0^1 (1-y^2)y^2 \,\mathrm{d}y = \int_0^1 (y^2 - y^4)\,\mathrm{d}y \\
&= \left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 = \frac{1}{3} - \frac{1}{5} \\
&= \frac{2}{15}
\end{aligned}$$

(c) The integrand is the product of two different kinds of functions, with no obvious substitution or simplification. If we differentiate $\log x$, it will match better with the polynomial nature of the rest of the integrand. So, integrate by parts with $u(x) = \log x$ and $\mathrm{d}v = x^3\,\mathrm{d}x$, then $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$ and $v = x^4/4$.

$$\begin{aligned}
\int_1^e x^3 \log x \,\mathrm{d}x = \frac{x^4}{4}\log x\bigg|_1^e - \int_1^e \frac{x^4}{4}\cdot\frac{1}{x}\,\mathrm{d}x = \frac{e^4}{4} - \int_1^e \frac{x^3}{4}\,\mathrm{d}x \\
= \frac{e^4}{4} - \frac{x^4}{16}\bigg|_1^e = \frac{3e^4}{16} + \frac{1}{16}
\end{aligned}$$

1.13.9. $*$. Solution. (a) Integrate by parts with $u = x$ and $\mathrm{d}v = \sin x\,\mathrm{d}x$ so that $\mathrm{d}u = \mathrm{d}x$ and $v = -\cos x$.

$$\int x\sin x\,\mathrm{d}x = -x\cos x - \int (-\cos x)\,\mathrm{d}x = -x\cos x + \sin x + C$$

So

$$\int_0^{\pi/2} x\sin x\,\mathrm{d}x = \Big[-x\cos x + \sin x\Big]_0^{\pi/2} = 1$$

(b) The power of cosine is odd, so we can reserve one cosine for $\mathrm{d}u$ and change the rest into sines. Make the substitution $u = \sin x$, $\mathrm{d}u = \cos x\,\mathrm{d}x$.

$$\begin{aligned}
\int_0^{\pi/2} \cos^5 x\,\mathrm{d}x &= \int_0^{\pi/2} \left(1-\sin^2 x\right)^2 \cos x\,\mathrm{d}x \\
&= \int_0^1 \left(1-u^2\right)^2 \mathrm{d}u = \int_0^1 \left(1 - 2u^2 + u^4\right)\mathrm{d}u \\
&= \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^1 = 1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}
\end{aligned}$$

1.13.10. $*$. Solution. (a) This is a classic integration-by-parts example. If we integrate e^x, it doesn't change, and if we differentiate x it becomes a constant. So, let $u = x$ and $\mathrm{d}v = e^x\,\mathrm{d}x$, so that $\mathrm{d}u = \mathrm{d}x$ and $v = e^x$.

$$\int_0^2 xe^x\,\mathrm{d}x = \Big[xe^x\Big]_0^2 - \int_0^2 e^x\,\mathrm{d}x = 2e^2 - \Big[e^x\Big]_0^2 = e^2 + 1$$

(b) We have a quadratic function underneath a square root. In the absence of an easier substitution, we can get rid of the square root with a trigonometric substitution. Substitute $x = \tan y$, $\mathrm{d}x = \sec^2 y\,\mathrm{d}y$. When $x = 0$, $\tan y = 0$ so $y = 0$. When $x = 1$, $\tan y = 1$ so $y = \frac{\pi}{4}$. Also $\sqrt{1+x^2} = \sqrt{1+\tan^2 y} = \sqrt{\sec^2 y} = \sec y$, since $\sec y \geq 0$ for all $0 \leq y \leq \frac{\pi}{4}$.

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x &= \int_0^{\pi/4} \frac{\sec^2 y\,\mathrm{d}y}{\sec y} = \int_0^{\pi/4} \sec y\,\mathrm{d}y \\
&= \Big[\log|\sec y + \tan y|\Big]_0^{\pi/4} \\
&= \log\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \log|\sec 0 + \tan 0| \\
&= \log\left|\sqrt{2}+1\right| - \log|1+0| = \log(\sqrt{2}+1)
\end{aligned}$$

So, $$\int_0^1 \frac{1}{\sqrt{1+x^2}}\,\mathrm{d}x = \log(\sqrt{2}+1)$$

(c) The integral is a rational function. In the absence of an obvious substitution, we use partial fractions.

$$\begin{aligned}
\frac{4x}{(x^2-1)(x^2+1)} &= \frac{4x}{(x-1)(x+1)(x^2+1)} \\
&= \frac{a}{x-1} + \frac{b}{x+1} + \frac{cx+d}{x^2+1}
\end{aligned}$$

Multiplying by the denominator,

$$\begin{aligned}
4x = a(x+1)(x^2+1) &+ b(x-1)(x^2+1) \\
&+ (cx+d)(x-1)(x+1) \qquad (*)
\end{aligned}$$

Setting $x = 1$ gives $4a = 4$, so $a = 1$. Setting $x = -1$ gives $-4b = -4$, so $b = 1$. Substituting in $a = b = 1$ in $(*)$ gives:

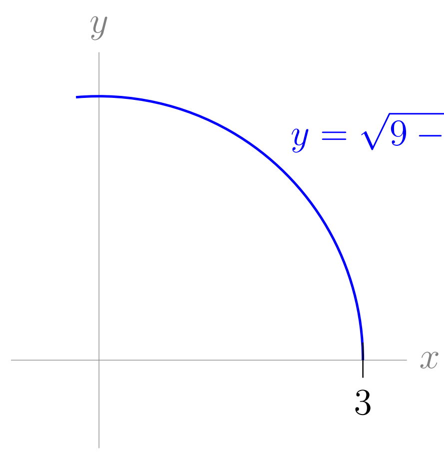
$$\begin{aligned}
4x &= (x+1)(x^2+1) + (x-1)(x^2+1) \\
&\quad + (cx+d)(x-1)(x+1) \\
4x &= 2x(x^2+1) + (cx+d)(x-1)(x+1) \\
4x - 2x(x^2+1) &= (cx+d)(x-1)(x+1) \\
-2x(x^2-1) &= (cx+d)(x^2-1) \\
-2x &= cx+d \\
c &= -2,\ d = 0
\end{aligned}$$

So,

$$\begin{aligned}\int_3^5 \frac{4x}{(x^2-1)(x^2+1)}\,\mathrm{d}x &= \int_3^5 \left(\frac{1}{x-1}+\frac{1}{x+1}-\frac{2x}{x^2+1}\right)\mathrm{d}x\\ &= \Big[\log|x-1|+\log|x+1|-\log(x^2+1)\Big]_3^5\\ &= \log 4+\log 6-\log 26-\log 2-\log 4+\log 10\\ &= \log\frac{6\times 10}{26\times 2} = \log\frac{15}{13}\approx 0.1431\end{aligned}$$

1.13.11. $*$. **Solution**. (a) $\int_0^3 \sqrt{9-x^2}\,\mathrm{d}x$ is the area of the portion of the disk $x^2+y^2\le 9$ that lies in the first quadrant. It is $\frac{1}{4}\pi 3^3 = \frac{9}{4}\pi$. Alternatively, you could also evaluate this integral using the substitution $x=3\sin y$, $\mathrm{d}x = 3\cos y\,\mathrm{d}y$.

$$\begin{aligned}\int_0^3 \sqrt{9-x^2}\,\mathrm{d}x &= \int_0^{\pi/2}\sqrt{9-9\sin^2 y}\,(3\cos y)\,\mathrm{d}y = 9\int_0^{\pi/2}\cos^2 y\,\mathrm{d}y\\ &= \frac{9}{2}\int_0^{\pi/2}[1+\cos(2y)]\,\mathrm{d}y = \frac{9}{2}\Big[y+\frac{\sin(2y)}{2}\Big]_0^{\pi/2}\\ &= \frac{9}{4}\pi\end{aligned}$$



(b) It's not immediately obvious what to do with this one, but remember we found $\int \log x\mathrm{d}x$ using integration by parts with $u=\log x$ and $\mathrm{d}v=\mathrm{d}x$. Let's hope a similar trick works here. Integrate by parts, using $u=\log(1+x^2)$ and $\mathrm{d}v=\mathrm{d}x$, so that $\mathrm{d}u=\frac{2x}{1+x^2}\,\mathrm{d}x$, $v=x$.

$$\begin{aligned}\int_0^1 \log(1+x^2)\,\mathrm{d}x &= \Big[x\log(1+x^2)\Big]_0^1 - \int_0^1 x\frac{2x}{1+x^2}\,\mathrm{d}x\\ &= \log 2 - 2\int_0^1 \frac{x^2}{1+x^2}\,\mathrm{d}x\\ &= \log 2 - 2\int_0^1\left(1-\frac{1}{1+x^2}\right)\mathrm{d}x\\ &= \log 2 - 2\big[x-\arctan x\big]_0^1\end{aligned}$$

$$= \log 2 - 2 + \frac{\pi}{2} \approx 0.264$$

(c) The integrand is a rational function with no obvious substitution, so we use partial fractions.

$$\begin{aligned}\frac{x}{(x-1)^2(x-2)} &= \frac{a}{(x-1)^2} + \frac{b}{x-1} + \frac{c}{x-2}\\ &= \frac{a(x-2) + b(x-1)(x-2) + c(x-1)^2}{(x-1)^2(x-2)}\end{aligned}$$

Multiply by the denominator.

$$x = a(x-2) + b(x-1)(x-2) + c(x-1)^2$$

Setting $x = 1$ gives $a = -1$. Setting $x = 2$ gives $c = 2$. Substituting in $a = -1$ and $c = 2$ gives

$$\begin{aligned}b(x-1)(x-2) &= x + (x-2) - 2(x-1)^2\\ &= -2x^2 + 6x - 4 = -2(x-1)(x-2)\\ &\implies b = -2\end{aligned}$$

Hence

$$\begin{aligned}&\int_3^\infty \frac{x}{(x-1)^2(x-2)}\,\mathrm{d}x\\ &= \lim_{M\to\infty} \int_3^M \left(-\frac{1}{(x-1)^2} - \frac{2}{x-1} + \frac{2}{x-2}\right)\mathrm{d}x\\ &= \lim_{M\to\infty} \left[\frac{1}{x-1} - 2\log|x-1| + 2\log|x-2|\right]_3^M\\ &= \lim_{M\to\infty} \left[\frac{1}{x-1} + 2\log\left|\frac{x-2}{x-1}\right|\right]_3^M\\ &= \lim_{M\to\infty} \left[\frac{1}{M-1} + 2\log\left|\frac{M-2}{M-1}\right|\right] - \left[\frac{1}{3-1} + 2\log\left|\frac{3-2}{3-1}\right|\right]\\ &= 2\log 2 - \frac{1}{2} \approx 0.886\end{aligned}$$

since

$$\begin{aligned}\lim_{M\to\infty} \log\frac{M-2}{M-1} &= \lim_{M\to\infty} \log\frac{1-2/M}{1-1/M} = \log 1 = 0\\ \text{and} \qquad \log\frac{1}{2} &= -\log 2\end{aligned}$$

1.13.12. Solution. This looks quite a lot like a rational function, but with variable $\sin\theta$ instead of x. So, we use the substitution $x = \sin\theta$, $\mathrm{d}x = \cos\theta\mathrm{d}\theta$.

$$\int \frac{\sin^4\theta - 5\sin^3\theta + 4\sin^2\theta + 10\sin\theta}{\sin^2\theta - 5\sin\theta + 6}\cos\theta\mathrm{d}\theta$$
$$= \int \frac{x^4 - 5x^3 + 4x^2 + 10x}{x^2 - 5x + 6}\mathrm{d}x$$

Since the numerator does not have smaller degree than the denominator, we need to do some long division before we can set up our partial fractions decomposition.

$$\begin{array}{r|l} & x^2 \qquad\qquad - 2 \\ \hline x^2 - 5x + 6 & x^4 - 5x^3 + 4x^2 + 10x \\ & -x^4 + 5x^3 - 6x^2 \\ \hline & -2x^2 + 10x \\ & 2x^2 - 10x + 12 \\ \hline & 12 \end{array}$$

That is,

$$\frac{x^4 - 5x^3 + 4x^2 + 10x}{x^2 - 5x + 6} = x^2 - 2 + \frac{12}{x^2 - 5x + 6}$$
$$= x^2 - 2 + \frac{12}{(x-2)(x-3)}$$

We use partial fractions decomposition on the rightmost term.

$$\frac{12}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$
$$12 = A(x-3) + B(x-2)$$

Setting $x = 3$ and $x = 2$ gives us

$$B = 12, \quad A = -12$$

Now we can evaluate our integral.

$$\int \frac{\sin^4\theta - 5\sin^3\theta + 4\sin^2\theta + 10\sin\theta}{\sin^2\theta - 5\sin\theta + 6}\cos\theta\mathrm{d}\theta$$
$$= \int \frac{x^4 - 5x^3 + 4x^2 + 10x}{x^2 - 5x + 6}\mathrm{d}x$$
$$= \int \left(x^2 - 2 + \frac{12}{(x-2)(x-3)}\right)\mathrm{d}x$$
$$= \int \left(x^2 - 2 - \frac{12}{x-2} + \frac{12}{x-3}\right)\mathrm{d}x$$
$$= \frac{1}{3}x^3 - 2x - 12\log|x-2| + 12\log|x-3| + C$$
$$= \frac{1}{3}x^3 - 2x + 12\log\left|\frac{x-3}{x-2}\right| + C$$
$$= \frac{1}{3}\sin^3\theta - 2\sin\theta + 12\log\left|\frac{\sin\theta - 3}{\sin\theta - 2}\right| + C$$

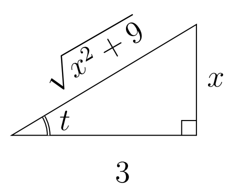
1.13.13. *. Solution. (a) It doesn't matter to us right now that the arguments of sine and cosine are $2x$ rather than x. This is still the integral of powers of products of sines and cosines. Since cosine has an odd power, we make the substitution $u = \sin(2x)$, $\mathrm{d}u = 2\cos(2x)\,\mathrm{d}x$.

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \sin^2(2x)\cos^3(2x)\ \mathrm{d}x &= \int_0^{\frac{\pi}{4}} \sin^2(2x)\big[1-\sin^2(2x)\big]\cos(2x)\ \mathrm{d}x\\ &= \frac{1}{2}\int_0^1 u^2\big[1-u^2\big]\ \mathrm{d}u\\ &= \frac{1}{2}\int_0^1 \left(u^2-u^4\right)\ \mathrm{d}u = \frac{1}{2}\left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_0^1\\ &= \frac{1}{15}\end{aligned}$$

(b) Make the substitution $x = 3\tan t$, $\mathrm{d}x = 3\sec^2 t\,\mathrm{d}t$ and use the trig identity $9 + 9\tan^2 t = 9\sec^2 t$.

$$\begin{aligned}\int \left(9+x^2\right)^{-\frac{3}{2}}\ \mathrm{d}x &= \int \left(9+9\tan^2 t\right)^{-\frac{3}{2}}\ 3\sec^2 t\,\mathrm{d}t\\ &= \int \left(3\sec t\right)^{-3}\ 3\sec^2 t\,\mathrm{d}t\\ &= \frac{1}{9}\int \cos t\ \mathrm{d}t = \frac{1}{9}\sin t + C\\ &= \frac{1}{9}\frac{x}{\sqrt{x^2+9}} + C\end{aligned}$$

To convert back to x, in the last step, we used the triangle below, which is rigged to have $\tan t = \frac{x}{3}$.



(c) Seeing a rational function with no obvious substitutions, we use partial fractions.

$$\frac{1}{(x-1)(x^2+1)} = \frac{a}{x-1} + \frac{bx+c}{x^2+1} = \frac{a(x^2+1)+(bx+c)(x-1)}{(x-1)(x^2+1)}$$

Multiply by the original denominator.

$$1 = a(x^2+1) + (bx+c)(x-1) \tag{$*$}$$

Setting $x = 1$ gives $2a = 1$ or $a = \frac{1}{2}$. Substituting in $a = \frac{1}{2}$ in $(*)$ gives

$$\begin{aligned}& \frac{1}{2}(x^2+1) + (bx+c)(x-1) = 1\\ \iff\quad & (bx+c)(x-1) = \frac{1}{2}(1-x^2) = -\frac{1}{2}(x-1)(x+1)\end{aligned}$$

$$\begin{aligned} \iff \qquad & (bx+c) = -\frac{1}{2}(x+1) \\ \iff \qquad & b = c = -\frac{1}{2} \end{aligned}$$

So,

$$\int \frac{\mathrm{d}x}{(x-1)(x^2+1)} = \int \Big[\frac{1/2}{x-1} - \frac{\frac{1}{2}(x+1)}{x^2+1}\Big]\,\mathrm{d}x$$

To antidifferentiate the second piece, we split it into two integrals: one that can be handled with the substitution $u = x^2+1$, and another that looks like the derivative of arctangent.

$$\begin{aligned} &= \int \Big(\frac{1/2}{x-1} - \frac{x/2}{x^2+1} - \frac{1/2}{x^2+1}\Big)\,\mathrm{d}x \\ &= \int \Big(\frac{1/2}{x-1} - \frac{1}{4}\cdot\frac{2x}{x^2+1} - \frac{1/2}{x^2+1}\Big)\,\mathrm{d}x \\ &= \frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) - \frac{1}{2}\arctan x + C \end{aligned}$$

(d) We know the derivative of arctangent, and it would integrate nicely if multiplied to the antiderivative of x. So, we integrate by parts with $u = \arctan x$ and $\mathrm{d}v = x\,\mathrm{d}x$ so that $\mathrm{d}u = \frac{1}{1+x^2}\mathrm{d}x$ and $v = \frac{1}{2}x^2$. Then

$$\begin{aligned} \int x\arctan x\,\mathrm{d}x &= \frac{1}{2}x^2\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,\mathrm{d}x \\ &= \frac{1}{2}x^2\arctan x - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2}\,\mathrm{d}x \\ &= \frac{1}{2}x^2\arctan x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right)\,\mathrm{d}x \\ &= \frac{1}{2}\big[x^2\arctan x - x + \arctan x\big] + C \end{aligned}$$

1.13.14. $*$. Solution. (a) We substitute $y = \sin(2x)$, $\mathrm{d}y = 2\cos(2x)\ \mathrm{d}x$. Note $\sin(2\cdot 0) = 0$ and $\sin(2\cdot\frac{\pi}{4}) = 1$.

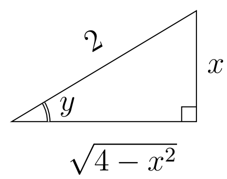
$$\int_0^{\pi/4} \sin^5(2x)\ \cos(2x)\ \mathrm{d}x = \int_0^1 y^5\,\frac{\mathrm{d}y}{2} = \frac{1}{12}\Big[y^6\Big]_0^1 = \frac{1}{12}$$

(b) We can get rid of the square root with a trig substitution. Substituting $x = 2\sin y$, $\mathrm{d}x = 2\cos y\ \mathrm{d}y$,

$$\begin{aligned} \int \sqrt{4-x^2}\ \mathrm{d}x &= \int \sqrt{4-4\sin^2 y}\ 2\cos y\ \mathrm{d}y = 4\int \cos^2 y\ \mathrm{d}y \\ &= 2\int \big[1+\cos(2y)\big]\ \mathrm{d}y \end{aligned}$$

$$\begin{aligned}&= 2y + \sin(2y) + C = 2y + 2\sin y \cos y + C\\ &= 2\sin^{-1}\frac{x}{2} + x\sqrt{1 - \frac{x^2}{4}} + C\end{aligned}$$

since $\sin y = \frac{x}{2}$ and $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - \frac{x^2}{4}}$. Alternately, we can draw a triangle with $\sin y = \frac{x}{2}$, and use the Pythagorean theorem to find the adjacent side.



(c) Seeing a rational function with no obvious substitution, we use the method of partial fractions. The denominator is already completely factored.

$$\begin{aligned}\frac{x+1}{x^2(x-1)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}\\ x + 1 &= Ax(x-1) + B(x-1) + Cx^2\end{aligned}$$

Setting $x = 1$ gives us $C = 2$. Setting $x = 0$ gives us $B = -1$. Furthermore, the coefficient of x^2 on the left hand side (after collecting like terms), namely $A + C$, must be the same as the coefficient of x^2 on the right hand side, namely 0. So $A + C = 0$ and $A = -2$. Checking,

$$-2x(x-1) - (x-1) + 2x^2 = -2x^2 + 2x - x + 1 + 2x^2 = x + 1$$

as desired. Thus,

$$\begin{aligned}\int \frac{x+1}{x^2(x-1)}\,\mathrm{d}x &= \int \left[-\frac{2}{x} - \frac{1}{x^2} + \frac{2}{x-1}\right]\mathrm{d}x\\ &= -2\log|x| + \frac{1}{x} + 2\log|x-1| + C\end{aligned}$$

1.13.15. $*$. **Solution**. (a) Define

$$I_1 = \int_0^\infty e^{-x}\sin(2x)\,\mathrm{d}x \qquad I_2 = \int_0^\infty e^{-x}\cos(2x)\,\mathrm{d}x$$

We integrate by parts, with $u = \sin(2x)$ or $\cos(2x)$ and $\mathrm{d}v = e^{-x}\,\mathrm{d}x$. That is, $v = -e^{-x}$.

$$\begin{aligned}I_1 = \int_0^\infty e^{-x}\sin(2x)\,\mathrm{d}x &= \lim_{R\to\infty}\int_0^R e^{-x}\sin(2x)\,\mathrm{d}x\\ &= \lim_{R\to\infty}\left(\Big[-e^{-x}\sin(2x)\Big]_0^R + 2\int_0^R e^{-x}\cos(2x)\,\mathrm{d}x\right)\\ &= 2I_2\end{aligned}$$

$$
\begin{aligned}
I_2 &= \int_0^{\infty} e^{-x}\cos(2x)\,\mathrm{d}x = \lim_{R\to\infty}\int_0^R e^{-x}\cos(2x)\,\mathrm{d}x \\
&= \lim_{R\to\infty}\left(\Big[-e^{-x}\cos(2x)\Big]_0^R - 2\int_0^R e^{-x}\sin(2x)\,\mathrm{d}x\right) \\
&= 1 - 2I_1
\end{aligned}
$$

Substituting $I_2 = \frac{1}{2}I_1$ into $I_2 = 1 - 2I_1$ gives $\frac{5}{2}I_1 = 1$, or $\int_0^{\infty} e^{-x}\sin(2x)\,\mathrm{d}x = \frac{2}{5}$.

(b) We can cancel out the square root if we use a trig substitution. Substitute $x = \sqrt{2}\tan y$, $\mathrm{d}x = \sqrt{2}\sec^2 y\,\mathrm{d}y$.

$$
\begin{aligned}
\int_0^{\sqrt{2}} \frac{1}{(2+x^2)^{3/2}}\,\mathrm{d}x &= \sqrt{2}\int_0^{\pi/4} \frac{\sec^2 y}{(2+2\tan^2 y)^{3/2}}\,\mathrm{d}y \\
&= \frac{1}{2}\int_0^{\pi/4}\cos y\,\mathrm{d}y = \frac{1}{2}\Big[\sin y\Big]_0^{\pi/4} \\
&= \frac{1}{2\sqrt{2}}
\end{aligned}
$$

(c)

- Solution 1: Integrate by parts, using $u = \log(1+x^2)$ and $\mathrm{d}v = x\,\mathrm{d}x$, so that $\mathrm{d}u = \frac{2x}{1+x^2}$, $v = \frac{x^2}{2}$.

$$
\begin{aligned}
\int_0^1 x\log(1+x^2)\,\mathrm{d}x &= \Big[\frac{1}{2}x^2\log(1+x^2)\Big]_0^1 - \int_0^1 \frac{x^3}{1+x^2}\,\mathrm{d}y \\
&= \frac{1}{2}\log 2 - \int_0^1 \Big[x - \frac{x}{1+x^2}\Big]\,\mathrm{d}x \\
&= \frac{1}{2}\log 2 - \Big[\frac{x^2}{2} - \frac{1}{2}\log(1+x^2)\Big]_0^1 \\
&= \log 2 - \frac{1}{2} \approx 0.193
\end{aligned}
$$

- Solution 2: First substitute $y = 1 + x^2$, $\mathrm{d}y = 2x\,\mathrm{d}x$.

$$
\int_0^1 x\log(1+x^2)\,\mathrm{d}x = \frac{1}{2}\int_1^2 \log y\ \mathrm{d}y
$$

Then integrate by parts, using $u = \log y$ and $\mathrm{d}v = \mathrm{d}y$, so that $\mathrm{d}u = \frac{1}{y}$, $v = y$.

$$
\begin{aligned}
\int_0^1 x\log(1+x^2)\,\mathrm{d}x &= \frac{1}{2}\int_1^2 \log y\ \mathrm{d}y \\
&= \Big[\frac{1}{2}y\log y\Big]_1^2 - \frac{1}{2}\int_1^2 y\frac{1}{y}\,\mathrm{d}y = \log 2 - \frac{1}{2} \\
&\approx 0.193
\end{aligned}
$$

(d) Seeing a rational function with no obvious substitution, we use partial fractions.

$$\frac{1}{(x-1)^2(x-2)} = \frac{a}{(x-1)^2} + \frac{b}{x-1} + \frac{c}{x-2}$$
$$1 = a(x-2) + b(x-1)(x-2) + c(x-1)^2 \qquad (*)$$

Setting $x = 1$ gives $a = -1$. Setting $x = 2$ gives $c = 1$. Substituting in $a = -1$ and $c = 1$ to $(*)$ gives

$$\begin{aligned} b(x-1)(x-2) &= 1 + (x-2) - (x-1)^2 \\ &= -x^2 + 3x - 2 \\ &= -(x-1)(x-2) \\ \implies \qquad b &= -1 \end{aligned}$$

Hence:

$$\begin{aligned} &\int_3^\infty \frac{1}{(x-1)^2(x-2)}\,\mathrm{d}x \\ &\quad = \lim_{M\to\infty} \int_3^M \Big(-\frac{1}{(x-1)^2} - \frac{1}{x-1} + \frac{1}{x-2} \Big)\,\mathrm{d}x \\ &\quad = \lim_{M\to\infty} \left[\frac{1}{x-1} - \log(x-1) + \log(x-2) \right]_3^M \\ &\quad = \lim_{M\to\infty} \left[\frac{1}{M-1} + \log\frac{M-2}{M-1} \right] - \left[\frac{1}{3-1} + \log\frac{3-2}{3-1} \right] \\ &\quad = \log 2 - \frac{1}{2} \approx 0.193 \end{aligned}$$

since

$$\lim_{M\to\infty} \log\frac{M-2}{M-1} = \lim_{M\to\infty} \log\frac{1-2/M}{1-1/M} = \log 1 = 0$$

1.13.16. *. Solution. (a) Integrate by parts with $u = \log x$ and $\mathrm{d}v = x\,\mathrm{d}x$, so that $\mathrm{d}u = \frac{\mathrm{d}x}{x}$ and $v = \frac{1}{2}x^2$.

$$\int x\log x\,\mathrm{d}x = \frac{1}{2}x^2\log x - \frac{1}{2}\int x^2 \cdot \frac{1}{x}\,\mathrm{d}x = \frac{1}{2}x^2\log x - \frac{1}{4}x^2 + C$$

(b) The denominator is an irreducible quadratic, so partial fractions can't get us any further. To integrate a function whose denominator is quadratic, we split the numerator up so that one piece can be evaluated with a u-substitution, and the other piece looks like arctangent.

$$\begin{aligned} \int \frac{(x-1)\mathrm{d}x}{x^2+4x+5} &= \int \frac{x+2-3}{x^2+4x+5}\mathrm{d}x \\ &= \frac{1}{2}\int \frac{2x+4}{x^2+4x+5}\mathrm{d}x - \int \frac{3}{x^2+4x+5}\mathrm{d}x \\ &= \frac{1}{2}\int \frac{2x+4}{x^2+4x+5}\mathrm{d}x - 3\int \frac{1}{(x+2)^2+1}\mathrm{d}x \end{aligned}$$

$$= \frac{1}{2}\log[x^2+4x+5] - 3\arctan(x+2) + C$$

For the last step, you can guess the antiderivative, or use the substitutions $u_1 = x^2+4x+5$ and $u_2 = x+2$, respectively, for the two integrals.
(c) We use partial fractions.

$$\frac{1}{x^2-4x+3} = \frac{1}{(x-3)(x-1)} = \frac{a}{x-3} + \frac{b}{x-1}$$
$$1 = a(x-1) + b(x-3)$$

Setting $x=3$ gives $a = \frac{1}{2}$. Setting $x=1$ gives $b = -\frac{1}{2}$. So,

$$\int \frac{\mathrm{d}x}{x^2-4x+3} = \int \left(\frac{1/2}{x-3} - \frac{1/2}{x-1}\right)\mathrm{d}x$$
$$= \frac{1}{2}\log|x-3| - \frac{1}{2}\log|x-1| + C$$

(d) Substitute $y = x^3$, $\mathrm{d}y = 3x^2\,\mathrm{d}x$.

$$\int \frac{x^2\,\mathrm{d}x}{1+x^6} = \frac{1}{3}\int \frac{\mathrm{d}y}{1+y^2} = \frac{1}{3}\arctan y + C = \frac{1}{3}\arctan x^3 + C$$

1.13.17. $*$. **Solution**. (a) Integrate by parts with $u = \arctan x$, $\mathrm{d}v = \mathrm{d}x$, $\mathrm{d}u = \frac{\mathrm{d}x}{1+x^2}$ and $v = x$. This gives

$$\int_0^1 \arctan x\ \mathrm{d}x = \big[x\arctan x\big]_0^1 - \int_0^1 \frac{x}{1+x^2}\mathrm{d}x$$
$$= \arctan 1 - \left[\frac{1}{2}\log(1+x^2)\right]_0^1$$
$$= \frac{\pi}{4} - \frac{1}{2}\log 2$$

(b) Note that the derivative of the denominator is $2x-2$, which differs from the numerator only by 1.

$$\int \frac{2x-1}{x^2-2x+5}\,\mathrm{d}x = \int \frac{2x-2}{x^2-2x+5}\,\mathrm{d}x + \int \frac{1}{x^2-2x+5}\,\mathrm{d}x$$
$$= \int \frac{2x-2}{x^2-2x+5}\,\mathrm{d}x + \int \frac{1}{(x-1)^2+4}\,\mathrm{d}x$$
$$= \log|x^2-2x+5| + \frac{1}{2}\arctan\frac{x-1}{2} + C$$

In the last step, you can guess the antiderivative, or use the substitutions $u_1 = x^2-2x+5$ and $u_2 = (x-1)/2$, respectively.

1.13.18. $*$. **Solution**. (a) Substituting $u = x^3+1$, $\mathrm{d}u = 3x^2\,\mathrm{d}x$

$$\int \frac{x^2}{(x^3+1)^{101}}\mathrm{d}x = \int \frac{1}{u^{101}}\cdot\frac{\mathrm{d}u}{3} = \frac{u^{-100}}{-100}\cdot\frac{1}{3} + C$$

$$= -\frac{1}{300(x^3+1)^{100}} + C$$

(b) Substituting $u = \sin x$, $\mathrm{d}u = \cos x\ \mathrm{d}x$, $\cos^2 x = 1 - \sin^2 x = 1 - u^2$,

$$\begin{aligned}\int \cos^3 x\ \sin^4 x\ \mathrm{d}x &= \int \cos^2 x\ \sin^4 x\ \cos x\ \mathrm{d}x = \int (1-u^2)u^4\ \mathrm{d}u \\ &= \int (u^4 - u^6)\ \mathrm{d}u = \frac{u^5}{5} - \frac{u^7}{7} + C \\ &= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C\end{aligned}$$

1.13.19. Solution. First, we note that the integral is improper, because $\sin\pi = 0$. So, we'll have to use a limit.
Second, we need to antidifferentiate. The substitution $u = \sin x$, $\mathrm{d}u = \cos x \mathrm{d}x$ fits just right.

$$\begin{aligned}\int_{\pi/2}^{\pi} \frac{\cos x}{\sqrt{\sin x}}\mathrm{d}x &= \lim_{b\to\pi^-} \int_{\pi/2}^{b} \frac{\cos x}{\sqrt{\sin x}}\mathrm{d}x = \lim_{b\to\pi^-} \int_1^{\sin b} \frac{1}{\sqrt{u}}\mathrm{d}u \\ &= \lim_{b\to\pi^-} \left[2\sqrt{u}\right]_1^{\sin b} = 2\sqrt{0} - 2\sqrt{1} = -2\end{aligned}$$

1.13.20. $*$**. Solution**. (a) If the integrand had x's instead of e^x's it would be a rational function, ripe for the application of partial fractions. So let's start by making the substitution $u = e^x$, $\mathrm{d}u = e^x\ \mathrm{d}x$:

$$\int \frac{e^x}{(e^x+1)(e^x-3)}\mathrm{d}x = \int \frac{\mathrm{d}u}{(u+1)(u-3)}$$

Now, we follow the partial fractions protocol, starting with expressing

$$\frac{1}{(u+1)(u-3)} = \frac{A}{u+1} + \frac{B}{u-3}$$

To find A and B, the sneaky way, we cross multiply by the denominator

$$1 = A(u-3) + B(u+1)$$

and find A and B by evaluating at $u = -1$ and $u = 3$, respectively.

$$\begin{aligned}1 = A(-1-3) + B(-1+1) &\iff A = -\frac{1}{4} \\ 1 = A(3-3) + B(3+1) &\iff B = \frac{1}{4}\end{aligned}$$

Finally, we can do the integral:

$$\begin{aligned}\int \frac{e^x}{(e^x+1)(e^x-3)}\mathrm{d}x &= \int \frac{\mathrm{d}u}{(u+1)(u-3)} = \int \Big(\frac{-1/4}{u+1} + \frac{1/4}{u-3}\Big)\mathrm{d}u \\ &= -\frac{1}{4}\log|u+1| + \frac{1}{4}\log|u-3| + C\end{aligned}$$

$$= -\frac{1}{4}\log|e^x+1| + \frac{1}{4}\log|e^x-3| + C$$

(b) The argument of the square root is

$$12+4x-x^2 = 12-(x-2)^2+4 = 16-(x-2)^2$$

Hmmm. The numerator is $x^2-4x+4 = (x-2)^2$. So let's make the integral look somewhat simpler by substituting $u = x-2$, $\mathrm{d}u = \mathrm{d}x$. When $x=2$ we have $u=0$, and when $x=4$ we have $u=2$, so:

$$\int_{x=2}^{x=4} \frac{x^2-4x+4}{\sqrt{12+4x-x^2}}\mathrm{d}x = \int_{u=0}^{u=2} \frac{u^2}{\sqrt{16-u^2}}\mathrm{d}u$$

This is perfect for the trig substitution $u = 4\sin\theta$, $\mathrm{d}u = 4\cos(\theta)\,\mathrm{d}\theta$. When $u=0$ we have $4\sin\theta = 0$ and hence $\theta = 0$. When $u=2$ we have $4\sin\theta = 2$ and hence $\theta = \frac{\pi}{6}$. So

$$\begin{aligned}
\int_{u=0}^{u=2} \frac{u^2}{\sqrt{16-u^2}}\mathrm{d}u &= \int_{\theta=0}^{\theta=\pi/6} \frac{16\sin^2\theta}{\sqrt{16-16\sin^2\theta}}4\cos\theta\mathrm{d}\theta \\
&= 16\int_0^{\pi/6} \sin^2\theta\ \mathrm{d}\theta \\
&= 8\int_0^{\pi/6} \big(1-\cos(2\theta)\big)\ \mathrm{d}\theta \\
&= 8\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\pi/6} = 8\left[\frac{\pi}{6} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right] \\
&= \frac{4\pi}{3} - 2\sqrt{3}
\end{aligned}$$

1.13.21. $*$. **Solution**. (a) Substituting $y = \cos x$, $\mathrm{d}y = -\sin x\,\mathrm{d}x$, $\sin^2 x = 1-\cos^2 x = 1-y^2$

$$\begin{aligned}
\int \frac{\sin^3 x}{\cos^3 x}\ \mathrm{d}x &= \int \frac{\sin^2 x}{\cos^3 x}\ \sin x\,\mathrm{d}x = \int \frac{1-y^2}{y^3}\ (-\mathrm{d}y) \\
&= -\int \left(y^{-3}-y^{-1}\right)\ \mathrm{d}y = -\frac{y^{-2}}{-2} + \log|y| + C \\
&= \frac{1}{2}\sec^2 x + \log|\cos x| + C
\end{aligned}$$

(b) The integrand is an even function, and the limits of integration are symmetric. So, we can slightly simplify the integral by replacing the lower limit with 0, and doubling the integral.

We'd rather not use partial fractions here, because it would be pretty complicated. Instead, notice that the numerator is only off by a constant from the derivative of x^5. Substituting $x^5 = 4y$, $5x^4\,\mathrm{d}x = 4\,\mathrm{d}y$, and using that $x = 2 \implies 2^5 = 4y \implies y = 8$,

$$\int_{-2}^{2} \frac{x^4}{x^{10}+16}\ \mathrm{d}x = 2\int_0^2 \frac{x^4}{x^{10}+16}\ \mathrm{d}x = 2\cdot\frac{4}{5}\int_0^8 \frac{1}{16y^2+16}\ \mathrm{d}y$$

$$= \frac{1}{10}\int_0^8 \frac{1}{y^2+1}\,\mathrm{d}y$$
$$= \frac{1}{10}\arctan 8 \approx 0.1446$$

1.13.22. Solution.

- Solution 1: Let's use the substitution $u = x-1$, $\mathrm{d}u = \mathrm{d}x$.

$$\begin{aligned}\int x\sqrt{x-1}\mathrm{d}x &= \int (u+1)\sqrt{u}\mathrm{d}u\\ &= \int \left(u^{3/2}+u^{1/2}\right)\mathrm{d}u\\ &= \frac{2}{5}u^{5/2}+\frac{2}{3}u^{3/2}+C\\ &= \frac{2}{5}(x-1)^{5/2}+\frac{2}{3}(x-1)^{3/2}+C\end{aligned}$$

- Solution 2: We have an integrand with x multiplied by something integrable. So, if we use integration by parts with $u = x$ and $\mathrm{d}v = \sqrt{x-1}\mathrm{d}x$, then $\mathrm{d}u = \mathrm{d}x$ (that is, the x goes away) and $v = \frac{2}{3}(x-1)^{3/2}$.

$$\begin{aligned}\int x\sqrt{x-1}\mathrm{d}x &= \frac{2}{3}x\sqrt{x-1}^3 - \frac{2}{3}\int (x-1)^{3/2}\mathrm{d}x\\ &= \frac{2}{3}x\sqrt{x-1}^3 - \frac{2}{3}\left(\frac{2}{5}(x-1)^{5/2}\right)+C\\ &= \frac{2}{3}\sqrt{x-1}\left(x(x-1)-\frac{2}{5}(x-1)^2\right)+C\\ &= \frac{2}{15}\sqrt{x-1}\cdot(3x^2-x-2)+C\\ &= \frac{2}{15}\sqrt{x-1}\cdot(3(x^2-2x+1)+5x-5)+C\\ &= \frac{2}{15}\sqrt{x-1}\cdot(3(x-1)^2+5(x-1))+C\\ &= \frac{2}{15}\cdot 3\sqrt{x-1}^5+\frac{2}{15}\cdot 5\sqrt{x-1}^3+C\\ &= \frac{2}{5}\sqrt{x-1}^5+\frac{2}{3}\sqrt{x-1}^3+C\end{aligned}$$

1.13.23. Solution.

$$\int \frac{\sqrt{x^2-2}}{x^2}\mathrm{d}x$$

We notice that there is a quadratic function under the square root. If that equation were a perfect square, we could get rid of the square root: so we'll mould it into a perfect square using a trig substitution.
Our candidates will use one of the following identities:

$$1-\sin^2\theta = \cos^2\theta \qquad \tan^2\theta+1 = \sec^2\theta \qquad \sec^2\theta - 1 = \tan^2\theta$$

We'll be substituting $x =$(something), so we notice that x^2-2 has the general form of (function)$-$(constant), as does $\sec^2\theta - 1$. In order to get the constant right, we multiply through by two:

$$2\sec^2\theta - 2 = 2\tan^2\theta$$

or:

$$(\sqrt{2}\sec\theta)^2 - 2 = 2\tan^2\theta$$

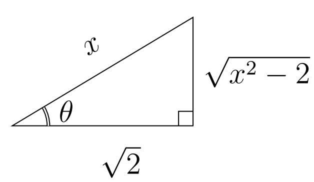
so we decide to use the substitution

$$x = \sqrt{2}\sec\theta \qquad \mathrm{d}x = \sqrt{2}\sec\theta\tan\theta\mathrm{d}\theta$$

Now that we've chosen the substitution, we evaluate the integral.

$$\begin{aligned}
\int \frac{\sqrt{x^2-2}}{x^2}\mathrm{d}x &= \int \frac{\sqrt{2\sec^2\theta - 2}}{2\sec^2\theta}\sqrt{2}\sec\theta\tan\theta\mathrm{d}\theta \\
&= \int \frac{\sqrt{2\tan^2\theta}}{2\sec^2\theta}\sqrt{2}\sec\theta\tan\theta\mathrm{d}\theta \\
&= \int \frac{\sqrt{2}\tan\theta}{2\sec^2\theta}\sqrt{2}\sec\theta\tan\theta\mathrm{d}\theta \\
&= \int \frac{\tan^2\theta}{\sec\theta}\mathrm{d}\theta \\
&= \int \frac{\sec^2\theta - 1}{\sec\theta}\mathrm{d}\theta \\
&= \int \big(\sec\theta - \cos\theta\big)\,\mathrm{d}\theta \\
&= \log|\sec\theta + \tan\theta| - \sin\theta + C
\end{aligned}$$

Now we need everything back in terms of x. We need a triangle. Since $x = \sqrt{2}\sec\theta$, that means if we label an angle θ, its secant (hypotenuse over adjacent side) is $\dfrac{x}{\sqrt{2}}$. By Pythagoras, the opposite side is $\sqrt{x^2-2}$.



So $\tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{\sqrt{x^2-2}}{\sqrt{2}}$, and $\sin\theta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{\sqrt{x^2-2}}{x}$. Then the value of the integral is:

$$\begin{aligned}
\log|\sec\theta + \tan\theta| - \sin\theta + C &= \log\left|\frac{x}{\sqrt{2}} + \frac{\sqrt{x^2-2}}{\sqrt{2}}\right| - \frac{\sqrt{x^2-2}}{x} + C \\
&= \log\left|x + \sqrt{x^2-2}\right| - \log\sqrt{2} - \frac{\sqrt{x^2-2}}{x} + C
\end{aligned}$$

$$= \log \left| x + \sqrt{x^2 - 2} \right| - \frac{\sqrt{x^2 - 2}}{x} + C$$

Note the simplification in the last step is due to our convention that C is an arbitrary constant. So, $C - \log\sqrt{2}$ can be re-written as simply C.

1.13.24. Solution. This is the product of secants and tangents, as in Section 1.8.2. If $u = \tan x$, then $\mathrm{d}u = \sec^2 x \mathrm{d}x$. We can get the remaining two secants to turn into tangents with the identity $\sec^2 x = 1 + \tan^2 x$, so we'll use this substitution.

$$\begin{aligned}
\int_0^{\pi/4} \sec^4 x \tan^5 x \mathrm{d}x &= \int_0^{\pi/4} \sec^2 x \tan^5 x \sec^2 x \mathrm{d}x \\
&= \int_0^{\pi/4} (1 + \tan^2 x) \tan^5 x \underbrace{\sec^2 x \mathrm{d}x}_{\mathrm{d}u} \\
&= \int_{\tan(0)}^{\tan(\pi/4)} (1 + u^2) u^5 \mathrm{d}u \\
&= \int_0^1 (u^5 + u^7) \mathrm{d}u \\
&= \left[\frac{1}{6} u^6 + \frac{1}{8} u^8 \right]_0^1 \\
&= \frac{1}{6} + \frac{1}{8} - 0 = \frac{7}{24}
\end{aligned}$$

1.13.25. Solution. We can use partial fraction decomposition to break this into chunks that we can deal with. The denominator has a repeated linear factor, so it can be decomposed as the sum of constants divided by powers of that factor.

$$\begin{aligned}
\frac{3x^2 + 4x + 6}{(x+1)^3} &= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} \\
&= \frac{A(x+1)^2 + B(x+1) + C}{(x+1)^3} \\
\Rightarrow \quad 3x^2 + 4x + 6 &= A(x+1)^2 + B(x+1) + C \\
&= Ax^2 + (2A + B)x + (A + B + C)
\end{aligned}$$

So, by matching coefficients:

$$A = 3, 2A + B = 4, \text{ and } A + B + C = 6$$
$$A = 3, \quad B = -2, \quad C = 5$$

Therefore:

$$\frac{3x^2 + 4x + 6}{(x+1)^3} = \frac{3}{x+1} + \frac{-2}{(x+1)^2} + \frac{5}{(x+1)^3}$$

Now, the integration is easy, with a substitution of $u = x + 1$ and $\mathrm{d}u = \mathrm{d}x$:

$$\int \frac{3x^2 + 4x + 6}{(x+1)^3} \mathrm{d}x = \int \left(\frac{3}{x+1} + \frac{-2}{(x+1)^2} + \frac{5}{(x+1)^3} \right) \mathrm{d}x$$

$$
\begin{aligned}
&= \int \left(3u^{-1} - 2u^{-2} + 5u^{-3}\right) \mathrm{d}u \\
&= 3\log|u| + 2u^{-1} - \frac{5}{2}u^{-2} + C \\
&= 3\log|x+1| + \frac{2}{x+1} - \frac{5}{2(x+1)^2} + C
\end{aligned}
$$

1.13.26. Solution. If the denominator were x^2+1, the antiderivative would be arctangent. So, by completing the square, let's aim for the fraction to look like $\frac{1}{u^2+1}$, for some u. This is a good strategy for integrating an irreducible quadratic under a constant.

First: complete the square

$$
\int \frac{1}{x^2+x+1}\mathrm{d}x = \int \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}\mathrm{d}x = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}x
$$

Second: get the denominator in the form u^2+1. To do this, we need to fix the constant

$$
\begin{aligned}
&= \int \left(\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\right)\left(\frac{\frac{4}{3}}{\frac{4}{3}}\right)\mathrm{d}x \\
&= \frac{4}{3}\int \frac{1}{\frac{4}{3}\cdot\left(x+\frac{1}{2}\right)^2+1}\mathrm{d}x
\end{aligned}
$$

Now a quick wiggle to make that first part of the denominator into something squared again:

$$
= \frac{4}{3}\int \frac{1}{\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right)^2+1}\mathrm{d}x
$$

Now we see that $u = \frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}$, $\mathrm{d}u = \frac{2}{\sqrt{3}}\mathrm{d}x$ will do the job

$$
\begin{aligned}
&= \frac{4}{3}\int \frac{1}{u^2+1}\cdot\frac{\sqrt{3}}{2}\mathrm{d}u = \frac{2}{\sqrt{3}}\int \frac{1}{u^2+1}\mathrm{d}u \\
&= \frac{2}{\sqrt{3}}\arctan u + C \\
&= \frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}x+\frac{1}{\sqrt{3}}\right) + C
\end{aligned}
$$

1.13.27. Solution. Since $\tan x = \frac{\sin x}{\cos x}$,

$$\begin{aligned}\int \sin x \cos x \tan x \mathrm{d}x &= \int \sin^2 x \mathrm{d}x = \int \frac{1}{2}\big(1-\cos(2x)\big)\mathrm{d}x\\ &= \frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) + C\\ &= \frac{1}{2}\left(x - \sin x \cos x\right) + C\end{aligned}$$

1.13.28. Solution. We have the integral of a rational function with no obvious substitution, so we use partial fractions. That means we need to factor the denominator. We see that $x=-1$ is a root of the denominator, so $x+1$ is a factor. You might be able to figure out the rest of the factorization by inspection, or from having seen this common expression before; alternately, we can use long division.

$$\begin{array}{r@{}l} & \quad\; x^2 - x + 1 \\ x+1 \big) & \; x^3 \qquad\qquad + 1 \\ & -x^3 - x^2 \\ \hline & \qquad -x^2 \\ & \qquad\; x^2 + x \\ \hline & \qquad\qquad x + 1 \\ & \qquad\quad -x - 1 \\ \hline & \qquad\qquad\quad 0 \end{array}$$

Note $x^2 - x + 1$ is an irreducible quadratic.

$$\begin{aligned}\frac{1}{x^3+1} &= \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}\\ 1 &= A(x^2-x+1) + (Bx+C)(x+1) \qquad (*)\end{aligned}$$

When $x=-1$, we see $1 = 3A$, so $\frac{1}{3} = A$. We plug this into $(*)$.

$$\begin{aligned}1 &= \frac{1}{3}(x^2-x+1) + (Bx+C)(x+1)\\ -\frac{1}{3}x^2 + \frac{1}{3}x + \frac{2}{3} &= Bx^2 + (B+C)x + C\end{aligned}$$

Matching up coefficients of corresponding power of x, we see $B = -\frac{1}{3}$ and $C = \frac{2}{3}$.

$$\int \frac{1}{x^3+1}\mathrm{d}x = \int\left(\frac{1/3}{x+1} - \frac{\frac{1}{3}x - \frac{2}{3}}{x^2-x+1}\right)\mathrm{d}x$$

To integrate the second fraction, we break it up into two pieces: one we can integrate using the substitution $u = x^2 - x + 1$, the other will look like the derivative of arctangent.

$$\begin{aligned}&= \frac{1}{3}\log|x+1| - \int \frac{\frac{1}{3}x - \frac{1}{6} - \frac{1}{2}}{x^2-x+1}\mathrm{d}x\\ &= \frac{1}{3}\log|x+1| - \frac{1}{6}\int \frac{2x-1}{x^2-x+1}\mathrm{d}x + \frac{1}{2}\int \frac{1}{(x-\frac{1}{2})^2 + \frac{3}{4}}\mathrm{d}x\end{aligned}$$

$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log|x^2-x+1| + \frac{1}{2}\int \frac{1}{\frac{3}{4}\left(\left(\frac{2x-1}{\sqrt{3}}\right)^2+1\right)}\mathrm{d}x$$
$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log|x^2-x+1| + \frac{2}{3}\int \frac{1}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1}\mathrm{d}x$$

Let $u = \frac{2x-1}{\sqrt{3}}$, $\mathrm{d}u = \frac{2}{\sqrt{3}}\mathrm{d}x$.

$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log|x^2-x+1| + \frac{1}{\sqrt{3}}\int \frac{1}{u^2+1}\mathrm{d}x$$
$$= \frac{1}{3}\log|x+1| - \frac{1}{6}\log|x^2-x+1| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

1.13.29. Solution. By process of elimination, we decide to use integration by parts. We won't get anything better by antidifferentiating arcsine, so let's plan on differentiating it:

$$u = \arcsin x \qquad\qquad \mathrm{d}v = (3x)^2\mathrm{d}x$$
$$\mathrm{d}u = \frac{1}{\sqrt{1-x^2}}\mathrm{d}x \qquad\qquad v = 3x^3$$

$$\int (3x)^2 \arcsin x\mathrm{d}x = \underbrace{\arcsin x}_{u}\cdot\underbrace{3x^3}_{v} - \int \underbrace{3x^3}_{v}\cdot\underbrace{\frac{1}{\sqrt{1-x^2}}\mathrm{d}x}_{\mathrm{d}u}$$
$$= 3x^3\arcsin x - \int \frac{3x^3}{\sqrt{1-x^2}}\mathrm{d}x$$

So: we've gotten rid of the ugly pairing of arcsine with a polynomial, but now we're in another pickle. From here, two options present themselves. We could use the substitution $u = 1-x^2$, or we could use a trig substitution.

- Option 1: Let $u = 1-x^2$. Then $-\frac{1}{2}\mathrm{d}u = \mathrm{d}x$, and $x^2 = 1-u$.

$$\int (3x)^2 \arcsin x\mathrm{d}x = 3x^3\arcsin x - \int \frac{3x^3}{\sqrt{1-x^2}}\mathrm{d}x$$
$$= 3x^3\arcsin x - 3\int \frac{x^2}{\sqrt{1-x^2}}\cdot x\mathrm{d}x$$
$$= 3x^3\arcsin x + \frac{3}{2}\int \frac{1-u}{\sqrt{u}}\mathrm{d}u$$
$$= 3x^3\arcsin x + \frac{3}{2}\int \left(u^{-1/2} - u^{1/2}\right)\mathrm{d}u$$
$$= 3x^3\arcsin x + \frac{3}{2}\left(2u^{1/2} - \frac{2}{3}u^{3/2}\right) + C$$
$$= 3x^3\arcsin x + 3\sqrt{1-x^2} - \sqrt{1-x^2}^3 + C$$

- Option 2: If we let $x = \sin\theta$, then $\sqrt{1-x^2} = \sqrt{\cos^2\theta} = \cos\theta$. So let's use the substitution $x = \sin\theta$, $\mathrm{d}x = \cos\theta\mathrm{d}\theta$.

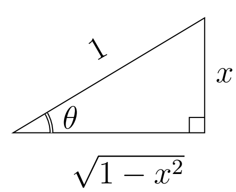
$$\begin{aligned}\int (3x)^2 \arcsin x\mathrm{d}x &= 3x^3 \arcsin x - \int \frac{3x^3}{\sqrt{1-x^2}}\mathrm{d}x \\ &= 3x^3 \arcsin x - \int \frac{3\sin^3\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\mathrm{d}\theta \\ &= 3x^3 \arcsin x - \int 3\sin^3\theta\mathrm{d}\theta\end{aligned}$$

And now: a substitution from Section 1.8.1, $u = \cos x$ and $\mathrm{d}u = -\sin x\mathrm{d}x$

$$\begin{aligned}3x^3 \arcsin x - \int 3\sin^3\theta\mathrm{d}\theta &= 3x^3 \arcsin x - 3\int \sin^2\theta\sin\theta\mathrm{d}\theta \\ &= 3x^3 \arcsin x - 3\int (1-\cos^2\theta)\sin\theta\mathrm{d}\theta \\ &= 3x^3 \arcsin x + 3\int (1-u^2)\mathrm{d}u \\ &= 3x^3 \arcsin x + 3\left(u - \frac{1}{3}u^3\right) + C \\ &= 3x^3 \arcsin x + 3u - u^3 + C \\ &= 3x^3 \arcsin x + 3\cos\theta - \cos^3\theta + C\end{aligned}$$

Recall $x = \sin\theta$; so we draw a triangle with angle θ, opposite side x, hypotenuse 1. Then by Pythagoras, adjacent side is $\sqrt{1-x^2}$, so $\cos\theta = \sqrt{1-x^2}$.

$$\begin{aligned}&\int (3x)^2 \arcsin x\mathrm{d}x \\ &\qquad = 3x^3 \arcsin x + 3\sqrt{1-x^2} - (1-x^2)^{3/2} + C\end{aligned}$$



Exercises — Stage 3

1.13.30. Solution. We would like to not have that square root there. Luckily, there's a way of turning cosine into cosine squared: the identity $\cos(2x) = 2\cos^2 x - 1$. If we take $2x = t$, then $\cos t = 2\cos^2(t/2) - 1$.

$$\int_0^{\pi/2} \sqrt{\cos t + 1}\ \mathrm{d}t = \int_0^{\pi/2} \sqrt{2\cos^2(t/2)}\mathrm{d}t = \sqrt{2}\int_0^{\pi/2} |\cos(t/2)|\mathrm{d}t$$

Over the interval $[0, \frac{\pi}{2}]$, $\cos(t/2) > 0$, so we can drop the absolute values.

$$\begin{aligned}
&= \sqrt{2}\int_0^{\pi/2} \cos(t/2)\mathrm{d}t = \sqrt{2}\left[2\sin\left(\frac{t}{2}\right)\right]_0^{\pi/2} \\
&= 2\sqrt{2}\sin\left(\frac{\pi}{4}\right) = 2
\end{aligned}$$

1.13.31. Solution.

- Solution 1: Using logarithm rules, $\log\sqrt{x} = \log\left(x^{1/2}\right) = \frac{1}{2}\log x$, so we can simplify:
$$\int_1^e \frac{\log\sqrt{x}}{x}\mathrm{d}x = \int_1^e \frac{\log x}{2x}\mathrm{d}x$$
We use the substitution $u = \log x$, $\mathrm{d}u = \frac{1}{x}\mathrm{d}x$:
$$\begin{aligned}
\int_1^e \frac{\log x}{2x}\mathrm{d}x &= \frac{1}{2}\int_1^e \underbrace{\log(x)}_{u}\cdot\underbrace{\frac{1}{x}\,\mathrm{d}x}_{\mathrm{d}u} \\
&= \frac{1}{2}\int_{\log(1)}^{\log(e)} u\,\mathrm{d}u \\
&= \frac{1}{2}\int_0^1 u\,\mathrm{d}u \\
&= \frac{1}{2}\left[\frac{1}{2}u^2\right]_0^1 \\
&= \frac{1}{2}\left[\frac{1}{2} - 0\right] = \frac{1}{4}
\end{aligned}$$
- Solution 2: We use the substitution $u = \log\sqrt{x}$. Then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{\sqrt{x}}\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2x}$, hence $2\mathrm{d}u = \dfrac{1}{x}\mathrm{d}x$. This fits our integral nicely!
$$\begin{aligned}
\int_1^e \frac{\log\sqrt{x}}{x}\mathrm{d}x &= \int_{\log\sqrt{1}}^{\log\sqrt{e}} u\cdot 2\mathrm{d}u \\
&= \left[u^2\right]_0^{1/2} \\
&= \left(\frac{1}{2}\right)^2 - 0^2 = \frac{1}{4}
\end{aligned}$$

1.13.32. Solution.

$$\int_{0.1}^{0.2} \frac{\tan x}{\log(\cos x)}\mathrm{d}x$$

It might not be immediately obvious how to proceed on this one, so this is another example of an integral where you should not be discouraged by finding methods that don't work. One thing that's worked for us in the past is to use a u-substitution with the denominator. With that in mind, let's find the derivative of the denominator.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\log(\cos x)\right\} = \frac{1}{\cos x}\cdot(-\sin x) = \frac{-\sin x}{\cos x} = -\tan x$$

So, if we let $u = \log(\cos x)$, we see $-\mathrm{d}u = \tan x\mathrm{d}x$, which will work for a substitution.

$$\begin{aligned}
\int_{0.1}^{0.2}\frac{\tan x}{\log(\cos x)}\mathrm{d}x &= \int_{\log(\cos(0.1))}^{\log(\cos(0.2))}\frac{-\mathrm{d}u}{u}\\
&= \Big[-\log|u|\Big]_{\log(\cos(0.1))}^{\log(\cos(0.2))}\\
&= -\log|\log(\cos 0.2)| + \log|\log(\cos 0.1)|\\
&= \log\left|\frac{\log(\cos(0.1))}{\log(\cos(0.2))}\right|\\
&= \log\left(\frac{\log(\cos(0.1))}{\log(\cos(0.2))}\right)
\end{aligned}$$

Things to notice: the integrand is only defined when $\log(\cos x)$ exists AND is nonzero. So, for instance, it is not defined when $x = 0$, because then $\log\cos x = \log 1 = 0$, and we can't divide by zero.
In the final simplification, since 0.1 and 0.2 are between 0 and $\pi/2$, the cosine term is positive but less than one, so $\log(\cos 0.1)$ and $\log(\cos 0.2)$ are both negative; then their quotient is positive, so we can drop the absolute value signs.
Using the base change formula, we can also write the final answer as $\log\big(\log_{\cos(0.2)}\cos(0.1)\big)$.

1.13.33. $*$. Solution. (a) Without any other ideas, we see we have a compound function—a function of a function. We often find it useful to substitute for the "inside" function. So, we substitute $u = \log x$, $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$. Then $\mathrm{d}x = x\,\mathrm{d}u = e^u\,\mathrm{d}u$.

$$\int \sin(\log x)\ \mathrm{d}x = \int \sin(u)\ e^u\ \mathrm{d}u$$

We have already seen, in Example 1.7.11, that

$$\int \sin(u)\ e^u\ \mathrm{d}u = \frac{1}{2}e^u\big(\sin u - \cos u\big) + C$$

So,

$$\int \sin(\log x)\ \mathrm{d}x = \frac{1}{2}x\big[\sin(\log x) - \cos(\log x)\big] + C$$

(b) The integrand is of the form $N(x)/D(x)$ with $N(x)$ of lower degree than $D(x)$. So we factor $D(x) = (x-2)(x-3)$ and look for a partial fractions decomposition:

$$\frac{1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

Multiplying through by the denominator yields

$$1 = A(x-3) + B(x-2)$$

Setting $x = 2$ we find:

$$1 = A(2-3) + 0 \implies A = -1$$

Setting $x = 3$ we find:

$$1 = 0 + B(3-2) \implies B = 1$$

So we have found that $A = -1$ and $B = 1$. Therefore

$$\begin{aligned}\int \frac{1}{(x-2)(x-3)}\,\mathrm{d}x &= \int \left(\frac{1}{x-3} - \frac{1}{x-2}\right)\mathrm{d}x \\ &= \log|x-3| - \log|x-2| + C\end{aligned}$$

and the definite integral

$$\begin{aligned}\int_0^1 \frac{1}{(x-2)(x-3)}\,\mathrm{d}x &= \Big[\log|x-3| - \log|x-2|\Big]_0^1 \\ &= \Big[\log 2 - \log 1\Big] - \Big[\log 3 - \log 2\Big] \\ &= 2\log 2 - \log 3 = \log\frac{4}{3}\end{aligned}$$

1.13.34. ***. Solution**. (a) If we expand the integrand, one part of it is quite familiar—a portion of a circle. So, we split the specified integral in two.

$$\int_0^3 (x+1)\sqrt{9-x^2}\,\mathrm{d}x = \int_0^3 \sqrt{9-x^2}\,\mathrm{d}x + \int_0^3 x\sqrt{9-x^2}\,\mathrm{d}x$$

The first piece represents the area above the x-axis and below the curve $y = \sqrt{9-x^2}$, i.e. $x^2 + y^2 = 9$, with $0 \le x \le 3$. That's the area of one quadrant of a disk of radius 3. So

$$\int_0^3 \sqrt{9-x^2}\,\mathrm{d}x = \frac{1}{4}(\pi \cdot 3^2) = \frac{9}{4}\pi$$

For the second part, we substitute $u = 9 - x^2$, $\mathrm{d}u = -2x\,\mathrm{d}x$. Note $u(0) = 9$ and $u(3) = 0$. So,

$$\int_0^3 x\sqrt{9-x^2}\,\mathrm{d}x = \int_9^0 \sqrt{u}\,\frac{\mathrm{d}u}{-2} = -\frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_9^0 = -\frac{1}{2}\left[-\frac{27}{3/2}\right] = 9$$

All together,

$$\int_0^3 (x+1)\sqrt{9-x^2}\,\mathrm{d}x = \frac{9}{4}\pi + 9$$

(b) The integrand is of the form $N(x)/D(x)$ with $D(x)$ already factored and $N(x)$

of lower degree. We immediately look for a partial fractions decomposition:

$$\frac{4x+8}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}.$$

Multiplying through by the denominator yields

$$4x+8 = A(x^2+4) + (Bx+C)(x-2) \tag{$*$}$$

Setting $x = 2$ we find:

$$8+8 = A(4+4)+0 \implies 16 = 8A \implies A = 2$$

Substituting $A = 2$ in $(*)$ gives

$$\begin{aligned} & 4x+8 = A(x^2+4)+(Bx+C)(x-2) \\ \implies \quad & -2x^2+4x = (x-2)(Bx+C) \\ \implies \quad & (-2x)(x-2) = (Bx+C)(x-2) \\ \implies \quad & B = -2, C = 0 \end{aligned}$$

So we have found that $A = 2$, $B = -2$, and $C = 0$. Therefore

$$\begin{aligned} \int \frac{4x+8}{(x-2)(x^2+4)}\,\mathrm{d}x &= \int \left(\frac{2}{x-2} - \frac{2x}{x^2+4}\right)\mathrm{d}x \\ &= 2\log|x-2| - \log(x^2+4) + C \end{aligned}$$

Here the second integral was found just by guessing an antiderivative. Alternatively, one could use the substitution $u = x^2+4$, $\mathrm{d}u = 2x\,\mathrm{d}x$.

(c) The given integral is improper, but only because of its infinite limits of integration. (The integrand is continuous for all real numbers.) So, we'll have to take two limits. Before we do that, though, let's find the antiderivative. We would like to use the substitution $u = e^x$, $\mathrm{d}u = e^x\mathrm{d}x$. That is, $\frac{1}{u}\mathrm{d}u = \mathrm{d}x$.

$$\begin{aligned} \int \frac{1}{e^x+e^{-x}}\mathrm{d}x &= \int \frac{1}{u(u+\frac{1}{u})}\mathrm{d}u = \int \frac{1}{u^2+1} = \arctan u + C \\ &= \arctan(e^x) + C \end{aligned}$$

Now we can deal with the limits of integration.

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\mathrm{d}x &= \int_{-\infty}^{0} \frac{1}{e^x+e^{-x}}\mathrm{d}x + \int_{0}^{\infty} \frac{1}{e^x+e^{-x}}\mathrm{d}x \\ &= \lim_{a\to-\infty}\left[\int_a^0 \frac{1}{e^x+e^{-x}}\mathrm{d}x\right] + \lim_{b\to\infty}\left[\int_0^b \frac{1}{e^x+e^{-x}}\mathrm{d}x\right] \\ &= \lim_{a\to-\infty}\left[\arctan(e^x)\right]_a^0 + \lim_{b\to\infty}\left[\arctan(e^x)\right]_0^b \\ &= \lim_{a\to-\infty}\left[\arctan(e^0) - \arctan(e^a)\right] + \lim_{b\to\infty}\left[\arctan(e^b) - \arctan(e^0)\right] \\ &= \lim_{a\to-\infty}\left[-\arctan(e^a)\right] + \lim_{b\to\infty}\left[\arctan(e^b)\right] \\ &= -\arctan(0) + \frac{\pi}{2} = \frac{\pi}{2} \end{aligned}$$

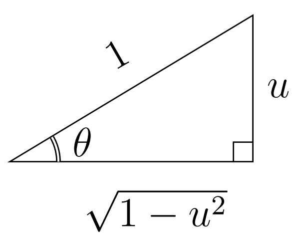
1.13.35. Solution. It's not immediately clear where to start, but a common method we've seen is to use the denominator in a u-substitution, especially when

square roots are involved.
Let $u = \sqrt{1-x}$, $\mathrm{d}u = -\frac{1}{2\sqrt{1-x}}\mathrm{d}x$. Then $u^2 = 1-x$, so $x = 1-u^2$.

$$\int \sqrt{\frac{x}{1-x}}\mathrm{d}x = 2\int \frac{\sqrt{x}}{2\sqrt{1-x}}\mathrm{d}x = -2\int \sqrt{1-u^2}\mathrm{d}u$$

Now we're back in familiar territory. Let $u = \sin\theta$, $\mathrm{d}u = \cos\theta\mathrm{d}\theta$.

$$\begin{aligned}
&= -2\int \sqrt{1-\sin^2\theta}\cos\theta\mathrm{d}\theta \\
&= -2\int \cos^2\theta\mathrm{d}\theta \\
&= -\int \big(1+\cos(2\theta)\big)\mathrm{d}\theta \\
&= -\theta - \frac{1}{2}\sin(2\theta) + C \\
&= -\theta - \sin\theta\cos\theta + C \\
&= -\arcsin u - u\sqrt{1-u^2} + C \qquad (*) \\
&= -\arcsin(\sqrt{1-x}) - \sqrt{1-x}\sqrt{x} + C
\end{aligned}$$



In $(*)$, to convert from θ to u, our substitution $u = \sin\theta$ tells us $\theta = \arcsin u$. To find $\cos\theta$, we can either trace our work backwards to see that we already simplified $\sqrt{1-u^2}$ into $\cos\theta$, or we can draw a right triangle with angle θ and $\sin\theta = u$, then use the Pythagorean theorem to find the length of the adjacent side of the triangle and $\cos\theta$.

1.13.36. Solution. Let's use the substitution $u = e^x$. There are a few reasons to think this is a good choice. It's an "inside function," in that if we let $f(x) = e^x$, then $f(e^x) = e^{e^x}$, which is a piece of our integrand. Also its derivative, e^x, is multiplied by the rest of the integrand, since $e^{2x} = e^x \cdot e^x$.
Let $u = e^x$, $\mathrm{d}u = e^x\mathrm{d}x$. When $x = 0$, $u = 1$, and when $x = 1$, $u = e$.

$$\int_0^1 e^{2x}e^{e^x}\mathrm{d}x = \int_0^1 e^x e^{e^x} e^x\mathrm{d}x = \int_1^e ue^u\mathrm{d}u$$

This is more familiar. We use integration by parts with $\mathrm{d}v = e^u \mathrm{d}u$, $v = e^u$. Conveniently, the "u" we brought in with the substitution is what we want to use for the "u" in integration by parts, so we don't have to change the names of our variables.

$$\begin{aligned}&= \left[ue^u\right]_1^e - \int_1^e e^u \mathrm{d}u\\&= e \cdot e^e - e - e^e + e = e^e(e-1)\end{aligned}$$

1.13.37. Solution. The substitution $u = x + 1$ looks promising at first, but doesn't result in something easily integrable. We can't use partial fractions because our integration isn't rational. This doesn't look like something from the trig-substitution family. So, let's think about integration by parts. There's a lot of different ways we could break up the integrand into two parts. For example, we could view it as $\left(\frac{x}{(x+1)^2}\right)\left(e^x\right)$, or we could view it as $\left(\frac{x}{x+1}\right)\left(\frac{e^x}{x+1}\right)$. After some trial and error, we settle on $u = xe^x$ and $\mathrm{d}v = (x+1)^{-2}\mathrm{d}x$. Then $\mathrm{d}u = e^x(x+1)$ and $v = \frac{-1}{x+1}$.

$$\begin{aligned}\int \frac{xe^x}{(x+1)^2}\mathrm{d}x &= -\frac{xe^x}{x+1} + \int \frac{e^x(x+1)}{x+1}\mathrm{d}x\\&= -\frac{xe^x}{x+1} + \int e^x \mathrm{d}x\\&= -\frac{xe^x}{x+1} + e^x + C\\&= \frac{e^x}{x+1} + C\end{aligned}$$

1.13.38. Solution. It would be nice to use integration by parts with $u = x$, because then we would integrate $\int v\mathrm{d}u$, and $\mathrm{d}u = \mathrm{d}x$. That is, the x would go away, and we'd be left with a pure trig integral. If we use $u = x$, then $\mathrm{d}v = \frac{\sin x}{\cos^2 x}$. We need to find v:

$$v = \int \frac{\sin x}{\cos^2 x}\mathrm{d}x = \int \tan x \sec x \mathrm{d}x = \sec x$$

Now we use integration by parts.

$$\int \frac{x \sin x}{\cos^2 x}\mathrm{d}x = x\sec x - \int \sec x \mathrm{d}x = x \sec x - \log|\sec x + \tan x| + C$$

1.13.39. Solution. If the unknown exponent gives you the jitters, think about what this looks like in easier cases. If n is a whole number, the integrand is a polynomial. Not so scary, right? However, it's a little complicated to expand. (You can do it using the very handy binomial theorem.) Let's think of an easier way.
If we had simply the variable x raised to the power n, rather than the binomial $x + a$, that might be nicer. So, let's use the substitution $u = x + a$, $\mathrm{d}u = \mathrm{d}x$. Note $x = u - a$.

$$\int x(x+a)^n \mathrm{d}x = \int (u-a)u^n \mathrm{d}x = \int \left(u^{n+1} - au^n\right)\mathrm{d}u$$

Now, if $n \neq -1$ and $n \neq -2$, we can just use the power rule:

$$\begin{aligned} &= \frac{u^{(n+2)}}{n+2} - a\frac{u^{n+1}}{n+1} + C \\ &= \frac{(x+a)^{(n+2)}}{n+2} - a\frac{(x+a)^{n+1}}{n+1} + C \end{aligned}$$

If $n = -1$, then

$$\begin{aligned} \int x(x+a)^n \mathrm{d}x = \int \left(u^{n+1} - au^n\right)\mathrm{d}u &= \int \left(1 - \frac{a}{u}\right)\mathrm{d}u \\ &= u - a\log|u| + C = (x+a) - a\log|x+a| + C \end{aligned}$$

If $n = -2$, then

$$\begin{aligned} \int x(x+a)^n \mathrm{d}x = \int \left(u^{n+1} - au^n\right)\mathrm{d}u &= \int \left(\frac{1}{u} - au^{-2}\right)\mathrm{d}u \\ &= \log|u| + \frac{a}{u} + C = \log|x+a| + \frac{a}{x+a} + C \end{aligned}$$

All together,

$$\int x(x+a)^n \mathrm{d}x = \begin{cases} \frac{(x+a)^{(n+2)}}{n+2} - a\frac{(x+a)^{n+1}}{n+1} + C & \text{if } n \neq -1, -2 \\ (x+a) - a\log|a+x| + C & \text{if } n = -1 \\ \log|x+a| + \frac{a}{x+a} + C & \text{if } n = -2 \end{cases}$$

1.13.40. Solution. We've seen how to antidifferentiate $\arctan x$: integration by parts. Let's hope the same thing will work here.

Step 1: integration by parts.

Let $u = \arctan(x^2)$ and $\mathrm{d}v = \mathrm{d}x$. Then $\mathrm{d}u = \frac{2x}{x^4+1}\mathrm{d}u$ and $v = x$.

$$\int \arctan(x^2)\mathrm{d}x = x\arctan(x^2) - \int \frac{2x^2}{x^4+1}\mathrm{d}x$$

Now we have a rational function. There's no obvious substitution, but we can use partial fractions. The degree of the numerator is strictly less than the degree of the denominator, so we don't need to long divide first. We do, however, need to factor the denominator. It's a common function, so you might already know the factorization, or you might be able to guess it. Below, we show another way to find the factorization, similar to the method of partial fractions.

Step 2: factor $\mathbf{x^4+1}$. For any real x, note $x^4 + 1 > 0$. Since it has no roots, it has no linear factors. That means it factors as the product of two irreducible quadratics. That is,

$$x^4 + 1 = (ax^2 + bx + c)(dx^2 + ex + f)$$

Since the coefficient of x^4 on the left-hand is 1, we may assume $a = d = 1$.

$$x^4 + 1 = (x^2 + bx + c)(x^2 + ex + f)$$

Since the constant term is 1, $cf = 1$. That is, $f = \frac{1}{c}$.

$$\begin{aligned} x^4 + 1 &= (x^2 + bx + c)(x^2 + ex + 1/c) \\ &= x^4 + \underbrace{(b+e)}_{(1)} x^3 + \underbrace{\left(\frac{1}{c} + be + c\right)}_{(3)} x^2 + \underbrace{\left(\frac{b}{c} + ec\right)}_{(2)} x + 1 \end{aligned}$$

1 The coefficient of x^3 tells us $e = -b$.

2 Then the coefficient of x tells us $0 = \frac{b}{c} + ec = \frac{b}{c} - bc$. So, $c = \frac{1}{c}$, hence $c = \pm 1$.

3 Finally, the coefficient of x^2 tells us $0 = \frac{1}{c} + be + c = \frac{1}{c} - b^2 + c$. Since $-b^2$ is negative (or zero), $\frac{1}{c} + c$ is positive, so $c = 1$. That is, $0 = 1 - b^2 + 1$. So, $b = \sqrt{2}$.

All together,

$$x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$$

Step 3: partial fraction decomposition.
Now that we have the denominator factored into irreducible quadratics, we can find the partial fraction decomposition of the integrand.

$$\begin{aligned} \frac{2x^2}{x^4+1} &= \frac{Ax + B}{x^2 + \sqrt{2}x + 1} + \frac{Cx + D}{x^2 - \sqrt{2}x + 1} \\ 2x^2 &= (Ax + B)(x^2 - \sqrt{2}x + 1) + (Cx + D)(x^2 + \sqrt{2}x + 1) \\ &= (A + C)x^3 + (B + D - \sqrt{2}A + \sqrt{2}C)x^2 \\ &\qquad + (A + C - \sqrt{2}B + \sqrt{2}D)x + (B + D) \end{aligned}$$

From the coefficient of x^3, we see $C = -A$.

$$2x^2 = (B + D - 2\sqrt{2}A)x^2 + (-\sqrt{2}B + \sqrt{2}D)x + (B + D)$$

From the constant term, we see $D = -B$.

$$2x^2 = (-2\sqrt{2}A)x^2 + (-2\sqrt{2}B)x$$

From the coefficient of x^2, we see $-2\sqrt{2}A = 2$, so $A = -1/\sqrt{2}$. Since $C = -A$, then $C = 1/\sqrt{2}$.
From the coefficient of x, we see $B = 0$. Since $D = -B$, also $D = 0$.
Step 4: integration.

$$\begin{aligned} \int \frac{2x^2}{x^4+1}\mathrm{d}x &= \int \left(\frac{(-1/\sqrt{2})x}{x^2 + \sqrt{2}x + 1} + \frac{(1/\sqrt{2})x}{x^2 - \sqrt{2}x + 1} \right) \mathrm{d}x \\ &= \frac{1}{\sqrt{2}} \int \left(\frac{-x}{x^2 + \sqrt{2}x + 1} + \frac{x}{x^2 - \sqrt{2}x + 1} \right) \mathrm{d}x \end{aligned}$$

To integrate, we want to break the fractions into two pieces each: one we can integrate with a substitution $u = x^2 \pm \sqrt{2}x + 1$, $du = \left(2x \pm \sqrt{2}\right)dx$ (shown in blue), and one that looks like the derivative of arctangent (shown in red).

$$= \frac{1}{\sqrt{2}} \int \left(\frac{-x - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}{x^2 + \sqrt{2}x + 1} + \frac{x - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}{x^2 - \sqrt{2}x + 1} \right) dx$$

$$= \frac{1}{\sqrt{2}} \int \left(\frac{-\frac{1}{2}(2x + \sqrt{2})}{x^2 + \sqrt{2}x + 1} + \frac{\frac{\sqrt{2}}{2}}{x^2 + \sqrt{2}x + 1} \right.$$
$$\left. + \frac{\frac{1}{2}(2x - \sqrt{2})}{x^2 - \sqrt{2}x + 1} + \frac{\frac{\sqrt{2}}{2}}{x^2 - \sqrt{2}x + 1} \right) dx$$

$$= \frac{1}{\sqrt{2}} \left(-\frac{1}{2} \log \left| x^2 + \sqrt{2}x + 1 \right| + \int \frac{\frac{\sqrt{2}}{2}}{x^2 + \sqrt{2}x + 1} dx \right.$$
$$\left. + \frac{1}{2} \log |x^2 - \sqrt{2}x + 1| + \int \frac{\frac{\sqrt{2}}{2}}{x^2 - \sqrt{2}x + 1} dx \right)$$

We use logarithm rules to compress our work. In order to evaluate the remaining integrals, we complete the squares of the denominators.

$$= \frac{1}{\sqrt{2}} \left(\frac{1}{2} \log \left| \frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1} \right| + \int \frac{\frac{\sqrt{2}}{2}}{\left(x + \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}} dx \right.$$
$$\left. + \int \frac{\frac{\sqrt{2}}{2}}{\left(x - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}} dx \right)$$

$$= \frac{1}{\sqrt{2}} \left(\frac{1}{2} \log \left| \frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1} \right| + \int \frac{\sqrt{2}}{\left(\sqrt{2}x + 1\right)^2 + 1} dx \right.$$
$$\left. + \int \frac{\sqrt{2}}{\left(\sqrt{2}x - 1\right)^2 + 1} dx \right)$$

Now, we can either guess the antiderivatives of the remaining integrals, or use the substitutions $u = (\sqrt{2}x \pm 1)$.

$$= \frac{1}{\sqrt{2}} \left(\frac{1}{2} \log \left| \frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1} \right| + \arctan\left(\sqrt{2}x + 1\right) + \arctan\left(\sqrt{2}x - 1\right) \right) + C$$

Step 5: finishing touches.

Finally, we can put our work together. (Remember way back in Step 1, we used integration by parts.)

$$\int \arctan(x^2) dx = x \arctan(x^2) - \int \frac{2x^2}{x^4 - 1} dx$$
$$= x \arctan(x^2) - \frac{1}{\sqrt{2}} \left(\frac{1}{2} \log \left| \frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1} \right| \right.$$

$$+\arctan\left(\sqrt{2}x+1\right)+\arctan\left(\sqrt{2}x-1\right)\Bigg)+C$$

Remark: although this integral calculation was longer than average, it didn't use any new ideas (except for the factoring of x^4+1 mentioned in the hint). It's good exercise to apply familiar techniques in challenging situations, to deepen your mastery.

2 · Applications of Integration

2.1 · Work

2.1.2 · Exercises

Exercises — Stage 1

2.1.2.1. Solution. Force is mass × acceleration (with acceleration equal to g in this problem), and both in this scenario are constant, so we don't need an integral — only a product — to calculate the force acting on the block.

To find the force in newtons, recall one newton is one $\frac{\text{kg}\cdot\text{m}}{\text{sec}^2}$, so we need the mass of our block in kg. Specifically, our block has mass $\frac{3}{1000}$ kg. So, the force involved is

$$F=\left(\frac{3}{1000}\text{ kg}\right)\times\left(9.8\frac{\text{m}}{\text{sec}^2}\right)=0.0294\frac{\text{kg}\cdot\text{m}}{\text{sec}^2}=0.0294\text{N}$$

To find the work in joules, recall one joule is one newton-metre: that is, one newton of force acting over one metre. So, we need our distance in metres.

$$W=\left(0.0294\text{N}\right)\times\left(\frac{1}{10}\text{m}\right)=0.00294\text{N}\cdot\text{m}=0.00294\text{J}$$

2.1.2.2. Solution. The force of the rock is one newton, or one kilogram-metre per second squared, so

$$1\frac{\text{kg}\cdot\text{m}}{\text{sec}^2}=(x\text{kg})\left(9.8\frac{\text{m}}{\text{sec}^2}\right)$$

Therefore, the mass of the rock is $\frac{1}{9.8}$ kg, or about 102 grams.

Now, since one joule is one newton-metre, the amount of work required to counteract 1 N of gravitational force for one metre is precisely one joule.

Remark: having an idea of how much work a joule is, and how much force a newton is, is a good tool for checking the reasonableness of your work. For example, after this question, if you calculate that a marble weighs 100 N, you can be pretty sure there's an error in your calculation.

2.1.2.3. Solution.

a We defined $\Delta x=\frac{b-a}{n}$: that is, the length of one interval, when we chop $[a,b]$ into n of them. If b and a are measured in metres, then Δx is measured in metres as well. So, the units of Δx are metres.

Put another way, since a and b both describe a quantity in metres, $b-a$ describes a quantity in metres as well. (When we add or subtract quantities of the same units, their sum or difference is given in the same units.) Since n is a unitless quantity (simply a number: not "n kg" or "n m"), $\frac{b-a}{n}$ still describes a quantity in metres. (If I have 6 metres of cloth, and I cut it into 3 pieces, each piece has $\frac{6}{3} = 2$ metres — not 2 kilograms, or 2 metres per second.)

b Since $F(x)$ is measured in kilogram-metres per second squared (newtons), the units of $F(x_i)$ are kilogram-metres per second squared (newtons).

c W is calculated by adding up summands of the form $F(x_i)\Delta x$. The units of $F(x_i)\Delta x$ are the products of the units of $F(x_i)$ with the units of Δx. That is, the units of $F(x_i)\Delta x$ are $\left(\frac{\text{kg}\cdot\text{m}}{\text{sec}^2}\right)(\text{m}) = \frac{\text{kg}\cdot\text{m}^2}{\text{sec}^2} = J$. The sum of terms given in joules is itself given in joules, so the units of W are joules.

2.1.2.4. Solution. As we saw in Question 3, the units of $\int_a^b f(x)\,\mathrm{d}x$ are simply the units of the integrand, $f(x)$, multiplied by the units of the variable of integration, x. In this case, that yields $\frac{\text{smoot}\cdot\text{barn}}{\text{megaFonzie}}$ (that is, smoot-barns per megaFonzie).

2.1.2.5. Solution. Hooke's law says that the force required to stretch a spring x units past its natural length is proportional to x; that is, there is some constant k associated with the individual spring such that the force required to stretch it x m past its natural length is kx.

- Solution 1: Since the force required to stretch the spring is proportional to the amount stretched, and the force acting on the spring is proportional to the mass hanging from it, we conclude the amount the spring stretches is proportional to the mass hung from it. So, if 1 kg stretches it 1 cm, then 10 kg will stretch it 10 cm. We should mark the wall 10 cm below the bottom of the spring as it hangs unloaded.

- Solution 2: We can find k from the test with the bag of water. The force exerted by the bag of water was $(1\text{ kg})(9.8\text{ m/sec}^2) = 9.8\text{ N} = k(1\text{ cm})$. So,

$$k = \frac{9.8\frac{\text{kg}\cdot\text{m}}{\text{sec}^2}}{0.01\text{ m}} = 980\frac{\text{kg}}{\text{sec}^2}$$

If we hang 10 kg from the spring, gravity exerts a force of $(10\text{ kg})(9.8\text{ m/sec}^2) = 98\frac{\text{kg}\cdot\text{m}}{\text{sec}^2}$. This will be matched by the spring with a force of kx newtons, where k is the spring constant and x is the amount stretched.

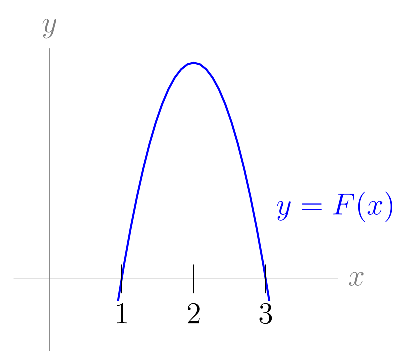
$$\begin{aligned} kx &= 98\frac{\text{kg}\cdot\text{m}}{\text{sec}^2} \\ \left(980\frac{\text{kg}}{\text{sec}^2}\right)(x\text{ m}) &= 98\frac{\text{kg}\cdot\text{m}}{\text{sec}^2} \\ x &= \frac{1}{10}\text{ m} = 10\text{ cm} \end{aligned}$$

So, we should put the mark at 10 cm below the natural length of the spring.

2.1.2.6. Solution. Definition 2.1.1 tells us the work done by the force is $W(b) = \int_1^b F(x)\mathrm{d}x$, where $F(x)$ is the force on the object at position x. So, by the Fundamental Theorem of Calculus Part 1,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}b}\{W(b)\} = \frac{\mathrm{d}}{\mathrm{d}b}\left\{\int_1^b F(x)\mathrm{d}x\right\} &= F(b) \\ \frac{\mathrm{d}}{\mathrm{d}b}\left\{-b^3 + 6b^2 - 9b + 4\right\} &= F(b) \\ -3b^2 + 12b - 9 &= F(b) \\ -3(b-1)(b-3) &= F(b) \end{aligned}$$

So, $F(x)$ is the quadratic polynomial $-3(x-1)(x-3)$.



The largest absolute value of $F(x)$ over $[1,3]$ occurs at $x = 2$. At this point, we have our strongest force.

Exercises — Stage 2

2.1.2.7. *. Solution. By Definition 2.1.1, the work done in moving the object from $x = 1$ meters to $x = 16$ meters by the force $F(x)$ is

$$W = \int_1^{16} F(x)\mathrm{d}x = \int_1^{16} \frac{a}{\sqrt{x}}\mathrm{d}x = \left[2a\sqrt{x}\right]_{x=1}^{x=16} = 6a$$

To have $W = 18$, we need $a = 3$.
As a side remark, $F(x) = \frac{a}{\sqrt{x}}$ should have units Newtons. Since x, a distance, is measured in meters, a has to have the bizarre units newton-$\sqrt{\text{meters}}$.

2.1.2.8. Solution.

a Since $\frac{c}{\ell - x}$ is measured in newtons, and ℓ and x (and therefore $\ell - x$) are measured in metres, the units of c are newton-metres, i.e. joules.

b Following Definition 2.1.1, the work done compressing the air is

$$W = \int_1^{1.5} F(x)\,\mathrm{d}x$$

where $F(x)$ is the amount of force applied when the plunger is x metres past its natural position. The amount of force applied is equal in magnitude to the amount of force supplied by the tube: $\frac{c}{\ell - x}$ N. Note ℓ and c are constants. We can guess the antiderivative, or use the substitution $u = \ell - x$, $\mathrm{d}u = -\mathrm{d}x$.

$$\begin{aligned} W &= \int_1^{1.5} \frac{c}{\ell - x}\,\mathrm{d}x = \big[-c\log|\ell - x| \big]_1^{1.5} \\ &= -c\big[\log|\ell - 1.5| - \log|\ell - 1| \big] \\ &= -c\log\left(\frac{\ell - 1.5}{\ell - 1} \right) \\ &= c\log\left(\frac{\ell - 1}{\ell - 1.5} \right) \text{ J} \end{aligned}$$

Note that, because $\ell > 1.5$, the argument of logarithm is positive, so we don't need the absolute value signs. Furthermore, $\ell - 1 > \ell - 1.5$, so $\frac{\ell-1}{\ell-1.5} > 1$, hence $\log\left(\frac{\ell-1}{\ell-1.5}\right) > 0$.

2.1.2.9. *. Solution. By Hooke's Law, the force exerted by the spring at displacement x m from its natural length is $F = kx$, where k is the spring constant. Measuring distance in meters and force in newtons (since one joule is one newton-metre), the total work is

$$\int_0^{0.1\text{ m}} kx\,\mathrm{d}x = \left[\frac{1}{2}kx^2\right]_0^{0.1\text{ m}} = \frac{1}{2}\cdot \underbrace{50}_{\text{N/m}} \cdot \underbrace{(0.1)^2}_{\text{m}^2} = \frac{1}{4}\text{J}.$$

Note the units of the integrand (kx) are newtons, and the units of the variable of integration, x, are metres. So, the evaluated integral has units newton-metres, or joules.

2.1.2.10. *. Solution. First note that newtons and joules are SI units with one joule equal to one newton-metre, so we should measure distances in meters rather than centimeters. Next recall that a spring with spring constant k exerts a force $F(x) = kx$ when the spring is stretched x m beyond its natural length. So in this case $(0.05\text{ m})(k) = 10$ N, or $k = 200$ N/m. The work done is:

$$\int_0^{0.5\text{ m}} F(x)\mathrm{d}x = \int_0^{0.5} 200x\mathrm{d}x = \left[100x^2\right]_0^{0.5} = 25\text{ J}$$

Note the units of the integrand ($F(x) = kx = 200x$) are newtons (k is given in N/m, and x is given in m). The units of the variable of integration, x are metres. So, the evaluated integral has units newton-metres, or joules.

2.1.2.11. *. Solution. Note that the cable has mass density $\frac{8}{5}$ kg/m. When the bucket is at height y, the cable that remains to be lifted has length $(5 - y)$ m and mass $\frac{8}{5}(5 - y) = 8\big(1 - \frac{y}{5}\big)$ kg. So, at height y, the cable is subject to a downward gravitational force of $8\big(1 - \frac{y}{5}\big) \cdot 9.8$ N; to raise the cable we need to apply a compensating upward force of $8\big(1-\frac{y}{5}\big)\cdot 9.8$ N. So, the work required is

$$\begin{aligned}\int_0^5 8\Big(1-\frac{y}{5}\Big)\cdot 9.8\,\mathrm{d}y &= 8\bigg[\Big(y-\frac{y^2}{10}\Big)\cdot 9.8\bigg]_0^5\\ &= 8\cdot 2.5\cdot 9.8 \quad \mathrm{N}\cdot\mathrm{m} = 196\ \mathrm{J}.\end{aligned}$$

Alternatively, the cable has linear density 8 kg/5 m = 1.6 kg/m, and so the work required to lift a small piece of the cable (of length Δy) from height y m to height 5 m is

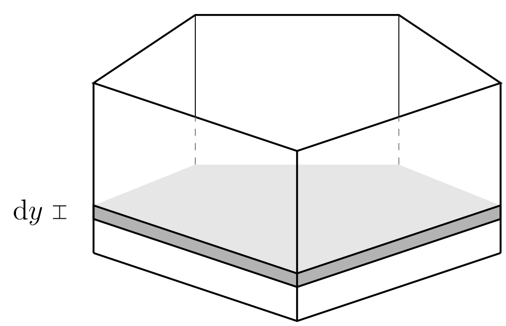
$$\underbrace{\underbrace{1.6\Delta y}_{\text{mass}}\cdot\underbrace{9.8}_{\text{gravity}}}_{\text{force}}\cdot\underbrace{(5-y)}_{\text{distance}}.$$

The total work required is therefore

$$\begin{aligned}\int_0^5 1.6\cdot 9.8(5-y)\,\mathrm{d}y &= 1.6\cdot 9.8\left[5y-\frac{1}{2}y^2\right]_0^5\\ &= 1.6\cdot 9.8\cdot\left(25-\frac{25}{2}\right) = 196\ \mathrm{J}\end{aligned}$$

as before.

2.1.2.12. Solution. Imagine pumping out a thin, horizontal layer of water that is at height y — that is, y metres above the bottom of the tank. Let the width of the layer be $\mathrm{d}y$.



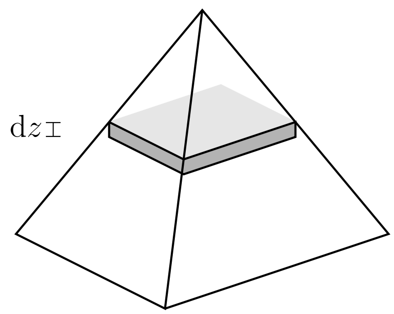
- The volume of water in the layer is $3\mathrm{d}y$ m^3 (since the cross-section has area 3 m^3).
- One cubic metre is equal to 100^3 cubic centimetres. So, the mass of water in one cubic metre is $\dfrac{100^3}{1000} = 1\,000$ kg.
- Therefore, the mass of water in our layer is $(3\,000\,\mathrm{d}y)$ kg.
- The force of gravity acting on it is $(-9.8\times 3\,000\,\mathrm{d}y)$ N, so we need to pump with a compensating force of $(9.8\times 3\,000\,\mathrm{d}y)$ N.
- The water needs to be pumped a distance of $1-y$ metres.

- So, the work required to pump out the thin layer of water at height y is $(9.8 \times 3\,000 \times (1-y)\,\mathrm{d}y)$ J.

So, all together, the work to pump out the entire tank is

$$\int_0^1 9.8 \times 3\,000 \times (1-y)\,\mathrm{d}y = 9.8 \times 3000 \times \left[y - \frac{1}{2}y^2\right]_0^1 = 14\,700 \text{ J}$$

2.1.2.13. $*$. **Solution**. We can model the sculpture as a collection of thin horizontal plates of width $\mathrm{d}z$. Remember work is force times distance; a horizontal plate at height z moved $z+2$ metres from the basement to its final position. So, we need to know the force acting on the plate, which is the product of the mass of the plate with the acceleration due to gravity. Since we are given the density of iron, if we find the volume of the plate, then we can calculate its mass.



The plate at height z

- has side length $3-z$ m and hence
- has area $(3-z)^2$ m^2 and hence
- has volume $(3-z)^2\,\mathrm{d}z$ m^3 and hence
- has mass $8000(3-z)^2\,\mathrm{d}z$ kg and hence
- is subject to a gravitational force of $9.8 \times 8000(3-z)^2\,\mathrm{d}z$ N and hence
- requires work $9.8 \times 8000(2+z)(3-z)^2\,\mathrm{d}z$ J to raise it from 2 m below ground level to z m above ground level.

So the total work is

$$\int_0^3 9.8 \times 8000(2+z)(3-z)^2\,\mathrm{d}z \text{ joules}$$

2.1.2.14. Solution. From the information given about the hanging kilogram, we can find the spring constant k. One kilogram generates a force of 9.8 N under gravity. (We find this by the calculation (1 kg) $\times$ (9.8m/sec^2) = 9.8 N.) This force is matched by the force of the spring, which by Hooke's law is equal to $k(\frac{1}{20}\text{m})$. So,

$$k = \frac{9.8 \text{ N}}{\frac{1}{20} \text{ m}} = 196\frac{\text{N}}{\text{m}}$$

Again by Hooke's law, the force required to stretch the spring x metres past its natural length is $196x$ N (when x is measured in metres).
So, the work required to stretch the spring from 5 cm past its natural length to 7 cm past its natural length is

$$W = \int_{0.05}^{0.07} 196x \, \mathrm{d}x = \left[98x^2\right]_{0.05}^{0.07} = 0.2352 \text{ J}$$

2.1.2.15. Solution. Let M be the mass of the rope. Then its density is $\frac{M}{4}$ kg/m. Following the method of Example 2.1.6, we let y be the height of the firewood above the ground, so the wood is raised from $y = 0$ to $y = 4$. When the wood is at height y,

- the rope that remains to be lifted has length $4-y$, and so it has mass $\frac{M}{4}(4-y)$ kg,
- and the firewood still has mass 10 kg.
- The remaining rope and the wood are subject to a downward gravitational force of magnitude $\underbrace{\left[\frac{M}{4}(4-y)+10\right]}_{\text{mass}} \times 9.8 \quad \text{N}.$
- So, to raise the firewood from height y to height $(y + \mathrm{d}y)$, we need to apply a compensating upward force of $\left[\frac{M}{4}(4-y)+10\right] \times 9.8$ through distance $\mathrm{d}y$. This takes work $\left[\frac{M}{4}(4-y)+10\right] \times 9.8 \, \mathrm{d}y$ J.

All together, the work involved in hauling up the wood is

$$\begin{aligned}\int_0^4 \left(\left[\frac{M}{4}(4-y)+10\right] \times 9.8\right) \mathrm{d}y &= 9.8 \int_0^4 \left((M+10) - \frac{M}{4}y\right) \mathrm{d}y \\ &= 9.8(2M+40) \text{ J}\end{aligned}$$

Since the work was 400 joules, solving $400 = 9.8(2M + 40)$ for M tells us the mass of the rope is $\frac{200}{9.8} - 20 = \frac{20}{49}$ kg, or about 408 g.
Alternately, the work involved in lifting up the wood is $10 \times 9.8 \times 4 = 392$ J, so the work in lifting up the rope is 8 J. A small section of rope of length $\mathrm{d}y$, that starts at height y above the ground, has mass $\frac{M}{4}\mathrm{d}y$ kg and is lifted $(4 - y)$ metres, so the work involved in lifting this section of rope is $9.8 \times (4-y) \times \frac{M}{4}\mathrm{d}y$. Then the amount of work to lift the whole rope (but not the wood) is

$$\begin{aligned}8 \text{ J} &= \int_0^4 \left(9.8 \times (4-y) \times \frac{M}{4}\right) \mathrm{d}y = \frac{9.8 \times M}{4} \int_0^4 (4-y)\mathrm{d}y \\ &= \frac{9.8 \times M}{4} \times 8\end{aligned}$$

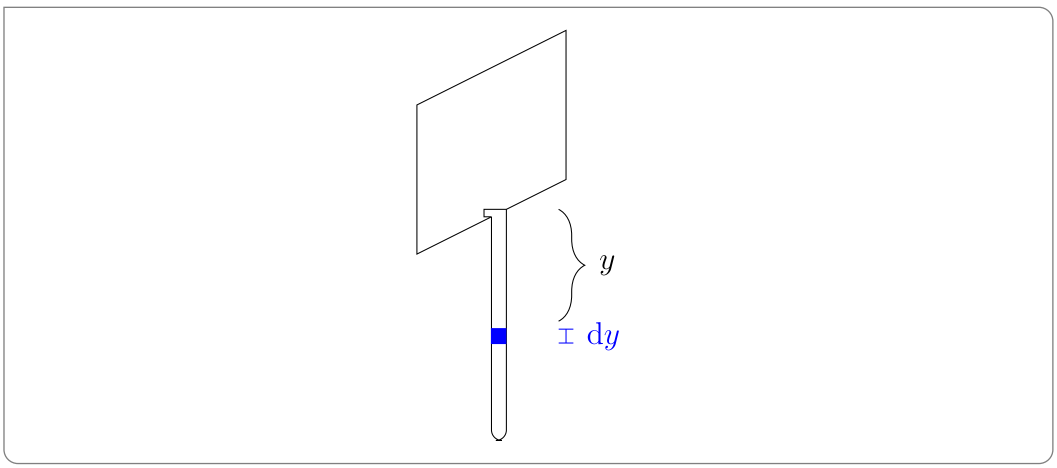
which again results in $M = \frac{4}{9.8} = \frac{20}{49}$ kg.

2.1.2.16. Solution. For Questions 11 and 15 in this section, we gave two methods for finding the work involved in pulling up a cable: one where we consider pulling up the entire remaining cable a tiny distance of $\mathrm{d}y$, and one where we consider pulling a tiny slice of cable of length $\mathrm{d}y$ the entire distance up.

There is another variation we can consider with the weight: we can either calculate the work done on the weight and the work done on the rope separately, or we can calculate them together. If we calculate them together, then there are two cases to consider: the work done pulling up the first 5 metres of rope involves the weight, while the last 5 metres does not. These two choices (how to model the rope, and how to deal with the weight) actually lead to four solutions, but to avoid unnecessary repetition only two are presented below.

- Solution 1: In this solution, we consider the work on the rope separately from work on the weight, and we imagine lifting a tiny piece of rope the entire distance to the window.

  The weight has a mass of 5 kg, and is lifted a distance of 5 m to the window. The force of gravity acting on the weight is (5 kg)(9.8 m/sec^2) = 49 N, so the work to lift it 5 metres is (49 N)(5 m) = 245J.

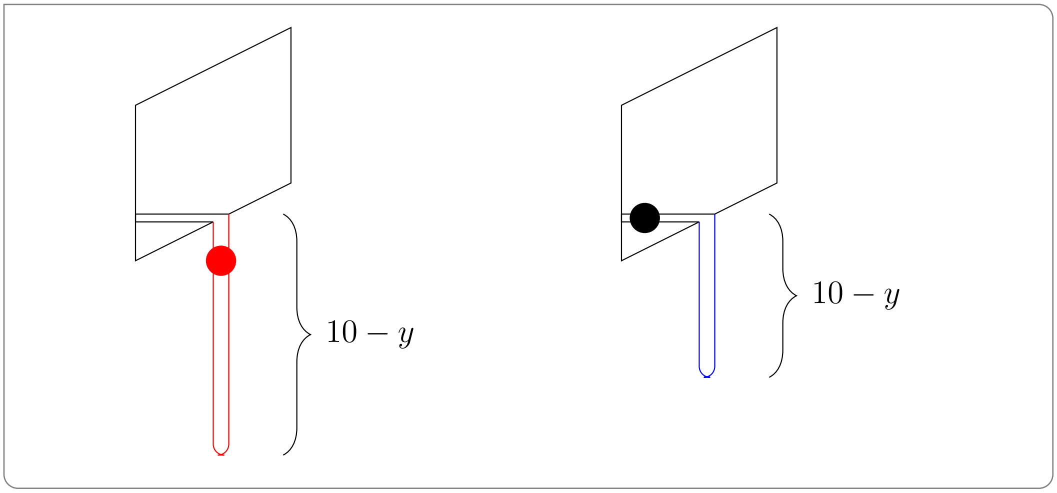
  The density of the rope is $\frac{1}{10}$ kg/m. A tiny piece of rope of length $\mathrm{d}y$, hanging y metres from the window, has mass $(\frac{1}{10}\,\mathrm{d}y)$ kg, and needs to be lifted y metres. So, the force of gravity acting on the piece of rope is $\left(\frac{1}{10}\,\mathrm{d}y \text{ kg}\right)\left(9.8 \text{ m/sec}^2\right) = 0.98\,\mathrm{d}y$ N, and the work to pull it up to the window is $(0.98y\,\mathrm{d}y)$ J. So, the total work to pull up the rope is

$$\int_0^{10} 0.98y\,\mathrm{d}y = 0.98\left[\frac{y^2}{2}\right]_0^{10} = 49 \text{ J}$$

  All together, the work to pull up the rope with the weight is $245 + 49 = 294$ J.

- Solution 2: In this solution, we consider the work on the rope together with the weight, and we imagine lifting the remaining rope a tiny distance to the window.

  Suppose y metres of the rope have been pulled in, and $0 \le y \le 5$ (shown on the left, below). Then the remaining rope has length $10-y$, and contains the weight, so the mass remaining to be pulled up is $\underbrace{\frac{1}{10}(10-y)}_{\text{rope}}+\underbrace{5}_{\text{weight}}=6-\frac{y}{10}$ kg. Then the force of gravity acting on the dangling rope and weight is $(9.8\ \mathrm{m/sec}^2)((6-\frac{y}{10})\ \mathrm{kg})=(58.8-0.98y)$ N. The work needed to lift this rope $\mathrm{d}y$ metres is $(58.8-0.98y)\,\mathrm{d}yJ$.



Now, suppose y metres of the rope have been pulled in, and $5<y\le 10$ (shown above, right). Then the remaining rope has length $10-y$, but *does not contain* the weight, so the mass remaining to be pulled up is $\frac{1}{10}(10-y)=1-\frac{y}{10}$ kg. Then the force of gravity acting on the dangling rope is $(9.8\ \mathrm{m/sec}^2)((1-\frac{y}{10})\ \mathrm{kg})=(9.8-0.98y)$ N. The work needed to lift this rope $\mathrm{d}y$ metres is $(9.8-0.98y)\,\mathrm{d}yJ$.

All together, the work needed to lift the rope is

$$\begin{aligned}W&=\int_0^{10}F(y)\mathrm{d}y\\&=\int_0^5(58.8-0.98y)\,\mathrm{d}y+\int_5^{10}(9.8-0.98y)\,\mathrm{d}y\\&=\left[58.8y-0.49y^2\right]_0^5+\left[9.8y-0.49y^2\right]_5^{10}\\&=294\ \mathrm{J}\end{aligned}$$

2.1.2.17. Solution.

a The frictional force is $\mu\times m\times g=0.4\,(10\ \mathrm{kg})\left(9.8\ \frac{\mathrm{m}}{\mathrm{sec}^2}\right)=39.2\ \frac{\mathrm{kg\cdot m}}{\mathrm{sec}^2}=39.2$ N. Since this constant force acts over a distance of 3 metres, the work is $3\times 39.2=117.6$ J.

In the case of a constant force, we don't need to use an integral, but we could if we wanted:

$$W=\int_0^3 39.2\mathrm{d}x=\left[39.2x\right]_0^3=39.2\times 3=117.6\ \mathrm{J}.$$

b Since the box is moving at a speed of 1 m/sec, at time t we can say the box is at position t, $0 \le t \le 3$. At position t, the mass of the box is $(10-\sqrt{t})$ kg, so the frictional force is $0.4 \times m \times g = 0.4\left(10-\sqrt{t}\text{kg}\right)\left(9.8\frac{\text{m}}{\text{sec}^2}\right) = 3.92(10-\sqrt{t})\text{N}$. Now that we know the force, to find the work we simply integrate, following Definition 2.1.1:

$$\begin{aligned} W &= \int_0^3 3.92(10-\sqrt{t})\mathrm{d}t = 3.92\left[10t - \frac{2}{3}t^{3/2}\right]_0^3 \\ &= 3.92\left[30 - \frac{2}{3}\sqrt{3}^3\right] = 3.92\left[30 - 2\sqrt{3}\right] \approx 104 \text{ J} \end{aligned}$$

2.1.2.18. Solution. Definition 2.1.1 in the text is justified by showing that the work done by a force acting on a particle is equal to the change in the kinetic energy of that particle. We can use Hooke's law to calculate the work done stretching the spring. That work will be equal to the change in kinetic energy of the ball.
The ball initially has kinetic energy $\frac{1}{2}(1 \text{ kg})(v_0 \text{ m/sec})^2 = \frac{v_0^2}{2}\frac{\text{kg}\cdot\text{m}^2}{\text{sec}^2} = \frac{v_0^2}{2}$ J. At the time the spring is stretched its farthest, the ball's velocity is 0 m/sec, so its kinetic energy is $\frac{1}{2}(1 \text{ kg})(0 \text{ m/sec})^2 = 0$J. So, the change in kinetic energy of the ball is $\frac{v_0^2}{2}$ J.
Now let's find the work done by the spring. Its spring constant is $k = 5$ N/m, so, the force on the spring when it is stretched x metres past its natural length is $5x$ N. The spring is stretched from its natural length to 10 cm, which is 0.1 m. Then the work done by the spring is

$$W = \int_0^{0.1} kx\mathrm{d}x = \int_0^{0.1} 5x\,\mathrm{d}x = \left[\frac{5}{2}x^2\right]_0^{0.1} = \frac{1}{40} \text{ J}$$

Now we can find v_0.

$$\frac{v_0^2}{2} = \frac{1}{40} \qquad \Longrightarrow \qquad v_0 = \frac{1}{\sqrt{20}} \text{ m/sec} \approx 22.36 \text{ cm/sec}$$

2.1.2.19. Solution. The setup to answer this question is similar to Question 18 in this section: the work done by a spring on the occupied vehicle will be equal to the change in kinetic energy of that occupied vehicle. So, we need to find the work done by the spring, and the kinetic energy lost by the falling car. In order to find the work done by the spring, we need to find the spring constant.

- Spring constant: The car's mass of 2000 kg compresses the struts 2 cm past their natural length. The force of the car under gravity is $(2000 \text{ kg}) \times (9.8 \text{ m/sec}^2) = 19\,600$ N. This force is exactly the same as that exerted by the spring, $k(0.02 \text{ m})$. So, $k = 980\,000$ N/m.

- Work done by spring: The spring can safely compress 20 cm. So, the amount of work done by the spring compressing that far gives us the *maximum* amount of work the spring can safely do. While the car is falling, the spring is at its natural length, so the work done to compress it to 20 cm (0.2 m) shorter is:

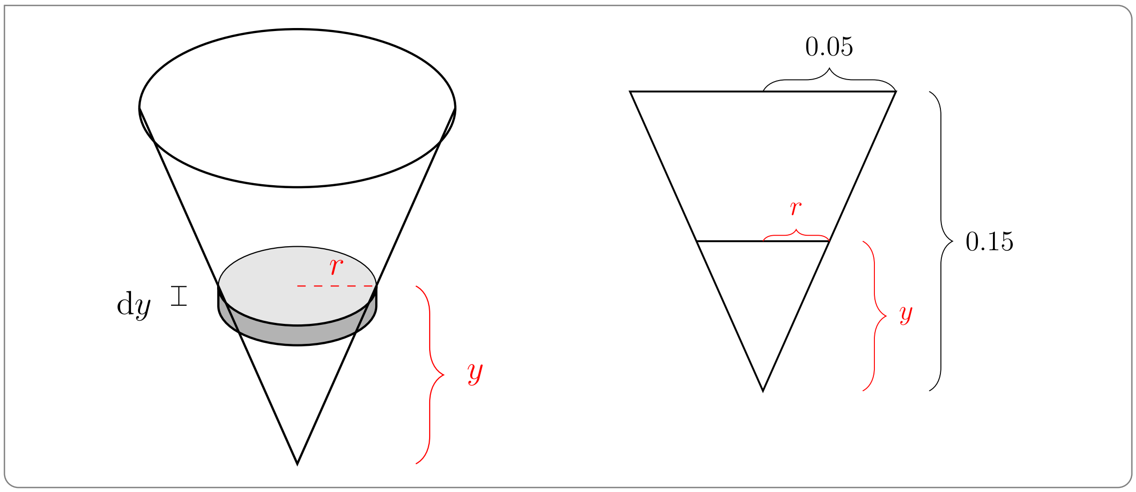
$$\int_0^{0.2} kx\,\mathrm{d}x = \int_0^{0.2} 980\,000x\,\mathrm{d}x = \left[490\,000x^2\right]_0^{0.2} = 19\,600 \text{ J}$$

- Change in kinetic energy: When the car first hits the pavement, it's falling at 4 m/sec, so it has kinetic energy $\frac{1}{2}(2100 \text{ kg})(4 \text{ m/sec})^2 = 16\,800$ J. When the car compresses the springs as far as they go and it starts to rebound, it has kinetic energy 0, since its instantaneous velocity is zero. So, the change in kinetic energy is 16 800 J.

Since the change in kinetic energy is 16 800 J, and the struts can safely do a work of (up to) 19 600 J, the jump is within the (meagre) safety limits set by the question.

Exercises — Stage 3

2.1.2.20. Solution. Let's consider sucking up a flat, horizontal layer of water. If the water is y metres above bottom of the cone, then it needs to be raised $0.15 - y$ metres. So, if its mass is m kg, then the force of gravity acting on it is $9.8m$ N and the work involved in slurping it to the top of the cone is $9.8m(0.15 - y)$ J. So, what we need to find is the mass of a layer of water y metres from the bottom of the cone.



A horizontal cross-section of the cone is a circle. To find its radius, we use similar triangles: $\frac{r}{y} = \frac{0.05}{0.15}$, so $r = \frac{1}{3}y$. Therefore, the area of the cross-section of the cone y metres above its bottom is $\pi\left(\frac{1}{3}y\right)^2 = \frac{\pi}{9}y^2$ m^2. If this layer has height dy, then its volume is $\frac{\pi}{9}y^2\,\mathrm{d}y$ m^3, and its mass is $1000\frac{\pi}{9}y^2\,\mathrm{d}y$ kg.

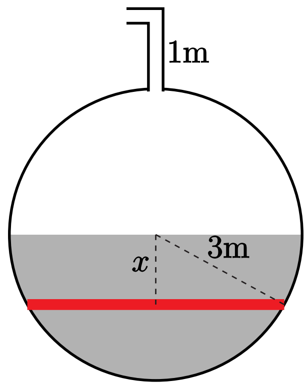
Now, we know that the work to suck up the layer of water y metres from the bottom of the cup is $9.8(0.15 - y)\left(1000\frac{\pi}{9}y^2\,\mathrm{d}y\right)$ J. So, the work involved in drinking all the water is:

$$\begin{aligned} W &= \int_0^{0.15} 9.8(0.15 - y)\left(1000\frac{\pi}{9}y^2\right)\mathrm{d}y \\ &= \frac{9800\pi}{9}\int_0^{0.15}\left(0.15y^2 - y^3\right)\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&= \frac{9800\pi}{9}\left[\frac{0.15}{3}y^3 - \frac{1}{4}y^4\right]_0^{0.15} \\
&= \frac{9800\pi}{9}\left[0.05(0.15)^3 - \frac{1}{4}(0.15)^4\right] \\
&\approx 0.144 \text{ J}
\end{aligned}
$$

Even drinking water takes work. Life is hard.

2.1.2.21. *. Solution. Imagine slicing the water into horizontal pancakes of thickness dx as in the sketch below.



Denote by x the distance of a pancake below the surface of the water. (So, x runs from 0 to 3.) Each pancake:

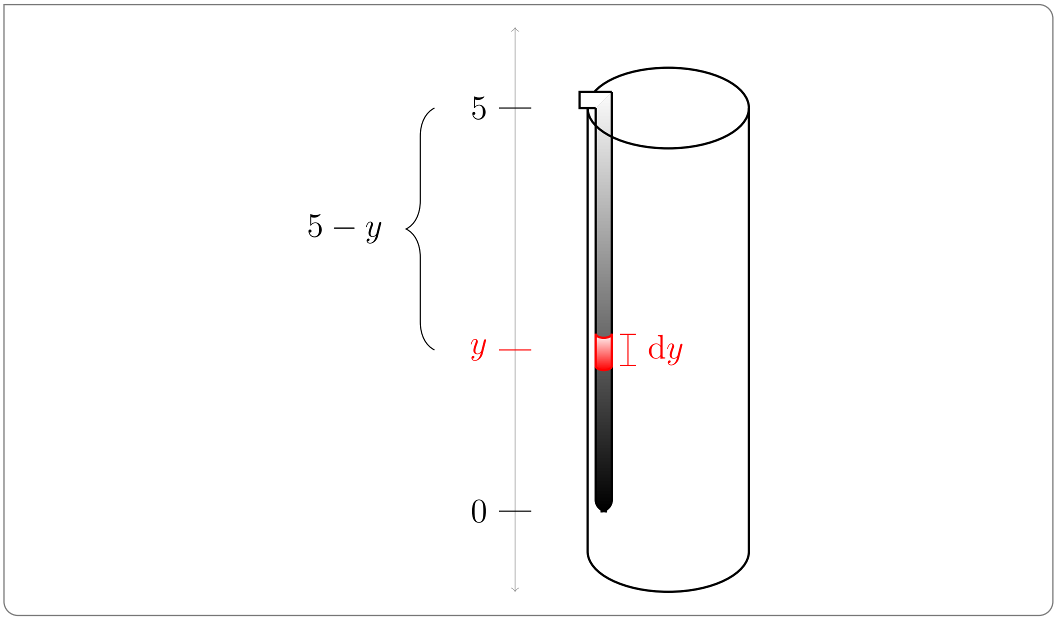
- has radius $\sqrt{3^2 - x^2}$ m (by Pythagoras) and hence
- has cross-sectional area $\pi(9 - x^2)$ m^2 and hence
- has volume $\pi(9 - x^2)\,\mathrm{d}x$ m^3 and hence
- has mass $1000\pi(9 - x^2)\,\mathrm{d}x$ kg and hence
- is subject to a gravitational force of $9.8 \times 1000\pi(9 - x^2)\,\mathrm{d}x$ N and hence
- requires work $9800\pi(9 - x^2)(x + 4)\,\mathrm{d}x$ J to raise it to the spout. (It has to be raised x m to bring it to the height of the centre of the sphere, then 3 m more to bring it to the top of the sphere, and finally 1 m more to bring it to the spout.)

The total work is:

$$
\begin{aligned}
\int_0^3 9800\pi(9 - x^2)(x + 4)\,\mathrm{d}x &= \int_0^3 9800\pi\big(36 + 9x - 4x^2 - x^3\big)\,\mathrm{d}x \\
&= 9800\pi\left[36x + \frac{9}{2}x^2 - \frac{4}{3}x^3 - \frac{1}{4}x^4\right]_0^3 \\
&= 9800 \cdot \frac{369}{4}\pi = 904{,}050\pi \text{ joules}
\end{aligned}
$$

2.1.2.22. Solution.

- Solution 1: Let's consider the work involved in lifting up a small section of cable, with length $\mathrm{d}y$, distance y from the bottom end of the cable.



The distance this section must travel is $(5-y)$ metres, so if its mass is $M(y)$, then the work involved is

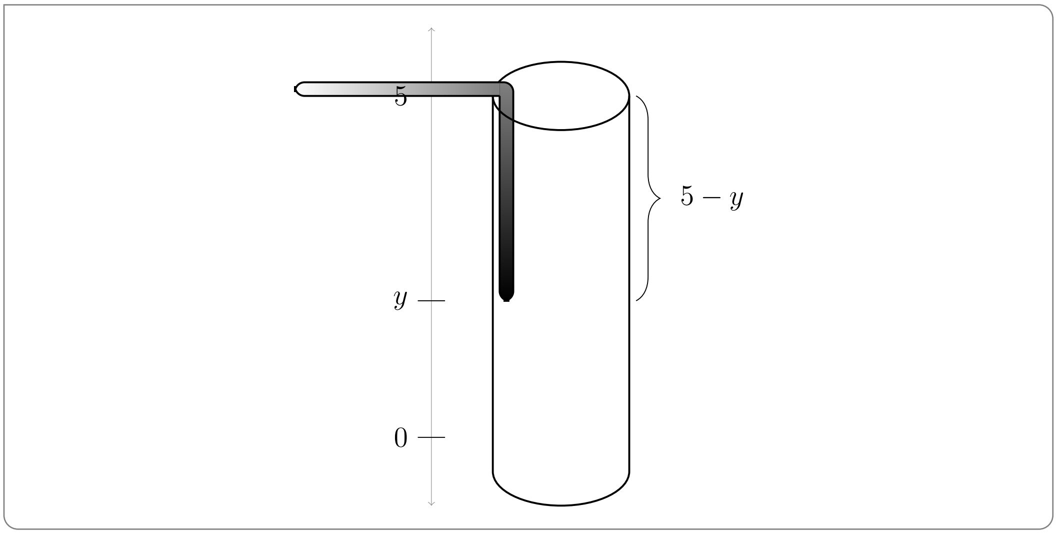
$$W = \int_0^5 9.8 \times (5-y) \times M(y)$$

So, we need to find $M(y)$. The length of the section of cable is $\mathrm{d}y$, and its distance from the end of the cable is y, so the mass of the section is $(10-y)\,\mathrm{d}y$. Therefore,

$$\begin{aligned}
W &= \int_0^5 9.8 \times (5-y) \times M(y) \\
&= \int_0^5 9.8 \times (5-y) \times (10-y)\,\mathrm{d}y \\
&= 9.8 \int_0^5 \left(50 - 15y + y^2\right) \mathrm{d}y \\
&= 9.8 \left[50y - \frac{15}{2}y^2 + \frac{1}{3}y^3\right]_0^5 \\
&= \frac{6125}{6} = 1020\tfrac{5}{6} \text{ J}
\end{aligned}$$

- Solution 2: Alternately, we can continue to use the basic method of Example 2.1.6 in the text, noticing that the density of the cable is no longer constant.

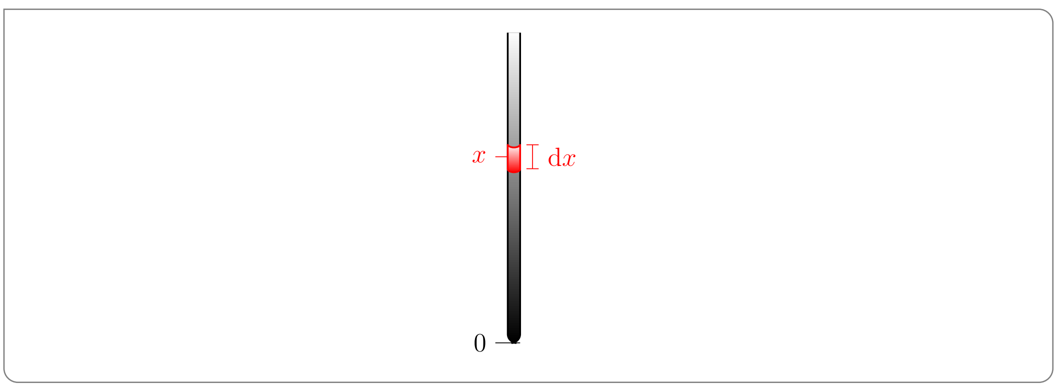
  Let's consider pulling the cable up a tiny distance of $\mathrm{d}y$ metres, after we have already lifted it y metres (so $(5-y)$ metres of the cable is still in the hole).



If $R(y)$ is the mass of the remaining cable (in kg), then the force of gravity is $-9.8 \times R(y)$, so the work done is $9.8 \times R(y) \times \mathrm{d}y$. Once we find $R(y)$, we can calculate the total work done:

$$W = \int_0^5 9.8 \times R(y) \times \mathrm{d}y \tag{$*$}$$

As given in the question statement, the density of the cable is $(10 - x)$ kg/m, where x is the distance from the bottom end of the cable. Consider a tiny section of cable x metres from the bottom end, of length $\mathrm{d}x$.



The mass of this tiny section is $\left(10 - x \ \frac{\mathrm{kg}}{\mathrm{m}}\right) \times (\mathrm{d}x \ \mathrm{m}) = (10 - x)\mathrm{d}x$ kg. The section of cable dangling is the last $(5 - y)$ metres of cable. So, the combined mass of the section of cable dangling, after we've already pulled up y metres of it, is

$$R(y) = \int_0^{5-y} (10 - x)\mathrm{d}x = \left[10x - \frac{1}{2}x^2\right]_0^{5-y} = \frac{75}{2} - 5y - \frac{1}{2}y^2$$

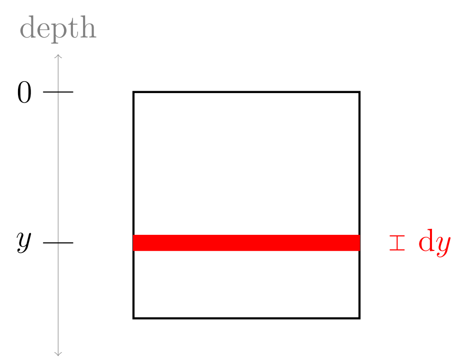
Now we can calculate the total work involved in pulling up the entire cable, using equation $(*)$.

$$W = \int_0^5 9.8 \times R(y) \times \mathrm{d}y$$

$$= \int_0^5 9.8 \times \left(\frac{75}{2} - 5y - \frac{1}{2}y^2\right) \times \mathrm{d}y$$
$$= 9.8\left[\frac{75}{2}y - \frac{5}{2}y^2 - \frac{1}{6}y^3\right]_0^5$$
$$= \frac{6125}{6} = 1020\frac{5}{6} \text{ J}$$

2.1.2.23. Solution.

a The force of the depends on depth, which varies. So, consider a thin rectangle of the plunger at depth y, with height $\mathrm{d}y$ and width 1 m (the width of the entire plunger). Let the area of this rectangle be $\mathrm{d}A$.



The area of this rectangle is $1\mathrm{d}y$ m^2, so the force of the water acting on it is $F = P \cdot \mathrm{d}A = \underbrace{\left(9800\frac{\mathrm{N}}{\mathrm{m}^3}\right)}_{c}\underbrace{(y \text{ m})}_{d}\underbrace{\left(\mathrm{d}y \text{ m}^2\right)}_{\mathrm{d}A} = 9800y\,\mathrm{d}y$ N.

The depth at the top of the plunger is $y = 0$. To find the depth at the bottom of the plunger, note that the water has a volume of 3 m^3, and is in a rectangular container with base 1 m by 3 m. So, its height is 1 m.

The force over the entire plunger, from depth $y = 0$ to $y = 1$, is

$$\int_0^1 9800y\,\mathrm{d}y = \left[4900y^2\right]_0^1 = 4900 \text{ N}$$

b Let's follow our work from part (a), but with the width of the length of the base as x m.

Still, a thin rectangle of plunger has width 1 m and height $\mathrm{d}y$ m, so it has area $\mathrm{d}y$ m^2. At depth y, it has a force from the water of $9800y\,\mathrm{d}y$ N. This hasn't changed from (a).

Now, let's consider the depth of the water. The volume of water is 3 m^3, and it is in a rectangular container with base 1 m by x m. So, its depth is $3/x$ m. Therefore, the force on the entire plunger must be calculated from $y = 0$ to $y = 3/x$.

$$F(x) = \int_0^{3/x} 9800y\,\mathrm{d}y = \left[4900y^2\right]_0^{3/x} = \frac{9}{x^2}4900 = \frac{44100}{x^2} \text{ N}$$

Let's check that this answer makes sense: $F(3) = 4900$ N, which matches our answer from (a).

c If the force of water acting on the plunger, when the length of the base is x metres, is given by $F(x)$, then we push the plunger with a force of $-F(x)$. Then the work we're looking for is

$$W = \int_3^1 -F(x)\,\mathrm{d}x = \int_1^3 F(x)\,\mathrm{d}x$$

$F(x)$ is exactly what we found in (b): $F(x) = \frac{44100}{x^2}$ N.

$$\begin{aligned} W &= \int_1^3 F(x)\,\mathrm{d}x = \int_1^3 \frac{44100}{x^2}\,\mathrm{d}x = 44100\left[-\frac{1}{x}\right]_1^3 \\ &= 44100 \cdot \frac{2}{3} = 29\,400 \text{ J} \end{aligned}$$

2.1.2.24. Solution. Let's start by converting from time spent pulling to amount pulled. When y metres of rope have been pulled up, $2y$ seconds have passed, so $\frac{1}{5}y$ litres of water have leaked out of the bucket, leaving $5 - \frac{1}{5}y$ litres. (This only makes sense when $\frac{1}{5}y \le 5$, but we only consider values of y from 0 to 5, so it's not a problem. That is, we're never hauling up an empty bucket that can't leak any more.)

When we've pulled up y metres of rope, the mass in the bucket is $(5 - \frac{1}{5}y)$ kg, so the force of gravity acting on it is $9.8(5 - \frac{1}{5}y)$ N. Since we pull up 5 metres of rope, the work done is:

$$\begin{aligned} W &= \int_0^5 9.8\left(5 - \frac{1}{5}y\right)\mathrm{d}y = 9.8\left[5y - \frac{1}{10}y^2\right]_0^5 = 9.8\,[25 - 2.5] \\ &= 220.5 \text{ J} \end{aligned}$$

2.1.2.25. Solution. According to the formula for gravity between two objects, the earth and moon will gravitationally attract one another no matter how far apart they are, so what we're looking for is the work to separate them infinitely far. That is, we want to calculate $\displaystyle\int_a^\infty F(r)\,\mathrm{d}r$, where $a = 400\,000\,000$ m.

If we take m_1 and m_2 to be the mass of the earth and moon as given in the question statement, then:

$$\begin{aligned} Gm_1m_2 &= \left(6.7 \times 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{sec}^2}\right)\left(6 \times 10^{24}\ \mathrm{kg}\right)\left(7 \times 10^{22}\ \mathrm{kg}\right) \\ &\approx 2.8 \times 10^{37}\frac{\mathrm{kg}\cdot\mathrm{m}^3}{\mathrm{sec}^2} \end{aligned}$$

With that out of the way, let's calculate our work.

$$W = \int_a^\infty F(r)\,\mathrm{d}r = \lim_{b\to\infty}\int_a^b G\frac{m_1m_2}{r^2}\,\mathrm{d}r$$

$$\begin{aligned}
&= (Gm_1m_2) \lim_{b\to\infty} \left[-\frac{1}{r}\right]_a^b \\
&= (Gm_1m_2) \lim_{b\to\infty} \left(\frac{1}{a} - \frac{1}{b}\right) \\
&= \frac{Gm_1m_2}{a} \\
&\approx \frac{2.8 \times 10^{37} \frac{\text{kg}\cdot\text{m}^3}{\text{sec}^2}}{4 \times 10^8 \text{ m}} \\
&= 7 \times 10^{28} \text{ J}
\end{aligned}$$

Remark: since the force of gravity between the earth and the moon gets weaker as they are farther apart, it takes less and less work to move them each kilometre. If we move them a finite distance apart, the work involved will always be less than 7×10^{28} joules, no matter how huge that finite distance is. If we move them a very, very long (but finite) distance apart, the work we did will be quite close to (but still less than) 7×10^{28} joules.

2.1.2.26. Solution. A ball of mass m experiences a gravitational force of mg, so lifting it a height of $\ell/2$ involves a work of $\frac{1}{2}mg\ell$.
The cable has density m/ℓ. A tiny section of cable with length $\mathrm{d}y$ has mass $\frac{m}{\ell}\,\mathrm{d}y$, and so gravity acts on it with a force of $\frac{mg}{\ell}\,\mathrm{d}y$. If the tiny section of cable is y units from the top of the cable, it needs to be pulled up y units, so the work on that section is $\frac{mg}{\ell}y\,\mathrm{d}y$. Therefore, the work to pull up the entire cable is

$$\int_0^\ell \frac{mg}{\ell} y\,\mathrm{d}y = \left[\frac{mg}{2\ell}y^2\right]_0^\ell = \frac{mg}{2\ell}\ell^2 = \frac{1}{2}mg\ell$$

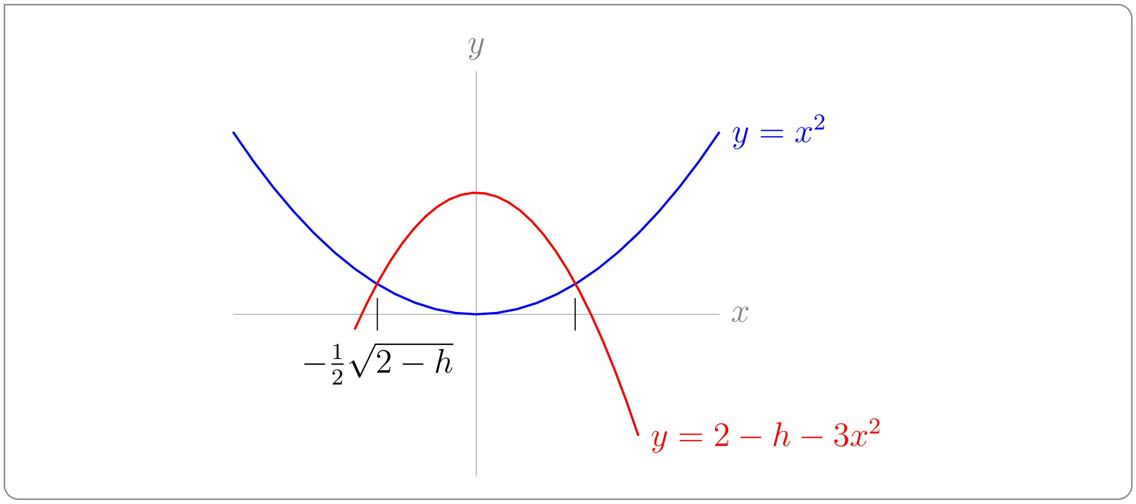
So, the work to pull up a cable with uniform density is the same as the work to pull up a ball with the same mass from the middle height of the cable.
Remark: this is a nice fact to use when you're checking your computations for "pulling up cable" problems, but keep in mind it depends on the cable being of uniform density.

2.1.2.27. Solution. Like our other tank-pumping problems (e.g. Questions 12, 20, and 21 in this section, and Example 2.1.4 in the text), we can find the work done by considering thin layers of liquid. If the layer of liquid h metres above the bottom of the tank with thickness $\mathrm{d}h$ has mass $M(h)$, then the force of gravity acting on it is $-9.8M(h)$ N and the work required to pump it to the top of the tank ($1-h$ metres away) is $9.8(1-h)M(h)$ J. So, the work to empty the entire tank is

$$W = \int_0^1 9.8(1-h)M(h) \tag{$*$}$$

Our remaining task is to find $M(h)$. There are two things that vary with height: the density of the liquid, and the area of the cross-section of the tank.
At height h metres, the cross-section of the tank is shaped like the finite region bounded by the curves $y = x^2$ and $y = 2 - h - 3x^2$. To find this area, we need an

integral (see Section 1.5 for a refresher), and to find the limits of integration, we need to know where the two curves meet. By solving $x^2 = 2-h-3x^2$, we find that they meet at $x = \pm\frac{1}{2}\sqrt{2-h}$. (Recall h is between 0 and 1, so $\sqrt{2-h}$ is a real number, i.e. the curves do indeed meet.) Furthermore, when $-\frac{1}{2}\sqrt{2-h} \le x \le \frac{1}{2}\sqrt{2-h}$, then $x^2 \le 2-h-3x^2$, so $y = x^2$ is the bottom function and $2-h-3x^2$ is the top function.



So, (taking advantage of the fact that our region has even symmetry) the area of the cross-section of the tank at height h is

$$\begin{aligned}
A(h) &= \int_{-\frac{1}{2}\sqrt{2-h}}^{\frac{1}{2}\sqrt{2-h}} \left(\left[2-h-3x^2\right] - x^2\right)\,\mathrm{d}x \\
&= 2\int_0^{\frac{1}{2}\sqrt{2-h}} \left(2-h-4x^2\right)\,\mathrm{d}x \\
&= 2\left[(2-h)x - \frac{4}{3}x^3\right]_0^{\frac{1}{2}\sqrt{2-h}} \\
&= 2\left[(2-h)\cdot\frac{1}{2}\sqrt{2-h} - \frac{4}{3}\cdot\frac{1}{8}\sqrt{2-h}^3\right] \\
&= (2-h)^{3/2}\left[1-\frac{1}{3}\right] = \frac{2}{3}(2-h)^{3/2}
\end{aligned}$$

Now, we can calculate the volume of a slice at height h of thickness $\mathrm{d}h$.

$$V(h) = \frac{2}{3}(2-h)^{3/2}\,\mathrm{d}h$$

The density of the liquid at height h is $1000\sqrt{2-h}$ kg/m^3, so

$$\begin{aligned}
M(h) &= 1000\sqrt{2-h} \times \frac{2}{3}(2-h)^{3/2}\,\mathrm{d}h \\
&= \frac{2000}{3}(2-h)^2\mathrm{d}h
\end{aligned}$$

Now we use $(*)$ to find the work done pumping out the tank.

$$W = \int_0^1 9.8(1-h)M(h) = \int_0^1 9.8(1-h)\cdot\frac{2000}{3}(2-h)^2\mathrm{d}h$$

$$\begin{aligned}
&= \frac{19600}{3}\int_0^1 \left(4-8h+5h^2-h^3\right) \mathrm{d}h \\
&= \frac{19600}{3}\left[4h-4h^2+\frac{5}{3}h^3-\frac{1}{4}h^4\right]_0^1 \\
&= \frac{19600}{3}\left[4-4+\frac{5}{3}-\frac{1}{4}\right] \\
&= \frac{19600}{3}\times\frac{17}{12} \\
&= \frac{83300}{9} = 9255\tfrac{5}{9} \text{ J}
\end{aligned}$$

2.1.2.28. Solution. Since the only work done is against the force of gravity, we only need to know how high the sand was lifted, not how it got there. So, we don't really need to worry about its semicircular path: we can imagine that every grain of sand was lifted from its old position to its new position.
Consider a thin, horizontal layer of sand in the hourglass, y metres below the vertical centre of the hourglass.

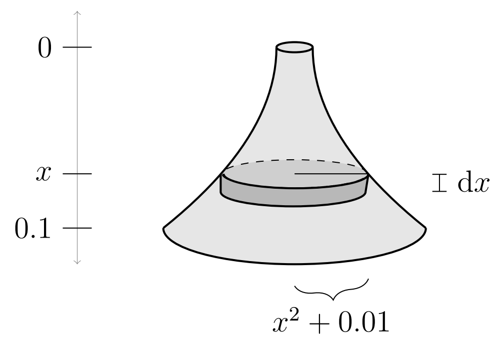
- Its final position is y metres above the centre of the hourglass. That is, it was lifted $2y$ metres against the force of gravity.
- The layer is shaped like a circle with radius $y^2+0.01$ and height $\mathrm{d}y$, so its volume is $\pi\left(y^2+0.01\right)^2 \mathrm{d}y$ cubic metres.
- To find the mass of the layer, we need to know the density of the sand. Let the volume of sand in the hourglass be V. We are given its mass M. Then $\pi\left(y^2+0.01\right)^2 \mathrm{d}y$ cubic metres has a mass of $\frac{M}{V}\pi\left(y^2+0.01\right)^2 \mathrm{d}y$ kilograms.
- So, the force of gravity acting on the layer is $9.8\frac{M}{V}\pi\left(y^2+0.01\right)^2 \mathrm{d}y$ N, acting vertically downwards.
- To lift the layer to its final position, we apply a compensating force over a distance of $2y$ metres, for a total work of $9.8\frac{M}{V}\pi\left(y^2+0.01\right)^2 2y\,\mathrm{d}y$ J.
- Since the hourglass has height 0.2 m, and exactly half of it is filled with sand, the top layer of sand is exactly at the vertical centre of the hourglass, and the bottom layer of sand is 0.1 metres below.

Using V is the volume of sand in the hourglass, and M is its mass (we're given $M=\frac{1}{7}$ kg) then the total work flipping *all* the sand is:

$$\begin{aligned}
W &= \int_0^{0.1} 9.8\frac{M}{V}\pi\left(y^2+0.01\right)^2 2y\,\mathrm{d}y \\
&= \frac{19.6M\pi}{V}\int_0^{0.1}\left(y^5+0.02y^3+0.0001y\right)\mathrm{d}y \\
&= \frac{19.6M\pi}{V}\int_0^{1/10}\left(y^5+\frac{2}{10^2}y^3+\frac{1}{10^4}y\right)\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&= \frac{19.6M\pi}{V}\left[\frac{1}{6}y^6 + \frac{1}{2\cdot 10^2}y^4 + \frac{1}{2\cdot 10^4}y^2\right]_0^{1/10} \\
&= \frac{19.6M\pi}{V}\left[\left(\frac{1}{6\cdot 10^6} + \frac{1}{2\cdot 10^6} + \frac{1}{2\cdot 10^6}\right) - 0\right] \\
&= \frac{19.6M\pi}{V}\left[\frac{7}{6\cdot 10^6}\right] \qquad (*)
\end{aligned}
$$

It remains to find V: the volume of sand in the hourglass. We know the sand is in the shape of a solid of rotation. Recall from Section 1.6 that we can find the volume of such shapes by slicing them into thin disks.



In the picture above, we've used an axis that matches the way we've been describing our solid: 0 is the vertical centre of the hourglass, which is where the top of the sand is, and the bottom of the sand is 0.1 metres from 0.

To find the volume of this solid, we slice it into horizontal disks. The disk that is x metres from the centre of the hourglass has radius $x^2 + 0.01$ and thickness $\mathrm{d}x$, so it has volume $\pi(x^2 + 0.01)^2\,\mathrm{d}x$. The volume of the entire solid, i.e. the volume of the sand, is:

$$
\begin{aligned}
V &= \int_0^{0.1} \pi(x^2 + 0.01)^2\,\mathrm{d}x = \pi\int_0^{0.1}(x^4 + 0.02x^2 + 0.0001)\,\mathrm{d}x \\
&= \pi\left[\frac{1}{5}x^5 + \frac{2}{3\times 10^2}x^3 + \frac{1}{10^4}x\right]_0^{0.1} \\
&= \pi\left[\left(\frac{1}{5\times 10^5} + \frac{2}{3\times 10^5} + \frac{1}{10^5}\right) - 0\right] \\
&= \frac{28\pi}{15\times 10^5} = \frac{28\pi}{1.5\times 10^6}
\end{aligned}
$$

So, the volume of sand in the hourglass is $V = \frac{28\pi}{1.5\times 10^6}$ cubic metres.

Using $(*)$, we can find the total work done quickly flipping the hourglass.

$$
\begin{aligned}
W &= \frac{19.6M\pi}{V}\left[\frac{7}{6\cdot 10^6}\right] \\
&= \frac{19.6\times\frac{1}{7}\pi}{\frac{28\pi}{1.5\times 10^6}}\left[\frac{7}{6\cdot 10^6}\right] \\
&= \frac{19.6\times 1.5}{28\times 6} = \frac{29.4}{168} = \frac{7}{40} = 0.175\ \mathrm{J}
\end{aligned}
$$

2.1.2.29. Solution. Using Definition 2.1.1, the work involved is

$$W = \int_0^{1/2} \sqrt{1-x^4}\,\mathrm{d}x \quad \text{J.}$$

However, the function $F(x) = \sqrt{1-x^4}$ happens to not have an antiderivative that can be expressed as an elementary function. That means we can't use the Fundamental Theorem of Calculus Part 2 to evaluate this integral (at least, not without knowing a bit more about functions than is prerequisite for this course). Instead, we can use numerical methods, like the midpoint rule or Simpson's rule, to approximate its value.

It's not immediately clear which rule (Simpson's, midpoint, or trapezoidal) will lead us down the easiest path. For Simpson's rule, we need to know the fourth derivative of $F(x)$, which is not a simple task. But, we often need fewer intervals for Simpson's rule than for the midpoint or trapezoid rules. In this case, we'll show below that $n = 2$ intervals suffice to guarantee a low enough error using the midpoint rule, so Simpson's rule won't let us get away with fewer intervals. Below, we find the approximation using the midpoint rule — but there are other ways as well.

In order to decide how many intervals we should use with the midpoint rule, we need to know the second derivative of $F(x)$.

$$\begin{aligned}
F(x) &= (1-x^4)^{1/2}\\
F'(x) &= \frac{1}{2}\left(1-x^4\right)^{-1/2}\cdot(-4x^3) = -2x^3\left(1-x^4\right)^{-1/2}\\
F''(x) &= (-2x^3)\left(-\frac{1}{2}\right)\left(1-x^4\right)^{-3/2}(-4x^3) + (-6x^2)\left(1-x^4\right)^{-1/2}\\
&= -4x^6\left(1-x^4\right)^{-3/2} + (-6x^2)\left(1-x^4\right)^{-1/2}\\
&= 2x^2\left(1-x^4\right)^{-3/2}\left(-2x^4 - 3\left(1-x^4\right)\right)\\
&= 2x^2\left(1-x^4\right)^{-3/2}\left(x^4-3\right)\\
&= \frac{2x^2\left(x^4-3\right)}{\left(1-x^4\right)^{3/2}}
\end{aligned}$$

For values of x between 0 and $\frac{1}{2}$,

- the denominator $\left(1-x^4\right)^{3/2}$ is always at least as big as $\left(1-\left(\frac{1}{2}\right)^4\right)^{3/2}$,
- the factor x^2 in the numerator is never bigger than $\left(\frac{1}{2}\right)^2$, and
- the factor $x^4 - 3$ in the numerator has magnitude at most 3,

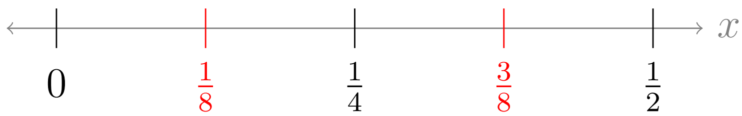
so that

$$|F''(x)| = \frac{2x^2\left|x^4-3\right|}{\left(1-x^4\right)^{3/2}} \le \frac{2(\frac{1}{2})^2(3)}{\left(1-(\frac{1}{2})^4\right)^{3/2}} = \frac{\frac{3}{2}}{\frac{15^{3/2}}{64}} = \frac{32}{5\sqrt{15}} < 2$$

Using Theorem 1.11.13 with $M = 2$, $a = 0$, and $b = 1/2$, the error in a midpoint approximation with n intervals is at most

$$\frac{M}{24} \cdot \frac{(b-a)^3}{n^2} = \frac{2}{24} \cdot \frac{1/8}{n^2} = \frac{1}{96n^2}$$

If $n \geq 2$, then our error is certainly less than $\frac{1}{100}$.
If we use the midpoint rule with $n = 2$, then $\bar{x}_1 = \frac{1}{8}$ and $\bar{x}_2 = \frac{3}{8}$.



The midpoint rule approximation of $\int_0^{1/2} \sqrt{1-x^4}\,\mathrm{d}x$ with $n = 2$ and $\Delta x = \frac{1}{4}$ is:

$$\begin{aligned}\int_0^{1/2} \sqrt{1-x^4}\,\mathrm{d}x &\approx \Delta x \left[\sqrt{1-\bar{x}_1^4} + \sqrt{1-\bar{x}_2^4}\right] \\ &= \frac{1}{4}\left[\sqrt{1-\left(\frac{1}{8}\right)^4} + \sqrt{1-\left(\frac{3}{8}\right)^4}\right]\end{aligned}$$

Since we used $n = 2$, by our previous work the error in this approximation is less than $\frac{1}{96\times 2^2} = \frac{1}{384}$, which is certainly less than 0.01, as required.

2.2 · Averages

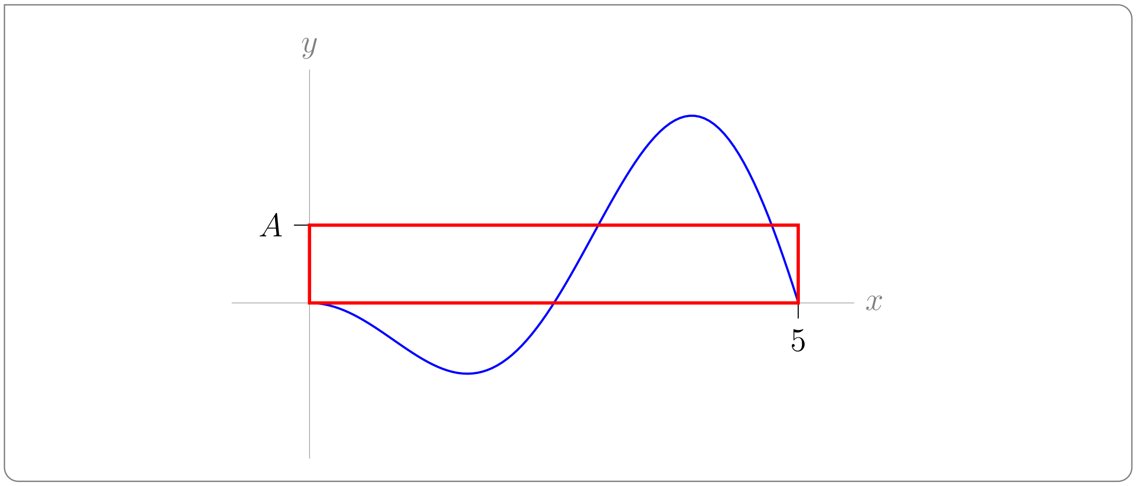
2.2.2 · Exercises

Exercises — Stage 1

2.2.2.1. Solution. Since the average of $f(x)$ on the interval $[0,5]$ is A, using Definition 2.2.2,

$$\begin{aligned}A &= \frac{1}{5}\int_0^5 f(x)\,\mathrm{d}x \\ 5A &= \int_0^5 f(x)\,\mathrm{d}x\end{aligned}$$

So, a rectangle with width 5 and height A has area $\int_0^5 f(x)\,\mathrm{d}x$.
That is: if we replace $f(x)$ with the constant function $g(x) = A$, then on the interval $[0,5]$, the area under the curve is unchanged.



(There are many rectangles with area $5A$; we drew the one we consider to be the most straightforward in this context.)

2.2.2.2. Solution. Average velocity, as discussed in Example 2.2.5, is change in position divided by change in time. So, the change in position (i.e. distance travelled) is (100 km/h)(5 h) = 500 km.

2.2.2.3. Solution. The work done is

$$W = \int_a^b F(x)\,\mathrm{d}x$$

so the average value of $F(x)$ is

$$\frac{1}{b-a}\int_a^b F(x)\,\mathrm{d}x = \frac{1}{b-a}(W).$$

We can quickly check our units: since W is in joules (that is, newton-metres), and $b-a$ is in metres, so $\frac{W}{b-a}$ is in newtons.

2.2.2.4. Solution.

a The entire interval has length $b-a$, and we're cutting it into n pieces, so the length of one piece (and hence the distance between two consecutive samples) is $\frac{b-a}{n}$.

b The first sample, as given in the question statement, is taken at $x = a$. The second sample, then, is at $x = a + \frac{b-a}{n}$, this third is at $x = 1 + 2\frac{b-a}{n}$, and the fourth is at $a + 3\frac{b-a}{n}$.

c The y-value of the fourth sample is simply $f\left(a + 3\frac{b-a}{n}\right)$. Note this is the number we use in our average, not the x-value.

d Our samples are $f(a)$, $f\left(a + \frac{b-a}{n}\right)$, $f\left(a + 2\frac{b-a}{n}\right)$, $f\left(a + 3\frac{b-a}{n}\right)$, etc. Since there are n of them, we divide their sum by n. So, the average is:

$$\frac{f(a) + f\left(a+\frac{b-a}{n}\right) + f\left(a+2\frac{b-a}{n}\right) + \cdots + f\left(a+(n-1)\frac{b-a}{n}\right)}{n}$$

$$\begin{aligned}
&= \frac{1}{n}\left[f(a) + f\left(a+\frac{b-a}{n}\right) f\left(a+2\frac{b-a}{n}\right) + \cdots \right. \\
&\qquad\qquad\qquad\qquad \left. + f\left(a+(n-1)\frac{b-a}{n}\right)\right] \\
&= \frac{1}{n}\sum_{i=1}^{n} f\left(a+(i-1)\frac{b-a}{n}\right)
\end{aligned}$$

Remark: if we multiply and divide by $b-a$, we see this expression is equivalent to a left Riemann sum, divided by the length of our interval.

$$\begin{aligned}
&= \frac{1}{b-a}\sum_{i=1}^{n} f\left(a+(i-1)\frac{b-a}{n}\right)\frac{b-a}{n} \\
&= \frac{1}{b-a}\sum_{i=1}^{n} f\left(a+(i-1)\Delta x\right)\Delta x
\end{aligned}$$

As n gets larger and larger, using the definition of a definite integral, this expression gets closer and closer to $\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$. This is one way of justifying our definition of an average of a function on an interval.

2.2.2.5. Solution.

a Yes, the average of $f(x)$ is less than or equal to the average of $g(x)$ on $[0, 10]$. The reason is that, if $f(x) \le g(x)$ for all x in $[0, 10]$, then:

$$\frac{1}{10}\int_0^{10} f(x)\,\mathrm{d}x \le \frac{1}{10}\int_0^{10} g(x)\,\mathrm{d}x.$$

b There is not enough information to tell. It's certainly possible: for instance, take $f(x) = 0$ and $g(x) = 1$ for all x in $[0, 10]$. Then $f(x) \le g(x)$ and the average of $f(x)$ is 0, which is less than 1, the average of $g(x)$.

However, consider $f(x) = \begin{cases} 100 & \text{if } 0 \le x \le 0.01 \\ 0 & \text{else} \end{cases}$ and $g(x) = 0$. Then $f(x) \le g(x)$ for all x in $[0.01, 10]$, but the average of $f(x)$ is 0.1, while the average of $g(x)$ is 0.

2.2.2.6. Solution. Recall the definition of an odd function: $f(-x) = -f(x)$. Since the domain of integration is symmetric, the signed area on one side of the y-axis "cancels out" the signed area on the other — this is Theorem 1.2.12 in the text.

$$\frac{1}{20}\int_{-10}^{10} f(x)\,\mathrm{d}x = \frac{1}{20}(0) = 0$$

Exercises — Stage 2

2.2.2.7. *. Solution. By definition, the average value is

$$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\big(\sin(5x)+1\big)\,\mathrm{d}x$$

We now observe that $\sin(5x)$ is an odd function, and hence its integral over the symmetric interval $[-\frac{\pi}{2},\frac{\pi}{2}]$ equals zero. So the average value of $f(x)$ on this interval is 1:

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\big(\sin(5x)+1\big)\,\mathrm{d}x &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\sin(5x)\,\mathrm{d}x+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}1\,\mathrm{d}x\\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}1\,\mathrm{d}x=1\end{aligned}$$

Alternatively, using the fundamental theorem of calculus, the average equals:

$$\begin{aligned}&\frac{1}{\pi}\left[\frac{-\cos(5x)}{5}+x\right]_{-\pi/2}^{\pi/2}\\ &\quad=\frac{1}{\pi}\left\{\left[\frac{-\cos(5\pi/2)}{5}+\frac{\pi}{2}\right]-\left[\frac{-\cos(-5\pi/2)}{5}+\frac{-\pi}{2}\right]\right\}\\ &\quad=\frac{\pi}{\pi}=1\end{aligned}$$

2.2.2.8. *. Solution. By definition, the average is

$$\frac{1}{e-1}\int_1^e x^2\log x\mathrm{d}x$$

To antidifferentiate, we use integration by parts with $u=\log x$ and $\mathrm{d}v=x^2\,\mathrm{d}x$, hence $\mathrm{d}u=\frac{1}{x}\,\mathrm{d}x$ and $v=\frac{1}{3}x^3$.

$$\begin{aligned}\frac{1}{e-1}\int_1^e x^2\log x\mathrm{d}x &= \frac{1}{e-1}\left(\left[\frac{1}{3}x^3\log x\right]_1^e-\int_1^e\frac{1}{3}x^2\,\mathrm{d}x\right)\\ &= \frac{1}{e-1}\left[\frac{x^3}{3}\log x-\frac{x^3}{9}\right]_{x=1}^{x=e}\\ &= \frac{1}{e-1}\left[\frac{e^3}{3}-\frac{e^3}{9}+\frac{1}{9}\right]\\ &= \frac{1}{e-1}\left[\frac{2}{9}e^3+\frac{1}{9}\right]\end{aligned}$$

2.2.2.9. *. Solution. By definition, the average value in question equals

$$\begin{aligned}&\frac{1}{\pi/2-0}\int_0^{\pi/2}(3\cos^3x+2\cos^2x)\,\mathrm{d}x\\ &\quad=\frac{2}{\pi}\left(\int_0^{\pi/2}3\cos^3x\,\mathrm{d}x+\int_0^{\pi/2}2\cos^2x\,\mathrm{d}x\right)\end{aligned}$$

For the first integral we use the substitution $u = \sin x$, $\mathrm{d}u = \cos x\,\mathrm{d}x$, $\cos^2 x = 1 - \sin^2 x = 1 - u^2$. Note that the endpoints $x = 0$ and $x = \frac{\pi}{2}$ become $u = 0$ and $u = 1$, respectively.

$$\begin{aligned}\int_0^{\pi/2} 3\cos^3 x\,\mathrm{d}x &= \int_0^{\pi/2} 3\cos^2 x\cos x\,\mathrm{d}x\\ &= \int_0^1 3(1-u^2)\,\mathrm{d}u\\ &= (3u - u^3)\Big|_0^1 = 2.\end{aligned}$$

For the second integral we use the trigonometric identity $\cos^2 x\,\mathrm{d}x = \frac{1+\cos(2x)}{2}$.

$$\begin{aligned}2\int_0^{\pi/2} \cos^2 x\,\mathrm{d}x &= \int_0^{\pi/2} \big(1+\cos(2x)\big)\,\mathrm{d}x\\ &= \left[x + \frac{1}{2}\sin(2x)\right]_0^{\pi/2} = \frac{\pi}{2}\end{aligned}$$

Therefore, the average value in question is

$$\frac{2}{\pi}\left(\int_0^{\pi/2} 3\cos^3 x\,\mathrm{d}x + \int_0^{\pi/2} 2\cos^2 x\,\mathrm{d}x\right) = \frac{2}{\pi}\left(2 + \frac{\pi}{2}\right) = \frac{4}{\pi} + 1.$$

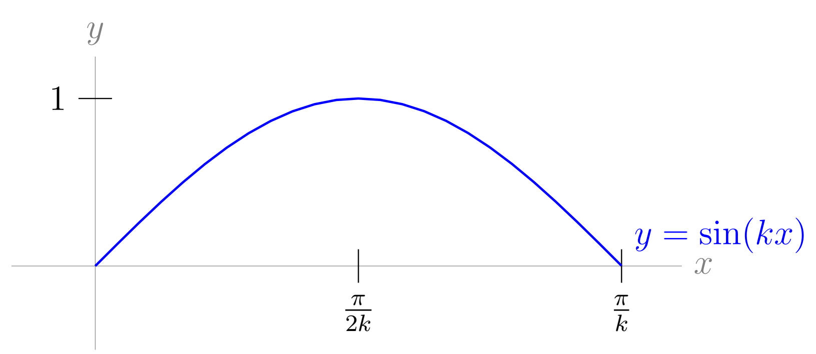
2.2.2.10. *. Solution. By definition, the average value in question equals

$$\mathrm{Ave} = \frac{1}{\pi/k - 0}\int_0^{\pi/k} \sin(kx)\,\mathrm{d}x$$

To evaluate the integral, we use the substitution $u = kx$, $\mathrm{d}u = k\,\mathrm{d}x$. Note that the endpoints $x = 0$ and $x = \pi/k$ become $u = 0$ and $u = \pi$, respectively. So

$$\mathrm{Ave} = \frac{k}{\pi}\int_0^{\pi} \sin(u)\,\frac{\mathrm{d}u}{k} = \frac{1}{\pi}\Big[-\cos(u)\Big]_0^{\pi} = \frac{2}{\pi}$$

Remark: the average does not depend on k. To see why this is, note that $\sin(kx)$ runs between -1 and 1 as x changes. When $x = 0$, $kx = 0$, and when $x = \pi/k$, $kx = \pi$. So, our function $\sin(kx)$ runs exactly from $\sin 0 = 0$ to $\sin(\pi/2) = 1$, then back down to $\sin\pi = 0$.



2.2.2.11. *. Solution. By definition, the average temperature is

$$\frac{1}{3}\int_0^3 T(x)\,\mathrm{d}x = \frac{1}{3}\int_0^3 \frac{80}{16-x^2}\,\mathrm{d}x$$

We don't see an obvious substitution, but integrand is a rational function. The degree of the numerator is strictly less than the degree of the denominator, so we factor the denominator and use a partial fraction decomposition.

$$\begin{aligned}\frac{80}{16-x^2} = \frac{80}{(4-x)(4+x)} &= \frac{A}{4-x} + \frac{B}{4+x} \\ 80 &= A(4+x) + B(4-x)\end{aligned}$$

Setting $x = 4$, we see $80 = 8A$, so $A = 10$. Setting $x = -4$, we see $80 = 8B$, so $B = 10$.

$$\begin{aligned}\frac{1}{3}\int_0^3 \frac{80}{16-x^2}\,\mathrm{d}x &= \frac{1}{3}\int_0^3 \frac{80}{(4-x)(4+x)}\,\mathrm{d}x \\ &= \frac{1}{3}\int_0^3 \left[\frac{10}{4-x} + \frac{10}{4+x}\right]\mathrm{d}x \\ &= \frac{1}{3}\int_0^3 \left[-\frac{10}{x-4} + \frac{10}{4+x}\right]\mathrm{d}x \\ &= \frac{10}{3}\Big[-\log|x-4| + \log|x+4|\Big]_0^3 \\ &= \frac{10}{3}\log\left|\frac{x+4}{x-4}\right|\Bigg|_0^3 = \frac{10}{3}[\log 7 - \log 1] \\ &= \frac{10}{3}\log 7 \quad \text{degrees Celsius}\end{aligned}$$

2.2.2.12. *. Solution. By definition, the average value is

$$\frac{1}{e-1}\int_1^e \frac{\log x}{x}\mathrm{d}x$$

To integrate, we use the substitution $u = \log x$, $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$. Then the limits of integration become 0 and 1, respectively.

$$\frac{1}{e-1}\int_1^e \frac{\log x}{x}\mathrm{d}x = \frac{1}{e-1}\int_0^1 u\mathrm{d}u = \frac{1}{e-1}\left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2(e-1)}$$

2.2.2.13. *. Solution. By definition, the average value is:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \cos^2 x\,\mathrm{d}x &= \frac{1}{2\pi}\cdot\frac{1}{2}\int_0^{2\pi}\big(\cos(2x)+1\big)\,\mathrm{d}x \\ &= \frac{1}{4\pi}\left[\frac{\sin(2x)}{2} + x\right]_0^{2\pi}\end{aligned}$$

$$= \frac{1}{4\pi} \cdot 2\pi = \frac{1}{2}$$

2.2.2.14. Solution. Before we start answering questions, let's look at our function a little more carefully. The term $50\cos\left(\frac{t}{12}\pi\right)$ has a period of 24 hours, while the term $200\cos\left(\frac{t}{4380}\pi\right)$ has a period of one year. So, the former term describes a standard daily variation, while the latter gives a seasonal variation over the year.
(a) Using the definition of an average, the average concentration over one year ($t = 0$ to 8760) is:

$$\begin{aligned}
&\frac{1}{8760}\int_0^{8760}\left(400 + 50\cos\left(\frac{t}{12}\pi\right) + 200\cos\left(\frac{t}{4380}\pi\right)\right)\,\mathrm{d}t \\
&= \frac{1}{8760}\int_0^{8760} 400\,\mathrm{d}t + \frac{50}{8760}\int_0^{8760}\cos\left(\frac{t}{12}\pi\right)\,\mathrm{d}t \\
&\qquad + \frac{200}{8760}\int_0^{8760}\cos\left(\frac{t}{4380}\pi\right)\,\mathrm{d}t \\
&= 400 + \frac{5}{876}\left[\frac{12}{\pi}\sin\left(\frac{t}{12}\pi\right)\right]_0^{8760} + \frac{5}{219}\left[\frac{4380}{\pi}\sin\left(\frac{t}{4380}\pi\right)\right]_0^{8760}
\end{aligned}$$

Since $\frac{8760}{12} = 730$, which is even, $\sin\left(\frac{8760}{12}\pi\right) = \sin(0) = 0$. Also, $\sin\left(\frac{8760}{4380}\pi\right) = \sin(2\pi) = 0$.

$$\begin{aligned}
&= 400 + \frac{5}{876}(0) + \frac{5}{219}(0) \\
&= 400 \quad \text{ppm}
\end{aligned}$$

Remark: for the portions of the integral in red and blue, we also could have noticed that the integrand goes through a whole (integer) number of periods. For every period, the net signed area between the curve and the x-axis is zero, so we could have seen from the very beginning these terms would contribute 0 to the final average.
(b) Using the definition of an average, the average concentration over the first day ($t = 0$ to $t = 24$) is:

$$\begin{aligned}
&\frac{1}{24}\int_0^{24}\left(400 + 50\cos\left(\frac{t}{12}\pi\right) + 200\cos\left(\frac{t}{4380}\pi\right)\right)\,\mathrm{d}t \\
&= \frac{1}{24}\int_0^{24} 400\,\mathrm{d}t + \frac{50}{24}\int_0^{24}\cos\left(\frac{t}{12}\pi\right)\,\mathrm{d}t + \frac{200}{24}\int_0^{24}\cos\left(\frac{t}{4380}\pi\right)\,\mathrm{d}t
\end{aligned}$$

Note $t = 0$ to $t = 24$ is one complete period for the integrand in red, so the red integral will evaluate to zero. However, $t = 0$ to $t = 24$ is *less* than one cycle for the integrand in blue, so we expect this will contribute some nonzero quantity to the average.

$$= 400 + 0 + \frac{200}{24}\left[\frac{4380}{\pi}\sin\left(\frac{t}{4380}\pi\right)\right]_0^{24}$$

$$\begin{aligned}
&= 400 + \frac{25}{3} \cdot \frac{4380}{\pi} \sin\left(\frac{24}{4380}\pi\right) \\
&= 400 + \frac{25}{3} \cdot \frac{4380}{\pi} \sin\left(\frac{2}{365}\pi\right) \\
&\approx 400 + 199.99 \\
&= 599.99 \quad \text{ppm}
\end{aligned}$$

Remark: $C(0) = 400 + 50 + 200$. The red term comes from the daily variation, and over the first day this will have an average of 0. The blue term comes from the seasonal variation, which changes dramatically over the course of an entire year but won't change very much over the course of a single day. So, it is reasonable that the average concentration over the first day should be close to (but not exactly) $400 + 200$ ppm.

(c) The average of $N(t)$ over $[0, 8760]$ is:

$$\begin{aligned}
\frac{1}{8760}\int_0^{8760} &\left(350 + 200\cos\left(\frac{t}{4380}\pi\right)\right)\,\mathrm{d}t \\
&= 350 + \frac{200}{8760}\left[\frac{4380}{\pi}\sin\left(\frac{t}{4380}\pi\right)\right]_0^{8760} \\
&= 350 + \frac{200}{8760}\left[\frac{4380}{\pi}\sin\left(\frac{8760}{4380}\pi\right)\right] \\
&= 350 + \frac{100}{\pi}\sin(2\pi) \\
&= 350
\end{aligned}$$

Since the average of $C(t)$ was 400, this gives us an absolute error of $|400 - 350| = 50$ ppm, for a relative error of

$$\frac{50}{400} = 0.125,$$

or 12.5%.

That is: sampling at the same time every day, rather than throughout the day, lead to an error of 12.5% in the yearly average concentration of carbon dioxide.

2.2.2.15. Solution.

a The cross-section of S at x is a circle with radius x^2, so area πx^4. The average of these values, $0 \le x \le 2$, is

$$A = \frac{1}{2-0}\int_0^2 \pi x^4\,\mathrm{d}x = \frac{1}{2}\left[\frac{\pi}{5}x^5\right]_0^2 = \frac{16\pi}{5}$$

b To find the volume of S, imagine cutting it into thin circular disks of radius x^2 and thickness $\mathrm{d}x$. The volume of one such disk is $\pi x^4\,\mathrm{d}x$, so the volume of S is

$$\int_0^2 \pi x^4\,\mathrm{d}x = \left[\frac{\pi}{5}x^5\right]_0^2 = \frac{32\pi}{5}$$

c The volume of a cylinder is the product of its base area with its length. A cylinder with circular cross-sections of area $\frac{16\pi}{5}$ and length 2 has volume $\frac{32\pi}{5}$.

Remark: this is the same as the volume of S, so the average cross-sectional area of S tells us the cross-sectional area of a cylinder with the same length and volume as S. Compare this to Question 1, where we saw the average value of a function gave the height of a rectangle with the same area as the function over the given interval.

2.2.2.16. Solution.

a We can see without calculation that the average will be zero, since $f(x) = x$ is an odd function and $[-3, 3]$ is a symmetric interval. Alternately, we can use the definition of an average to calculate

$$\frac{1}{6}\int_{-3}^{3} x\,\mathrm{d}x = \left[\frac{1}{12}x^2\right]_{-3}^{3} = \frac{1}{12}(9-9) = 0$$

b Using the definition provided for root mean square:

$$\mathrm{RMS} = \sqrt{\frac{1}{6}\int_{-3}^{3} x^2\,\mathrm{d}x} = \sqrt{\left[\frac{1}{18}x^3\right]_{-3}^{3}} = \sqrt{\frac{27}{18} - \frac{-27}{18}} = \sqrt{3}$$

2.2.2.17. Solution. Using the definition provided,

$$\begin{aligned}\mathrm{RMS} &= \sqrt{\frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \tan^2 x\,\mathrm{d}x} = \sqrt{\frac{2}{\pi}\int_{-\pi/4}^{\pi/4} (\sec^2 x - 1)\,\mathrm{d}x} \\ &= \sqrt{\frac{2}{\pi}\Big[\tan x - x\Big]_{-\pi/4}^{\pi/4}} = \sqrt{\frac{2}{\pi}\left[\left(1-\frac{\pi}{4}\right) - \left(-1+\frac{\pi}{4}\right)\right]} \\ &= \sqrt{\frac{2}{\pi}\left(2-\frac{\pi}{2}\right)} = \sqrt{\frac{4}{\pi} - 1} \approx 0.52\end{aligned}$$

2.2.2.18. Solution.

a Using Hooke's law, when the spring is stretched (or compressed) $f(t)$ metres past its natural length, the force exerted is $kf(t)$, where k is the spring constant. In this case, the force is

$$F(x) = (3\ \mathrm{N/cm})(f(t)\ \mathrm{cm}) = 3\sin(t\pi)\ \mathrm{N}$$

b Our interval encompasses three full periods of sine, so the average will be zero.

Alternately, we can compute, using the definition of an average:

$$\begin{aligned}\mathrm{Avg} &= \frac{1}{6}\int_{0}^{6} 3\sin(t\pi)\,\mathrm{d}t = \frac{1}{6}\left[-\frac{3}{\pi}\cos(t\pi)\right]_{0}^{6} \\ &= \frac{1}{2\pi}\left[\cos 0 - \cos(6\pi)\right] = 0\end{aligned}$$

This it doesn't tell us very much about the "normal" amount of force from the spring during our time period. It only tells us that force in one direction at is "cancelled out" by force in the opposite direction at another time.

c Using the definition given for root mean square,

$$\begin{aligned}
\text{RMS} &= \sqrt{\frac{1}{6}\int_0^6 (3\sin(t\pi))^2\, \mathrm{d}t} = \sqrt{\frac{3}{2}\int_0^6 \sin^2(t\pi)\,\mathrm{d}t} \\
&= \sqrt{\frac{3}{4}\int_0^6 (1-\cos(2t\pi))\, \mathrm{d}t} \\
&= \sqrt{\frac{3}{4}\left[t - \frac{1}{2\pi}\sin(2t\pi)\right]_0^6} \\
&= \sqrt{\frac{3}{4}\left[6 - \frac{1}{2\pi}\sin(12\pi) - 0\right]} \\
&= \sqrt{\frac{3}{4}(6)} = \frac{3}{\sqrt{2}} \approx 2.12
\end{aligned}$$

Exercises — Stage 3

2.2.2.19. *. Solution. (a) Let $v(t)$ be the speed of the car at time t. Then, by the trapezoidal rule with $a = 0$, $b = 2$, $\Delta t = 1/3$, the distance traveled is

$$\begin{aligned}
\int_0^2 v(t)\,\mathrm{d}t &\approx \Delta t\Big[\frac{1}{2}v(0) + v(1/3) + v(2/3) + v(3/3) + v(4/3) \\
&\qquad + v(5/3) + \frac{1}{2}v(2)\Big] \\
&= \frac{1}{3}\Big[\frac{1}{2}50 + 70 + 80 + 55 + 60 + 80 + \frac{1}{2}40\Big] = 130 \text{ km}
\end{aligned}$$

(b) The average speed is $\frac{\text{dist}}{\text{time}} \approx \frac{130 \text{ km}}{2 \text{ hr}} = 65$ km/hr.

2.2.2.20. Solution.

a Using the definition of an average,

$$A = \frac{1}{1-0}\int_0^1 e^t\,\mathrm{d}t = e - 1$$

b Since $s(t) - A = e^t - e + 1$, its average on $[0, 1]$ is

$$\begin{aligned}
\frac{1}{1-0}\int_0^1 \left(e^t - e + 1\right)\,\mathrm{d}t &= \left[e^t - et + t\right]_0^1 \\
&= (e - e + 1) - (1) = 0
\end{aligned}$$

Remark: what's happening here is that the average difference between $s(t)$ and A is zero, because the values of $s(t)$ that are larger than A (and give a positive value of $s(t)-A$) exactly cancel out the values of $s(t)$ that are smaller than A (and give a negative value of $s(t)-A$). However, knowing how far the average value is from our calculated average is a reasonable thing to measure. That's where (c) comes in.

c Using the definition of an average, the quantity we want is:

$$\frac{1}{1-0}\int_0^1 \left|e^t - e + 1\right| \,\mathrm{d}t$$

To deal with the absolute value, we consider the integral over two intervals: one where $e^t - e + 1$ is positive, and one where it's negative. To decide where to break the limits of integration, notice $e^t - e + 1 > 0$ exactly when $e^t > e - 1$, so $t > \log(e-1)$.

$$\begin{aligned}
&\frac{1}{1-0}\int_0^1 \left|e^t - e + 1\right| \,\mathrm{d}t \\
&= \int_0^{\log(e-1)} |\underbrace{e^t - e + 1}_{\text{negative}}| \,\mathrm{d}t + \int_{\log(e-1)}^1 |\underbrace{e^t - e + 1}_{\text{positive}}| \,\mathrm{d}t \\
&= \int_0^{\log(e-1)} \big(-e^t + e - 1\big) \,\mathrm{d}t + \int_{\log(e-1)}^1 \big(e^t - e + 1\big) \,\mathrm{d}t \\
&= \Big[-e^t + (e-1)t\Big]_0^{\log(e-1)} + \Big[e^t - (e-1)t\Big]_{\log(e-1)}^1 \\
&= [-(e-1) + (e-1)\log(e-1) + 1] \\
&\qquad\qquad + [e - (e-1) - (e-1) + (e-1)\log(e-1)] \\
&= 4 - 2e + 2(e-1)\log(e-1) \\
&\approx 0.42
\end{aligned}$$

Remark: what we just measured is how far $s(t)$ is, on average, from A. We had to neglect whether $s(t)$ was above or below A, because (as we saw in (b)) the values above A "cancel out" the values below A. That's where the absolute value came in.

Knowing how well most of your function's values match the average is an important measure, but dealing with absolute values can be a little clumsy. Therefore, the ***variance*** of a function squares the differences, rather than taking their absolute value. (In our example, that means looking at $(s(t)-A)^2$, rather than $|s(t)-A|$.) To compensate for the change in magnitude involve in squaring, the ***standard deviation*** is the square root of the variance. These are two very commonly used measures of how similar a function is to its average. Compare standard deviation to root-square-mean voltage from Example 2.2.6 and Questions 16 to 18.

2.2.2.21. Solution.

a Neither: the average of both these functions is zero. We saw this with a particular function in Question 20 (b), but it's actually true in general. It's a quick calculation to prove.

The average of $f(x) - A$ is:

$$\frac{1}{4-0}\int_0^4 \big(f(x)-A\big)\,\mathrm{d}x = \underbrace{\frac{1}{4}\int_0^4 f(x)\,\mathrm{d}x}_{A} - A = A - A = 0$$

Similarly, the average of $g(x) - A$ is:

$$\frac{1}{4-0}\int_0^4 \big(g(x)-A\big)\,\mathrm{d}x = \underbrace{\frac{1}{4}\int_0^4 g(x)\,\mathrm{d}x}_{A} - A = A - A = 0$$

b The function $|f(x) - A|$ tells us how far $f(x)$ is from A, without worrying whether $f(x)$ is larger or smaller. Looking at our graph, for most values of x in $[0, 4]$, $f(x)$ is quite far away from A, so $|f(x)-A|$ is usually a large, positive quantity.

By contrast, $|g(x) - A|$ is a small positive quantity for most values of x. The function $g(x)$ is quite close to A for all values of x in $[0, 4]$.

So, since $|g(x) - A|$ generally has much smaller values than $|f(x) - A|$, the average of $|f(x) - A|$ on $[0, 4]$ will be larger than the average of $|g(x) - A|$ on $[0, 4]$.

As discussed in Question 20(c), the average of $|f(x) - A|$ is a measure of how closely $f(x)$ resembles its average. We see from the graph that $f(x)$ doesn't resemble the constant function $y = A$ much at all, while $g(x)$ seems much more similar to the constant function $y = A$.

This kind of measure — how similar a function is to its average — is also the idea behind the root square mean.

2.2.2.22. Solution. When we rotate $f(x)$ about the x-axis, we form a solid whose radius at x is $|f(x)|$. So, its circular cross-sections have area $\pi|f(x)|^2 = \pi f^2(x)$. If we slice this solid into circular disks of thickness $\mathrm{d}x$, then the disks have volume $\pi f^2(x)\,\mathrm{d}x$. Therefore, the volume of the entire solid is $\displaystyle\int_a^b \pi f^2(x)\,\mathrm{d}x$. All we need to do now is get this into a form where we can replace the integral with the root mean square, R.

$$\begin{aligned} V &= \int_a^b \pi f^2(x)\,\mathrm{d}x = \pi\frac{b-a}{b-a}\int_a^b f^2(x)\,\mathrm{d}x \\ &= \pi(b-a)\left(\sqrt{\frac{1}{b-a}\int_a^b f^2(x)\,\mathrm{d}x}\right)^2 \\ &= \pi(b-a)R^2 \end{aligned}$$

Remark: the volume of a cylinder with length $b-a$ and radius r is $\pi(b-a)r^2$. So, the root mean square of $f(x)$ gave us the radius of a cylinder with the same volume as the solid formed by rotating $f(x)$. Recall the average of $f(x)$ gave us the height of a rectangle with the same area as $f(x)$. Compare this to the geometric interpretations of averages in Questions 1 and 15.

2.2.2.23. Solution. The question tells you $\frac{1}{1-0}\int_0^1 f(x)\,\mathrm{d}x = \frac{f(0)+f(1)}{2}$. Note $f(0)=c$, and $f(1)=a+b+c$.

$$\begin{aligned}
\frac{1}{1-0}\int_0^1 f(x)\,\mathrm{d}x &= \frac{f(0)+f(1)}{2} = \frac{c+(a+b+c)}{2}\\
\int_0^1 \big(ax^2+bx+c\big)\,\mathrm{d}x &= \frac{a+b+2c}{2}\\
\left[\frac{a}{3}x^3+\frac{b}{2}x^2+cx\right]_0^1 &= \frac{a}{2}+\frac{b}{2}+c\\
\frac{a}{3}+\frac{b}{2}+c &= \frac{a}{2}+\frac{b}{2}+c\\
\frac{a}{3} &= a\\
a &= 0
\end{aligned}$$

That is, $f(x)$ is linear.

2.2.2.24. Solution. The information given in the question is:

$$\begin{aligned}
\frac{\big(as^2+bs+c\big)+\big(at^2+bt+c\big)}{2} &= \frac{1}{t-s}\int_s^t \big(ax^2+bx+c\big)\,\mathrm{d}x\\
&= \frac{1}{t-s}\left[\frac{a}{3}x^3+\frac{b}{2}x^2+cx\right]_s^t\\
&= \frac{1}{t-s}\left[\frac{a}{3}(t^3-s^3)+\frac{b}{2}(t^2-s^2)\right.\\
&\qquad\left.+c(t-s)\right]\\
&= \frac{a}{3}(t^2+st+s^2)+\frac{b}{2}(t+s)+c\\
\frac{a}{2}(s^2+t^2)+\frac{b}{2}(s+t)+c &= \frac{a}{3}(t^2+st+s^2)+\frac{b}{2}(t+s)+c\\
\frac{a}{2}(s^2+t^2) &= \frac{a}{3}(t^2+st+s^2)\\
a\left[\frac{s^2+t^2}{2}-\frac{t^2+st+s^2}{3}\right] &= 0\\
a\left[\frac{s^2-2st+t^2}{6}\right] &= 0\\
a(s-t)^2 &= 0\\
a=0 \quad \text{OR} \quad s &= t
\end{aligned}$$

So, unless $s = t$ (and we're taking the very boring average of a single point!) then $a = 0$. That is: $f(x)$ is linear whenever $s \neq t$.

2.2.2.25. Solution. The function $g(x) = f(a+b-x)$, on the interval $[a,b]$, is a mirror of the function $f(x)$, with $g(a) = f(b)$ and $g(b) = f(a)$. So, $\int_a^b f(a+b-x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$, and hence $\frac{1}{b-a}\int_a^b f(a+b-x)\,\mathrm{d}x = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$, so the average value of $f(a+b-x)$ on $[a,b]$ is A.

Alternately, we can evaluate $\frac{1}{b-a}\int_a^b f(a+b-x)\,\mathrm{d}x$ directly, using the substitution $u = a+b-x$, $\mathrm{d}x = -\mathrm{d}x$:

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(a+b-x)\,\mathrm{d}x &= \frac{-1}{b-a}\int_{u(a)}^{u(b)} f(u)\,\mathrm{d}u \\
&= \frac{-1}{b-a}\int_b^a f(u)\,\mathrm{d}u \\
&= \frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}u \\
&= A
\end{aligned}$$

2.2.2.26. Solution.

a The function $A(x)$ only gives us information about an integral when one limit of integration is zero. We can get around this by using properties of definite integrals from Section 1.2 to break our integral into two integrals, each of which has 0 as one limit of integration. So, we find the average of $f(t)$ on $[a,b]$ as follows:

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t &= \frac{1}{b-a}\left(\int_a^0 f(t)\,\mathrm{d}t + \int_0^b f(t)\,\mathrm{d}t\right) \\
&= \frac{1}{b-a}\left(-\int_0^a f(t)\,\mathrm{d}t + \int_0^b f(t)\,\mathrm{d}t\right) \\
&= \frac{1}{b-a}\left(-a\cdot\underbrace{\frac{1}{a}\int_0^a f(t)\,\mathrm{d}t}_{A(a)} + b\cdot\underbrace{\frac{1}{b}\int_0^b f(t)\,\mathrm{d}t}_{A(b)}\right) \\
&= \frac{1}{b-a}\left(-aA(a) + bA(b)\right) = \frac{bA(b) - aA(a)}{b-a}
\end{aligned}$$

b From the definition of $A(x)$, we know

$$A(x) = \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t$$

That is,

$$xA(x) = \int_0^x f(t)\,\mathrm{d}t$$

To find $f(x)$, we differentiate both sides. For the left side, we use the product rule; for the right side, we use the Fundamental Theorem of Calculus part 1.

$$A(x) + xA'(x) = f(x)$$

So, $f(t) = A(t) + tA'(t)$.

2.2.2.27. Solution.

a One of many possible answers: $f(x) = \begin{cases} -1 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$.

b No such function exists.

- Note 1: Suppose $f(x) > 0$ for all x in $[-1,1]$. Then $\frac{1}{2}\int_{-1}^{1} f(x)\,\mathrm{d}x > \frac{1}{2}\int_{-1}^{1} 0\,\mathrm{d}x = 0$. That is, the average value of $f(x)$ on the interval $[-1,1]$ is not zero — it's something greater than zero.
- Note 2: Suppose $f(x) < 0$ for all x in $[-1,1]$. Then $\frac{1}{2}\int_{-1}^{1} f(x)\,\mathrm{d}x < \frac{1}{2}\int_{-1}^{1} 0\,\mathrm{d}x = 0$. That is, the average value of $f(x)$ on the interval $[-1,1]$ is not zero — it's something less than zero.

So, if the average value of $f(x)$ is zero, then $f(x) \ge 0$ for some x in $[-1,1]$, and $f(y) \le 0$ for some $y \in [-1,1]$. Since f is a continuous function, and 0 is between $f(x)$ and $f(y)$, by the intermediate value theorem (see the CLP-1 text) there is some value c between x and y such that $f(c) = 0$. Since x and y are both in $[-1,1]$, then c is as well. Therefore, no function exists as described in the question.

2.2.2.28. Solution. This seems like it might be true: if f is getting closer and closer to zero, as x grows towards infinity, then over time the later values will become a larger and larger portion of the total interval we're looking at, and so the average should look more and more like $f(x)$ when x is large — that is, like 0. That's some intuition to start us out, but it isn't a rigorous argument. To be sure we haven't overlooked something, let's use the definition of an average to express $A(x)$.

$$\begin{aligned} A(x) &= \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t \\ \lim_{x\to\infty} A(x) &= \lim_{x\to\infty}\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t \end{aligned}$$

If $\int_0^\infty f(t)\,\mathrm{d}t$ converges, then this limit is 0, and the statement is true. So, suppose it does not converge. Since $f(x)$ is positive, that means $\lim\limits_{x\to\infty}\int_0^x f(t)\,\mathrm{d}(t) = \infty$, so we can use l'Hôpital's rule. To differentiate the numerator, we use the Fundamental Theorem of Calculus part 1.

$$\lim_{x\to\infty} A(x) = \lim_{x\to\infty}\frac{\int_0^x f(t)\,\mathrm{d}t}{x} = \lim_{x\to\infty}\frac{f(t)}{1} = 0$$

So, the statement is true whether $\int_0^x f(t)\,\mathrm{d}t$ converges or not.

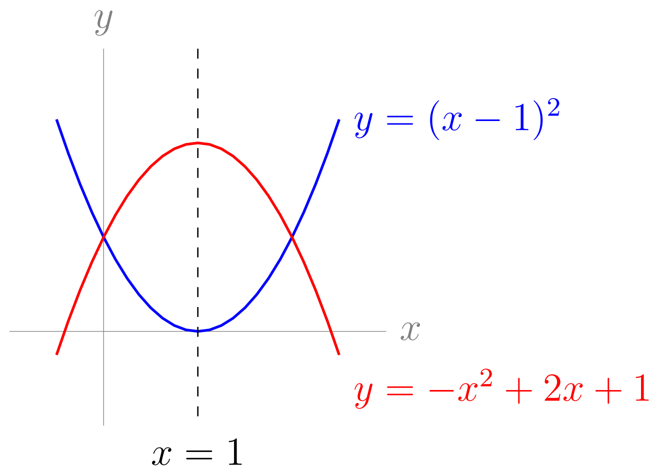
2.2.2.29. Solution. Note $f(t)$ is a continuous function that takes only positive values, and $\lim_{t\to\infty} f(t) = 0$. By the result of Question 28, $\lim_{x\to\infty} A(x) = 0$.

2.3 · Centre of Mass and Torque

2.3.3 · Exercises

Exercises — Stage 1

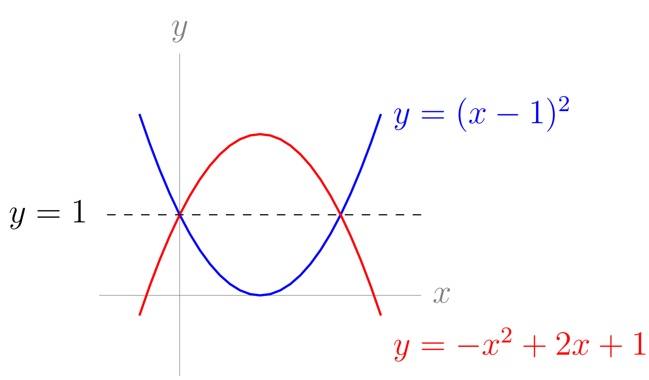
2.3.3.1. Solution. Note $-x^2 + 2x + 1 = 2 - (x-1)^2$. So, both parabolas are symmetric about the line $x = 1$, and the x-coordinate of the centroid is $x = 1$.



The parabolas meet when:

$$\begin{aligned}
(x-1)^2 &= 2 - (x-1)^2 \\
2(x-1)^2 &= 2 \\
|x-1| &= 1 \\
x = 0, &\quad x = 2
\end{aligned}$$

At both these points, $y = 1$, so we see the figure is symmetric about the line $y = 1$. Then the y-coordinate of the centroid is 1.



Therefore, the centroid is at $(1, 1)$.

2.3.3.2. Solution. The circle and the cut-out rectangle are symmetric about the x-axis, and about the y-axis, so the centroid is the origin.
Remark: the centroid of a region doesn't have to be a point in the region!

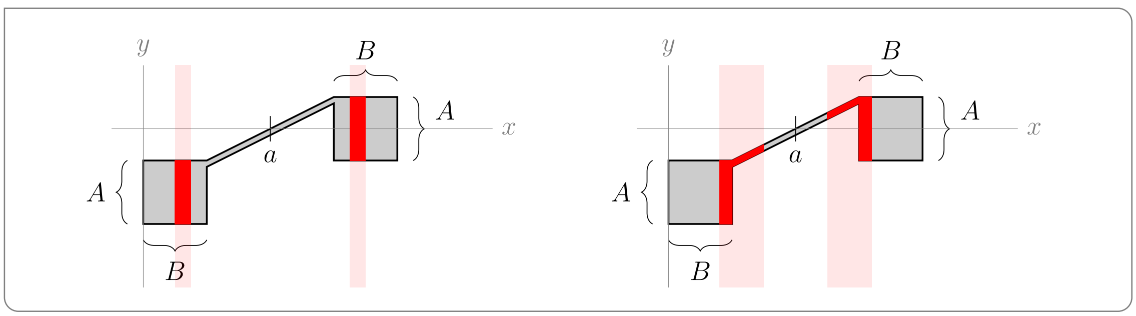
2.3.3.3. Solution. In general, this is false: weights farther out from the centre "count more" when we calculate the centre of mass. For instance, a rod with a 1-kg weight at $x = -10$ and a 10-kg weight at $x = 1$ will balance at $x = 0$. There's far more mass to one side of $x = 0$ than the other.

2.3.3.4. Solution. Following Equation 2.3.1, the centre of mass of the rod is at:

$$\bar{x} = \frac{\sum(\text{mass}) \times (\text{position})}{\sum(\text{mass})} = \frac{1 \times 1 + 2 \times 3 + 2 \times 4 + 1 \times 6}{1 + 2 + 2 + 1} = \frac{21}{6} = \frac{7}{2}$$

That is, the centre of mass is 3.5 metres from the left end.

2.3.3.5. Solution. (a) If we were to set this figure on a pencil lined up along the vertical line $x = a$, it seems pretty clear that it would fall to the left. So, the centre of mass is to the left of the line $x = a$. The same is true in (b): the added density on the left makes it only more lopsided. However, in (c), the right side is denser than the left, which could counterbalance the left. Without knowing more about the dimensions and the density, we can't say where the centre of mass is in relation to the line $x = a$.
(d) Consider a section of the figure, consisting of all points (x, y) in the figure with $b \leq x \leq c$, and its "mirror" section on the other side of the line $x = a$. These two sections, which are drawn in red in the sketches below, will have the same area, at the same distance from $x = a$. Since we only care about the x-coordinate of the centre of mass, it doesn't matter that the two halves are at different y-coordinates. The centre of mass falls along the line $x = a$.



(e) There is the same amount of area to the left and right of the line $x = a$, as in part (d). However, the area to the right is "stretched out" more, so that it occupies space *farther away* from the line $x = a$. So, the centre of mass will be to the right of the line $x = a$.

2.3.3.6. Solution.

- The volume of water in Tank A is $\frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$ cubic metres.
- The mass of water is $\frac{4000}{3}\pi$ kg.
- By symmetry, the centre of mass of the water when it fills Tank A is exactly in the centre of the sphere, at height $\bar{y}_1 = 4$ metres above the ground (one

metre above the bottom of Tank A, which is three metres above the ground).

- When the water is entirely in Tank B, its height is $\frac{2}{3}\pi$ metres. (The base of Tank B has area 2 m^2, and the volume of water is $\frac{4}{3}\pi$ m^3.) By symmetry, the centre of mass is exactly halfway up, at height $\bar{y}_2 = \frac{1}{3}\pi$ metres.
- So, the point mass in our model is moved from $\bar{y}_1 = 4$ to $\bar{y}_2 = \frac{1}{3}\pi$, a distance of $4 - \frac{1}{3}\pi$ metres, by gravity.
- The work involved is:

$$\begin{aligned}\left(\frac{4000}{3}\pi\mathrm{kg}\right) \times \left(4 - \frac{1}{3}\pi\mathrm{m}\right) \times \left(9.8\frac{\mathrm{m}}{\mathrm{sec}^2}\right) &= \frac{39200\pi}{9}(12-\pi)\\ &\approx 121{,}212\mathrm{J}\end{aligned}$$

2.3.3.7. Solution.

a A thin slice of S at position x has height $\frac{1}{x}$, so if its width is $\mathrm{d}x$, its area is $\frac{1}{x}\,\mathrm{d}x$.

b A small piece of R at position x has density $\frac{1}{x}$, so if its length is $\mathrm{d}x$, its mass is $\frac{1}{x}\,\mathrm{d}x$.

c Adding up all our tiny slices from (a) gives us the total area of S:

$$\int_1^3 \frac{1}{x}\,\mathrm{d}x = \log 3$$

d Adding up all our tiny pieces from (b) gives us the total mass of R:

$$\int_1^3 \frac{1}{x}\,\mathrm{d}x = \log 3$$

e Using Equation 2.3.5, the x-coordinate of the centroid of S is

$$\frac{\int_1^3 x\cdot\frac{1}{x}\,\mathrm{d}x}{\int_1^3 \frac{1}{x}\,\mathrm{d}x} = \frac{\int_1^3 1\,\mathrm{d}x}{\log 3} = \frac{2}{\log 3}$$

f Using Equation 2.3.4, the centre of mass of R is

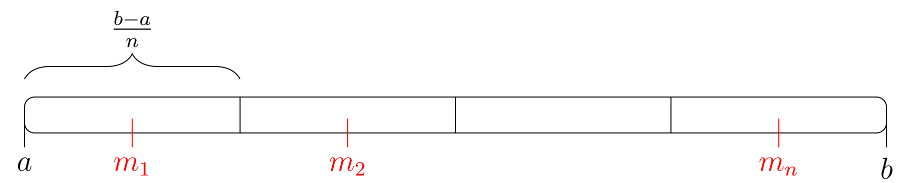
$$\frac{\int_1^3 x\cdot\frac{1}{x}\,\mathrm{d}x}{\int_1^3 \frac{1}{x}\,\mathrm{d}x} = \frac{\int_1^3 1\,\mathrm{d}x}{\log 3} = \frac{2}{\log 3}$$

Remark: following the derivation of Equation 2.3.5, if we wanted to find the x-coordinate of the centroid of S, we would set up a rod that had exactly the characteristics of R. That's why all the answers were repeated.

2.3.3.8. Solution.

a If we chop R into n pieces, each piece has length $\frac{b-a}{n}$. Then our ith cut is at position $a+i\left(\frac{b-a}{n}\right)$, so our ith piece runs from $a+(i-1)\left(\frac{b-a}{n}\right)$ to $a+i\left(\frac{b-a}{n}\right)$. The approximation of the mass of this piece comes from the density at its midpoint,

$$\begin{aligned} m_i &= \frac{\left[a+(i-1)\left(\frac{b-a}{n}\right)\right]+\left[a+i\left(\frac{b-a}{n}\right)\right]}{2} \\ &= a+(i-\tfrac{1}{2})\left(\frac{b-a}{n}\right) \end{aligned}$$



So, the ith piece has length $\frac{b-a}{n}$, with approximate density $\rho\left(m_i\right) = \rho\left(a+(i-\frac{1}{2})\left(\frac{b-a}{n}\right)\right)$. We approximate that the ith piece has mass $\left(\frac{b-a}{n}\right) \cdot \rho\left(m_i\right)$ and position m_i. Using Equation 2.3.1, the centre of mass of R is approximately at position:

$$\begin{aligned} \bar{x}_n &= \frac{\sum_{i=1}^{n} (\text{mass of } i\text{th piece}) \times (\text{position of } i\text{th piece})}{\sum_{i=1}^{n} (\text{mass of } i\text{th piece})} \\ &= \frac{\sum_{i=1}^{n} \left[\frac{b-a}{n}\rho(m_i) \times m_i\right]}{\sum_{i=1}^{n} \frac{b-a}{n}\rho(m_i)} \\ &= \frac{\sum_{i=1}^{n} \left[\frac{b-a}{n}\rho\left(a+\left(i-\frac{1}{2}\right)\left(\frac{b-a}{n}\right)\right) \times \left(a+\left(i-\frac{1}{2}\right)\left(\frac{b-a}{n}\right)\right)\right]}{\sum_{i=1}^{n} \frac{b-a}{n}\rho\left(a+\left(i-\frac{1}{2}\right)\left(\frac{b-a}{n}\right)\right)} \end{aligned}$$

b Remember the definition of a midpoint Riemann sum:

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{i=1}^{n} \frac{b-a}{n} \cdot f\left(a+(i-\tfrac{1}{2})\left(\frac{b-a}{n}\right)\right)$$

The numerator of our approximation in part (a) is, therefore, a midpoint Riemann sum of $\int_a^b \rho(x) \times x\,\mathrm{d}x$, and the denominator is a midpoint Riemann sum of $\int_a^b \rho(x)\,\mathrm{d}x$.

Using the definition of a definite integral (Definition 1.1.9), we see the limit of the approximation in (a) as x goes to infinity is

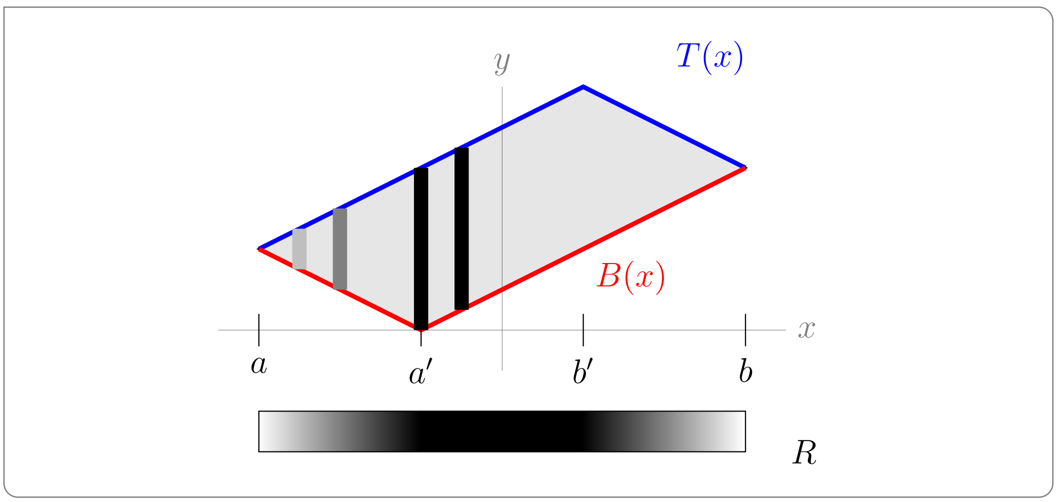
$$\bar{x} = \frac{\int_a^b x\rho(x)\,\mathrm{d}x}{\int_a^b \rho(x)\,\mathrm{d}x}$$

This gives us the exact centre of mass of our rod.

Remark: this is Equation 2.3.4 in the text.

2.3.3.9. Solution.

a On the left-most corner of S, $T(x) = B(x)$, so the height of S is zero; that is, the area of a very small vertical strip is very close to zero, so the density of R is close to 0. As we move closer to the position labeled a', the height of the strips increases, so the areas of the strips increases, so the density of R increases. Then, between the points labeled a' and b', the height of S remains constant, since $T(x)$ and $B(x)$ are parallel here, so the areas of the strips of S remain constant, and the density of R remains constant. Then, between b' and b, the height of S decreases, so the area of the strips decrease, so the density of R decreases.



b At position x, the height of S is $T(x) - B(x)$, so a rectangle with width $\mathrm{d}x$ and this height would have area $(T(x) - B(x))\,\mathrm{d}x$.

c According to our model, the tiny section of R at position x with width $\mathrm{d}x$ has mass $(T(x) - B(x))\,\mathrm{d}x$ (that is, the area of S over this same tiny interval), so its density is $\rho(x) = \frac{\text{mass}}{\text{length}} = \frac{(T(x)-B(x))\,\mathrm{d}x}{\mathrm{d}x} = T(x) - B(x)$.

d Imagine S were a solid, of constant density. The the mass of a portion of S is proportional to the area of that portion. To find the x-coordinate where the solid would balance, we imagine compressing together the vertical dimension of S until it's a rod. That is, we would take a very thin vertical strip of S, and turn it into a small segment of a rod, with the same mass. Then the centre of mass of that rod would be exactly the x-coordinate of the centre of mass of the solid — that is, the x-coordinate of the centroid of S.

The compressed rod we form in this way is exactly R (perhaps multiplied by a constant, to account for the density of S, but this doesn't affect where R balances). So, the x-coordinate of the centroid has the same position as the centre of mass of R.

Our result from Question 8(b) tells us the centre of mass of R is

$$\frac{\int_a^b x\rho(x)\,\mathrm{d}x}{\int_a^b \rho(x)\,\mathrm{d}x}$$

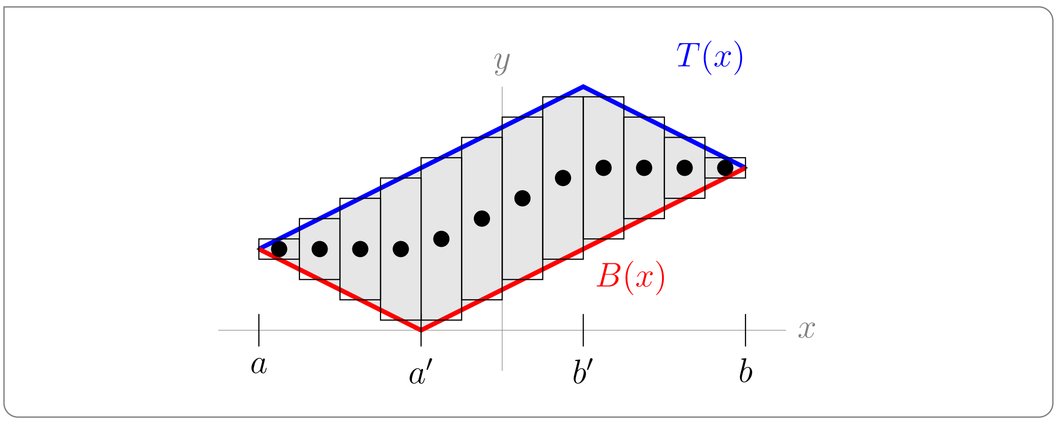
In (c), we found $\rho(x) = T(x) - B(x)$. So, for the solid S bounded by $T(x)$ and $B(x)$ on the interval $[a, b]$,

$$\bar{x} = \frac{\int_a^b x(T(x) - B(x))\,\mathrm{d}x}{\int_a^b (T(x) - B(x))\,\mathrm{d}x}$$

Remark: the denominator is the area of S. This formula is the same as the formula found in Equation 2.3.5.

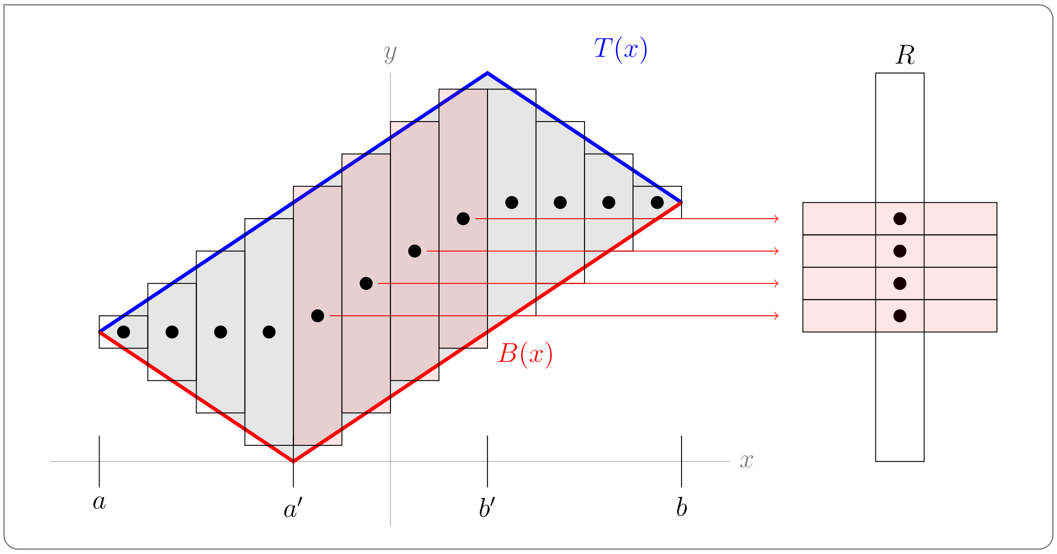
2.3.3.10. Solution.

a To begin with, we'll sketch some strips, and put a dot at the centre of mass of each one (its vertical centre).



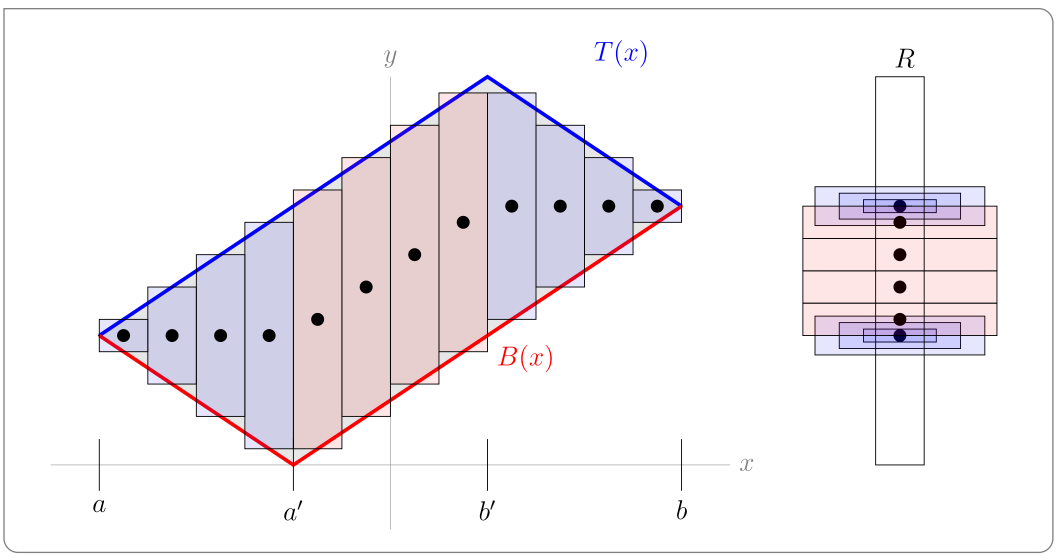
In our model, each of these strips corresponds to a weight on R, positioned at its centre of mass (the height of the dot), and with a mass equal to the strip's area. For the portion of S with $a' \le x \le b'$, each centre of mass is at a slightly different height, but the areas of the slices are the same. So, the corresponding weights along R are at different heights, but all have the same mass, as shown below. (Note the rod R below only contains the weights from the middle of S — we'll add the rest later.)

For clarity, the diagrams below are zoomed in.

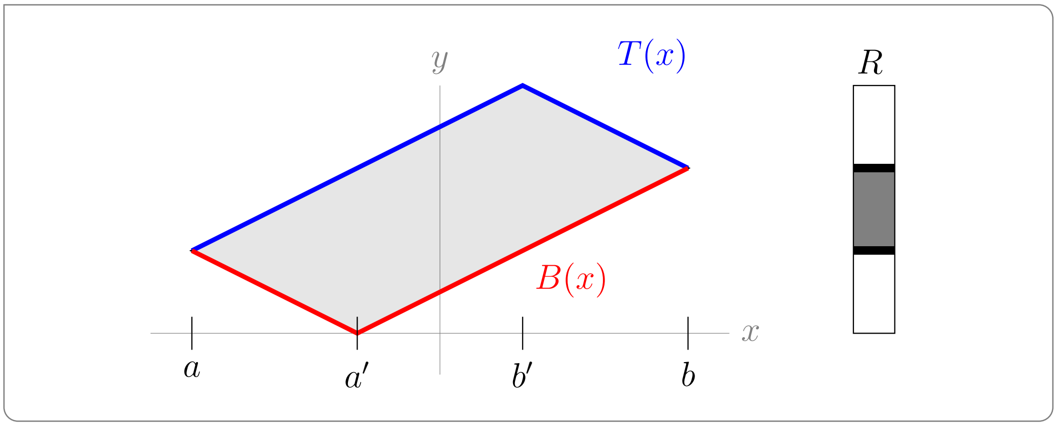


By contrast to the slices in the interval $[a', b']$, the slices of S along $[a, a']$ all have the same centre of mass, but different areas. So, there is one position along R that has a number of weights all stacked on top of one another, of varying masses.

The same situation applies to the slices of S along $[b, b']$. So, all together, our rod looks something like this:



Remark: if we had sketched the density of R, it would have looked something like this:



because from our sketch, we see that the density of R:

- is 0 at either end,
- is suddenly very high where the blue weights are, and
- is constant and lower between the blue weights.

b At position x, the height of S is $T(x) - B(x)$, and the width of the strip is $\mathrm{d}x$, so the area of the strip is $(T(x) - B(x))\,\mathrm{d}x$.

Since the density of S is uniform, the centre of mass of the strip is halfway up: at $\dfrac{T(x)+B(x)}{2}$.

c If we cut S into n strips, then the strip at position x_i has area $(T(x_i) - B(x_i))\Delta x$, where $\Delta x = \frac{b-a}{n}$, and its centre of mass is at height $\dfrac{T(x_i)+B(x_i)}{2}$. So, our approximation of the centre of mass of the rod is:

$$\begin{aligned}\bar{y}_n &= \frac{\sum_{i=1}^{n}(M_i \times y_i)}{\sum_{i=1}^{n} M_i}\\ &= \frac{\sum_{i=1}^{n}\left((T(x_i) - B(x_i))\Delta x\right) \times \left(\dfrac{T(x_i)+B(x_i)}{2}\right)}{\sum_{i=1}^{n}(T(x_i) - B(x_i))\Delta x}\\ &= \frac{\sum_{i=1}^{n}(T(x_i)^2 - B(x_i)^2)\Delta x}{2\sum_{i=1}^{n}(T(x_i) - B(x_i))\Delta x}\end{aligned}$$

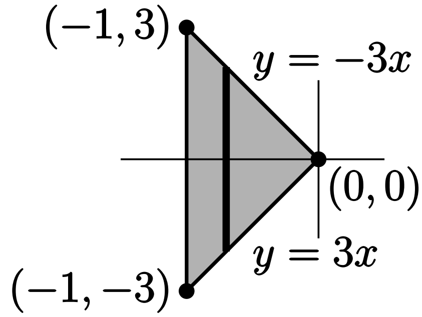
We use the definition of a definite integral (Definition 1.1.9) to re-write the limit of the above function.

$$\bar{y} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n}(T(x_i)^2 - B(x_i)^2)\Delta x}{2\sum_{i=1}^{n}(T(x_i) - B(x_i))\Delta x}$$

$$= \frac{\int_a^b \left(T(x)^2 - B(x)^2\right)\,\mathrm{d}x}{2\int_a^b \left(T(x) - B(x)\right)\,\mathrm{d}x}$$

Remark: the denominator is twice the area of S. This equation for the y-coordinate of the centroid is the same as the one given in Equation 2.3.5.

2.3.3.11. $*$. Solution. We use vertical strips, as in the sketch below. (To use horizontal strips we would have to split the domain of integration in two: $-3 \le y \le 0$ and $0 \le y \le 3$.)



The equations of the top and bottom of the triangle are

$$y = T(x) = -3x \qquad \text{and} \qquad y = B(x) = 3x.$$

The area of the triangle is $A = \frac{1}{2}(6)(1) = 3$. Now, we can apply the vertical-slice versions of Equation 2.3.5.

$$\begin{aligned}\bar{x} &= \frac{1}{A}\int_{-1}^{0} x\big[T(x) - B(x)\big]\mathrm{d}x = \frac{1}{3}\int_{-1}^{0} x\big[(-3x) - (3x)\big]\mathrm{d}x\\ &= -\frac{1}{3}\int_{-1}^{0} 6x^2\mathrm{d}x\end{aligned}$$

Exercises — Stage 2

2.3.3.12. Solution. Applying Equation 2.3.4,

$$\bar{x} = \frac{\int_0^7 x \cdot x\,\mathrm{d}x}{\int_0^7 x\,\mathrm{d}x} = \frac{\left[\frac{1}{3}x^3\right]_0^7}{\left[\frac{1}{2}x^2\right]_0^7} = \frac{\frac{1}{3}(7^3)}{\frac{1}{2}(7^2)} = \frac{14}{3}$$

2.3.3.13. Solution. Applying Equation 2.3.4,

$$\bar{x} = \frac{\int_{-3}^{10} x \cdot \frac{1}{1+x^2}\,\mathrm{d}x}{\int_{-3}^{10} \frac{1}{1+x^2}\,\mathrm{d}x}$$

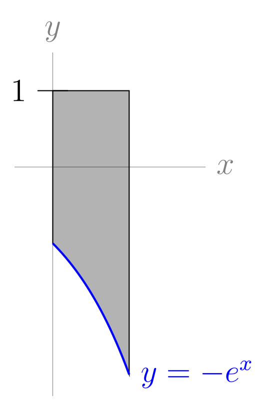
For the numerator, we use the substitution $u = 1 + x^2$, $\mathrm{d}u = 2x\,\mathrm{d}x$.

$$= \frac{\frac{1}{2}\int_{10}^{101}\frac{1}{u}\,\mathrm{d}u}{\Big[\arctan x\Big]_{-3}^{10}} = \frac{\frac{1}{2}\Big[\log u\Big]_{10}^{101}}{\arctan 10 - \arctan(-3)}$$

$$= \frac{\Big[\log 101 - \log 10\Big]}{2(\arctan 10 + \arctan(3))} = \frac{\log 10.1}{2(\arctan 10 + \arctan(3))} \approx 0.43$$

Since arctangent is an odd function, $\arctan(-3) = -\arctan(3)$; using logarithm rules, $\log 101 - \log 10 = \log\frac{101}{10} = \log 10.1$.

2.3.3.14. *. Solution. If we use horizontal strips, then we need to break the region into two pieces: $y \geq -1 = -e^0$, and $y \leq -1$. However, if we use vertical strips, the equation of the top of the region is $y = T(x) = 1$, and the equation of the bottom of the region is $y = B(x) = -e^x$, for all x from $a = 0$ to $b = 1$. So, we use vertical strips.



Using Equation 2.3.5, the y-coordinate of the centre of mass is

$$\begin{aligned}\bar{y} &= \frac{1}{2A}\int_0^1 \left[T(x)^2 - B(x)^2\right]\mathrm{d}x = \frac{1}{2e}\int_0^1\left(1 - e^{2x}\right)\mathrm{d}x \\ &= \frac{1}{2e}\left[x - \frac{1}{2}e^{2x}\right]_0^1 = \frac{1}{2e}\left[1 - \frac{e^2}{2} - 0 + \frac{1}{2}\right] \\ &= \frac{3}{4e} - \frac{e}{4}\end{aligned}$$

2.3.3.15. *. Solution. (a) The lines $y = 0$, $x = 0$, and $x = 2$ are easy enough to sketch. Let's get some basic information about $y = T(x) = \frac{1}{\sqrt{16-x^2}}$ on the interval $[0, 2]$.

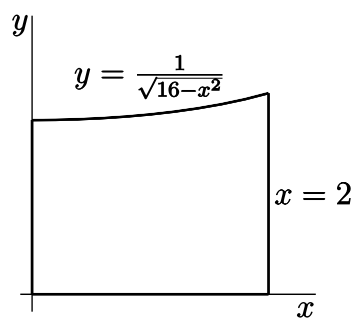
- For all x in its domain, $T(x) \geq 0$. In particular, it's always the top of our region (so $T(x)$ is a reasonable name for it), while the bottom is $B(x) = 0$.
- $T(0) = \frac{1}{4}$, and $T(2) = \frac{1}{2\sqrt{3}}$

- $T'(x) = \frac{x}{(16-x^2)^{3/2}}$, which is positive on $[0,2]$, so $T(x)$ is increasing.

  Remark: to see that $T(x)$ is increasing, we can also just break it into pieces:

  - When $x \geq 0$, x^2 is increasing, so
  - $16 - x^2$ is decreasing, so
  - $\sqrt{16-x^2}$ is decreasing, so
  - $\frac{1}{\sqrt{16-x^2}} = T(x)$ is increasing.

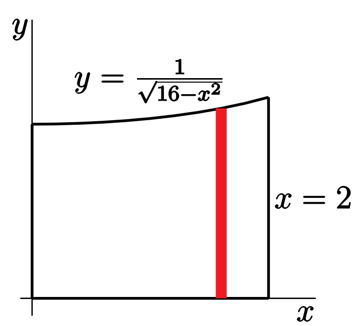
- $T''(x) = \frac{2x^2+16}{(16-x^2)^{5/2}}$, which is positive, so $T(x)$ is concave up.



Remark: If we only wanted to solve (b), it would still be nice to have a sketch of the region, but it wouldn't need to be so detailed. Knowing that $T(x)$ is always greater than 0 would be enough to tell us we could use vertical slices with $T(x)$ as the top and $y = 0$ as the bottom.
If we wanted to use horizontal slices (we don't... but we could!) we would additionally want to know that $T(x)$ is increasing over $[0,2]$, $T(0) = \frac{1}{4}$, and $T(2) = \frac{1}{2\sqrt{3}}$.
This would tell us that:

- the right endpoint of a horizontal strip is always $x = 2$,
- the left endpoint is determined by $T(x)$ from $y = \frac{1}{4}$ to $y = \frac{1}{2\sqrt{3}}$, and
- the left endpoint is $x = 0$ for $0 \leq y \leq \frac{1}{4}$.

(b)



The part of the region with x coordinate between x and $x + \mathrm{d}x$ is a strip of width $\mathrm{d}x$ running from $y = 0$ to $y = \frac{1}{\sqrt{16-x^2}}$. It is illustrated in red in the figure above. So, the area of the region is

$$A = \int_0^2 \frac{1}{\sqrt{16-x^2}}\,\mathrm{d}x = \int_0^{\arcsin(1/2)} \frac{1}{4\cos t} 4\cos t\,\mathrm{d}t = \arcsin\frac{1}{2} = \frac{\pi}{6}$$

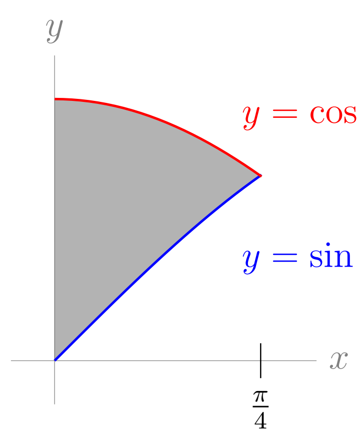
where we made the substitution $x = 4\sin t, \mathrm{d}x = 4\cos t\mathrm{d}t, \sqrt{16-x^2} = 4\cos t$. Using Equation 2.3.5,

$$\begin{aligned}\bar{y} &= \frac{\int_0^2 \left[T(x)^2 - B(x)^2\right]\mathrm{d}x}{2A} = \frac{\displaystyle\int_0^2 \left[\left(\frac{1}{\sqrt{16-x^2}}\right)^2 - 0^2\right]\mathrm{d}x}{2A} \\ &= \frac{1}{2A}\int_0^2 \frac{1}{16-x^2}\,\mathrm{d}x = \frac{1}{2A}\int_0^2 \frac{1}{(4-x)(4+x)}\,\mathrm{d}x\end{aligned}$$

Using the method of partial fractions, we see $\dfrac{1}{16-x^2} = \dfrac{1/8}{4+x} + \dfrac{1/8}{4-x}$.

$$\begin{aligned}&= \frac{1}{2A}\int_0^2 \left[\frac{1/8}{4+x} + \frac{1/8}{4-x}\right]\mathrm{d}x = \frac{1}{16A}\int_0^2 \left[\frac{1}{x+4} - \frac{1}{x-4}\right]\mathrm{d}x \\ &= \frac{1}{16A}\Big[\log|x+4| - \log|x-4|\Big]_0^2 = \frac{6}{16\pi}\big[\log 6 - \log 2 - \log 4 + \log 4\big] \\ &= \frac{3\log 3}{8\pi}\end{aligned}$$

2.3.3.16. *. Solution.



The top of the region is $y = T(x) = \cos(x)$ and the bottom of the region is $y = B(x) = \sin(x)$. So, the area of the region is

$$\begin{aligned}A &= \int_0^{\pi/4} \big(T(x) - B(x)\big)\,\mathrm{d}x = \int_0^{\pi/4} \big(\cos(x) - \sin(x)\big)\,\mathrm{d}x \\ &= \Big[\sin(x) + \cos(x)\Big]_0^{\pi/4}\end{aligned}$$

$$= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] - [0+1] = \sqrt{2} - 1$$

If we use horizontal slices, we'll need to break up the object into two regions, so let's use vertical slices. Using Equation 2.3.5, the region has centroid $(\bar{x}, \bar{y})$ with:

$$\bar{x} = \frac{1}{A}\int_0^{\pi/4} x\big(T(x) - B(x)\big)\,\mathrm{d}x = \frac{1}{A}\int_0^{\pi/4} x\big(\cos(x) - \sin(x)\big)\,\mathrm{d}x$$

We use integration by parts with $u = x$, $\mathrm{d}v = (\cos x - \sin x)\mathrm{d}x$; $\mathrm{d}u = \mathrm{d}x$, $v = \sin x + \cos x$.

$$\begin{aligned}
&= \frac{1}{A}\left(\Big[x(\sin x + \cos x)\Big]_0^{\pi/4} - \int_0^{\pi/4} (\sin x + \cos x)\,\mathrm{d}x\right)\\
&= \frac{1}{A}\Big[x\sin(x) + x\cos(x) + \cos x - \sin x\Big]_0^{\pi/4}\\
&= \frac{1}{A}\left[\left(\frac{\pi}{4}\cdot\frac{1}{\sqrt{2}} + \frac{\pi}{4}\cdot\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - 1\right]\\
&= \frac{\frac{\pi}{4}\sqrt{2} - 1}{A} = \frac{\frac{\pi}{4}\sqrt{2} - 1}{\sqrt{2} - 1}
\end{aligned}$$

Again using Equation 2.3.5,

$$\begin{aligned}
\bar{y} &= \frac{1}{2A}\int_0^{\pi/4} \big(T(x)^2 - B(x)^2\big)\,\mathrm{d}x = \frac{1}{2A}\int_0^{\pi/4} \big(\cos^2(x) - \sin^2(x)\big)\,\mathrm{d}x\\
&= \frac{1}{2A}\int_0^{\pi/4} \cos(2x)\,\mathrm{d}x = \frac{1}{2A}\left[\frac{1}{2}\sin(2x)\right]_0^{\pi/4} = \frac{1}{4(\sqrt{2} - 1)}
\end{aligned}$$

2.3.3.17. $*$. Solution. (a) Since k is positive, $\frac{k}{\sqrt{1+x^2}} > 0$ for every x. Then the top of our region is defined by $T(x) = \frac{k}{\sqrt{1+x^2}}$, and the bottom is defined by $B(x) = 0$.

If we make vertical slices, we don't have to turn our region into two parts, so let's use vertical slices. The question asks for our final answer in terms of the area A of the region, so we don't need to find A explicitly.

Using Equation 2.3.5, the x-coordinate of the centroid is

$$\bar{x} = \frac{1}{A}\int_0^1 x(T(x) - B(x))\,\mathrm{d}x = \frac{1}{A}\int_0^1 x\frac{k}{\sqrt{1+x^2}}\,\mathrm{d}x$$

Although we have a quadratic function underneath a square root, we find an easier method than a trig substitution: the substitution $u = 1 + x^2, \mathrm{d}u = 2x\,\mathrm{d}x$. This changes the limits of integration to $1 + 0^2 = 1$ and $1 + 1^2 = 2$, respectively.

$$= \frac{1}{A}\int_1^2 \frac{k}{\sqrt{u}}\,\frac{\mathrm{d}u}{2} = \frac{k}{2A}\left[\frac{\sqrt{u}}{1/2}\right]_1^2 = \frac{k}{A}\big[\sqrt{2} - 1\big]$$

Again using Equation 2.3.5, the y-coordinate of the centroid is

$$\begin{aligned}\bar{y} &= \frac{1}{2A}\int_0^1 \left(T(x)^2 - B(x)^2\right)\mathrm{d}x = \frac{1}{2A}\int_0^1 \frac{k^2}{1+x^2}\,\mathrm{d}x \\ &= \frac{k^2}{2A}\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \frac{k^2}{2A}\Big[\arctan 1 - \arctan 0\Big] = \frac{k^2}{2A}\cdot\frac{\pi}{4} = \frac{k^2\pi}{8A}\end{aligned}$$

(b) We have $\bar{x} = \bar{y}$ if and only if

$$\frac{k}{A}\left[\sqrt{2}-1\right] = \frac{k^2\pi}{8A}$$

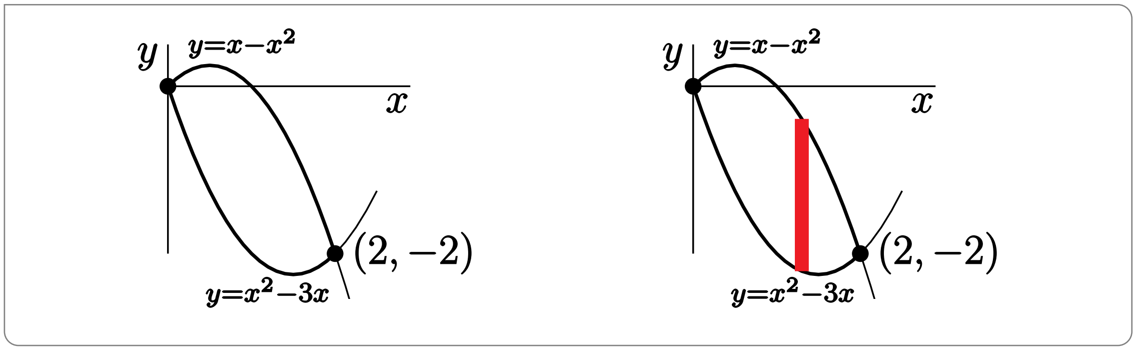
Since k and A are a positive constants (hence neither is equal to 0), we can divide both sides by k and multiply both sides by A:

$$\begin{aligned}\sqrt{2}-1 &= \frac{k\pi}{8} \\ k &= \frac{8}{\pi}\left[\sqrt{2}-1\right]\end{aligned}$$

2.3.3.18. *. Solution. (a)
The curve $y = x^2 - 3x$ is a parabola, pointing up, with x-intercepts at $x = 0$ and $x = 3$.
The curve $y = x - x^2$ is a parabola, pointing down, with x-intercepts at $x = 0$ and $x = 1$.
To find where the two curves meet, we set them equal to each other:

$$\begin{aligned}x^2 - 3x &= x - x^2 \\ 2x^2 - 4x &= 0 \\ 2x(x-2) &= 0 \\ x &= 0 \quad \text{and} \quad x = 2\end{aligned}$$

This is enough information to sketch the figure, on the left below.



(b) As we found in (a), the curves cross when $x = 0, x = 2$. The corresponding values of y are $y = 0$ and $y = 2 - 2^2 = -2$. Note the top curve is $T(x) = x - x^2$, and the bottom curve is $B(x) = x^2 - 3x$. Using vertical strips, as in the figure on the right above, the area of R is

$$\int_0^2 \left[(x-x^2)-(x^2-3x)\right]\mathrm{d}x = \int_0^2 \left[4x-2x^2\right]\mathrm{d}x = \left[2x^2-\frac{2}{3}x^3\right]_0^2$$
$$= 8-\frac{16}{3} = \frac{8}{3}$$

(c) Using Equation 2.3.5, the x-coordinate of the centroid of R (i.e. the weighted average of x over R) is

$$\begin{aligned}\bar{x} &= \frac{3}{8}\int_0^2 x\left[(x-x^2)-(x^2-3x)\right]\mathrm{d}x = \frac{3}{8}\int_0^2 \left[4x^2-2x^3\right]\mathrm{d}x\\ &= \frac{3}{8}\left[\frac{4}{3}x^3-\frac{1}{2}x^4\right]_0^2 = \frac{3}{8}\left[\frac{32}{3}-8\right]\\ &= 1\end{aligned}$$

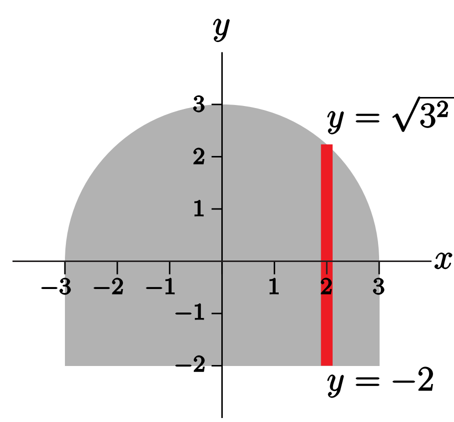
2.3.3.19. $*$. Solution. Using Equation 2.3.5, the x-coordinate of the centroid is

$$\bar{x} = \frac{\int_0^1 x\frac{1}{1+x^2}\,\mathrm{d}x}{\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x}$$

We can guess the antiderivative in the numerator, or use the substitution $u=1+x^2$, $\mathrm{d}u = 2x\,\mathrm{d}x$.

$$= \frac{\frac{1}{2}\log(1+x^2)\big|_0^1}{\arctan x\big|_0^1} = \frac{\frac{1}{2}\log 2}{\pi/4} = \frac{2}{\pi}\log 2 \approx 0.44127$$

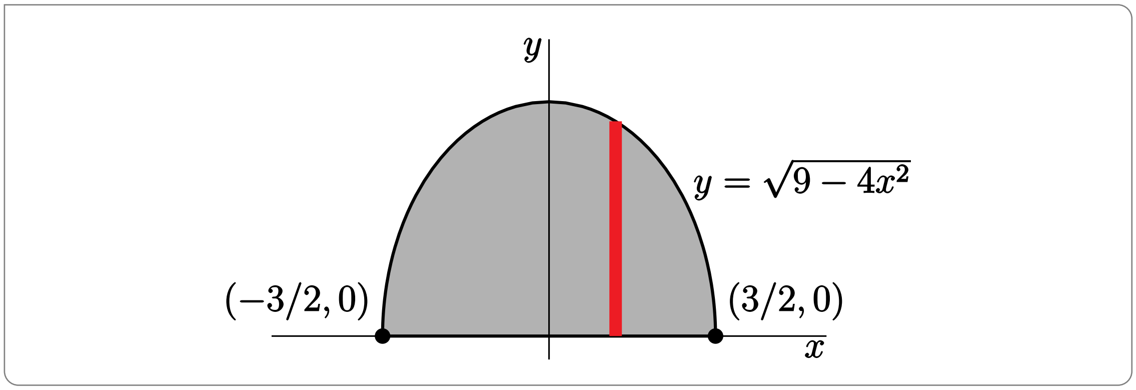
2.3.3.20. $*$. Solution. By symmetry, the centroid lies on the y-axis, so $\bar{x}=0$. The area of the figure is the area of a half-circle of radius 3, and a rectangle of width 6 and height 2. So, $A = \frac{1}{2}\pi(9)+6\times 2 = \frac{9}{2}\pi+12$.
We'll use vertical strips as in the sketch below.



The top function of our figure is $T(x)=\sqrt{9-x^2}$, and the bottom function of our figure is $B(x) = -2$. Using Equation 2.3.5, the y-coordinate of the centroid is:

$$\begin{aligned}
\bar{y} &= \frac{1}{2A}\int_a^b \big(T(x)^2 - B(x)^2\big)\,\mathrm{d}x \\
&= \frac{1}{2A}\int_{-3}^{3} \left(\sqrt{9-x^2}^{\,2} - (-2)^2\right)\,\mathrm{d}x \\
&= \frac{1}{2A}\int_{-3}^{3} \left(5 - x^2\right)\,\mathrm{d}x \\
&= \frac{1}{2A}\left[5x - \frac{1}{3}x^3\right]_{-3}^{3} \\
&= \frac{1}{2A}\left[15 - 9 + 15 - 9\right] \\
&= \frac{6}{A} = \frac{6}{\frac{9}{2}\pi + 12} = \frac{12}{9\pi + 24}
\end{aligned}$$

2.3.3.21. *. Solution. (a) Notice that when $x = 0$, $y = 3$ and as x^2 increases, y decreases until y hits zero at $x^2 = \frac{9}{4}$, i.e. at $x = \pm\frac{3}{2}$. For $x^2 > \frac{9}{4}$, y is not even defined. So, on D, x runs from $-\frac{3}{2}$ to $+\frac{3}{2}$ and, for each x, y runs from 0 to $\sqrt{9-4x^2}$. Here is a sketch of D.



As an aside, we can rewrite $y = \sqrt{9-4x^2}$ as $4x^2 + y^2 = 9$, $y \ge 0$, which is the top half of the ellipse which passes through $(\pm a, 0)$ and $(0, \pm b)$ with $a = \frac{3}{2}$ and $b = 3$. The area of the full ellipse is $\pi ab = \frac{9}{2}\pi$. The area of D is half of that, which is $\frac{9}{4}\pi$. But we are told to use an integral, so we will do so.
The area is

$$\text{Area} = \int_{-3/2}^{3/2} \sqrt{9-4x^2}\,\mathrm{d}x$$

We can evaluate this integral by substituting $x = \frac{3}{2}\sin\theta$, $\mathrm{d}x = \frac{3}{2}\cos\theta\,\mathrm{d}\theta$ and using

$$x = \pm\frac{3}{2} \iff \sin\theta = \pm 1$$

So $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and

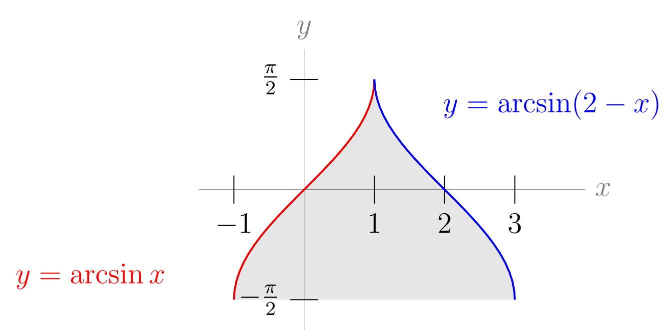
$$\text{Area} = \int_{-\pi/2}^{\pi/2} \sqrt{9 - 4\big(\tfrac{3}{2}\sin\theta\big)^2} \cdot \frac{3}{2}\cos\theta\,\mathrm{d}\theta$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{9 - 9\sin^2\theta} \cdot \frac{3}{2}\cos\theta \,\mathrm{d}\theta$$
$$= \frac{9}{2}\int_{-\pi/2}^{\pi/2} \cos^2\theta \,\mathrm{d}\theta = \frac{9}{2}\int_{-\pi/2}^{\pi/2} \frac{\cos(2\theta)+1}{2}\mathrm{d}\theta$$
$$= \frac{9}{4}\left[\frac{\sin(2\theta)}{2} + \theta\right]_{-\pi/2}^{\pi/2} = \frac{9}{4}\pi$$

(b) The region D is symmetric about the y axis. So the centre of mass lies on the y axis. That is, $\bar{x} = 0$. Since D has area $A = \frac{9}{4}\pi$, top equation $y = T(x) = \sqrt{9-4x^2}$ and bottom equation $y = B(x) = 0$, with x running from $a = -\frac{3}{2}$ to $b = \frac{3}{2}$, Equation 2.3.5 gives us $\bar{y}$:

$$\bar{y} = \frac{1}{2A}\int_a^b \left[T(x)^2 - B(x)^2\right]\mathrm{d}x = \frac{2}{9\pi}\int_{-3/2}^{3/2}\left[9-4x^2\right]\mathrm{d}x$$
$$= \frac{4}{9\pi}\int_0^{3/2}\left[9-4x^2\right]\mathrm{d}x = \frac{4}{9\pi}\left[9x - \frac{4}{3}x^3\right]_0^{3/2}$$
$$= \frac{4}{9\pi}\left[9\cdot\frac{3}{2} - \frac{4}{3}\cdot\frac{3^3}{2^3}\right] = \frac{4}{9\pi}\left[9\cdot\frac{3}{2} - 9\cdot\frac{1}{2}\right]$$
$$= \frac{4}{\pi}$$

2.3.3.22. Solution. Let's start by sketching the region at hand. We know the general shape of arcsine (it's like half a period of sine, if you swapped the x and y axes); we can sketch the curve $y = \arcsin(2-x)$ by mirroring $y = \arcsin x$ about the line $x = 1$.



If we use vertical strips, then we need two separate regions, because $T(x) = \arcsin x$ when $x \le 1$, and $T(x) = \arcsin(2-x)$ when $x > 1$. Also, we'd have to antidifferentiate functions that have arcsine in them. Let's think about horizontal strips. If $y = \arcsin x$, then $x = \sin y$, and if $y = \arcsin(2-x)$ then $x = 2 - \sin y$. For all y from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, the left endpoint of a strip is given by $L(y) = \sin y$, and the right endpoint is given by $R(y) = 2 - \sin y$.
First, let's use our horizontal slices [a] to find the area of our region, A.

$$A = \int_{-\pi/2}^{\pi/2} \big((2 - \sin y) - (\sin y)\big)\,\mathrm{d}y = \int_{-\pi/2}^{\pi/2} \big(2 - 2\sin y\big)\,\mathrm{d}y$$

$$= \Big[2y + 2\cos y\Big]_{-\pi/2}^{\pi/2} = (\pi + 0) - (-\pi + 0) = 2\pi$$

From symmetry, it is clear that $\bar{x} = 1$. We find $\bar{y}$ using Equation 2.3.5.

$$\begin{aligned}
\bar{y} &= \frac{\int_{-\pi/2}^{\pi/2} y\,[R(y) - L(y)]\,\mathrm{d}y}{A} \\
&= \frac{\int_{-\pi/2}^{\pi/2} y\,[(2 - \sin y) - (\sin y)]\,\mathrm{d}y}{2\pi} \\
&= \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} y(2 - 2\sin y)\,\mathrm{d}y \\
&= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} y\,\mathrm{d}y - \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} (y\sin y)\,\mathrm{d}y
\end{aligned}$$

Since y is an odd function, and the domain of integration is symmetric, the first integral evaluates to 0. Since $y\sin y$ is an even function (recall the product of two odd functions is an even function), we can simplify our limits of integration.

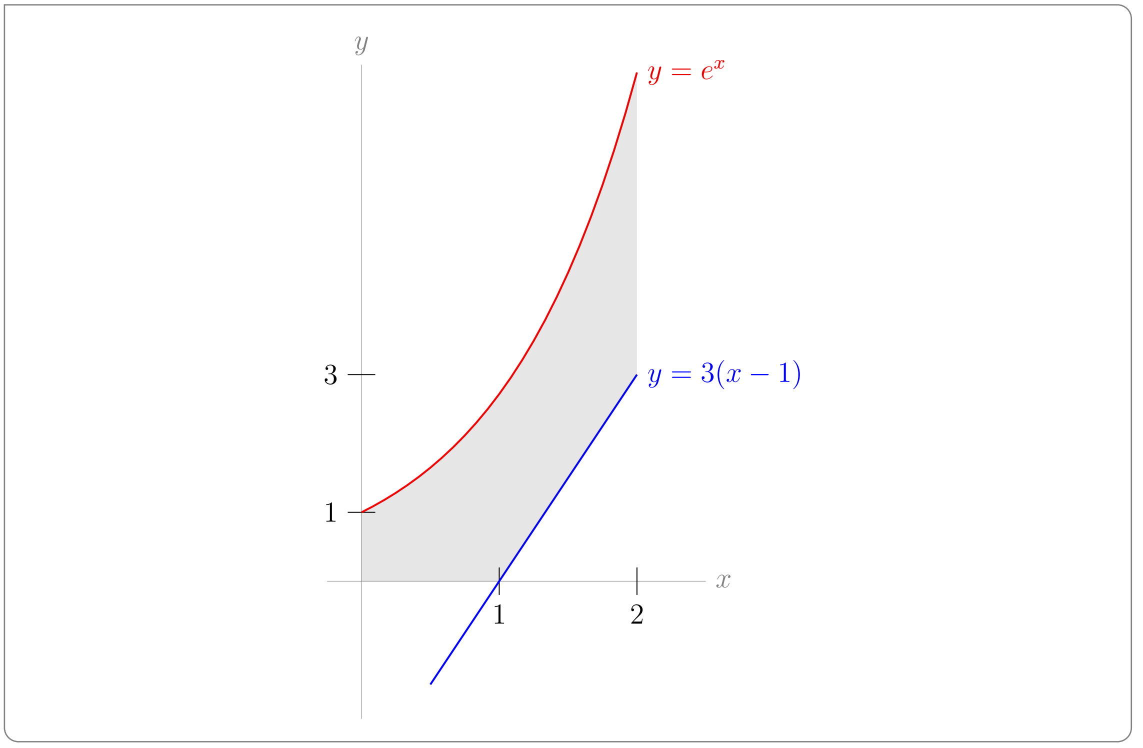
$$= -\frac{2}{\pi}\int_0^{\pi/2} y\sin y\,\mathrm{d}y$$

We use integration by parts with $u = y$, $\mathrm{d}v = \sin y\,\mathrm{d}y$; $\mathrm{d}u = \mathrm{d}y$, $v = -\cos y$.

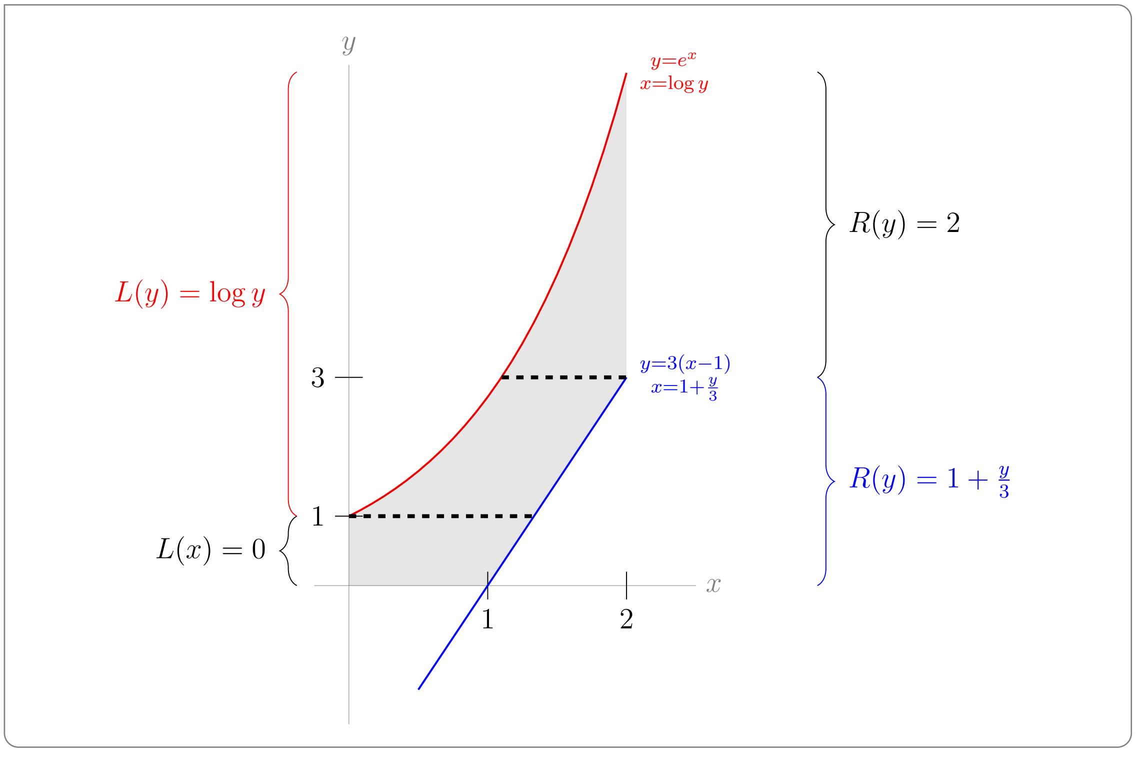
$$\begin{aligned}
&= -\frac{2}{\pi}\left(\big[-y\cos y\big]_0^{\pi/2} + \int_0^{\pi/2} \cos y\,\mathrm{d}y\right) \\
&= -\frac{2}{\pi}\big[-y\cos y + \sin y\big]_0^{\pi/2} \\
&= -\frac{2}{\pi}\,[(0 + 1) - 0] = -\frac{2}{\pi}
\end{aligned}$$

a There's also a sneaky way to find the area of A: look for a way to snip and rearrange bits of the figure to turn it into a rectangle!

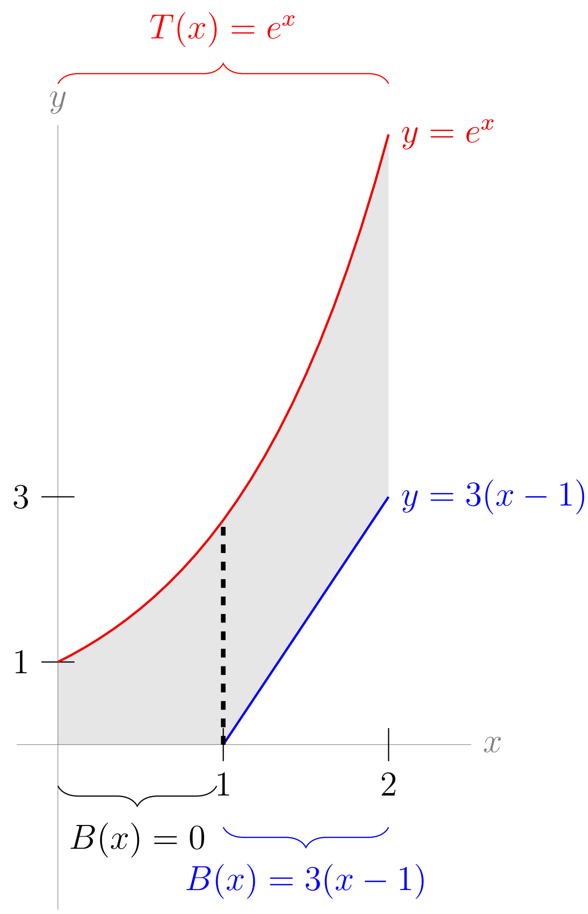
2.3.3.23. Solution. We'll start by sketching the region.



If we use horizontal slices, we need to divide our figure into three regions, as in the figure below, because the left and right functions change at the dashed lines.



If we use vertical slices, we only need two regions (shown below) to account for the different top and bottom functions. This seems easier than three regions, so we use vertical slices.



When $0 \le x \le 2$, $T(x) = e^x$. When $0 \le x \le 1$, $B(x) = 0$, and when $1 \le x \le 2$, $B(x) = 3(x-1)$.
The area of the figure is:

$$\begin{aligned}
A &= \int_0^2 (T(x) - B(x))\,\mathrm{d}x = \int_0^1 (e^x - 0)\,\mathrm{d}x + \int_1^2 (e^x - 3(x-1))\,\mathrm{d}x \\
&= \int_0^2 e^x\,\mathrm{d}x - \int_1^2 3(x-1)\,\mathrm{d}x \\
&= \Big[e^x\Big]_0^2 - \left[\frac{3}{2}(x-1)^2\right]_1^2 \\
&= e^2 - 1 - \frac{3}{2} = e^2 - \frac{5}{2}
\end{aligned}$$

Using Equation 2.3.5:

$$\begin{aligned}
\bar{x} &= \frac{\int_0^2 x(T(x) - B(x))\,\mathrm{d}x}{A} \\
&= \frac{1}{e^2 - 5/2}\left[\int_0^1 x\left(e^x - 0\right)\,\mathrm{d}x + \int_1^2 x\left(e^x - 3(x-1)\right)\,\mathrm{d}x\right] \\
&= \frac{1}{e^2 - 5/2}\left[\int_0^2 xe^x\,\mathrm{d}x - \int_1^2 3x(x-1)\,\mathrm{d}x\right]
\end{aligned}$$

For the left integral, we use integration by parts with $u = x$, $\mathrm{d}v = e^x\,\mathrm{d}x$; $\mathrm{d}u = \mathrm{d}x$, $v = e^x$.

$$\begin{aligned}
&= \frac{1}{e^2 - 5/2}\left[[xe^x]_0^2 - \int_0^2 e^x\,\mathrm{d}x - 3\int_1^2 (x^2 - x)\,\mathrm{d}x\right]\\
&= \frac{1}{e^2 - 5/2}\left([xe^x - e^x]_0^2 - 3\left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_1^2\right)\\
&= \frac{1}{e^2 - 5/2}\left((2e^2 - e^2) - (-1) - 3\left(\frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{2}\right)\right)\\
&= \frac{e^2 - 3/2}{e^2 - 5/2} \approx 1.2
\end{aligned}$$

Using Equation 2.3.5 again:

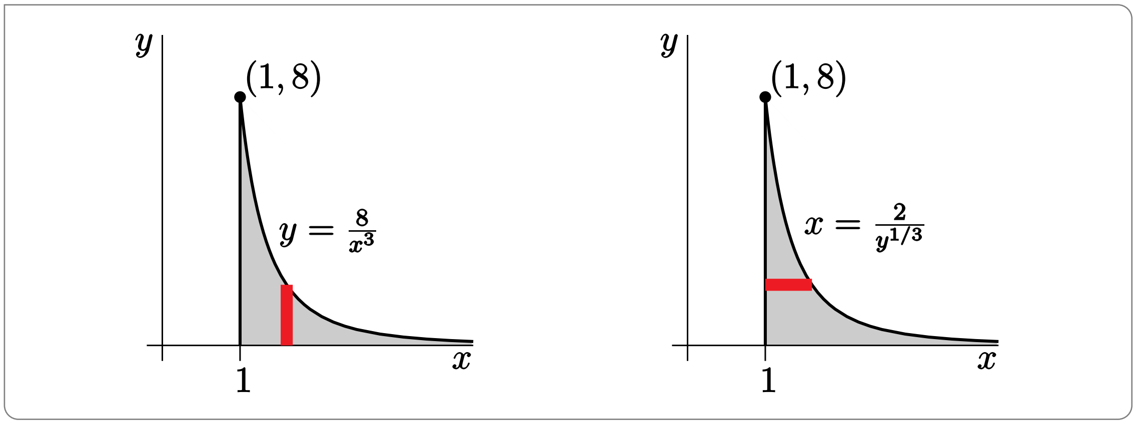
$$\begin{aligned}
\bar{y} &= \frac{\int_0^2 \left(T(x)^2 - B(x)^2\right)\,\mathrm{d}x}{2A}\\
&= \frac{1}{2(e^2 - 5/2)}\left[\int_0^1 \left(e^{2x} - 0\right)\,\mathrm{d}x + \int_1^2 \left(e^{2x} - 9(x-1)^2\right)\,\mathrm{d}x\right]\\
&= \frac{1}{2(e^2 - 5/2)}\left[\int_0^2 e^{2x}\,\mathrm{d}x - \int_1^2 9(x-1)^2\,\mathrm{d}x\right]\\
&= \frac{1}{2(e^2 - 5/2)}\left(\left[\frac{1}{2}e^{2x}\right]_0^2 - \left[3(x-1)^3\right]_1^2\right)\\
&= \frac{1}{2e^2 - 5}\left(\frac{1}{2}e^4 - \frac{1}{2} - 3\right)\\
&= \frac{e^4 - 7}{4e^2 - 10} \approx 2.4
\end{aligned}$$

Exercises — Stage 3

2.3.3.24. *. Solution. The area of the region is

$$\begin{aligned}
A &= \int_1^\infty \frac{8}{x^3}\,\mathrm{d}x = \lim_{t\to\infty}\left(\int_1^t \frac{8}{x^3}\,\mathrm{d}x\right) = \lim_{t\to\infty}\left[-\frac{4}{x^2}\right]_1^t\\
&= \lim_{t\to\infty}\left[-\frac{4}{t^2} + \frac{4}{1^2}\right] = 0 + 4
\end{aligned}$$

We'll now compute $\bar{y}$ twice, once with vertical strips, as in the figure in the left below, and once with horizontal strips as in the figure on the right below.



Vertical strips: The equation of the top of the region is $y = T(x) = \dfrac{8}{x^3}$ and the equation of the bottom of the region is $y = B(x) = 0$. Using vertical strips, as in the figure on the left above, the y-coordinate of the centre of mass is

$$\begin{aligned}
\bar{y} &= \frac{1}{2A}\int_1^\infty \big[T(x)^2 - B(x)^2\big]\,\mathrm{d}x \\
&= \frac{1}{8}\int_1^\infty \left(\frac{8}{x^3}\right)^2 \mathrm{d}x \\
&= \lim_{t\to\infty}\left(\int_1^t \frac{8}{x^6}\,\mathrm{d}x\right) \\
&= \lim_{t\to\infty}\left[-\frac{8}{5x^5}\right]_1^t \\
&= \lim_{t\to\infty}\left[-\frac{8}{5t^5} + \frac{8}{5\times 1^5}\right] = \frac{8}{5}
\end{aligned}$$

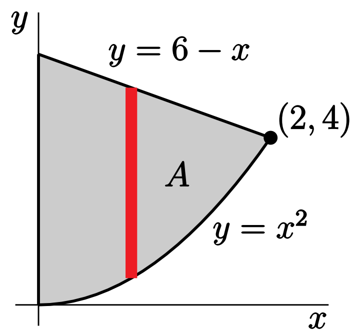
Vertical strips: Since $y = \dfrac{8}{x^3}$ is equivalent to $x = \sqrt[3]{\dfrac{8}{y}}$, the equation of the right-hand side of the region is $x = R(y) = \dfrac{2}{y^{1/3}}$ and the equation of the left hand side of the region is $x = L(y) = 1$. The point at the top of the region is $(1, 8)$. Thus y runs from 0 to 8. So, using horizontal strips, as in the figure on the right above, the y-coordinate of the centre of mass is

$$\begin{aligned}
\bar{y} &= \frac{1}{A}\int_0^8 y\big[R(y) - L(y)\big]\,\mathrm{d}y \\
&= \frac{1}{4}\int_0^8 y\big[2y^{-1/3} - 1\big]\,\mathrm{d}y \\
&= \frac{1}{4}\int_0^8 \big[2y^{2/3} - y\big]\,\mathrm{d}y \\
&= \frac{1}{4}\left[\frac{6}{5}y^{5/3} - \frac{y^2}{2}\right]_0^8 \\
&= \frac{1}{4}\left[\frac{6\times 32}{5} - \frac{8\times 8}{2}\right] = 8\left[\frac{6}{5} - 1\right] = \frac{8}{5}
\end{aligned}$$

2.3.3.25. *. Solution. (a) The two curves cross at points (x, y) that satisfy both $y = x^2$ and $y = 6 - x$, and hence

$$x^2 = 6 - x \iff x^2 + x - 6 = 0 \iff (x+3)(x-2) = 0$$

So we see that the two curves intersect at $x = 2$ (as well as $x = -3$, which is to the left of the y-axis and therefore irrelevant). Here is a sketch of A.



The top of A has equation $y = T(x) = 6 - x$, the bottom has equation $y = B(x) = x^2$ and x runs from 0 to 2. So, using vertical strips,

$$\begin{aligned}
\bar{x} &= \frac{1}{A}\int_0^2 x\big[T(x) - B(x)\big]\,\mathrm{d}x \\
&= \frac{1}{22/3}\int_0^2 x\big[(6-x) - x^2\big]\,\mathrm{d}x = \frac{3}{22}\int_0^2 (6x - x^2 - x^3)\,\mathrm{d}x \\
&= \frac{3}{22}\left[3x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 \\
&= \frac{3}{22}\left[12 - \frac{8}{3} - 4\right] = \frac{3}{22}\frac{16}{3} = \frac{8}{11}
\end{aligned}$$

and

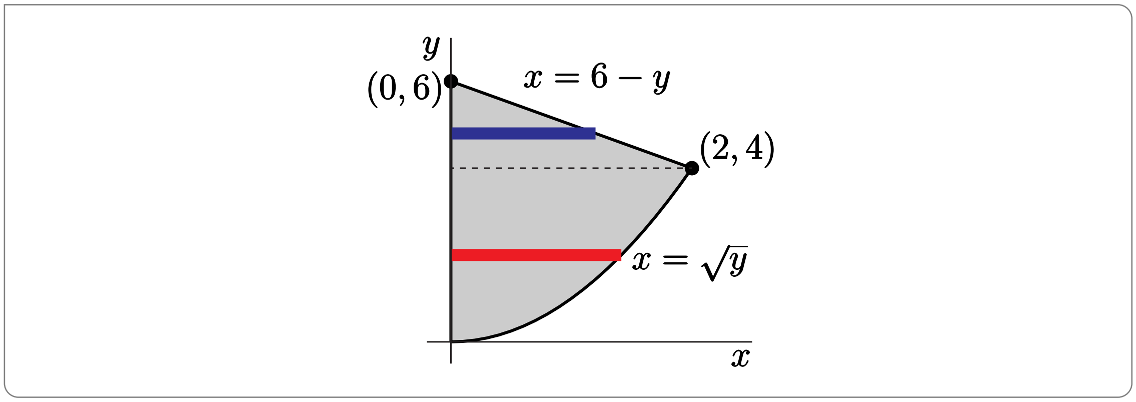
$$\begin{aligned}
\bar{y} &= \frac{1}{2A}\int_0^2 \big[T(x)^2 - B(x)^2\big]\,\mathrm{d}x \\
&= \frac{1}{2}\cdot\frac{1}{22/3}\int_0^2 \big((6-x)^2 - x^4\ \big)\,\mathrm{d}x = \frac{3}{44}\left[-\frac{(6-x)^3}{3} - \frac{x^5}{5}\right]_0^2 \\
&= \frac{3}{44}\left(-\frac{64-216}{3} - \frac{32}{5}\right) = \frac{3}{44}\cdot\frac{664}{15} = \frac{166}{55}
\end{aligned}$$

The integral was evaluated by guessing an antiderivative for the integrand. It could also be evaluated as

$$\begin{aligned}
\frac{3}{44}\int_0^2 \big(36 - 12x + x^2 - x^4\ \big)\,\mathrm{d}x &= \frac{3}{44}\left[36x - 6x^2 + \frac{x^3}{3} - \frac{x^5}{5}\right]_0^2 \\
&= \frac{3}{44}\left(72 - 24 + \frac{8}{3} - \frac{32}{5}\right)
\end{aligned}$$

$$= \frac{3}{44}\frac{664}{15} = \frac{166}{55}$$

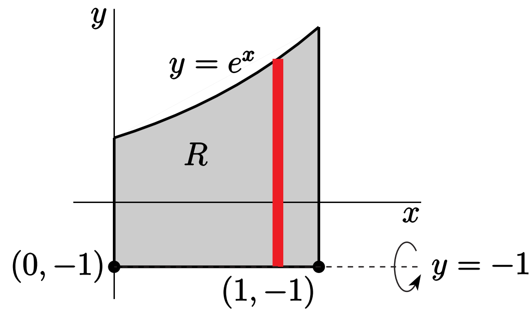
(b) The question specifies the use of horizontal slices (as in Example 1.6.5). The radius of the slice at height y is the x-value of the right-hand boundary of the region at that point. So, we start by converting both equations $y = 6 - x$ and $y = x^2$ into equations of the form $x = f(y)$. To do so we solve for x in both equations, yielding $x = \sqrt{y}$ and $x = 6 - y$.



- We use thin horizontal strips of width $\mathrm{d}y$ as in the figure above.
- When we rotate about the y-axis, each strip sweeps out a thin disk
  - whose radius is $r = 6-y$ when $4 \le y \le 6$ (see the blue strip in the figure above), and whose radius is $r = \sqrt{y}$ when $0 \le y \le 4$ (see the red strip in the figure above) and
  - whose thickness is $\mathrm{d}y$ and hence
  - whose volume is $\pi r^2\,\mathrm{d}y = \pi(6-y)^2\,\mathrm{d}y$ when $4 \le y \le 6$ and whose volume is $\pi r^2\,\mathrm{d}y = \pi y\,\mathrm{d}y$ when $0 \le y \le 4$.
- As our bottommost strip is at $y = 0$ and our topmost strip is at $y = 6$, the total volume is

$$\pi \int_0^4 y\,\mathrm{d}y + \pi \int_4^6 (6-y)^2\,\mathrm{d}y$$

2.3.3.26. ∗. Solution. (a) Here is a sketch of the specified region, which we shall call R.



The top of R has equation $y = T(x) = e^x$, the bottom has equation $y = B(x) = -1$ and x runs from 0 to 1. So, using vertical strips, we see that R has area

$$\begin{aligned} A &= \int_0^1 \big[T(x) - B(x)\big]\,\mathrm{d}x = \int_0^1 \big[e^x - (-1)\big]\,\mathrm{d}x = \int_0^1 \big[e^x + 1\big]\,\mathrm{d}x \\ &= \big[e^x + x\big]_0^1 = e \end{aligned}$$

and

$$\begin{aligned} \bar{y} &= \frac{1}{2A}\int_0^1 \big[T(x)^2 - B(x)^2\big]\,\mathrm{d}x \\ &= \frac{1}{2e}\int_0^1 \big[e^{2x} - 1\ \big]\,\mathrm{d}x = \frac{1}{2e}\left[\frac{e^{2x}}{2} - x\right]_0^1 \\ &= \frac{1}{2e}\left(\frac{e^2}{2} - 1 - \frac{1}{2}\right) = \frac{e}{4} - \frac{3}{4e} \end{aligned}$$

(b) To compute the volume when R is rotated about the line $y = -1$

- we use thin vertical strips of width $\mathrm{d}x$ as in the figure above.
- When we rotate about the line $y = -1$, each strip sweeps out a thin disk
  - whose radius is $r = T(x) - B(x) = e^x + 1$ and
  - whose thickness is $\mathrm{d}x$ and hence
  - whose volume is $\pi r^2\,\mathrm{d}x = \pi(e^x + 1)^2\,\mathrm{d}x$.
- As our leftmost strip is at $x = 0$ and our rightmost strip is at $x = 1$, the total volume is

$$\begin{aligned} \pi\int_0^1 (e^x + 1)^2\,\mathrm{d}x &= \pi\int_0^1 (e^{2x} + 2e^x + 1)\,\mathrm{d}x \\ &= \pi\left[\frac{e^{2x}}{2} + 2e^x + x\right]_0^1 \\ &= \pi\left[\left(\frac{e^2}{2} + 2e + 1\right) - \left(\frac{1}{2} + 2 + 0\right)\right] \\ &= \pi\left(\frac{e^2}{2} + 2e - \frac{3}{2}\right) \end{aligned}$$

2.3.3.27. Solution. By symmetry, $\bar{y} = 1.5$. We can't immediately use Equation 2.3.5 to find $\bar{x}$, because the density is not constant. Instead, we'll go through the derivation of Equation 2.3.5, to figure out what to do with a non-constant density. (This is a good time to review Questions 9 and 10 in this section.)

Our model is that we're making a rod R that reaches from $x = 0$ to $x = 4$, and the mass of the section of the rod along $[a, b]$ is equal to the mass of the strip of our rectangle along $[a, b]$. If we have a formula $\rho(x)$ for the density of R, we can find the centre of mass of R, which is also the x-coordinate of the centre of mass of the rectangle.

A thin vertical strip of the rectangle with length $\mathrm{d}x$ at position x has area $3\mathrm{d}x$ m^2 and density x^2 kg/m^2, so it has mass $3x^2\,\mathrm{d}x$ kg. Therefore, a short section of R at position x with length $\mathrm{d}x$ ought to have mass $3x^2\,\mathrm{d}x$ kg as well. Then its density at x is $\rho(x) = \frac{3x^2\,\mathrm{d}x\text{ kg}}{\mathrm{d}x\text{ m}} = 3x^2$ kg/m.

Now, we can use Equation 2.3.4 to find the centre of mass of the rod, which is also the x-coordinate of the centre of mass of our rectangle:

$$\bar{x} = \frac{\int_0^4 x\rho(x)\,\mathrm{d}x}{\int_0^4 \rho(x)\,\mathrm{d}x} = \frac{\int_0^4 3x^3\,\mathrm{d}x}{\int_0^4 3x^2\,\mathrm{d}x} = \frac{\left[\frac{3}{4}x^4\right]_0^4}{\left[x^3\right]_0^4} = \frac{3\cdot 4^3}{4^3} = 3$$

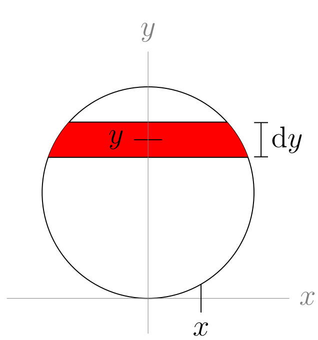
The centre of mass of our rectangle is $(3, 1.5)$.

2.3.3.28. Solution. By symmetry, the x-coordinate of the centre of mass will be $\bar{x} = 0$; that is, exactly in the middle, horizontally. To find the y-coordinate of the centre of mass, we need to consider the origin of Equation 2.3.5.

We can make vertical strips or horizontal strips. A vertical strip of the circle has a density that varies from the bottom of the strip to the top, but a horizontal strip has a constant density (assuming the strip is very thin). So it seems that horizontal strips in this case will be the easier route.

Following the derivation of Equation 2.3.5, we model our circle as a vertical rod R, filling the y-interval $[0, 6]$. A portion of the rod with $a \le y \le b$ should have the same mass as the portion of the circle with $a \le y \le b$. To achieve this, we slice the circle into thin horizontal strips of thickness $\mathrm{d}y$, calculate their mass, then use that to find $\rho(y)$, the density of R.

{First, let's find a formula for the mass of a thin horizontal strip of the circle at position y with height $\mathrm{d}y$.}



The circle with radius 3 centred at $(0, 3)$ has equation $x^2 + (y - 3)^2 = 9$. So, the

right half of the circle has equation $x = \sqrt{9-(y-3)^2}$, and the left half of the circle has equation $x = -\sqrt{9-(y-3)^2}$. So, the width of a strip at height y is $2\sqrt{9-(y-3)^2}$ m. Its height is $\mathrm{d}y$ m, so its area is $2\sqrt{9-(y-3)^2}\,\mathrm{d}y$ m^2. Its density is $2+y\frac{\mathrm{kg}}{\mathrm{m}^2}$, so its mass is $2(2+y)\sqrt{9-(y-3)^2}\,\mathrm{d}y$ kg.
Now we can find $\rho(y)$, the density of R at position y. The mass of the section of R at position y with length $\mathrm{d}y$ is $2(2+y)\sqrt{9-(y-3)^2}\,\mathrm{d}y$ kg (the mass of the strip in the paragraph above), so its density is $\frac{2(2+y)\sqrt{9-(y-3)^2}\,\mathrm{d}y\mathrm{kg}}{\mathrm{d}y\mathrm{m}} = 2(2+y)\sqrt{9-(y-3)^2}\frac{\mathrm{kg}}{\mathrm{m}} = \rho(y)$.
Now, Equation 2.3.4 will tell us the centre of mass of R, which is also the y-coordinate of the centre of mass of the circle.

$$\begin{aligned}\bar{y} &= \frac{\int_a^b y\rho(y)\,\mathrm{d}y}{\int_a^b \rho(y)\,\mathrm{d}y} = \frac{\int_0^6 y \times 2(2+y)\sqrt{9-(y-3)^2}\,\mathrm{d}y}{\int_0^6 2(2+y)\sqrt{9-(y-3)^2}\,\mathrm{d}y} \\ &= \frac{\int_0^6 y(2+y)\sqrt{9-(y-3)^2}\,\mathrm{d}y}{\int_0^6 (2+y)\sqrt{9-(y-3)^2}\,\mathrm{d}y}\end{aligned}$$

To make things look a little cleaner, we use the substitution $u = y-3$, $\mathrm{d}u = \mathrm{d}y$. Then the limits of integration become -3 and 3, respectively, and $y = u+3$. (Geometrically, we're re-centring the circle at the origin, instead of at the point (0,3).)

$$\begin{aligned} &= \frac{\int_{-3}^3 (u+3)(2+u+3)\sqrt{9-u^2}\,\mathrm{d}u}{\int_{-3}^3 \big(2+u+3\big)\sqrt{9-u^2}\,\mathrm{d}u} \\ &= \frac{\int_{-3}^3 \big(u^2+8u+15\big)\sqrt{9-u^2}\,\mathrm{d}u}{\int_{-3}^3 \big(u+5\big)\sqrt{9-u^2}\,\mathrm{d}u} = \frac{N}{D} \end{aligned} \tag{$*$}$$

Let's start by finding D, the integral of the denominator. If we break it into two pieces, we can use symmetry and geometry to evaluate it.

$$D = \int_{-3}^3 u\sqrt{9-u^2}\,\mathrm{d}u + 5\int_{-3}^3 \sqrt{9-u^2}\,\mathrm{d}u$$

The left integrand is odd, so its integral over a symmetric interval is 0. (You can also evaluate this using the substitution $w = 9-u^2$, $\mathrm{d}w = -2u\,\mathrm{d}u$.) The right integral represents the area underneath half a circle of radius 3, centred at the origin.

$$D = 0 + 5\cdot\frac{1}{2}\pi\cdot 3^2 = \frac{45}{2}\pi$$

Now, let's evaluate our numerator integral from $(*)$, $N = \int_{-3}^3 \big(u^2+8u+15\big)\sqrt{9-u^2}\,\mathrm{d}u$. If we break it into three pieces, we can simplify the integration somewhat.

$$N = \int_{-3}^3 u^2\sqrt{9-u^2}\,\mathrm{d}u + 8\int_{-3}^3 u\sqrt{9-u^2}\,\mathrm{d}u + 15\int_{-3}^3 \sqrt{9-u^2}\,\mathrm{d}u$$

The first integrand is even, with a symmetric interval of integration, so we can simplify its limits of integration a little bit. The middle integrand is odd, so its integral over the symmetric interval $[-3, 3]$ is zero. The last integral is the area of half a circle of radius 3.

$$\begin{aligned}
N &= 2\int_0^3 u^2\sqrt{9-u^2}\,\mathrm{d}u + 0 + 15\cdot\pi\cdot 3^2 \\
&= \frac{135}{2}\pi + 2\int_0^3 u^2\sqrt{9-u^2}\,\mathrm{d}u
\end{aligned}$$

The remaining integral has a quadratic function underneath a square root with no obvious substitution, so we use a trigonometric substitution. Let $u = 3\sin\theta$, $\mathrm{d}u = 3\cos\theta\,\mathrm{d}\theta$. Note $3\sin(0) = 0$ and $3\sin(\pi/2) = 3$, so the limits of integration become 0 and $\frac{\pi}{2}$.

$$\begin{aligned}
N &= \frac{135}{2}\pi + 2\int_0^{\pi/2} \big(3\sin\theta\big)^2\sqrt{9-\big(3\sin\theta\big)^2}\cdot 3\cos\theta\,\mathrm{d}\theta \\
&= \frac{135}{2}\pi + 2\int_0^{\pi/2} 9\sin^2\theta\cdot\sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\,\mathrm{d}\theta \\
&= \frac{135}{2}\pi + 54\int_0^{\pi/2} \sin^2\theta\cdot\sqrt{9\cos^2\theta}\cdot\cos\theta\,\mathrm{d}\theta \\
&= \frac{135}{2}\pi + 54\int_0^{\pi/2} \sin^2\theta\cdot 3\cos\theta\cdot\cos\theta\,\mathrm{d}\theta \\
&= \frac{135}{2}\pi + 162\int_0^{\pi/2} \sin^2\theta\cdot\cos^2\theta\,\mathrm{d}\theta
\end{aligned}$$

Using the identity $\sin(2\theta) = 2\sin\theta\cos\theta$, we see $\sin^2\theta\cos^2\theta = \big(\sin\theta\cos\theta\big)^2 = \frac{1}{4}\sin^2(2\theta)$

$$N = \frac{135}{2}\pi + 162\int_0^{\pi/2} \frac{1}{4}\sin^2(2\theta)\,\mathrm{d}\theta$$

Now, we use the identity $\sin^2 x = \frac{1}{2}(1-\cos(2x))$, with $x = 2\theta$.

$$\begin{aligned}
N &= \frac{135}{2}\pi + 162\int_0^{\pi/2} \frac{1}{8}\big(1-\cos(4\theta)\big)\,\mathrm{d}\theta \\
&= \frac{135}{2}\pi + \frac{81}{4}\int_0^{\pi/2} 1-\cos(4\theta)\,\mathrm{d}\theta \\
&= \frac{135}{2}\pi + \frac{81}{4}\left[\theta - \frac{1}{4}\sin(4\theta)\right]_0^{\pi/2} \\
&= \frac{135}{2}\pi + \frac{81}{4}\left(\frac{\pi}{2}\right) \\
&= \frac{621}{8}\pi
\end{aligned}$$

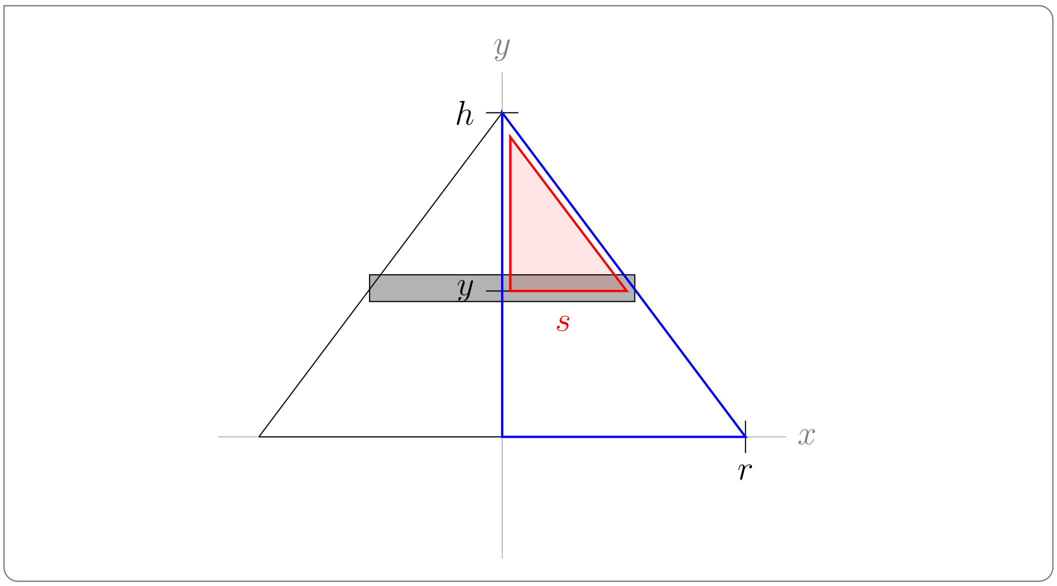
Now, using equation $(*)$, we find $\bar{y}$:

$$\bar{y} = \frac{N}{D} = \frac{\frac{621}{8}\pi}{\frac{45}{2}\pi} = \frac{69}{20} = 3.45$$

Let's quickly check that this makes sense: if the circle has uniform density, its centre of mass would lie at $(0,3)$. Since it's denser at the top, the centre of mass should be higher, and indeed 3.45 is higher than 3 (without being so high it's above the entire circle).

2.3.3.29. Solution.

a To find the centre of mass of the rod R, we need to know its density at height y, $\rho(y)$. Since the mass of a section of R is the same as the volume of a section of the cone, let's find the volume of a thin horizontal slice of the cone at height y, with thickness $\mathrm{d}y$. To find its radius s, we use similar triangles. The diagram below represents a vertical cross-section of the cone.



Since $\frac{r}{h} = \frac{s}{h-y}$, the radius of our slice at height y is $s = \frac{r}{h}(h-y)$. Then the volume of the slice is $\pi s^2 \mathrm{d}y = \pi\left(\frac{r}{h}(h-y)\right)^2 \mathrm{d}y$. Correspondingly, the mass of the piece of the rod at position y with length $\mathrm{d}y$ is $\pi\left(\frac{r}{h}(h-y)\right)^2 \mathrm{d}y$, so its density is

$$\rho(y) = \frac{\pi\left(\frac{r}{h}(h-y)\right)^2 \mathrm{d}y}{\mathrm{d}y} = \pi\left(\frac{r}{h}(h-y)\right)^2.$$

Now, we can find the centre of mass of R:

$$\begin{aligned}\bar{y} &= \frac{\int_0^h y\rho(y)\,\mathrm{d}y}{\int_0^h \rho(y)\,\mathrm{d}y} = \frac{\int_0^h y\pi\left(\frac{r}{h}(h-y)\right)^2 \mathrm{d}y}{\int_0^h \pi\left(\frac{r}{h}(h-y)\right)^2 \mathrm{d}y}\\ &= \frac{\frac{r^2}{h^2}\pi\int_0^h y\,(h-y)^2\,\mathrm{d}y}{\frac{r^2}{h^2}\pi\int_0^h (h-y)^2\,\mathrm{d}y}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\int_0^h (h^2y - 2hy^2 + y^3)\ \mathrm{d}y}{\int_0^h (h^2 - 2hy + y^2)\ \mathrm{d}y}\\
&= \frac{\left[\frac{h^2}{2}y^2 - \frac{2h}{3}y^3 + \frac{1}{4}y^4\right]_0^h}{\left[h^2y - hy^2 + \frac{1}{3}y^3\right]_0^h}\\
&= \frac{\frac{h^4}{2} - \frac{2h^4}{3} + \frac{h^4}{4}}{h^3 - h^3 + \frac{1}{3}h^3}\\
&= \frac{h^4}{h^3} \cdot \frac{\frac{1}{2} - \frac{2}{3} + \frac{1}{4}}{\frac{1}{3}} = \frac{h}{4}
\end{aligned}
$$

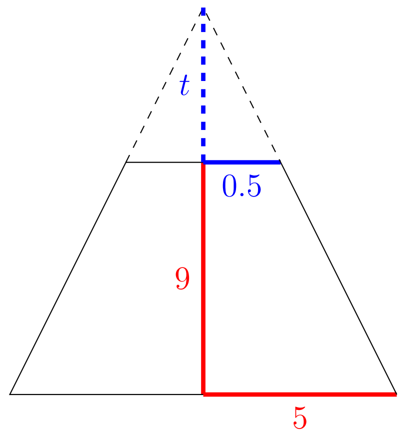
So, the centre of mass of the cone occurs $\frac{h}{4}$ metres above its base.

Remark: it is quite interesting that the centre of mass does not depend on the radius of the cone!

b To find the centre of mass of a truncated cone, we simply consider a truncated rod. If the top $h-k$ metres are missing, then the height of the cone (and also the rod) is k. Then the centre of mass has height:

$$
\begin{aligned}
\bar{y} &= \frac{\int_0^k y\rho(y)\,\mathrm{d}y}{\int_0^h \rho(y)\,\mathrm{d}y} = \frac{\int_0^k y\pi\left(\frac{r}{h}(h-y)\right)^2\ \mathrm{d}y}{\int_0^h \pi\left(\frac{r}{h}(h-y)\right)^2\ \mathrm{d}y}\\
&= \frac{\frac{r^2}{h^2}\pi\int_0^k y\,(h-y)^2\ \mathrm{d}y}{\frac{r^2}{h^2}\pi\int_0^k (h-y)^2\ \mathrm{d}y}\\
&= \frac{\int_0^k (h^2y - 2hy^2 + y^3)\ \mathrm{d}y}{\int_0^k (h^2 - 2hy + y^2)\ \mathrm{d}y}\\
&= \frac{\left[\frac{h^2}{2}y^2 - \frac{2h}{3}y^3 + \frac{1}{4}y^4\right]_0^k}{\left[h^2y - hy^2 + \frac{1}{3}y^3\right]_0^k}\\
&= \frac{\frac{1}{2}h^2k^2 - \frac{2}{3}hk^3 + \frac{1}{4}k^4}{h^2k - hk^2 + \frac{1}{3}k^3}\\
&= \frac{\frac{1}{2}h^2k - \frac{2}{3}hk^2 + \frac{1}{4}k^3}{h^2 - hk + \frac{1}{3}k^2}
\end{aligned}
$$

2.3.3.30. Solution. To use the result of Question 29, we need to know the dimensions of the cone that was truncated to make the hourglass. The bottom (or top) half of our hourglass has base radius 5 cm, height 9 cm, and top radius 0.5 cm. Imagine extending it to a full cone. Let t be the distance from the top of the half hourglass to the tip of the full cone.



Using similar triangles,

$$\frac{t}{0.5} = \frac{t+9}{5}$$

$$\text{so} \qquad 5t = \frac{1}{2}(t+9)$$

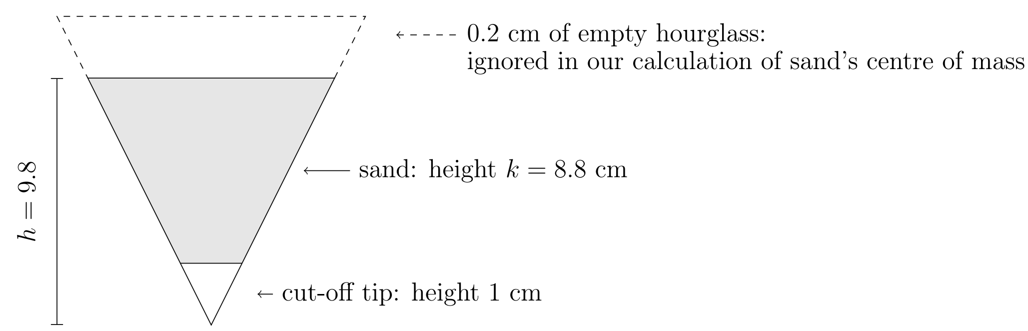
$$4.5t = 4.5$$

$$t = 1$$

Then the height of the full cone (that we imagined truncating to make half of the hourglass) is $h = 10$ cm.
Before the hourglass is turned over, the sand forms a truncated cone of height 6 cm. So, it's the bottom $k = 6$ cm of a cone of height $h = 10$ cm. Using the result of Question 29, its centre of mass is at height:

$$\frac{\frac{1}{2}h^2k - \frac{2}{3}hk^2 + \frac{1}{4}k^3}{h^2 - hk + \frac{1}{3}k^2} = \frac{\frac{1}{2}10^2 \cdot 6 - \frac{2}{3}10 \cdot 6^2 + \frac{1}{4}6^3}{10^2 - 10 \cdot 6 + \frac{1}{3}6^2} = \frac{57}{26} \approx 2.2$$

Next, let's find the centre of mass of the sand after it's been rotated. We have to be a little careful with our vocabulary here: usually we imagine a cone sitting on its base, with its tip pointing up. The upturned sand is in the opposite configuration. When we say the "base" of the cone, we mean the larger horizontal face — the top of the sand as it sits in the hourglass.
The formula we have from Question 29 gives us our centre of mass as a distance from the base of the truncated cone (that is, the distance from the top of the upturned sand). If k is the height the sand actually occupies, then we were told we may assume $k = 8.8$ cm. It's missing its "tip" of height 1 cm, so h, the height of the "untruncated" cone, is 9.8 cm. Using our model from Question 29, we don't care about the empty, uppermost piece of the hourglass. The shape of the sand is of a cone of height 9.8 cm (not 10 cm), with a tip of height 1 cm chopped off.



$$\frac{\frac{1}{2}h^2k - \frac{2}{3}hk^2 + \frac{1}{4}k^3}{h^2 - hk + \frac{1}{3}k^2} = \frac{\frac{1}{2}9.8^2 \cdot 8.8 - \frac{2}{3}9.8 \cdot 8.8^2 + \frac{1}{4}8.8^3}{9.8^2 - 9.8 \cdot 8.8 + \frac{1}{3}8.8^2} \approx 2.443$$

That is, the centre of mass of the upturned sand is about 2.443 centimetres below its top, which is at height $8.8 + 10 = 18.8$ cm above the very bottom of the hourglass. So, the centre of mass of the upturned sand is at height $y = 18.8 - 2.443 = 16.357$ cm.

Now, we have our model: the sand, viewed as a point mass, is moved from $y = \frac{57}{26}$ to $y = 16.357$ cm. That is, it moved about 14.165 cm, or about 0.14165 m. It has a mass of 0.6 kg, so the force required to lift it against gravity is

$$(0.6\text{kg}) \times \left(9.8\frac{\text{m}}{\text{sec}^2}\right) \times (0.14165\text{m}) \approx 0.833\text{newtons}$$

2.3.3.31. Solution. The techniques of Section 2.1 get pretty complicated here, so we will use the techniques we developed in Questions 6, 29 and 30 in this section. That is, (1) find the centre of mass of the water in its starting and ending positions, and then (2) compute the work done as the work moving a point mass with the weight of the water from the first centre of mass to the second. For the centre of mass, all we need to know is the height — for one thing, we could find the other coordinates by symmetry, but we don't need them. The height moved by the water is all that matters if we're calculating the work done opposing gravity.

Let's start by calculating the volume of the water. The volume of a sphere of radius 1 is $\frac{4}{3}\pi \cdot 1^3$, so the volume of water is $\frac{2}{3}\pi$ m^3.

Then the mass of the water is $\frac{2000}{3}\pi$ kg.

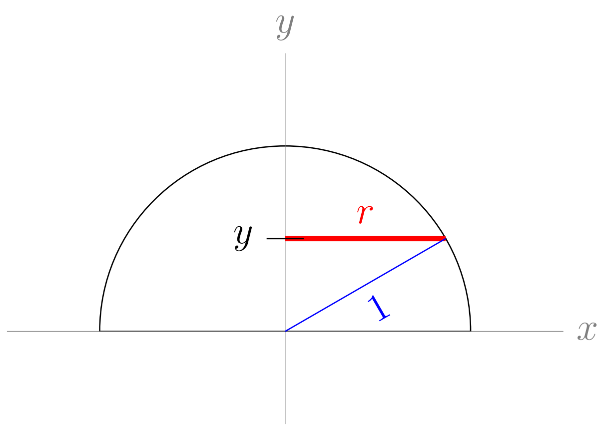
Next, we calculate the centre of mass of Tank A, and the work done to pump the water out of Tank A to a height of 3 metres. Symmetry alone won't tell us the height of the centre of mass. We'll show you two ways to go about this.

- Option 1: As in Question 29, we'll model the tank of water as a vertical rod, along the y-axis spanning the interval $[0, 1]$, such that the mass of a piece of the rod along $[a, b]$ is the same as the mass of the water from height $y = a$ to height $y = b$. Then, the centre of mass of the rod will be the same as the centre of mass of the water.

Consider a horizontal slice of water at height y, with thickness $\mathrm{d}y$. If the radius of this slice is $r(y)$, then the volume of the slice is $\pi r(y)^2\,\mathrm{d}y$ m^3, so its mass is $1000\pi r(y)^2\,\mathrm{d}y$ kg. Then the mass of the slice of the rod at position y with length $\mathrm{d}y$ is $1000\pi r(y)^2\,\mathrm{d}y$ kg, so its density $\rho(y)$ is

$$\rho(y) = \frac{1000\pi r(y)^2\,\mathrm{d}y\text{ kg}}{\mathrm{d}y\text{ m}} = 1000\pi r(y)^2\frac{\text{kg}}{\text{m}}.$$

So, let's find $r(y)$, the radius of the slice of water at height y.



Using the Pythagorean Theorem, $r = \sqrt{1-y^2}$. Therefore,

$$\rho(y) = 1000\pi(1-y^2)$$

We use Equation 2.3.4 to calculate the centre of mass of the rod, which is the height of the centre of mass of Tank A:

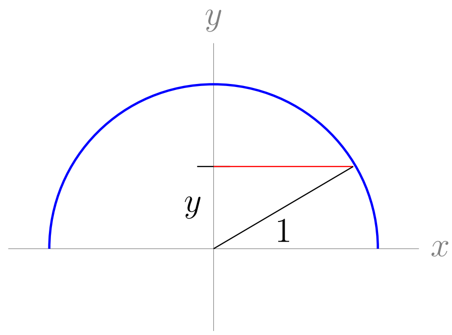
$$\begin{aligned}\bar{y}_A &= \frac{\int_0^1 y\rho(y)\,\mathrm{d}y}{\int_0^1 \rho(y)\,\mathrm{d}y} = \frac{\int_0^1 1000\pi y(1-y^2)\,\mathrm{d}y}{\int_0^1 1000\pi(1-y^2)\,\mathrm{d}y}\\ &= \frac{\int_0^1 (y-y^3)\,\mathrm{d}y}{\int_0^1 (1-y^2)\,\mathrm{d}y} = \frac{\left[\frac{1}{2}y^2-\frac{1}{4}y^4\right]_0^1}{\left[y-\frac{1}{3}y^3\right]_0^1} = \frac{\frac{1}{2}-\frac{1}{4}}{1-\frac{1}{3}} = \frac{3}{8}\text{m}\end{aligned}$$

From here, we can find the work done moving pumping the water to a height of 3 metres. We've moved the centre of mass from $\bar{y}_A = \frac{3}{8}$ metres to 3 metres.

$$\begin{aligned}W &= \left(\frac{2000}{3}\pi\text{kg}\right)\times\left(3-\frac{3}{8}\text{m}\right)\times\left(9.8\frac{\text{m}}{\text{sec}^2}\right)\\ &= 17,150\pi\text{J}\end{aligned}$$

- Option 2: We can use the techniques of Section 2.1 to calculate the amount of work it takes to pump the water from tank A to a height of 3 metres. That solves part (a), and we can use the amount of work to figure out the centre of gravity of the water in Tank A to help us solve part (b).

At height y, a horizontal layer of water in Tank A forms a disk with thickness $\mathrm{d}y$ and radius $\sqrt{1-y^2}$. (The radius comes from the Pythagorean Theorem — see the diagram below.)



The volume of the layer at height y is $\pi\left(\sqrt{1-y^2}\right)^2\,\mathrm{d}y = \pi(1-y^2)\,\mathrm{d}y$ m^3, so its mass is $1000\pi(1-y^2)\,\mathrm{d}y$ kg.

The layer at height y needs to be pumped a distance of $3-y$ metres. So, the work involved pumping the layer at height y is:

$$\begin{aligned}\mathrm{d}W &= \big(1000\pi(1-y^2)\,\mathrm{d}y\mathrm{kg}\big)\times(3-y\mathrm{m})\times\big(9.8\mathrm{m/sec}^2\big)\\ &= 9800\pi(y^3-3y^2-y+3)\,\mathrm{d}y\ \mathrm{J}\end{aligned}$$

Then the work involved pumping out the entire tank to a height of 3 metres is:

$$\begin{aligned}W &= \int_0^1 9800\pi(y^3-3y^2-y+3)\,\mathrm{d}y\\ &= 9800\pi\left[\frac{1}{4}y^4-y^3-\frac{1}{2}y^2+3y\right]_0^1\\ &= 17,150\pi\mathrm{J}\end{aligned}$$

This gives us an answer to part (a). To find the centre of mass of the water in Tank A, note that the work done is equivalent to moving a point mass from the centre of mass of the tank to a height of 3 metres. We know the water in Tank A has mass $\frac{2000}{3}\pi$ kg. So, if $\bar{y}_A$ is the centre of mass of the water in Tank A:

$$\begin{aligned}W &= \left(\frac{2000}{3}\pi\mathrm{kg}\right)\times(3-\bar{y}_A\mathrm{m})\times\big(9.8\mathrm{m/sec}^2\big)\\ 17,150\pi &= \left(\frac{2000}{3}\pi\right)(3-\bar{y}_A)\,(9.8)\\ \frac{21}{8} &= 3-\bar{y}_A\\ \bar{y}_A &= \frac{3}{8}\mathrm{m}\end{aligned}$$

Next let's calculate the centre of mass of the water in Tank B. Since the volume of the water in Tank B is $\frac{2}{3}\pi$ m^3, and the base of Tank B has area 1 m^2, the height of the water in Tank B is $\frac{2}{3}\pi$ m. Since the water is of uniform density, and Tank B has uniform horizontal cross-sections, by symmetry the centre of mass of the water in Tank B is at

$$\bar{y}_B = \frac{1}{3}\pi \text{ m}.$$

Now, we can calculate the work done by moving the water directly from Tank A to its final position in Tank B. The work done moving a point mass of $\frac{2000}{3}\pi$ kg a distance of $\bar{y}_B - \bar{y}_A = \frac{1}{3}\pi - \frac{3}{8}$ m against the gravity, $g = 9.8$ m/sec^2, is:

$$\begin{aligned} W &= \left(\frac{2000}{3}\pi \text{ kg}\right) \times \left(\frac{1}{3}\pi - \frac{3}{8} \text{ m}\right) \times \left(9.8\text{m/sec}^2\right) \\ &= \frac{2450}{9}\pi\,(8\pi - 9) \approx 13,797 \text{ J} \end{aligned}$$

Finally, the "wasted" work is:

$$\begin{aligned} \Delta W &= 17,150\pi - \frac{2450}{9}\pi\,(8\pi - 9) \\ &= 2450\pi\left(7 - \frac{8\pi - 9}{9}\right) \\ &= 2450\pi\left(8 - \frac{8\pi}{9}\right) \\ &= 19,600\pi\left(1 - \frac{\pi}{9}\right) \end{aligned}$$

As a percentage of 17,150π, this is:

$$\begin{aligned} \text{waste} &= \left(\frac{19,600\pi\left(1 - \frac{\pi}{9}\right)}{17,150\pi}\right) \times 100 \\ &= \frac{8}{7}\left(1 - \frac{\pi}{9}\right) \times 100 \approx 74\% \end{aligned}$$

2.3.3.32. Solution. Using Equation 2.3.5 with $T(x) = 2x\sin(x^2)$ and $B(x) = 0$,

$$\bar{x} = \frac{\displaystyle\int_0^{\sqrt{\pi/2}} 2x^2 \sin(x^2)\,\mathrm{d}x}{\displaystyle\int_0^{\sqrt{\pi/2}} 2x\sin(x^2)\,\mathrm{d}x}$$

We can evaluate the bottom integral exactly with the substitution $u = x^2$, $\mathrm{d}u = 2x\mathrm{d}x$. When $x = 0$, $u = 0$, and when $x = \sqrt{\pi/2}$, $u = \pi/2$.

$$\int_0^{\sqrt{\pi/2}} 2x\sin(x^2)\,\mathrm{d}x = \int_0^{\pi/2} \sin u\,\mathrm{d}u = \Big[-\cos u\Big]_0^{\pi/2} = 1$$

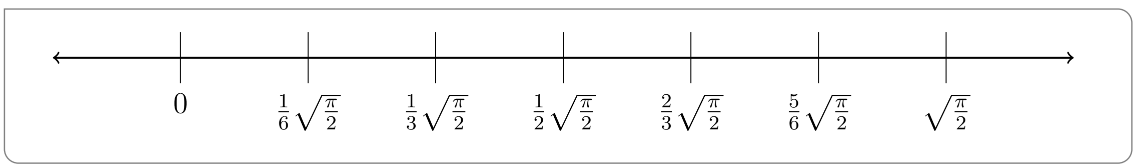
So,

$$\bar{x} = \int_0^{\sqrt{\pi/2}} 2x^2 \sin(x^2)\,\mathrm{d}x$$

Evaluating the integral $\int x^2 \sin(x^2)\,\mathrm{d}x$ is not so simple [a] , so we use a numerical approximation. Since we're given an upper bound on the fourth derivative, we decide to use Simpson's rule. The error involved using Simpson's rule with n intervals is at most $\frac{L(b-a)^5}{180n^4}$. For our approximation, $a = 0$ and $b = \sqrt{\pi/2}$. According to the information given in the problem statement, $\left|\frac{\mathrm{d}^4}{\mathrm{d}x^4}\left\{2x^2\sin(x^2)\right\}\right| \le 415$ over the interval $\left[0, \sqrt{\frac{\pi}{2}}\right]$, so we set $L = 415$.
We want our final error to be no more than $\frac{1}{100}$, so we want to find an even n such that:

$$\frac{415\left(\sqrt{\frac{\pi}{2}} - 0\right)^5}{180n^4} \le \frac{1}{100}$$
$$n^4 \ge \frac{415 \cdot 100\left(\frac{\pi}{2}\right)^{5/2}}{180} = \frac{2075\pi^{5/2}}{36\sqrt{2}}$$
$$n \ge \sqrt[4]{\frac{2075\pi^{5/2}}{36\sqrt{2}}} \approx 5.17$$

So, $n = 6$ intervals suffices. Then $\Delta x = \frac{b-a}{6} = \frac{1}{6}\sqrt{\frac{\pi}{2}}$ and our grid points are $x_0 = 0$, $x_1 = \frac{1}{6}\sqrt{\frac{\pi}{2}}$, $x_2 = \frac{1}{3}\sqrt{\frac{\pi}{2}}$, $x_3 = \frac{1}{2}\sqrt{\frac{\pi}{2}}$, $x_4 = \frac{2}{3}\sqrt{\frac{\pi}{2}}$, $x_5 = \frac{5}{6}\sqrt{\frac{\pi}{2}}$, and , $x_6 = \sqrt{\frac{\pi}{2}}$.



Following Equation 1.11.9, the Simpson's rule approximation of $\displaystyle\int_0^{\sqrt{\pi/2}} 2x^2 \sin(x^2)\,\mathrm{d}x$ is:

$$\begin{aligned}
&\frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\Big] \\
&= \frac{1}{6}\sqrt{\frac{\pi}{2}} \cdot \frac{1}{3}\Bigg[0 + 4 \times \frac{2\pi}{72}\sin\left(\frac{\pi}{72}\right) + 2 \times \frac{2\pi}{18}\sin\left(\frac{\pi}{18}\right) \\
&\qquad + 4 \times \frac{2\pi}{8}\sin\left(\frac{\pi}{8}\right) + 2 \times \frac{8\pi}{18}\sin\left(\frac{4\pi}{18}\right) \\
&\qquad + 4 \times \frac{50\pi}{72}\sin\left(\frac{25\pi}{72}\right) + \frac{2\pi}{2}\sin\left(\frac{\pi}{2}\right)\Bigg] \\
&= \frac{1}{18}\sqrt{\frac{\pi}{2}}\Bigg[\frac{\pi}{9}\sin\left(\frac{\pi}{72}\right) + \frac{2\pi}{9}\sin\left(\frac{\pi}{18}\right) + \pi\sin\left(\frac{\pi}{8}\right) + \frac{8\pi}{9}\sin\left(\frac{2\pi}{9}\right) \\
&\qquad + \frac{25\pi}{9}\sin\left(\frac{25\pi}{72}\right) + \pi\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\pi}{18}\sqrt{\frac{\pi}{2}}\left[\frac{1}{9}\sin\left(\frac{\pi}{72}\right) + \frac{2}{9}\sin\left(\frac{\pi}{18}\right) + \sin\left(\frac{\pi}{8}\right) + \frac{8}{9}\sin\left(\frac{2\pi}{9}\right)\right. \\
&\qquad\qquad\left. + \frac{25}{9}\sin\left(\frac{25\pi}{72}\right) + 1\right] \\
&= \frac{\pi}{162}\sqrt{\frac{\pi}{2}}\left[\sin\left(\frac{\pi}{72}\right) + 2\sin\left(\frac{\pi}{18}\right) + 9\sin\left(\frac{\pi}{8}\right) + 8\sin\left(\frac{2\pi}{9}\right)\right. \\
&\qquad\qquad\left. + 25\sin\left(\frac{25\pi}{72}\right) + 9\right] \\
&\approx 0.976
\end{aligned}
$$

The absolute error in our answer is at most:

$$\frac{L(b-a)^5}{180n^4} = \frac{415 \times \sqrt{\frac{\pi}{2}}^5}{180 \times 6^4} = \frac{82\sqrt{\pi}^5}{186624\sqrt{2}} \approx 0.005$$

Remark: combining the error with our approximation, we see the actual value of $\bar{x}$ is in the interval

$$[0.976 - 0.005, 0.976 + 0.005] = [0.971, .981]$$

A computer algebra system approximates $\bar{x}$ as 0.977451.

a Indeed, the antiderivative of $2x^2\sin(x^2)$ is not expressible as an elementary function.

2.4 · Separable Differential Equations
2.4.7 · Exercises

Exercises — Stage 1

2.4.7.1. Solution.

a If $y = 5(e^x - 3x^2 - 6x - 6)$, then $\frac{dy}{dx} = 5(e^x - 6x - 6)$. Let's see whether this is equal to $y + 15x^2$:

$$
\begin{aligned}
y + 15x^2 &= 5(e^x - 3x^2 - 6x - 6) + 15x^2 \\
&= 5(e^x - 3x^2 - 6x - 6 + 3x^2) \\
&= 5(e^x - 6x - 6) \\
&= \frac{dy}{dx}
\end{aligned}
$$

So, $y = 5(e^x - 3x^2 - 6x - 6)$ is indeed a solution to the differential equation $\frac{dy}{dx} = y + 15x^2$.

b If $y = \dfrac{-2}{x^2+1}$, then $\frac{dy}{dx} = \dfrac{4x}{(x+1)^2}$. Let's see whether this is equal to xy^2:

$$xy^2 = x\left(\frac{-2}{x^2+1}\right)^2$$

$$\begin{aligned} &= \frac{4x}{(x^2+1)^2} \\ &= \frac{\mathrm{d}y}{\mathrm{d}x} \end{aligned}$$

So, $y = \dfrac{-2}{x^2+1}$ is indeed a solution to the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = yx^2$.

c If $y = x^{3/2} + x$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}\sqrt{x} + 1$.

$$\begin{aligned} \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}y}{\mathrm{d}x} &= \left(\frac{3}{2}\sqrt{x}+1\right)^2 + \frac{3}{2}\sqrt{x} + 1 \\ &= \frac{9}{4}x + \frac{9}{2}\sqrt{x} + 2 \\ &\neq \frac{\mathrm{d}y}{\mathrm{d}x} \end{aligned}$$

So, $y = x^{3/2} + x$ is not a solution to the differential equation $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \frac{\mathrm{d}y}{\mathrm{d}x} = y$.

2.4.7.2. Solution.

a $3y\frac{\mathrm{d}y}{\mathrm{d}x} = x\sin y$ can be written as $\frac{\mathrm{d}y}{\mathrm{d}x} = x\left(\frac{\sin y}{3y}\right)$, which fits the form of a separable equation with $f(x) = x$, $g(y) = \frac{\sin y}{3y}$.

b $\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x+y} = e^x e^y$ which fits the form of a separable equation using $f(x) = e^x$, $g(y) = e^y$.

c $\frac{\mathrm{d}y}{\mathrm{d}x} + 1 = x$ can be written as $\frac{\mathrm{d}y}{\mathrm{d}x} = (x-1)$, which fits the form of a separable equation using $f(x) = x - 1$, $g(y) = 1$. (We can solve it by simply antidifferentiating.)

d Notice the left side of the equation $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + x^2 = 0$ is a perfect square. So, this equation is equivalent to $\left(\frac{\mathrm{d}y}{\mathrm{d}x} - x\right)^2 = 0$, that is, $\frac{\mathrm{d}y}{\mathrm{d}x} = x$. This has the form of a separable equation with $f(x) = x$, $g(y) = 1$.

2.4.7.3. Solution. The mnemonic allows us to skip from the separable differential equation we want to solve (very first line) to the equation

$$\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$$

So, the mnemonic is just a shortcut for the substitution we performed to get this point.
We also generally skip the explanation about C_1 and C_2 being replaced with C.

2.4.7.4. Solution. To say $y = f(x) + C$ is a solution to the differential equation means:

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f(x) + C\} = x(f(x) + C)$$

Since $y = f(x)$ is a solution, we know $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)\} = xf(x)$. Also, $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)+C\} = \frac{\mathrm{d}}{\mathrm{d}x}\{f(x)\}$. So, $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)+C\} = xf(x)$.

$$\begin{aligned} xf(x) &= x(f(x)+C) \\ 0 &= xC \end{aligned}$$

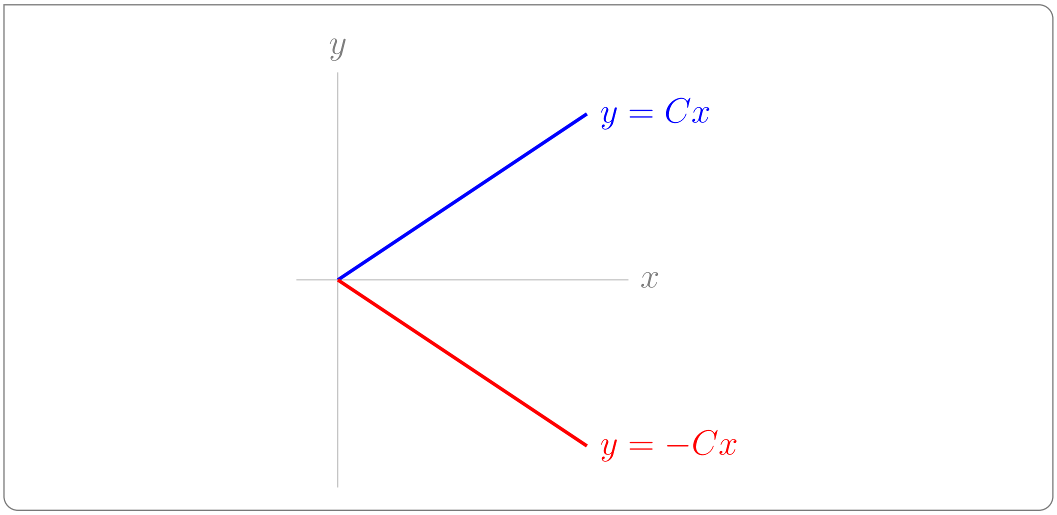
Our equation should hold for all x in our domain, and for the derivative to y with respect to x to make sense, our domain should not be a single point. So, there is some x in our domain such that $x \neq 0$. Therefore, the C *must be zero*. So, $f(x)+C$ is *not* a solution to the differential equation for any constant C.
When we're finding a general antiderivative, we add "$+C$" at the end. When we're finding a general solution to a differential equation, the "$+C$" gets added when we antidifferentiate–we don't add another one at the end of our work.

2.4.7.5. Solution.

a Since $|y| \geq 0$ no matter what y is, we see $Cx \geq 0$ for all x in the domain of $f(x)$. Since C is positive, that means the domain of $f(x)$ only includes nonnegative numbers. So, the largest possible domain of $f(x)$ is $[0, \infty)$.

b None exists.

The graph of Cx is given below for some positive constant C, also with the graph of $-Cx$. If $y = f(x)$ were sometimes the top function, and other times the bottom function, then there would be a jump discontinuity where it switched. Then the derivative of $f(x)$ would not exist, violating the second property.



A tiny technical note is that it's possible that $f(x) = Cx$ when $x = 0$ and $f(x) = -Cx$ when $x > 0$ (or vice-versa). This would not introduce a jump discontinuity, but it also does not satisfy that $f(x) > 0$ for some values of x.

Remark: in several instances below, solving a differential equation will lead us to conclude something like $|y| = g(x)$. In these cases, we choose either $y = g(x)$, or $y = -g(x)$, but not $y = \pm g(x)$ (which is not a function) or that y is sometimes

$g(x)$, and other times $-g(x)$. The reasoning above somewhat explains this choice: if y were sometimes positive and sometimes negative, then $\frac{\mathrm{d}y}{\mathrm{d}x}$ would not exist at the values of x where the sign of y switches, unless that switch occurs at a root of $g(x)$. Since that's a pretty specific occurrence, we usually feel safe ignoring it to avoid getting bogged down in technical details.

2.4.7.6. Solution. Let $Q(t)$ be the quantity of morphine in a patient's bloodstream at time t, where t is measured in minutes.
Using the definition of a derivative,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \lim_{h\to 0}\frac{Q(t+h)-Q(t)}{h} \approx \frac{Q(t+1)-Q(t)}{1}$$

So, $\frac{\mathrm{d}Q}{\mathrm{d}t}$ is roughly the change in the amount of morphine in one minute, from t to $t+1$.
The sentence tells us that the change in the amount of morphine in one minute is about $-0.003Q$, where Q is the quantity in the bloodstream. That is:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -0.003Q(t)$$

2.4.7.7. Solution. If $p(t)$ is the proportion of times speakers use the new form, measured between 0 and 1, then $1-p(t)$ is the proportion of times speakers use the old form.
The law, then, states that $\frac{\mathrm{d}p}{\mathrm{d}t}$ is proportional to $p(t)\times\big(1-p(t)\big)$. When we say two quantities are proportional, we mean that one is a constant multiple of the other. So, the law says

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \alpha p(t)\big(1-p(t)\big)$$

for some constant α.
Remark: it follows from this model that, when a new form is either very rare or entirely ubiquitous, the rate of change of its adoption is small. This makes sense: if the new form is used all the time ($p(t)\approx 1$), there's nobody left to convert; if the new form is almost never used ($p(t)\approx 0$) then people don't know about it, so they won't pick it up.

2.4.7.8. Solution.

a When $y=0$, $y' = \frac{0}{2}-1=-1$.

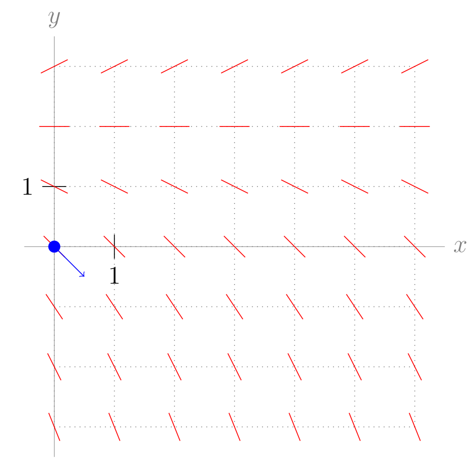
b When $y=2$, $y' = \frac{2}{2}-1=0$.

c When $y=3$, $y' = \frac{3}{2}-1=0.5$.

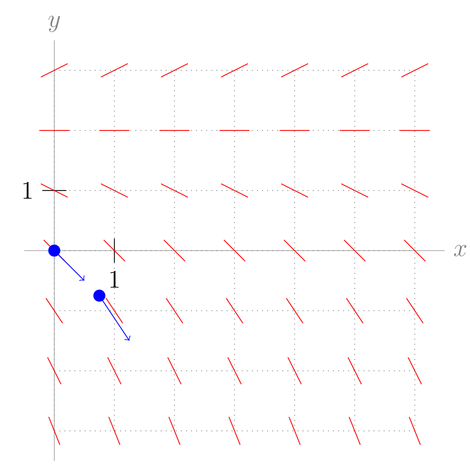
d The small red lines have varying slopes. The red lines on points with y-coordinate 2 have slopes of 0; this matches y' when $y=0$, as we saw above. The red lines on points with y-coordinate 0 have slopes of approximately -1; again, this matches what we found for y' when $y=0$.

  The red lines correspond to a tiny section of $y(x)$, *if* $y(x)$ passes through that point. So, we can sketch a possible curve $y(x)$ satisfying the equation by starting somewhere, then following the slopes.

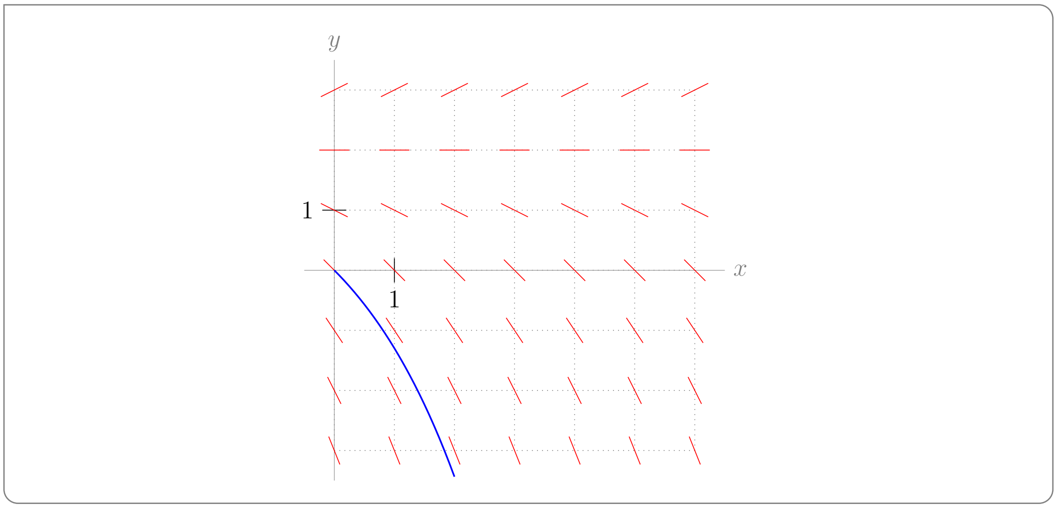
For example, suppose we start at the origin.



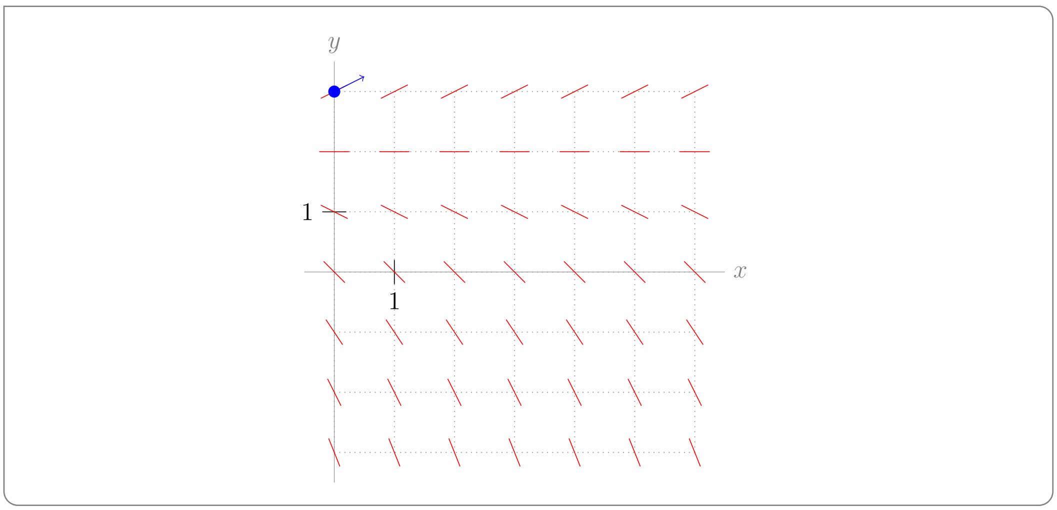
Then our function is decreasing at that point, which leads us to a coordinate where (as we see from the red marks) the function is decreasing slightly faster.



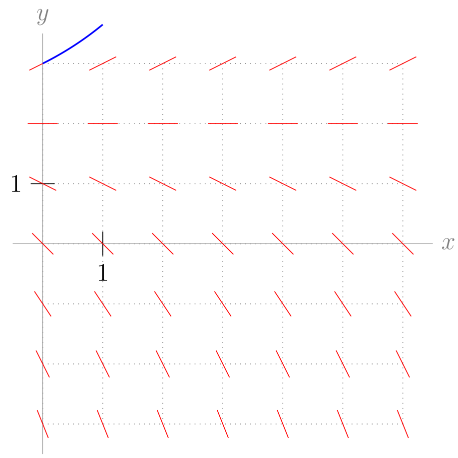
Following the red marks leads us down even further, so our function $y(x)$ might look something like this:



However, we didn't have to start at the origin. Suppose $y(0) = 3$. Then at $x = 0$, y is increasing, with slope $\frac{1}{2}$.



Our red marks run out that high up, but we now $y' = \frac{1}{2}y - 1$, so y' increases as y increases. That means our function keeps getting steeper and steeper, possibly something like this:



If $y(0) = 2$, we see another possible curve is the constant function $y(x) = 2$.

Remark: from Theorem 2.4.4, we see the solutions to the equation $y' = \frac{1}{2}y - 1 = \frac{1}{2}(y-2)$ are of the form $y(x) = Ce^{x/2} + 2$ for some constant C. Check that the curves you're sketching look exponential.

2.4.7.9. Solution.

a If $y(1) = 0$, then $y'(1) = 0 - \frac{1}{2} = -\frac{1}{2}$.

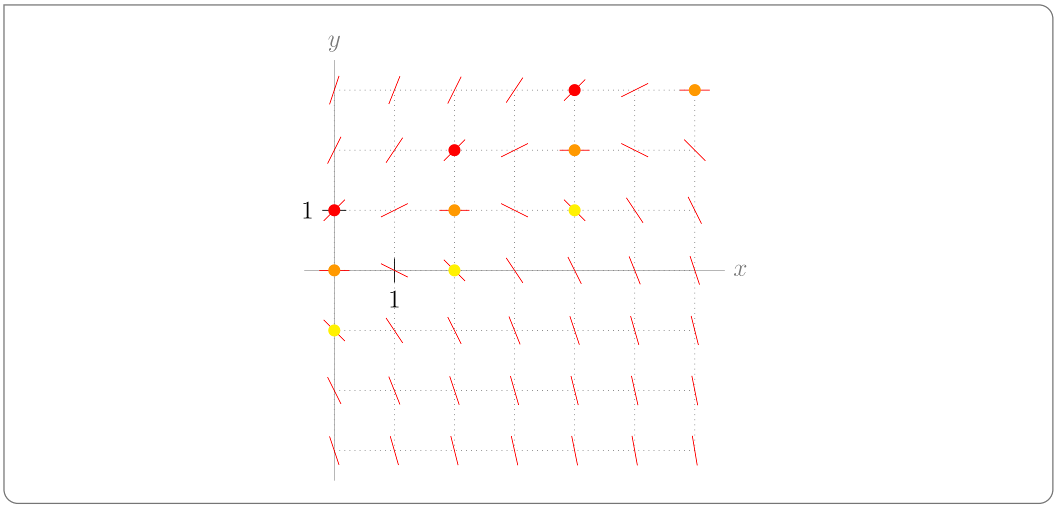
b If $y(1) = 2$, then $y'(1) = 2 - \frac{1}{2} = \frac{3}{2}$.

c If $y(1) = -2$, then $y'(1) = -2 - \frac{1}{2} = -\frac{5}{2}$.

d There are $7 \times 7 = 49$ points on the grid; we don't want to make 49 separate calculations. Let's find some shortcuts.

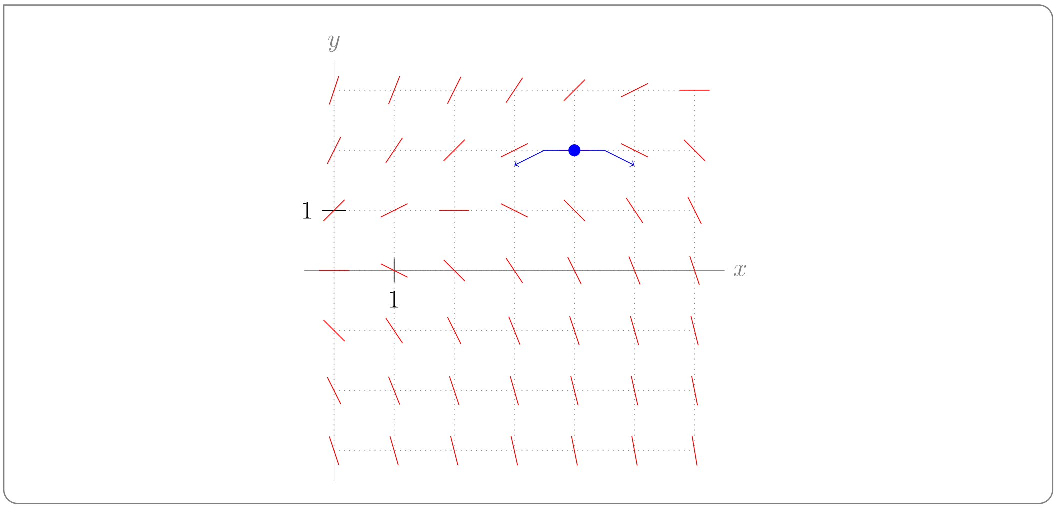
- If $y'(x) = 0$, then $y = \frac{x}{2}$, which applies to the points $(0,0)$, $(2,1)$, $(4,2)$ and $(6,3)$. These are the orange dots in the sketch below.
- If $y'(x) = 1$, then $y = 1 + \frac{x}{2}$, which applies to the points $(0,1)$, $(2,2)$, and $(4,3)$. (Note these are exactly 1 unit above the points with $y' = 0$.) These are the red dots in the sketch below.
- If $y'(x) = -1$, then $y = -1 + \frac{x}{2}$, which applies to the points $(0,-1)$, $(2,-2)$, and $(4,-3)$. (Note these are exactly 1 unit below the points with $y' = 0$.) These are the yellow dots in the sketch below.
- If x increases and y stays the same, y decreases.
- If y increases and x stays the same, y increases.
- If we draw a straight line of slope $\dfrac{1}{2}$ on our sketch, for every point on that line, our mark has the same slope: for instance, the points where we draw a mark with slope 0 are $(0,0)$, $(2,1)$, and $(4,2)$, and these all lie on the line $f(x) = \frac{x}{2}$.

This is enough to give us a pretty good sketch. The points whose slopes we found explicitly have dots; the rest can be sketched as either steeper or less steep than what's near them.

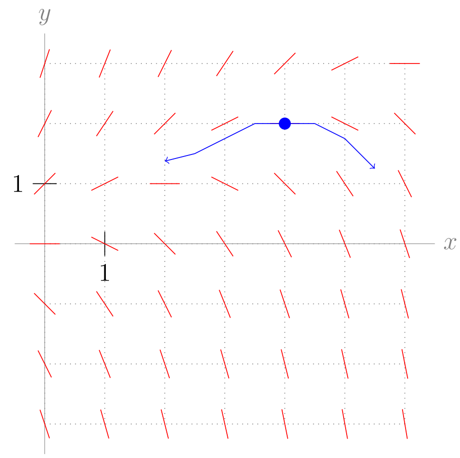


e To sketch a possible graph of $y(x)$, we choose a point $(x, y(x))$, then follow the red lines.

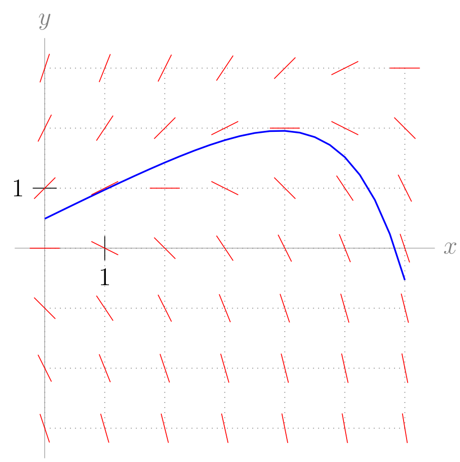
For example, if we suppose that $y(4) = 2$, then near $(4, 2)$, the lines tell us $y(x)$ is fairly flat; and it is increasing to the left of $x = 4$, and decreasing to the right.



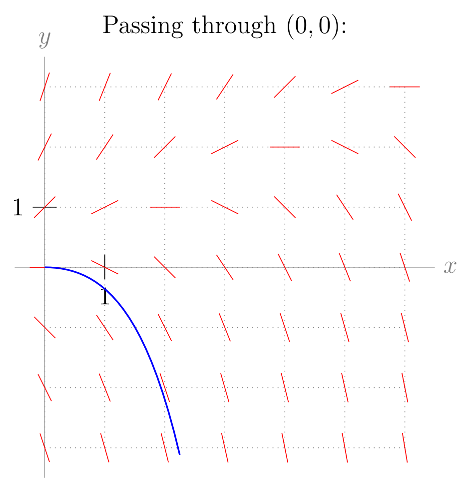
Following the red lines a little farther in each direction brings us somewhere like this:



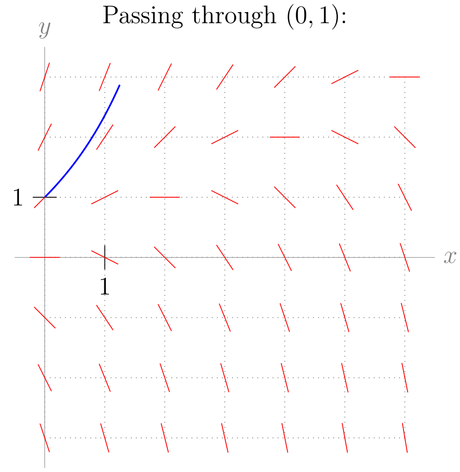
Extending yet further, we might sketch something like the following:



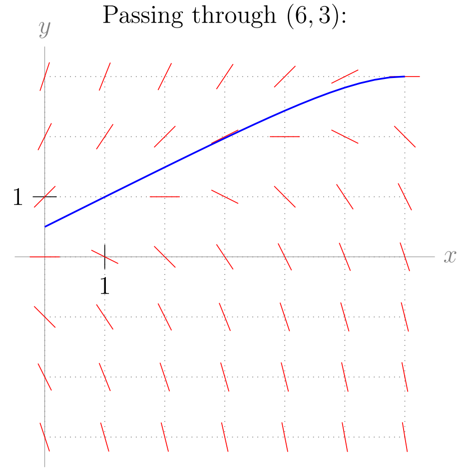
By choosing another point $(x, y(x))$ to be on the curve, we might find other potential curves. Some examples are shown below.



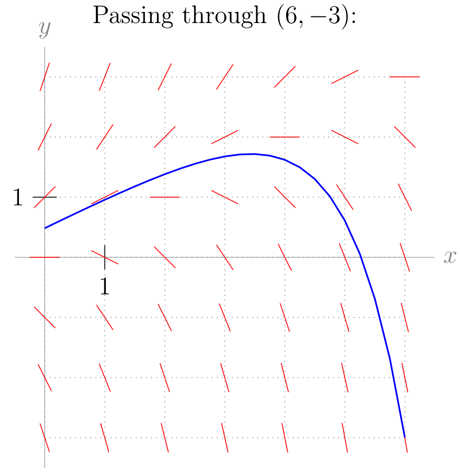
Passing through $(0, 1)$:



Passing through $(6, 3)$:



Passing through $(6, -3)$:



Remark: the differential equation $y' = y - \frac{x}{2}$ is not separable, so we haven't talked about how to solve it. The solutions have the form $y(x) = Ce^x + \frac{x+1}{2}$. You can verify that these functions satisfy $y' = y - \frac{x}{2}$.

Exercises — Stage 2

2.4.7.10. *. Solution. Rearranging, we have:

$$e^y \,\mathrm{d}y = 2x \,\mathrm{d}x.$$

Integrating both sides:

$$\begin{aligned} \int e^y \,\mathrm{d}y &= \int 2x \,\mathrm{d}x \\ e^y &= x^2 + C \end{aligned}$$

Since $y = \log 2$ when $x = 0$, we have

$$\begin{aligned} e^{\log 2} &= 0^2 + C \\ 2 &= C, \end{aligned}$$

and therefore

$$\begin{aligned} e^y &= x^2 + 2 \\ y &= \log(x^2 + 2) \end{aligned}$$

2.4.7.11. *. Solution. Using separation of variables:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{xy}{x^2+1} \\ \frac{\mathrm{d}y}{y} &= \frac{x}{x^2+1}\mathrm{d}x \\ \int \frac{\mathrm{d}y}{y} &= \int \frac{x}{x^2+1}\mathrm{d}x \\ \log|y| &= \frac{1}{2}\log(1+x^2) + C \end{aligned}$$

To satisfy $y(0) = 3$, we need $\log 3 = \frac{1}{2}\log(1+0) + C$, so $C = \log 3$. Thus:

$$\begin{aligned} \log|y| &= \frac{1}{2}\log(1+x^2) + \log 3 \\ &= \log\sqrt{1+x^2} + \log 3 \\ &= \log 3\sqrt{1+x^2} \end{aligned}$$

So,

$$|y| = 3\sqrt{1+x^2}$$

We are told to find a function $y(x)$. So far, we have two possible functions from the work above: maybe $y = 3\sqrt{1+x^2}$, and maybe $y = -3\sqrt{1+x^2}$. It's important to note that $y = \pm 3\sqrt{1+x^2}$ is not a function: for an equation to represent a function, for every input in the domain, there must only be one output. That is, functions pass the vertical line test. (See the CLP-1 text for a definition of the vertical line test and a formal definition of a function.) So, we need to decide whether our function is $y = 3\sqrt{1+x^2}$ or $y = -3\sqrt{1+x^2}$. Since $y(0) = 3$, we conclude

$$y(x) = 3\sqrt{1+x^2}$$

2.4.7.12. *. Solution. The given differential equation is separable and we solve it accordingly.

$$\begin{aligned}
y' &= e^{\frac{y}{3}}\cos t\\
e^{-y/3}\mathrm{d}y &= \cos t\,\mathrm{d}t\\
\int e^{-y/3}\mathrm{d}y &= \int \cos t\,\mathrm{d}t\\
-3e^{-y/3} &= \sin t + C\\
\frac{1}{e^{y/3}} &= \frac{\sin t + C}{-3}\\
e^{y/3} &= \frac{-3}{C+\sin t}\\
\frac{y}{3} &= \log\left(\frac{-3}{C+\sin t}\right)\\
y(t) &= 3\log\left(\frac{-3}{C+\sin t}\right)
\end{aligned}$$

for any constant C.
Since the domain of logarithm is $(0,\infty)$, the solution only exists when $C+\sin t < 0$.

2.4.7.13. *. Solution. The given differential equation is separable and we solve it accordingly.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= xe^{x^2-\log(y^2)} = \frac{xe^{x^2}}{y^2}\\
y^2\mathrm{d}y &= xe^{x^2}\,\mathrm{d}x\\
\int y^2\mathrm{d}y &= \int xe^{x^2}\,\mathrm{d}x
\end{aligned}$$

We can guess the antiderivative of xe^{x^2}, or use the substitution $u = x^2$, $\mathrm{d}u = 2x\mathrm{d}x$.

$$\begin{aligned}
\frac{y^3}{3} &= \frac{1}{2}e^{x^2} + C'\\
y^3 &= \frac{3}{2}e^{x^2} + 3C'
\end{aligned}$$

Since C' can be any constant in $(-\infty, \infty)$, then also $3C'$ can be any constant in $(-\infty, \infty)$, so we replace $3C'$ with the arbitrary constant C.

$$\begin{aligned} y^3 &= \frac{3}{2}e^{x^2} + C \\ y &= \sqrt[3]{\frac{3}{2}e^{x^2} + C} \end{aligned}$$

for any constant C.

2.4.7.14. $*$. Solution. The given differential equation is separable and we solve it accordingly.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= xe^y \\ \frac{\mathrm{d}y}{e^y} &= x\,\mathrm{d}x \\ \int \frac{\mathrm{d}y}{e^y} &= \int x\,\mathrm{d}x \\ -e^{-y} &= \frac{1}{2}x^2 + C \\ e^{-y} &= -\frac{1}{2}x^2 - C \end{aligned}$$

Since C can be any constant in $(-\infty, \infty)$, then also $-C$ can be any constant in $(-\infty, \infty)$, so we write C instead of $-C$.

$$\begin{aligned} e^{-y} &= C - \frac{1}{2}x^2 \\ -y &= \log\left(C - \frac{x^2}{2}\right) \\ y &= -\log\left(C - \frac{x^2}{2}\right) \end{aligned}$$

for any constant C.
The solution only exists for $C - \frac{x^2}{2} > 0$. For this to happen, we need $C > 0$, and then the domain of the function is those values x for which $|x| < \sqrt{2C}$.

2.4.7.15. $*$. Solution. The given differential equation is separable and we solve it accordingly. Cross–multiplying, we rewrite the equation as

$$\begin{aligned} y^2\frac{\mathrm{d}y}{\mathrm{d}x} &= e^x - 2x \\ y^2\,\mathrm{d}y &= (e^x - 2x)\,\mathrm{d}x. \end{aligned}$$

Integrating both sides, we find

$$\int y^2\,\mathrm{d}y = \int (e^x - 2x)\,\mathrm{d}x$$

$$\frac{1}{3}y^3 = e^x - x^2 + C$$

Setting $x = 0$ and $y = 3$, we find $\frac{1}{3}3^3 = e^0 - 0^2 + C$ and hence $C = 8$.

$$\begin{aligned}\frac{1}{3}y^3 &= e^x - x^2 + 8\\ y &= (3e^x - 3x^2 + 24)^{1/3}\end{aligned}$$

2.4.7.16. *. Solution. This is a separable differential equation that we solve in the usual way.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -xy^3\\ -\frac{\mathrm{d}y}{y^3} &= x\mathrm{d}x\\ \int -\frac{\mathrm{d}y}{y^3} &= \int x\mathrm{d}x\\ -\frac{y^{-2}}{-2} &= \frac{x^2}{2} + C\\ y^{-2} &= x^2 + 2C. \qquad (*)\end{aligned}$$

To have $y = -\frac{1}{4}$ when $x = 0$, we must choose C to obey

$$\begin{aligned}\left(-\frac{1}{4}\right)^{-2} &= 0 + 2C\\ 16 &= 2C\end{aligned}$$

So, from $(*)$,

$$\begin{aligned}y^{-2} &= x^2 + 2C = x^2 + 16\\ y^2 &= \frac{1}{x^2+16}\end{aligned}$$

Now, we have two potential candidates for $y(x)$:

$$y = \frac{1}{\sqrt{x^2+16}} \qquad \text{OR} \qquad y = -\frac{1}{\sqrt{x^2+16}}$$

We know $y = -\frac{1}{4}$ when $x = 0$. The only function above that fits this is

$$y = -\frac{1}{\sqrt{x^2+16}}$$

So, $f(x) = -\dfrac{1}{\sqrt{x^2+16}}$.

2.4.7.17. *. Solution. This is a separable differential equation that we solve in the usual way. Cross-multiplying and integrating,

$$\begin{aligned} y\,\mathrm{d}y &= (15x^2+4x+3)\,\mathrm{d}x \\ \int y\,\mathrm{d}y &= \int (15x^2+4x+3)\,\mathrm{d}x \\ \frac{y^2}{2} &= 5x^3+2x^2+3x+C. \end{aligned}$$

Plugging in $x=1$ and $y=4$ gives $\frac{4^2}{2} = 5+2+3+C$, and so $C=-2$. Therefore

$$\begin{aligned} \frac{y^2}{2} &= 5x^3+2x^2+3x-2 \\ y^2 &= 10x^3+4x^2+6x-4 \end{aligned}$$

This leaves us with two possible functions for y:

$$y = \sqrt{10x^3+4x^2+6x-4} \qquad \text{or} \qquad y = -\sqrt{10x^3+4x^2+6x-4}$$

When $x=1$, $y=4$. This only fits the first equation, so

$$y = \sqrt{10x^3+4x^2+6x-4}$$

2.4.7.18. *. Solution. The given differential equation is separable and we solve it accordingly.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= x^3 y \\ \frac{\mathrm{d}y}{y} &= x^3\,\mathrm{d}x \\ \int \frac{\mathrm{d}y}{y} &= \int x^3\,\mathrm{d}x \\ \log|y| &= \frac{x^4}{4}+C \\ |y| &= e^{x^4/4+C} = e^{x^4/4}e^C \end{aligned}$$

We are told that $y=1$ when $x=0$. That is, $1 = e^0 e^C$, so $e^C=1$. That is, $C=0$.

$$|y| = e^{x^4/4}$$

This leaves us with two potential functions:

$$y = e^{x^4/4} \qquad \text{or} \qquad y = -e^{x^4/4}$$

The first is always positive, and the second is always negative. Since $y=1$ (a positive number) when $x=0$, we see

$$y = e^{x^4/4}$$

2.4.7.19. *. Solution. This is a separable differential equation, even if it doesn't quite look like it. First move the y from the left hand side to the right hand side.

$$\begin{aligned} x\frac{\mathrm{d}y}{\mathrm{d}x} + y &= y^2 \\ x\frac{\mathrm{d}y}{\mathrm{d}x} &= y^2 - y = y(y-1) \\ \frac{\mathrm{d}y}{y(y-1)} &= \frac{\mathrm{d}x}{x} \end{aligned}$$

Using the method of partial fractions, we see $\frac{1}{y(y-1)} = \frac{1}{y-1} - \frac{1}{y}$.

$$\begin{aligned} \left(\frac{1}{y-1} - \frac{1}{y}\right)\,\mathrm{d}y &= \frac{\mathrm{d}x}{x} \\ \int \left(\frac{1}{y-1} - \frac{1}{y}\right)\,\mathrm{d}y &= \int \frac{\mathrm{d}x}{x} \\ \log|y-1| - \log|y| &= \log|x| + C \\ \log\frac{|y-1|}{|y|} &= \log|x| + C \end{aligned} \tag{$*$}$$

To determine C we set $x = 1$ and $y = -1$.

$$\begin{aligned} \log\frac{|-2|}{|-1|} &= \log|1| + C \\ \log 2 &= C \end{aligned}$$

Returning to $(*)$,

$$\begin{aligned} \log\frac{|y-1|}{|y|} &= \log|x| + \log 2 \\ \log\left|\frac{y-1}{y}\right| &= \log|2x| \\ \left|\frac{y-1}{y}\right| &= |2x| \end{aligned}$$

As $y(1) = -1$ is an initial condition, we have that $x \geq 1$ and $|2x| = 2x$. For $x = 1$, we have $y = -1$. So at least for x near 1, we have y near -1, so that $\frac{y-1}{y}$ is positive and we may drop the absolute value signs. There remains the possibility that $\frac{y(x)-1}{y(x)}$ changes sign for some larger $x > 1$. For now, we will simply ignore that possibility. At the end, we will explicitly check that the $y(x)$ we come up with really does satisfy the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = y^2$ and the initial condition $y(1) = -1$.

$$\begin{aligned} \frac{y-1}{y} &= 2x \\ y - 1 &= 2xy \\ y - 2xy &= 1 \end{aligned}$$

$$\begin{aligned} y(1-2x) &= 1 \\ y &= \frac{1}{1-2x} \end{aligned}$$

As a check, we compute:

$$\begin{aligned} x\frac{\mathrm{d}y}{\mathrm{d}x} + y &= x\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{1-2x}\right\} + y \\ &= x\frac{2}{(1-2x)^2} + \frac{1}{1-2x} \\ &= \frac{2x + (1-2x)}{(1-2x)^2} \\ &= \frac{1}{(1-2x)^2} \\ &= y^2 \end{aligned}$$

So, our differential equation is satisfied. Furthermore:

$$y(1) = \frac{1}{1 - 2 \times 1} = -1$$

as desired. This confirms that our solution is correct.

2.4.7.20. *. Solution. The unknown function $f(x)$ satisfies an equation that involves the derivative of f. That means we're in differential equation territory. Specifically, we are told that $y = f(x)$ obeys the separable differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = xy$.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= xy \\ \frac{\mathrm{d}y}{y} &= x\,\mathrm{d}x \\ \int \frac{\mathrm{d}y}{y} &= \int x\,\mathrm{d}x \\ \log|y| &= \frac{x^2}{2} + C \end{aligned}$$

To determine C we set $x = 0$ and $y = e$.

$$\begin{aligned} \log e &= \frac{0^2}{2} + C \\ 1 &= C \end{aligned}$$

So, the solution is

$$\log|y| = \frac{x^2}{2} + 1$$

We are told that $y = f(x) > 0$, so may drop the absolute value signs.

$$\log y = \frac{x^2}{2} + 1$$

$$y = e^{1+\frac{1}{2}x^2} = e \cdot e^{x^2/2}$$

2.4.7.21. *. Solution. This is a separable differential equation.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{(x^2+x)y} \\
y\,\mathrm{d}y &= \frac{\mathrm{d}x}{x(x+1)}
\end{aligned}$$

Using partial fractions decomposition, we find $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$.

$$\begin{aligned}
y\,\mathrm{d}y &= \left(\frac{1}{x} - \frac{1}{x+1}\right)\mathrm{d}x \\
\int y\,\mathrm{d}y &= \int \left(\frac{1}{x} - \frac{1}{x+1}\right)\mathrm{d}x \\
\frac{y^2}{2} &= \log|x| - \log|x+1| + C = \log\left|\frac{x}{x+1}\right| + C
\end{aligned}$$

To satisfy the initial condition $y(1) = 2$ we must choose C to obey

$$\begin{aligned}
\frac{2^2}{2} &= \log\left|\frac{1}{1+1}\right| + C \\
2 &= \log\frac{1}{2} + C \\
C &= 2 - \log\frac{1}{2}
\end{aligned}$$

So,

$$\begin{aligned}
\frac{y^2}{2} &= \log\left|\frac{x}{x+1}\right| + 2 - \log\frac{1}{2} \\
y^2 &= 2\log\left|\frac{x}{x+1}\right| + 4 - 2\log\frac{1}{2}
\end{aligned}$$

Note that the question specifies that $y(1) = 2$ is an initial condition. So we always have $x \geq 1$. Then $\frac{x}{x+1}$ is positive, and we can drop the absolute values.

$$y^2 = 2\log\frac{x}{x+1} + 4 - 2\log\frac{1}{2}$$

This leaves two options for $y(x)$: the positive or negative square root of the right hand side above. Since $y(1) = 1$, which is positive, we must choose the positive square root.

$$\begin{aligned}
y(x) &= \sqrt{2\left(\log\frac{x}{x+1} - \log\frac{1}{2} + 2\right)} \\
&= \sqrt{4 + 2\log\frac{2x}{x+1}}
\end{aligned}$$

You might worry that $y(x)$ could pass through zero, changing sign, at some $x > 1$. But the differential equation says that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{(x^2+x)y}$ is positive whenever $y > 0$ and $x \geq 1$. So $y(x)$ is an increasing function whenever $y > 0$ and $x \geq 1$. As $y(1) = 2$, we have $y(x) \geq 2$ for all $x \geq 1$.

2.4.7.22. $*$. Solution. This is a separable differential equation.

$$\begin{aligned} \frac{1+\sqrt{y^2-4}}{\tan x}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\sec x}{y} \\ y\big[1+\sqrt{y^2-4}\big]\mathrm{d}y &= \sec x\ \tan x\mathrm{d}x \\ \int y\big[1+\sqrt{y^2-4}\big]\mathrm{d}y &= \int \sec x\ \tan x\mathrm{d}x \end{aligned}$$

For the integral on the left, we use the substitution $u = y^2-4$, $\frac{1}{2}\mathrm{d}u = y\,\mathrm{d}y$.

$$\begin{aligned} \frac{1}{2}\int\left(1+\sqrt{u}\right)\mathrm{d}u &= \sec x + C \\ \frac{1}{2}\left(u+\frac{2}{3}u^{3/2}\right) &= \sec x + C \\ \frac{1}{2}\left(y^2-4+\frac{2}{3}(y^2-4)^{3/2}\right) &= \sec x + C \\ y^2+\frac{2}{3}(y^2-4)^{3/2} &= 2\sec x + 2C + 4 \end{aligned}$$

To find C we set $x=0$ and $y=2$.

$$\begin{aligned} 4+\frac{2}{3}\sqrt{4-4}^3 &= 2\sec(0)+2C+4 \\ 4 &= 2+2C+4 \\ 2 &= 2C+4 \end{aligned}$$

So,

$$y^2+\frac{2}{3}(y^2-4)^{3/2} = 2\sec x + 2$$

2.4.7.23. $*$. Solution. The given differential equation is separable and we solve it accordingly.

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}t} &= -k\sqrt{P} \\ \frac{\mathrm{d}P}{\sqrt{P}} &= -k\,\mathrm{d}t \\ \int\frac{\mathrm{d}P}{\sqrt{P}} &= \int -k\,\mathrm{d}t \\ 2\sqrt{P} &= -kt + C \end{aligned}$$

At $t=0$, $P=90,000$ so

$$\begin{aligned} 2\sqrt{90,000} &= -k\times 0 + C \\ C &= 2\times 300 = 600 \end{aligned}$$

Therefore,

$$2\sqrt{P} = -kt + 600 \tag{$*$}$$

Now, we find k. Let t be measured in weeks. Then when $t = 6$, $P = 40,000$.

$$\begin{aligned} 2\sqrt{40,000} &= -6k + 600 \\ 2 \cdot 200 &= -6k + 600 \\ k &= \frac{200}{6} = \frac{100}{3} \end{aligned}$$

Substituting our value of k into $(*)$:

$$2\sqrt{P} = -\frac{100}{3}t + 600$$

To find when the population will be 10,000, we set $P = 10,000$ and solve for t.

$$\begin{aligned} 2\sqrt{10,000} &= -\frac{100}{3}t + 600 \\ 2 \cdot 100 &= -\frac{100}{3}t + 600 \\ \frac{100}{3}t &= 400 \\ t &= 12 \end{aligned}$$

Since we measured t in weeks when we found k, we see that in 12 weeks the population will decrease to 10,000 individuals.

2.4.7.24. $*$. Solution. The given differential equation is separable and we solve it accordingly.

$$\begin{aligned} m\frac{\mathrm{d}v}{\mathrm{d}t} &= -(mg + kv^2) \\ \frac{m}{mg + kv^2}\,\mathrm{d}v &= -\mathrm{d}t \\ \int \frac{m}{mg + kv^2}\,\mathrm{d}v &= \int -\mathrm{d}t \end{aligned}$$

The left integral looks something like the antiderivative of arctangent. Let's factor out that mg from the denominator.

$$\begin{aligned} \frac{1}{mg}\int \frac{m}{1 + \frac{k}{mg}v^2}\,\mathrm{d}v &= -t + C \\ \frac{1}{g}\int \frac{1}{1 + \left(\sqrt{\frac{k}{mg}}v\right)^2}\,\mathrm{d}v &= -t + C \end{aligned}$$

Now it looks even more like the derivative of arctangent. We can guess the antiderivative from here, or use the substitution $u = \sqrt{\frac{k}{mg}}v$, $\mathrm{d}u = \sqrt{\frac{k}{mg}}\,\mathrm{d}v$.

$$\frac{1}{g}\sqrt{\frac{mg}{k}}\arctan\left(\sqrt{\frac{k}{mg}}v\right) = -t + C$$

$$\sqrt{\frac{m}{gk}}\arctan\left(\sqrt{\frac{k}{mg}}v\right) = -t + C \tag{$*$}$$

At $t = 0$, $v = v_0$, so:

$$\sqrt{\frac{m}{gk}}\arctan\left(\sqrt{\frac{k}{mg}}v_0\right) = C$$

Plug C into $(*)$.

$$\sqrt{\frac{m}{gk}}\arctan\left(\sqrt{\frac{k}{mg}}v\right) = \sqrt{\frac{m}{gk}}\arctan\left(\sqrt{\frac{k}{mg}}v_0\right) - t$$

At its highest point, the object has velocity $v = 0$. This happens when t obeys:

$$\begin{aligned}
\sqrt{\frac{m}{gk}}\arctan\left(\sqrt{\frac{k}{mg}}0\right) &= \sqrt{\frac{m}{gk}}\arctan\left(\sqrt{\frac{k}{mg}}v_0\right) - t\\
0 &= \sqrt{\frac{m}{gk}}\arctan\left(\sqrt{\frac{k}{mg}}v_0\right) - t\\
t &= \sqrt{\frac{m}{gk}}\arctan\left(\sqrt{\frac{k}{mg}}v_0\right)
\end{aligned}$$

2.4.7.25. $*$. Solution. (a) The given differential equation is separable and we solve it accordingly.

$$\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}t} &= -k\,v^2\\
-\frac{\mathrm{d}v}{v^2} &= k\,\mathrm{d}t\\
\int -\frac{\mathrm{d}v}{v^2} &= \int k\,\mathrm{d}t\\
\frac{1}{v} &= kt + C
\end{aligned}$$

At $t = 0$, $v = 40$ so

$$\begin{aligned}
\frac{1}{40} &= k \times 0 + C\\
C &= \frac{1}{40}
\end{aligned}$$

Therefore,

$$v(t) = \frac{1}{kt + C} = \frac{1}{kt + 1/40} = \frac{40}{40kt + 1} \tag{$*$}$$

The constant of proportionality k is determined by

$$\begin{aligned} v(10) &= 20 \\ 20 &= \frac{40}{40k \times 10 + 1} \\ \frac{1}{2} &= \frac{1}{400k + 1} \\ 400k + 1 &= 2 \\ k &= \frac{1}{400} \end{aligned}$$

(b) Subbing in the value of k to $(*)$,

$$v(t) = \frac{40}{40kt + 1} = \frac{40}{t/10 + 1}$$

We want to know the value of t that gives $v(t) = 5$.

$$\begin{aligned} 5 &= \frac{40}{t/10 + 1} \\ \frac{t}{10} + 1 &= 8 \\ t &= 70 \text{ sec} \end{aligned}$$

2.4.7.26. *. Solution. (a) The given differential equation is separable and we solve it accordingly.

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= k(3 - x)(2 - x) \\ \frac{\mathrm{d}x}{(x-2)(x-3)} &= k\mathrm{d}t \end{aligned}$$

Using the method of partial fractions, we find $\frac{1}{(x-2)(x-3)} = \frac{1}{x-3} - \frac{1}{x-2}$.

$$\begin{aligned} \int \left[\frac{1}{x-3} - \frac{1}{x-2}\right]\mathrm{d}x &= \int k\mathrm{d}t \\ \log|x-3| - \log|x-2| &= kt + C \\ \log\left|\frac{x-3}{x-2}\right| &= kt + C \\ \left|\frac{x-3}{x-2}\right| &= e^{kt+C} = e^{kt}e^{C} \\ \frac{x-3}{x-2} &= De^{kt} \end{aligned}$$

where $D = \pm e^C$. When $t = 0$, $x = 1$, forcing

$$\frac{1-3}{1-2} = De^0$$

$$D = 2$$

Hence

$$\begin{aligned}\frac{x-3}{x-2} &= 2e^{kt}\\ x-3 &= 2e^{kt}(x-2)\\ x-2e^{kt}x &= 3-4e^{kt}\\ x(t) &= \frac{3-4e^{kt}}{1-2e^{kt}}\end{aligned}$$

(b) To evaluate the limit, we could use l'Hôpital's rule, but we could also just multiply the numerator and denominator by e^{-kt}. Note $\lim\limits_{t\to\infty} e^{-tk} = 0$.

$$\begin{aligned}\lim_{t\to\infty} x(t) &= \lim_{t\to\infty} \underbrace{\frac{3-4e^{kt}}{1-2e^{kt}}}_{\substack{\text{num}\to-\infty\\ \text{den}\to-\infty}} = \lim_{t\to\infty} \frac{3-4e^{kt}}{1-2e^{kt}} \cdot \frac{e^{-kt}}{e^{-kt}}\\ &= \lim_{t\to\infty} \frac{3e^{-kt}-4}{e^{-kt}-2} = \frac{0-4}{0-2} = 2\end{aligned}$$

2.4.7.27. *. Solution. (a) The given differential equation is separable and we solve it accordingly.

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} &= 4P - P^2\\ \frac{\mathrm{d}P}{4P-P^2} &= \mathrm{d}t\\ \frac{\mathrm{d}P}{P(4-P)} &= \mathrm{d}t\end{aligned}$$

Using the method of partial fractions, we see $\frac{1}{P(4-P)} = \frac{1/4}{P} + \frac{1/4}{4-P}$.

$$\begin{aligned}\frac{1}{4}\left[\frac{1}{P} + \frac{1}{4-P}\right]\mathrm{d}P &= \mathrm{d}t\\ \int \frac{1}{4}\left[\frac{1}{P} + \frac{1}{4-P}\right]\mathrm{d}P &= \int \mathrm{d}t\\ \frac{1}{4}\Big[\log|P| - \log|4-P|\Big] &= t + C\end{aligned}$$

When $t = 0$, $P = 2$, so $\frac{1}{4}\big[\log|2| - \log|2|\big] = C \implies C = 0$. So,

$$\frac{1}{4}\log\left|\frac{P}{4-P}\right| = t$$

At time $t = 0$, $\frac{P}{4-P} = 1 > 0$. The ratio may not change sign at any finite time, because this could only happen if at some finite time P took either the value 0 or the value 4. But at this time $t = \frac{1}{4}\log\left|\frac{P}{4-P}\right|$ would have to be infinite. So $\frac{P}{4-P} > 0$ for all time and:

$$\frac{1}{4}\log\frac{P}{4-P} = t$$

$$\begin{aligned}
\log \frac{P}{4-P} &= 4t \\
\frac{P}{4-P} &= e^{4t} \\
P &= (4-P)e^{4t} \\
P + Pe^{4t} &= 4e^{4t} \\
P &= \frac{4e^{4t}}{1+e^{4t}} = \frac{4}{1+e^{-4t}}
\end{aligned}$$

(b) At $t = \frac{1}{2}$, $P = \frac{4}{1+e^{-2}} \approx 3.523$.

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{4}{1+e^{-4t}} = \frac{4}{1+0} = 4$$

2.4.7.28. *. Solution.

a The rate of change of speed at time t is $-kv(t)^2$ for some constant of proportionality k (to be determined–but we assume it is positive, since the speed is decreasing). So $v(t)$ obeys the differential equation $\frac{dv}{dt} = -kv^2$.

b The equation $\frac{dv}{dt} = -kv^2$ is a separable differential equation, which we can solve in the usual way.

$$\begin{aligned}
\frac{dv}{dt} &= -kv^2 \\
\frac{dv}{-v^2} &= k\,dt \\
\int -\frac{dv}{v^2} &= \int k\,dt \\
\frac{1}{v} &= kt + C
\end{aligned}$$

At time $t = 0$, $v = 400$, so $C = \frac{1}{400}$. Then:

$$\frac{1}{v} = kt + \frac{1}{400} \tag{$*$}$$

At time $t = 1$, $v = 200$, so

$$\begin{aligned}
\frac{1}{200} &= k + \frac{1}{400} \\
k &= \frac{1}{400}
\end{aligned}$$

Therefore, from $(*)$,

$$\begin{aligned}
\frac{1}{v} &= \frac{t}{400} + \frac{1}{400} = \frac{t+1}{400} \\
v &= \frac{400}{t+1}
\end{aligned}$$

c To find when the speed is 50, we set $v = 50$ in the equation from (b) and solve for t.

$$\begin{aligned} 50 &= \frac{400}{t+1} \\ 50(t+1) &= 400 \\ t+1 &= 8 \\ t &= 7 \end{aligned}$$

Exercises — Stage 3

2.4.7.29. *. Solution. (a) The given differential equation is separable and we solve it accordingly.

$$\begin{aligned} \frac{\mathrm{d}B}{\mathrm{d}t} &= (0.06 + 0.02\sin t)B \\ \frac{\mathrm{d}B}{B} &= (0.06 + 0.02\sin t)\,\mathrm{d}t \\ \int \frac{\mathrm{d}B}{B} &= \int (0.06 + 0.02\sin t)\,\mathrm{d}t \\ \log|B(t)| &= 0.06t - 0.02\cos t + C' \end{aligned}$$

Since $B(t)$ is our bank account balance and we're not withdrawing money, $B(t)$ is positive, so we can drop the absolute value signs.

$$\begin{aligned} \log B(t) &= 0.06t - 0.02\cos t + C' \\ B(t) &= e^{0.06t - 0.02\cos t}e^{C'} \\ B(t) &= Ce^{0.06t - 0.02\cos t} \end{aligned}$$

for arbitrary constants C' and $C = e^{C'} \geq 0$.
Remark: the function $B(t) = 0$ obeys the differential equation so that $C = 0$ is allowed, even though it is not of the form $C = e^{C'}$. This seeming discrepancy arose because, in our very first step of part (a), we divided both sides of the differential equation by B, which is only allowable if $B \neq 0$. So, in this step, we implicitly assumed B was nonzero.
(b) We are told that $B(0) = 1000$. This allows us to find C.

$$\begin{aligned} 1000 = B(0) &= Ce^{0 - 0.02\cos 0} = Ce^{-0.02} \\ C &= 1000e^{0.02} \end{aligned}$$

So, when $t = 2$,

$$B(2) = \underbrace{1000e^{0.02}}_{C}\, e^{0.06\times 2 - 0.02\cos 2} = \$1159.89$$

rounded to the nearest cent.
Note that $\cos 2$ is the cosine of 2 radians, $\cos 2 \approx -0.416$.

2.4.7.30. $*$. **Solution**. (a) The given differential equation is separable and we could solve it accordingly. In fact we have already done so. If we rewrite the equation in the form

$$\frac{\mathrm{d}B}{\mathrm{d}t} = a\Big(B - \frac{m}{a}\Big)$$

it is of the form covered by Theorem 2.4.4. So that theorem tells us that the solution is

$$B(t) = \Big(B(0) - \frac{m}{a}\Big)\, e^{at} + \frac{m}{a}$$

In this problem we are told that $a = 0.02 = \frac{1}{50}$, so

$$B(t) = \{B(0) - 50m\}\, e^{t/50} + 50m = \{30000 - 50m\}\, e^{t/50} + 50m$$

(b) The solution of part (a) is independent of time if and only if $30000 - 50m = 0$. So we need

$$m = \frac{30000}{50} = \$600$$

2.4.7.31. $*$. **Solution**. What we're given is an equation relating y to the integral of a function of y. What we know how to solve is an equation relating the *derivative* of y to a function of y. We can create this by differentiating the given integral equation. By the Fundamental Theorem of Calculus, part 1:

$$\begin{aligned} y'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_0^x \big(y(t)^2 - 3y(t) + 2\big)\sin t\,\mathrm{d}t\right\} \\ &= \big(y(x)^2 - 3y(x) + 2\big)\sin x \end{aligned}$$

So $y(x)$ satisfies the differential equation $y' = \big(y^2 - 3y + 2\big)\sin x = (y-2)(y-1)\sin x$ and the initial equation $y(0) = 3$ (just substitute $x = 0$ into $(*)$). For $y \neq 1, 2$:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= (y-2)(y-1)\sin x \\ \frac{\mathrm{d}y}{(y-2)(y-1)} &= \sin x\mathrm{d}x \\ \int \frac{\mathrm{d}y}{(y-2)(y-1)} &= \int \sin x\mathrm{d}x \end{aligned}$$

Using the method of partial fractions, we see $\frac{1}{(y-2)(y-1)} = \frac{1}{y-2} - \frac{1}{y-1}$.

$$\begin{aligned} \int \Big[\frac{1}{y-2} - \frac{1}{y-1}\Big]\mathrm{d}y &= \int \sin x\mathrm{d}x \\ \log|y-2| - \log|y-1| &= -\cos x + c \\ \log\left|\frac{y-2}{y-1}\right| &= -\cos x + c \end{aligned}$$

$$\left|\frac{y-2}{y-1}\right| = e^{c-\cos x}$$

The condition $y(0) = 3$ forces $\left|\frac{3-2}{3-1}\right| = e^{c-1}$ or $e^c = \frac{1}{2}e$, hence

$$\left|\frac{y-2}{y-1}\right| = \frac{1}{2}e^{1-\cos x}$$

Observe that, when $x = 0$, $\frac{y-2}{y-1} = \frac{1}{2} > 0$. Furthermore $\frac{1}{2}e^{1-\cos x}$, and hence $\left|\frac{y-2}{y-1}\right|$, can never take the value zero. As $y(x)$ varies continuously with x, $y(x)$ must remain larger than 2. Consquently, $\frac{y-2}{y-1}$ remains positive and we may drop the absolute value signs. Hence

$$\frac{y-2}{y-1} = \frac{1}{2}e^{1-\cos x}$$

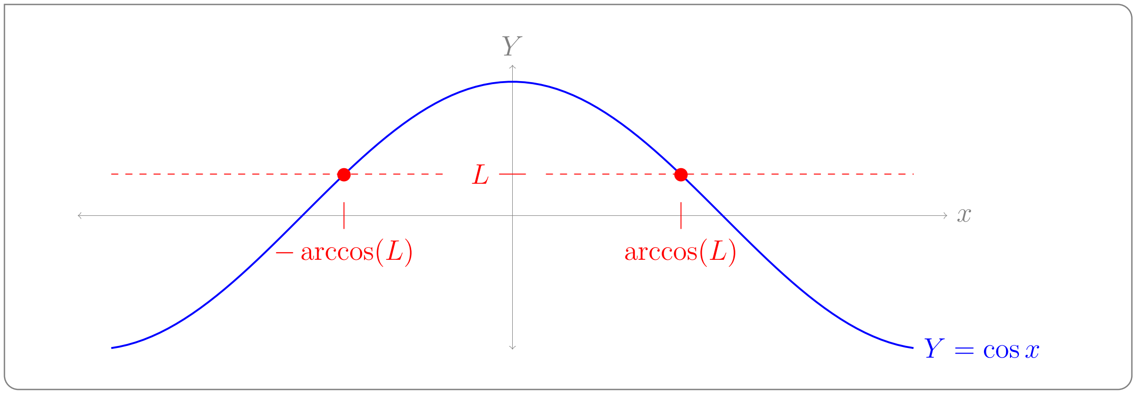
Solving for y,

$$\begin{aligned}
\frac{y-2}{y-1} &= \frac{1}{2}e^{1-\cos x}\\
2(y-2) &= e^{1-\cos x}(y-1)\\
2y - 4 &= ye^{1-\cos x} - e^{1-\cos x}\\
y\big(2 - e^{1-\cos x}\big) &= 4 - e^{1-\cos x}\\
y &= \frac{4 - e^{1-\cos x}}{2 - e^{1-\cos x}}
\end{aligned}$$

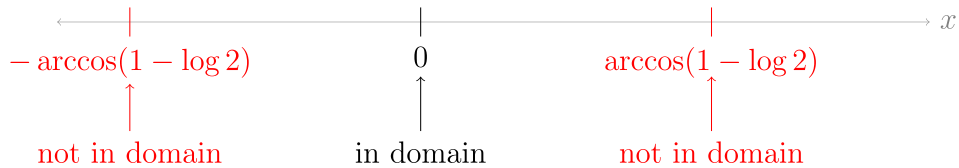
To avoid division by zero in the last step, we need

$$\begin{aligned}
e^{1-\cos x} &\neq 2\\
1 - \cos x &\neq \log 2\\
\cos x &\neq 1 - \log 2
\end{aligned}$$

Let $L = 1 - \log 2$, for brevity, and note that $L > 0$. (This can be seen by observing $2 < e$, so, $\log 2 < \log e = 1$, hence $1 - \log 2 > 0$.)



We know $x = 0$ is in the domain of our function, but the points $x = \pm\arccos(L) = \pm\arccos(1 - \log 2)$ are *not*.

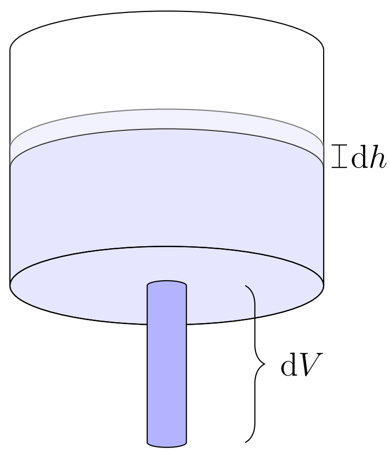


Therefore, the largest interval for which our answer makes sense is

$$-\arccos(1-\log 2)) > x > \arccos(1-\log 2)$$

or approximately $-1.259 < x < 1.259$.

2.4.7.32. *. Solution. Suppose that in a very short time interval $\mathrm{d}t$, the height of water in the tank changes by $\mathrm{d}h$ (which is negative). Then in this time interval the amount of the water in the tank decreases by $\mathrm{d}V = -\pi(3)^2\mathrm{d}h$. This must be the same as the amount of water that flows through the hole in this time interval. The water flowing through the hole makes a cylinder of radius 1 cm (that is, 0.01 m) with length $v(t)\mathrm{d}t$, the distance the water moves out of the hole in $\mathrm{d}t$ seconds. So, the amount of water leaving the hole over the time interval $\mathrm{d}t$ is $\pi(0.01)^2 v(t)\,\mathrm{d}t = \pi(0.01)^2\sqrt{2gh(t)}\,\mathrm{d}t$.



This gives us a separable differential equation. Recall g is a constant.

$$\begin{aligned}
-\pi(3)^2\mathrm{d}h &= \pi(0.01)^2\sqrt{2gh(t)}\,\mathrm{d}t\\
\frac{\mathrm{d}h}{\sqrt{h}} &= -\left(\frac{0.01}{3}\right)^2\sqrt{2g}\,\mathrm{d}t\\
\int\frac{\mathrm{d}h}{\sqrt{h}} &= \int -\left(\frac{0.01}{3}\right)^2\sqrt{2g}\,\mathrm{d}t\\
2\sqrt{h} &= -\left(\frac{0.01}{3}\right)^2\sqrt{2g}\,t + C
\end{aligned}$$

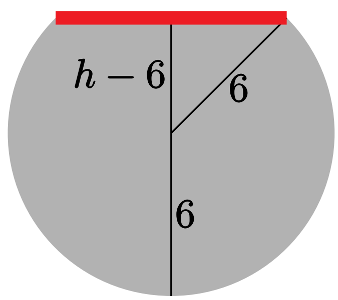
At time 0, the height is 6, so $C = 2\sqrt{6}$ and

$$2\sqrt{h} = -\left(\frac{0.01}{3}\right)^2\sqrt{2g}\,t + 2\sqrt{6}$$

We want to know when the height of the water in the tank is 0.

$$\begin{aligned}
0 &= -\left(\frac{0.01}{3}\right)^2\sqrt{2g}\,t + 2\sqrt{6}\\
\left(\frac{0.01}{3}\right)^2\sqrt{2g}\,t &= 2\sqrt{6}\\
t &= \frac{2\sqrt{6}}{\left(\frac{0.01}{3}\right)^2\sqrt{2g}}\\
&= 2\left(\frac{3}{0.01}\right)^2\sqrt{\frac{3}{g}}\\
&= 180,000\sqrt{\frac{3}{g}} \approx 99,591 \text{ sec} \approx 27.66 \text{ hr}
\end{aligned}$$

2.4.7.33. *. Solution. Suppose that at time t, the mercury in the tank has height h, which is between 0 and 12 feet.



At that time, the top surface of the mercury forms a circular disk of radius $\sqrt{6^2-(h-6)^2}$. (We found this by applying the Pythagorean Theorem to the triangle in the diagram above. In the diagram, h is shown as being larger than 6, but the same equation holds for all h in $[0, 12]$.) Now suppose that in a very short time interval $\mathrm{d}t$, the height of mercury in the tank changes by $\mathrm{d}h$ (which is negative). Then in this time interval the amount of the mercury in the tank decreases by $-\pi\left(\sqrt{6^2-(h-6)^2}\,\right)^2\mathrm{d}h$. (That's the volume of the red disk in the figure above.) This must be the same as the amount of mercury that flows through the hole in this time interval. The mercury comes out of the hole as a cylinder. Its radius is the radius of the hole, $\frac{1}{12}$ foot, and its length is the distance the mercury travels in $\mathrm{d}t$ seconds, $v(t)\mathrm{d}t$ feet. So, the volume of escaped mercury is $\pi\left(\frac{1}{12}\right)^2 v\,\mathrm{d}t = \pi\left(\frac{1}{12}\right)^2\sqrt{2gh}\,\mathrm{d}t$. This gives us a separable differential equation.

$$\begin{aligned}
-\pi\left(\sqrt{6^2-(h-6)^2}\,\right)^2\mathrm{d}h &= \pi\left(\frac{1}{12}\right)^2\sqrt{2gh}\,\mathrm{d}t\\
-\left(36-(h^2-12h+36)\,\right)\mathrm{d}h &= \left(\frac{1}{12}\right)^2\sqrt{2gh}\,\mathrm{d}t\\
\left(h^2-12h\right)\mathrm{d}h &= \frac{1}{144}\sqrt{2g}\sqrt{h}\,\mathrm{d}t\\
\left(h^{3/2}-12h^{1/2}\right)\mathrm{d}h &= \frac{1}{144}\sqrt{2g}\,\mathrm{d}t
\end{aligned}$$

$$\begin{aligned} \int \big(h^{3/2} - 12h^{1/2}\big)\mathrm{d}h &= \int \frac{1}{144}\sqrt{2g}\,\mathrm{d}t \\ \frac{h^{5/2}}{5/2} - 12\frac{h^{3/2}}{3/2} &= \frac{1}{144}\sqrt{2g}\,t + C \end{aligned}$$

At time 0, the height is 12, so $C = \frac{12^{5/2}}{5/2} - 12\frac{12^{3/2}}{3/2} = 12^{5/2}\big(\frac{2}{5} - \frac{2}{3}\big) = -\frac{4}{15}12^{5/2}$, which yields

$$\frac{h^{5/2}}{5/2} - 12\frac{h^{3/2}}{3/2} = \frac{1}{144}\sqrt{2g}\,t - \frac{4}{15}12^{5/2}$$

We want to find the time t when the height is $h = 0$.

$$\begin{aligned} 0 &= \frac{1}{144}\sqrt{2g}\,t - \frac{4}{15}12^{5/2} \\ \frac{1}{144}\sqrt{2g}\,t &= \frac{4}{15}12^{5/2} \\ t &= \frac{4 \times 144}{15}\sqrt{\frac{12^5}{2g}} \\ &= 38.4\sqrt{\frac{124416}{g}} \approx 2,394\,\mathrm{sec} \ \approx 0.665\,\mathrm{hr} \end{aligned}$$

2.4.7.34. *. Solution. (a) Setting $x = 0$ gives

$$f(0) = 3 + \int_0^0 \big(f(t) - 1\big)\big(f(t) - 2\big)\mathrm{d}t = 3$$

(b) By the Fundamental Theorem of Calculus part 1,

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \big(f(t) - 1\big)\big(f(t) - 2\big)\mathrm{d}t = \big(f(x) - 1\big)\big(f(x) - 2\big)$$

Thus $y = f(x)$ obeys the differential equation $y' = (y - 1)(y - 2)$.
(c) If $y \neq 1, 2$,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= (y - 1)(y - 2) \\ \frac{\mathrm{d}y}{(y - 1)(y - 2)} &= \mathrm{d}x \\ \int \frac{\mathrm{d}y}{(y - 1)(y - 2)} &= \int \mathrm{d}x \end{aligned}$$

Using the method of partial fractions,

$$\begin{aligned} \int \left(\frac{1}{y - 2} - \frac{1}{y - 1}\right)\mathrm{d}y &= \int \mathrm{d}x \\ \log|y - 2| - \log|y - 1| &= x + C \end{aligned}$$

$$\log\left|\frac{y-2}{y-1}\right| = x + C$$

Observe that $\frac{\mathrm{d}y}{\mathrm{d}x} = (y-1)(y-2) > 0$ for all $y \geq 2$. That is, $f(x)$ is increasing at all x for which $f(x) > 2$. As $f(0) = 3$, $f(x)$ increases for all $x \geq 0$, and $f(x) \geq 3$ for all $x \geq 0$. So we may drop the absolute value signs.

$$\begin{aligned} \log\frac{f(x)-2}{f(x)-1} &= x + C \\ \frac{f(x)-2}{f(x)-1} &= e^C e^x \end{aligned}$$

At $x = 0$, $\frac{f(x)-2}{f(x)-1} = \frac{1}{2}$ so $e^C = \frac{1}{2}$.

$$\begin{aligned} \frac{f(x)-2}{f(x)-1} &= \frac{1}{2}e^x \\ 2f(x) - 4 &= [f(x)-1]e^x \\ [2-e^x]f(x) &= 4 - e^x \\ f(x) &= \frac{4-e^x}{2-e^x} \end{aligned}$$

2.4.7.35. *. Solution. Suppose that at time t (measured in hours starting at, say, noon), the water in the tank has height y, which is between 0 and 2 metres. At that time, the top surface of the water forms a circular disk of radius $r = y^p$ and area $A(y) = \pi y^{2p}$. Thus, by Torricelli's law,

$$\begin{aligned} \pi y^{2p}\frac{\mathrm{d}y}{\mathrm{d}t} &= -c\sqrt{y} \\ -\frac{\pi}{c}\cdot y^{2p-\frac{1}{2}}\mathrm{d}y &= \mathrm{d}t \\ \int -\frac{\pi}{c}\cdot y^{2p-\frac{1}{2}}\mathrm{d}y &= \int \mathrm{d}t \\ -\frac{\pi}{c}\cdot\frac{y^{2p+\frac{1}{2}}}{2p+\frac{1}{2}} + d &= t \end{aligned}$$

for some constant d. At time $t = 0$, the height is $y = 2$, so $d = \dfrac{\pi}{c}\cdot\dfrac{2^{2p+\frac{1}{2}}}{2p+\frac{1}{2}}$.

$$\begin{aligned} t &= \frac{\pi}{c}\left(\frac{2^{2p+\frac{1}{2}}}{2p+\frac{1}{2}} - \frac{y^{2p+\frac{1}{2}}}{2p+\frac{1}{2}}\right) \\ &= \frac{\pi}{c(2p+\frac{1}{2})}\left(2^{2p+\frac{1}{2}} - y^{2p+\frac{1}{2}}\right) \end{aligned}$$

The time at which the height is 1 is obtained by subbing $y = 1$ into this formula. The time at which the height is 0 is obtained by subbing $y = 0$ into this formula. Thus the condition that the top half ($y = 2$ to $y = 1$) takes exactly the same amount of time to drain as the bottom half ($y = 1$ to $y = 0$) is:

$$\begin{aligned}
t(2) - t(1) &= t(1) - t(0)\\
0 - t(1) &= t(1) - t(0)\\
t(0) &= 2t(1)\\
\frac{\pi}{c(2p+\frac{1}{2})}\left(2^{2p+\frac{1}{2}} - 0^{2p+\frac{1}{2}}\right) &= 2\frac{\pi}{c(2p+\frac{1}{2})}\left(2^{2p+\frac{1}{2}} - 1^{2p+\frac{1}{2}}\right)\\
2^{2p+\frac{1}{2}} &= 2\left(2^{2p+\frac{1}{2}} - 1\right)\\
2^{2p+\frac{1}{2}} &= 2\cdot 2^{2p+\frac{1}{2}} - 2\\
2 &= 2^{2p+\frac{1}{2}}\\
1 &= 2p + \frac{1}{2}\\
p &= \frac{1}{4}
\end{aligned}$$

2.4.7.36. Solution.

a If we let $f(t) = 0$ for all t, then its average over any interval is 0, as is its root mean square.

b Let's start by simplifying the given equation.

$$\frac{1}{x-a}\int_a^x f(t)\,\mathrm{d}t = \sqrt{\frac{1}{x-a}\int_a^x f^2(t)\,\mathrm{d}t}$$

$$\frac{1}{\sqrt{x-a}}\int_a^x f(t)\,\mathrm{d}t = \sqrt{\int_a^x f^2(t)\,\mathrm{d}t} \tag{E1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{\sqrt{x-a}}\int_a^x f(t)\,\mathrm{d}t\right\} = \frac{\mathrm{d}}{\mathrm{d}x}\left\{\sqrt{\int_a^x f^2(t)\,\mathrm{d}t}\right\} \tag{E2}$$

For the derivative on the left, we use the product rule and the Fundamental Theorem of Calculus, part 1.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{\sqrt{x-a}}\int_a^x f(t)\,\mathrm{d}t\right\}\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{\sqrt{x-a}}\right\}\int_a^x f(t)\,\mathrm{d}t + \frac{1}{\sqrt{x-a}}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_a^x f(t)\,\mathrm{d}t\right\}\\
&= -\frac{1}{2\sqrt{x-a}^3}\int_a^x f(t)\,\mathrm{d}t + \frac{f(x)}{\sqrt{x-a}}\\
&= \frac{1}{\sqrt{x-a}}\left[f(x) - \frac{1}{2(x-a)}\int_a^x f(t)\,\mathrm{d}t\right]
\end{aligned}$$

For the derivative on the right in Equation (E2) we use the chain rule and the

Fundamental Theorem of Calculus part 1

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\sqrt{\int_a^x f^2(t)\,\mathrm{d}t}\right\} = \frac{1}{2}\left(\int_a^x f^2(t)\,\mathrm{d}t\right)^{-\frac{1}{2}} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_a^x f^2(t)\,\mathrm{d}t\right\}$$
$$= \frac{f^2(x)}{2\sqrt{\int_a^x f^2(t)\,\mathrm{d}t}}$$

So, Equation (E2) yields the following:

$$\frac{1}{\sqrt{x-a}}\left[f(x) - \frac{1}{2(x-a)}\int_a^x f(t)\,\mathrm{d}t\right] = \frac{f^2(x)}{2\sqrt{\int_a^x f^2(t)\,\mathrm{d}t}} \tag{E3}$$

c From Equation (E1), $\sqrt{\int_a^x f^2(t)\,\mathrm{d}t} = \frac{1}{\sqrt{x-a}}\int_a^x f(t)\,\mathrm{d}t$.

$$\frac{1}{\sqrt{x-a}}\left[f(x) - \frac{1}{2(x-a)}\int_a^x f(t)\,\mathrm{d}t\right] = \frac{f^2(x)}{2\frac{1}{\sqrt{x-a}}\int_a^x f(t)\,\mathrm{d}t}$$

and

$$\frac{2}{x-a}\int_a^x f(t)\,\mathrm{d}t\left[f(x) - \frac{1}{2(x-a)}\int_a^x f(t)\,\mathrm{d}t\right] = f^2(x)$$

d Now what we have is a differential equation, although it might not look like it. Let $Y(x) = \int_a^x f(t)\,\mathrm{d}t$. Then $\frac{\mathrm{d}Y}{\mathrm{d}x}(x) = f(x)$.

$$\frac{2}{x-a}Y\left[\frac{\mathrm{d}Y}{\mathrm{d}x} - \frac{1}{2(x-a)}Y\right] = \left(\frac{\mathrm{d}Y}{\mathrm{d}x}\right)^2 \tag{E4}$$

We're used to solving differential equations of the form $\frac{\mathrm{d}Y}{\mathrm{d}x}$ =(something). So, let's manipulate our equation until it has this form.

$$\left(\frac{\mathrm{d}Y}{\mathrm{d}x}\right)^2 - \left(\frac{2Y}{x-a}\right)\left(\frac{\mathrm{d}Y}{\mathrm{d}x}\right) + \left(\frac{Y}{x-a}\right)^2 = 0$$

This is a quadratic equation, with variable $\frac{\mathrm{d}Y}{\mathrm{d}x}$. Its solutions are:

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \frac{\left(\frac{2Y}{x-a}\right) \pm \sqrt{\left(\frac{2Y}{x-a}\right)^2 - 4\cdot\left(\frac{Y}{x-a}\right)^2}}{2}$$
$$= \frac{\frac{2Y}{x-a} \pm 0}{2}$$
$$= \frac{Y}{x-a}$$

This gives us the separable differential equation

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \frac{Y}{x-a}$$

$$\begin{aligned}
\frac{\mathrm{d}Y}{Y} &= \frac{\mathrm{d}x}{x-a} \qquad \text{(E5)}\\
\int \frac{\mathrm{d}Y}{Y} &= \int \frac{\mathrm{d}x}{x-a}\\
\log|Y| &= \log|x-a| + C\\
|Y| &= e^{\log|x-a|+C} = |x-a|e^C\\
Y &= D(x-a)
\end{aligned}$$

where D is some constant, e^C or $-e^C$. Note this covers all real constants except $D=0$. If $D=0$, then $Y(x)=0$ for all x. This function also satisfies Equation (E4), so indeed,

$$Y(x) = D(x-a) \qquad \text{(E6)}$$

for *any* constant D is the family of equations satisfying our differential equation.

Remark: the reason we "lost" the solution $Y(x)=0$ is that in Equation (E5), we divided by Y, thus tacitly assuming it was not identically 0.

e Remember $Y = \int_a^x f(t)\,\mathrm{d}t$. So, Equation (E6) tells us:

$$\begin{aligned}
\int_a^x f(t)\,\mathrm{d}t &= D(x-a)\\
\frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_a^x f(t)\,\mathrm{d}t\right\} &= \frac{\mathrm{d}}{\mathrm{d}x}\{D(x-a)\}\\
f(x) &= D
\end{aligned}$$

We should check that this function works.

$$\begin{aligned}
f_{\text{avg}} &= \frac{1}{x-a}\int_a^x D\,\mathrm{d}t = \frac{1}{x-a}\Big[Dt\Big]_{t=a}^{t=x} = \frac{Dx-Da}{x-a} = D\\
f_{\text{RMS}} &= \sqrt{\frac{1}{x-a}\int_a^x D^2\,\mathrm{d}t} = \sqrt{\frac{1}{x-a}\Big[D^2t\Big]_{t=a}^{t=x}}\\
&= \sqrt{\frac{D^2x-D^2a}{x-a}} = \sqrt{D^2} = |D|
\end{aligned}$$

So, $f(x)=D$ works only if D is nonnegative.

That is: the only functions whose average matches their root mean square over every interval are constant, nonnegative functions.

Remark: it was step (c) where we introduced the erroneous answer $f(x)=D$, $D<0$ to our solution. In Equation (E3), $f(x)=D$ is not a solution if $D<0$:

$$\frac{1}{\sqrt{x-a}}\left[f(x) - \frac{1}{2(x-a)}\int_a^x f(t)\,\mathrm{d}t\right] = \frac{f^2(x)}{2\sqrt{\int_a^x f^2(t)\,\mathrm{d}t}}$$

$$\begin{aligned}
\frac{1}{\sqrt{x-a}}\left[D-\frac{1}{2(x-a)}\int_a^x D\,\mathrm{d}t\right] &= \frac{D^2}{2\sqrt{\int_a^x D^2\,\mathrm{d}t}}\\
\frac{1}{\sqrt{x-a}}\left[D-\frac{1}{2(x-a)}D(x-a)\right] &= \frac{D^2}{2\sqrt{D^2(x-a)}}\\
\frac{1}{\sqrt{x-a}}\left[\frac{1}{2}D\right] &= \frac{D^2}{2|D|\sqrt{x-a}}\\
D &= \frac{D^2}{|D|} = |D|
\end{aligned}$$

In (c), we replace $\sqrt{\int_a^x f^2(t)\,\mathrm{d}t}$, which cannot be negative, with $\frac{1}{\sqrt{x-a}}\int_a^x f(t)\,\mathrm{d}t$, which could be negative if $f(t)=D<0$. Indeed, if $f(t)=D$, then $\sqrt{\int_a^x f^2(t)\,\mathrm{d}t} = |D|\sqrt{x-a}$, while $\frac{1}{\sqrt{x-a}}\int_a^x f(t)\,\mathrm{d}t = D\sqrt{x-a}$. It is at this point that negative functions creep into our solution.

2.4.7.37. Solution. We start by antidifferentiating both sides with respect to x.

$$\int\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)\mathrm{d}x = \int\left(\frac{2}{y^3}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\right)\mathrm{d}x$$

The right integral is in exactly the form we would use for a change of variables (substitution) to y.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \int\left(\frac{2}{y^3}\right)\mathrm{d}y = -\frac{1}{y^2}+C$$

When $y=1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=3$.

$$\begin{aligned}
3 &= -\frac{1}{1}+C\\
C &= 4
\end{aligned}$$

So,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{y^2}+4$$

This is a separable differential equation.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4y^2-1}{y^2}\\
\frac{y^2}{4y^2-1}\,\mathrm{d}y &= \mathrm{d}x\\
\int\frac{y^2}{4y^2-1}\,\mathrm{d}y &= \int\mathrm{d}x \qquad (*)
\end{aligned}$$

We can evaluate the left integral with partial fractions, but because the numerator has the same degree as the denominator, we have to simplify first. We do this by inspection, but you can also use long division.

$$\begin{aligned}
\frac{y^2}{4y^2-1} &= \frac{\frac{1}{4}(4y^2-1)+\frac{1}{4}}{4y^2-1} \\
&= \frac{1}{4}\left(1+\frac{1}{4y^2-1}\right) \\
&= \frac{1}{4}\left(1+\frac{1}{(2y-1)(2y+1)}\right) \\
&= \frac{1}{4}\left(1+\frac{1/2}{2y-1}-\frac{1/2}{2y+1}\right)
\end{aligned}$$

Now, we return to $(*)$.

$$\begin{aligned}
\int \mathrm{d}x &= \int \frac{y^2}{4y^2-1}\,\mathrm{d}y \\
&= \int \frac{1}{4}\left(1+\frac{1/2}{2y-1}-\frac{1/2}{2y+1}\right)\mathrm{d}y \\
&= \frac{1}{4}\left(y+\frac{1}{4}\log|2y-1|-\frac{1}{4}\log|2y+1|\right) \\
&= \frac{1}{4}\left(y+\frac{1}{4}\log\left|\frac{2y-1}{2y+1}\right|\right) \\
x+C &= \frac{1}{4}\left(y+\frac{1}{4}\log\left|\frac{2y-1}{2y+1}\right|\right)
\end{aligned}$$

When $x=-\frac{1}{16}\log 3$, $y=1$.

$$\begin{aligned}
-\frac{1}{16}\log 3+C &= \frac{1}{4}\left(1+\frac{1}{4}\log\left|\frac{2-1}{2+1}\right|\right) = \frac{1}{4}+\frac{1}{16}\log\frac{1}{3} \\
C &= \frac{1}{4}
\end{aligned}$$

So,

$$\begin{aligned}
x+\frac{1}{4} &= \frac{1}{4}\left(y+\frac{1}{4}\log\left|\frac{2y-1}{2y+1}\right|\right) \\
x &= \frac{1}{4}\left(y-1+\frac{1}{4}\log\left|\frac{2y-1}{2y+1}\right|\right)
\end{aligned}$$

We can check our answer by differentiating with respect to x.

$$\begin{aligned}
x &= \frac{1}{4}\left(y-1+\frac{1}{4}\log\left|\frac{2y-1}{2y+1}\right|\right) \\
4x &= y-1+\frac{1}{4}\log|2y-1|-\frac{1}{4}\log|2y+1|
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\{4x\} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{y - 1 + \frac{1}{4}\log|2y-1| - \frac{1}{4}\log|2y+1|\right\} \\
4 &= \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{4}\cdot\frac{2\frac{\mathrm{d}y}{\mathrm{d}x}}{2y-1} - \frac{1}{4}\cdot\frac{2\frac{\mathrm{d}y}{\mathrm{d}x}}{2y+1} \\
4 &= \frac{\mathrm{d}y}{\mathrm{d}x}\left(1 + \frac{1/2}{2y-1} - \frac{1/2}{2y+1}\right) = \frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{4y^2}{4y^2-1}\right) \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4y^2-1}{y^2} = 4 - \frac{1}{y^2} \qquad (**)
\end{aligned}$$

Differentiating with respect to x again, using the chain rule,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{2}{y^3}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}$$

This is exactly the differential equation we were meant to solve.

3 · Sequence and series

3.1 · Sequences

3.1.2 · Exercises

Exercises — Stage 1

3.1.2.1. Solution. (a) The values of the sequence seem to be getting closer and closer to -2, so we guess the limit of this sequence is -2.
(b) Overall, the values of the sequence seem to be getting extremely close to 0, so we approximate the limit of this sequence as 0. It doesn't matter that the sequence changes signs, or that the numbers are sometimes farther from 0, sometimes closer.
(c) This limit does not exist. The sequence is sometimes 0, sometimes -2, and not consistently staying extremely near to either one.

3.1.2.2. Solution. True. We consider the end behaviour of the sequences, which does not depend on any finite number of terms at their beginning.

3.1.2.3. Solution. (a) We follow the arithmetic of limits, Theorem 3.1.8: $\dfrac{A-B}{C}$
(b) Since $\lim\limits_{n\to\infty} c_n$ is some real number, and n grows without bound, $\lim\limits_{n\to\infty}\dfrac{c_n}{n} = 0$.
(c) We note $\lim\limits_{n\to\infty} a_{2n+5} = \lim\limits_{n\to\infty} a_n$, so $\dfrac{a_{2n+5}}{b_n} = \dfrac{A}{B}$.

3.1.2.4. Solution. There are many possible answers. One is:

$$a_n = \begin{cases} 3000 - n & \text{if } n \le 1000 \\ -2 + \frac{1}{n} & \text{if } n > 1000 \end{cases}$$

where we have a series that looks different before and after its thousandth term.

Note every term is smaller than the term preceding it.
Another sequence with the desired properties is:

$$a_n = \frac{1,002,001}{n} - 2$$

When $n \leq 1000$, $a_n \geq \frac{1{,}002{,}001}{1000} - 2 > \frac{1{,}002{,}000}{1{,}000} - 2 = 1000$. That is, $a_n > 1000$ when $n \leq 1000$. As n gets larger, a_n gets smaller, so $a_{n+1} < a_n$ for all n. Finally, $\lim\limits_{n\to\infty} a_n = 0 - 2 = -2$.

3.1.2.5. Solution. One possible answer is $a_n = (-1)^n = \{-1, 1, -1, 1, -1, 1, -1, \ldots\}$.
Another is $a_n = n(-1)^n = \{-1, 2, -3, 4, -5, 6, -7, \ldots\}$.

3.1.2.6. Solution. If the terms of a sequence are alternating sign, but the limit of the sequence exists, the limit must be zero. (If it were a positive number, the negative terms would not get very close to it; if it were a negative number, the positive terms would not get very close to it.)
This gives us the idea to modify an answer from Question 5. One possible sequence:

$$a_n = \frac{(-1)^n}{n} = \left\{-1,\ \frac{1}{2},\ -\frac{1}{3},\ \frac{1}{4},\ -\frac{1}{5},\ \frac{1}{6},\ \ldots\right\}$$

3.1.2.7. Solution.

a Since $-1 \leq \sin n \leq 1$ for all n, one potential set of upper and lower bound is

$$\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$$

Note $\lim\limits_{n\to\infty} \frac{-1}{n} = \lim\limits_{n\to\infty} \frac{1}{n}$, so these are valid comparison sequences for the squeeze theorem.

b Since $-1 \leq \sin n \leq 1$ and $-5 \leq -5\cos n \leq 5$ for all n, we see

$$\begin{aligned} 7-1-5 &\leq 7 + \sin n - 5\cos n \leq 7 + 1 + 5 \\ 1 &\leq 7 + \sin n - 5\cos n \leq 13 \end{aligned}$$

This gives us the idea to try the bounds

$$\frac{n^2}{13e^n} \leq \frac{n^2}{e^n(7 + \sin n - 5\cos n)} \leq \frac{n^2}{e^n}$$

We check that $\lim\limits_{n\to\infty} \frac{n^2}{13e^n} = \lim\limits_{n\to\infty} \frac{n^2}{e^n}$ (they're both 0 — you can verify using l'Hôpital's rule), so these are indeed reasonable bounds to choose to use with the squeeze theorem.

c Since $(-n)^{-n} = \frac{1}{(-n)^n} = \frac{(-1)^n}{n^n}$, we see

$$\frac{-1}{n^n} \leq (-n)^{-n} \leq \frac{1}{n^n}$$

Since both $\lim_{n\to\infty} \frac{-1}{n^n}$ and $\lim_{n\to\infty} \frac{1}{n^n}$ are 0, these are reasonable bounds to use with the squeeze theorem.

3.1.2.8. Solution.

a
- Note $a_n = b_n$, since (in the absence of evidence to the contrary) we assume n begins at one, hence $n = |n|$. Then $a_n = b_n = 1 + \frac{1}{n} = \frac{n+1}{n}$. So, whenever n is a whole number, a_n and b_n are the same as $h(n)$ and $i(n)$. (Be careful here: $h(x) \neq i(x)$ when x is not a whole number.)
- $c_n = e^{-n} = \frac{1}{e^n} = j(n)$
- For any integer n, $\cos(\pi n) = (-1)^n$. So, $d_n = f(n)$.
- Similarly, $e_n = g(n)$.

b According to Theorem 3.1.6, if any of the functions on the right have limits that exist as $x \to \infty$, then these limits match the limits of their corresponding sequences. So, we only have to be suspicious of $f(x)$ and $i(x)$, since these do not converge.

The limit $\lim_{x\to\infty} f(x)$ does not exist, and $f(n) = d_n$; the limit $\lim_{n\to\infty} d_n$ also does not exist. (We generally don't write equality for two things that don't exist: equality refers to numerical value, and these have none. [a])

The limit $\lim_{x\to\infty} i(x)$ does not exist, because $i(x) = 0$ when x is not a whole number, while $i(x)$ approaches 1 when x is a whole number. However, $\lim \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 1$.

So, using our answers from part (a), we match the following:

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{x\to\infty} h(x) = 1$
- $\lim_{n\to\infty} c_n = \lim_{n\to\infty} e_n = \lim_{x\to\infty} g(x) = \lim_{x\to\infty} j(x) = 0$
- $\lim_{n\to\infty} d_n$, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} i(x)$ do not exist.

[a] The idea "two things that both don't exist are equal" is also rejected because it can lead to contradictions. For example, in the real numbers $\sqrt{-1}$ and $\sqrt{-2}$ don't exist; if we write $\sqrt{-1} = \sqrt{-2}$, then squaring both sides yields the inanity $-1 = -2$.

3.1.2.9. Solution. (a) We want to find odd multiples of π that are close to integers.

- Solution 1: One way to do that is to remember that π is somewhat close to $\frac{22}{7}$. Then when we multiply π by a multiple of 7, we should get something close to an integer. In particular, 7π, 21π, and 35π should be reasonably close to $7\left(\frac{22}{7}\right) = 22$, $21\left(\frac{22}{7}\right) = 66$, and $35\left(\frac{22}{7}\right) = 110$, respectively. We check whether they are close enough:

$$7\pi \approx 21.99 \qquad 21\pi \approx 65.97 \qquad 35\pi \approx 109.96$$

So indeed, 22, 66, and 110 are all within 0.1 of some odd multiple of π.

Since the cosine of an odd multiple of π is -1, we expect all of the sequence values to be close to -1. Using a calculator:

$$\begin{aligned} a_{22} &= \cos(22) \approx -0.99996, \\ a_{66} &= \cos(66) \approx -0.99965, \\ a_{110} &= \cos(110) \approx -0.99902 \end{aligned}$$

- Solution 2: Alternately, we could have just listed odd multiple of π until we found three that are close to integers.

$\mathbf{2k+1}$	$(\mathbf{2k+1})\pi$
1	3.14
3	9.42
5	15.71
7	21.99
9	28.27
11	34.56
13	40.84
15	47.12
17	53.41
19	59.69
21	65.97
23	72.26
25	78.54
27	84.82
29	91.11
31	97.39
33	103.67
35	109.96

Some earlier odd multiples of π (like 15π and 29π) get fairly close to integers, but not within 0.1.

(b) If $x = \dfrac{2k+1}{2}\pi$ for some integer k (that is, x is an odd multiple of $\pi/2$), then $\cos x = 0$. So, we can either list out the first few terms of a_n until we find three that are very close to 0, or we can use our approximation $\pi \approx \dfrac{22}{7}$ to choose values of n that are close to $\dfrac{2k+1}{2}\pi$.

- Solution 1:

$$\frac{2k+1}{2}\pi \approx \frac{(2k+1)\times 22}{2\times 7} = 11\frac{2k+1}{7}$$

So, we expect our values to be close to integers when $2k+1$ is a multiple of 7. For example, $2k+1=7$, $2k+1=21$, and $2k+1=35$.

We check:

x	**n**	$\mathbf{a_n}$
$7\times\frac{\pi}{2}\approx 10.99557$	11	$a_{11}\approx 0.0044$
$21\times\frac{\pi}{2}\approx 32.98672$	33	$a_{33}\approx -0.0133$
$35\times\frac{\pi}{2}\approx 54.97787$	55	$a_{55}\approx 0.0221$

These seem like values of a_n that are all pretty close to 0.

- Solution 2: We could have listed the first several values of a_n, and looked for some that are close to 0.

n	$\mathbf{a_n}$
1	0.54
2	-0.42
3	-0.99
4	-0.65
5	0.28
6	0.96
7	0.75
8	-0.15
9	-0.91
10	-0.84

Oof. Nothing very close yet. Maybe a better way is to list values of $\frac{2k+1}{2}\pi$, and see which ones are close to integers.

$\mathbf{2k+1}$	$\frac{\mathbf{2k+1}}{\mathbf{2}}\pi$
1	1.57
3	4.71
5	7.85
7	10.996
9	14.14
11	17.28
13	20.42
15	23.56
17	26.70
19	29.85
21	32.99
23	36.13
25	39.27
27	42.41
29	45.55
31	48.69
33	51.84
35	54.98

We find roughly the same candidates we did in Solution 1, depending on what we're ready to accept as "close".

Remark: it is possible to turn the ideas of this question into a rigorous proof that $\lim\limits_{n\to\infty} \cos n$ is undefined.

- Let, for each integer $k \ge 1$, n_k be the integer that is closest to $2k\pi$. Then $2k\pi - \frac{1}{2} \le n_k \le 2k\pi + \frac{1}{2}$ so that $\cos(n_k) \ge \cos\frac{1}{2} \ge 0.8$. Consequently, if $\lim\limits_{n\to\infty} \cos n = c$ exists, we must have $c \ge 0.8$.
- Let, for each integer $k \ge 1$, n'_k be the integer that is closest to $(2k+1)\pi$. Then $(2k+1)\pi - \frac{1}{2} \le n'_k \le (2k+1)\pi + \frac{1}{2}$ so that $\cos(n'_k) \le -\cos\frac{1}{2} \le -0.8$. Consequently, if $\lim\limits_{n\to\infty} \cos n = c$ exists, we must have $c \le -0.8$.
- It is impossible to have both $c \ge 0.8$ and $c \le -0.8$, so $\lim\limits_{n\to\infty} \cos n$ does not exist.

Exercises — Stage 2

3.1.2.10. Solution. When determining the end behaviour of rational functions, recall from last semester that we can either cancel out the highest power of n from the numerator and denominator, or skip this step and compare the highest powers of the numerator and denominator.

a Since the numerator has a higher degree than the denominator, this sequence will diverge to positive or negative infinity; since its terms are positive for large n, its limit is (positive) infinity. (You can imagine that the numerator is growing much, much faster than the denominator, leading the terms to have a very, very large absolute value.)

Calculating the longer way:

$$a_n = \frac{3n^2-2n+5}{4n+3}\left(\frac{\frac{1}{n}}{\frac{1}{n}}\right) = \frac{3n-2+\frac{5}{n}}{4+\frac{3}{n}}$$
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n-2+\frac{5}{n}}{4+\frac{3}{n}} = \lim_{n\to\infty} \frac{3n-2+0}{4+0} = \infty$$

b Since the numerator has the same degree as the denominator, as n goes to infinity, this sequence will converge to the ratio of their leading coefficients: $\frac{3}{4}$. (You can imagine that the numerator is growing at roughly the same rate as the denominator, so the terms settle into an almost-constant ratio.)

Calculating the longer way:

$$b_n = \frac{3n^2-2n+5}{4n^2+3}\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right) = \frac{3-\frac{2}{n}+\frac{5}{n^2}}{4+\frac{3}{n^2}}$$
$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3-\frac{2}{n}+\frac{5}{n^2}}{4+\frac{3}{n^2}} = \frac{3-0+0}{4+0} = \frac{3}{4}$$

c Since the numerator has a lower degree than the denominator, this sequence will converge to 0 as n goes to infinity. (You can imagine that the denominator is growing much, much faster than the numerator, leading the terms to be very, very small.)

Calculating the longer way:

$$c_n = \frac{3n^2-2n+5}{4n^3+3}\left(\frac{\frac{1}{n^3}}{\frac{1}{n^3}}\right) = \frac{\frac{3}{n}-\frac{2}{n^2}+\frac{5}{n^3}}{4+\frac{3}{n^3}}$$
$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{\frac{3}{n}-\frac{2}{n^2}+\frac{5}{n^3}}{4+\frac{3}{n^3}} = \frac{0-0+0}{4+0} = 0$$

3.1.2.11. Solution. At first glance, we see both the numerator and denominator grow huge as n increases, so we'll need to think a little further to find the limit. We don't have a rational function, but we can still divide the top and bottom by n^e to get a clearer picture.

$$a_n = \frac{4n^3-21}{n^e+\frac{1}{n}}\left(\frac{\frac{1}{n^e}}{\frac{1}{n^e}}\right) = \frac{4n^{3-e}-\frac{21}{n^e}}{1+\frac{1}{n^{e+1}}}$$

Since $e < 3$, we see $3 - e$ is positive, so $\lim_{n\to\infty} n^{3-e} = \infty$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{4n^{3-e} - \frac{21}{n^e}}{1 + \frac{1}{n^{e+1}}} = \lim_{n\to\infty} \frac{4n^{3-e} - 0}{1+0} = \infty$$

3.1.2.12. Solution. This isn't a rational sequence, but factoring out $\sqrt{n}$ from the top and bottom will still clear things up.

$$b_n = \frac{\sqrt[4]{n}+1}{\sqrt{9n+3}}\left(\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}}\right) = \frac{\frac{1}{\sqrt[4]{n}} + \frac{1}{\sqrt{n}}}{\sqrt{9+\frac{3}{n}}}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\frac{1}{\sqrt[4]{n}} + \frac{1}{\sqrt{n}}}{\sqrt{9+\frac{3}{n}}} = \frac{0+0}{\sqrt{9+0}} = 0$$

3.1.2.13. Solution. First, let's start with a tempting fallacy.

> The denominator grows without bound, so $\lim_{n\to\infty} \frac{\cos(n+n^2)}{n} = 0$.

It's certainly true that *if the limit of the numerator is a real number*, and the denominator grows without bound, then the limit of the sequence is zero. However, in our case, the limit of the numerator does not exist. To apply the limit arithmetic rules from Theorem 3.1.8, our limits must actually exist.
A better reasoning looks something like this:

> The denominator grows without bound, and the numerator never gets very large, so $\lim_{n\to\infty} \frac{\cos(n+n^2)}{n} = 0$.

To quantify this reasoning more precisely, we use the squeeze theorem, Theorem 3.1.10. There are two parts to the squeeze theorem: finding two bounding functions, and making sure these functions have the same limit.

- Since $-1 \le \cos(n+n^2) \le 1$ for all n, we choose functions $a_n = \frac{-1}{n}$ and $b_n = \frac{1}{n}$. Then $a_n \le c_n \le b_n$ for all n.
- Both $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 0$.

So, by the squeeze theorem, $\lim_{n\to\infty} \frac{\cos(n+n^2)}{n} = 0$.

3.1.2.14. Solution. The denominator of this sequence grows without bound. The numerator is unpredictable: imagine that n is large. When $\sin n$ is close to -1, $n^{\sin n}$ puts a power of n "in the denominator," so we can have $n^{\sin n}$ very close to 0. When $\sin n$ is close to 1, $n^{\sin n}$ is close to n, which is large.
To control for these variations, we'll use the squeeze theorem.

- Since $-1 \le \sin n \le 1$ for all n, let $b_n = \frac{n^{-1}}{n^2} = \frac{1}{n^3}$ and $c_n = \frac{n}{n^2} = \frac{1}{n}$. Then $b_n \le a_n \le c_n$.

- Both $\lim\limits_{n\to\infty} b_n = 0$ and $\lim\limits_{n\to\infty} c_n = 0$.

So, by the squeeze theorem, $\lim\limits_{n\to\infty} \dfrac{n^{\sin n}}{n^2} = 0$ as well.
Remark: we also could have used $b_n = 0$ for our lower bound, since $a_n \geq 0$ for all n.

3.1.2.15. Solution.

$$d_n = e^{-1/n} = \frac{1}{e^{1/n}}$$
$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \frac{1}{e^{1/n}} = \frac{1}{e^0} = \frac{1}{1} = 1$$

3.1.2.16. Solution.

- Solution 1: Let's use the squeeze theorem. Since $\sin(n^2)$ and $\sin n$ are both between -1 and 1 for all n, we note:

$$1 + 3(-1) - 2(1) \leq 1 + 3\sin(n^2) - 2\sin n \leq 1 + 3(1) - 2(-1)$$
$$-4 \leq 1 + 3\sin(n^2) - 2\sin n \leq 6$$

  This allows us to choose suitable bounding functions for the squeeze theorem.

  - Let $b_n = -\dfrac{4}{n}$ and $c_n = \dfrac{6}{n}$. From the work above, we see $b_n \leq a_n \leq c_n$ for all n.
  - Both $\lim\limits_{n\to\infty} b_n = 0$ and $\lim\limits_{n\to\infty} c_n = 0$.

  So, by the squeeze theorem, $\lim\limits_{n\to\infty} \dfrac{1 + 3\sin(n^2) - 2\sin n}{n} = 0$.

- Solution 2: We simplify slightly to begin.

$$a_n = \frac{1 + 3\sin(n^2) - 2\sin n}{n} = \frac{1}{n} + 3 \cdot \frac{\sin(n^2)}{n} - 2 \cdot \frac{\sin n}{n}$$

  We apply the squeeze theorem to the pieces $\dfrac{\sin(n^2)}{n}$ and $\dfrac{\sin n}{n}$.

  - Let $b_n = \dfrac{-1}{n}$ and $c_n = \dfrac{1}{n}$. Then $b_n \leq \dfrac{\sin(n^2)}{n} \leq c_n$, and $b_n \leq \dfrac{\sin n}{n} \leq c_n$.
  - Both $\lim\limits_{n\to\infty} b_n = 0$ and $\lim\limits_{n\to\infty} c_n = 0$.

  So, by the squeeze theorem, $\lim\limits_{n\to\infty} \dfrac{\sin(n^2)}{n} = 0$ and $\lim\limits_{n\to\infty} \dfrac{\sin n}{n} = 0$.
  Now, using the arithmetic of limits from Theorem 3.1.8,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left[\frac{1}{n} + 3 \cdot \frac{\sin(n^2)}{n} - 2 \cdot \frac{\sin n}{n}\right]$$
$$= 0 + 3 \cdot 0 - 2 \cdot 0 = 0$$

3.1.2.17. Solution. First, we note that both numerator and denominator grow without bound. So, we have to decide whether one outstrips the other, or whether they reach a stable ratio.

- Solution 1: Let's try dividing the numerator and denominator by 2^n (the dominant term in the denominator; this is the same idea behind factoring out the leading term in rational expressions).

$$b_n = \frac{e^n}{2^n+n^2}\left(\frac{\frac{1}{2^n}}{\frac{1}{2^n}}\right) = \frac{\left(\frac{e}{2}\right)^n}{1+\frac{n^2}{2^n}}$$

Since $e > 2$, we see $\dfrac{e}{2} > 1$, and so $\displaystyle\lim_{n\to\infty}\left(\frac{e}{2}\right)^n = \infty$. Since exponential functions grow much, much faster than polynomial functions, we also see $\lim_{n\to\infty}\frac{n^2}{2^n} = 0$. So,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{\left(\frac{e}{2}\right)^n}{1+\frac{n^2}{2^n}} = \lim_{n\to\infty}\frac{\left(\frac{e}{2}\right)^n}{1+0} = \infty$$

- Solution 2: Since the numerator and denominator both increase without bound, we apply l'Hôpital's rule. Recall $\frac{\mathrm{d}}{\mathrm{d}x}\{2^x\} = 2^x\log 2$.

$$\begin{aligned}\lim_{n\to\infty} b_n &= \lim_{n\to\infty}\frac{e^n}{\underbrace{2^n+n^2}_{\substack{\text{num}\to\infty\\ \text{den}\to\infty}}}\\ &= \lim_{n\to\infty}\frac{e^n}{\underbrace{2^n\log 2+2n}_{\substack{\text{num}\to\infty\\ \text{den}\to\infty}}}\\ &= \lim_{n\to\infty}\frac{e^n}{\underbrace{2^n(\log 2)^2+2}_{\substack{\text{num}\to\infty\\ \text{den}\to\infty}}}\\ &= \lim_{n\to\infty}\frac{e^n}{2^n(\log 2)^3}\\ &= \frac{1}{(\log 2)^3}\lim_{n\to\infty}\left(\frac{e}{2}\right)^n\\ &= \infty\end{aligned}$$

Since $e > 2$, we see $\dfrac{e}{2} > 1$, and so $\displaystyle\lim_{n\to\infty}\left(\frac{e}{2}\right)^n = \infty$.

3.1.2.18. *. Solution. First, we simplify. Remember $n! = n(n-1)(n-2)\cdots(2)(1)$ for any whole number n, so $(k+1)! = (k+1)k!$.

$$a_k = \frac{k!\sin^3 k}{(k+1)!} = \frac{k!\sin^3 k}{(k+1)k!} = \frac{\sin^3 k}{k+1}$$

Now, we can use the squeeze theorem.

- $-1 \le \sin k \le 1$ for all k, so $-1 \le \sin^3 k \le 1$. Let $b_k = \frac{-1}{k+1}$ and $c_k = \frac{1}{k+1}$. Then $b_k \le a_k \le c_k$.
- Both $\lim_{k\to\infty} b_k = 0$ and $\lim_{k\to\infty} c_k = 0$.

So, by the squeeze theorem, also $\lim_{k\to\infty} a_k = 0$.

3.1.2.19. *. Solution. Note $\lim_{n\to\infty}(-1)^n$ doesn't exist, but $-1 \le (-1)^n \le 1$ for all n. Let's use the squeeze theorem.

- Let $a_n = -\sin\left(\frac{1}{n}\right)$ and $b_n = \sin\left(\frac{1}{n}\right)$. Then $a_n \le (-1)^n \sin\left(\frac{1}{n}\right) \le b_n$.
- Both $\lim_{n\to\infty} -\sin\left(\frac{1}{n}\right) = 0$ and $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$, since $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\sin 0 = 0$.

By the squeeze theorem, the sequence $\left\{(-1)^n \sin\frac{1}{n}\right\}$ converges to 0.

3.1.2.20. *. Solution. First, we note that $\lim_{n\to\infty} \dfrac{6n^2+5n}{n^2+1} = 6$. We see this either by comparing the leading terms in the numerator and denominator, or by factoring out n^2 from the top and the bottom.
Second, since $\lim_{n\to\infty} \dfrac{1}{n^2} = 0$, we see $\lim_{n\to\infty} \cos\left(\dfrac{1}{n^2}\right) = \cos 0 = 1$.
Using arithmetic of limits, Theorem 3.1.8, we conclude

$$\lim_{n\to\infty}\left[\frac{6n^2+5n}{n^2+1} + 3\cos(1/n^2)\right] = 6 + 3(1) = 9.$$

Exercises — Stage 3

3.1.2.21. *. Solution. Let's take stock: $\sin(1/n) \to \sin(0) = 0$ as $n \to \infty$, so $\log(\sin(1/n)) \to -\infty$. However, $\log(2n) \to \infty$. So, we have some tension here: the two pieces behave in ways that pull the terms of the sequence in different directions. (Recall we *cannot* conclude anything like "$-\infty + \infty = 0$.")
We try using logarithm rules to get a clearer picture.

$$\log\left(\sin\frac{1}{n}\right) + \log(2n) = \log\left(2n\sin\left(\frac{1}{n}\right)\right)$$

Still, we have indeterminate behaviour: $2n\sin(1/n)$ is the product of $2n$, which grows without bound, and $\sin(1/n)$, which approaches zero. In the past, we learned that we can handle the indeterminate form $0\cdot\infty$ with l'Hôpital's rule (after a little algebra), but there's a slicker way. Note $1/n \to 0$ as $n \to \infty$. If we write $\frac{1}{n} = x$, then this piece of our limit resembles something familiar.

$$2n\sin\left(\frac{1}{n}\right) = 2\left(\frac{\sin x}{x}\right)$$

If $n \to \infty$, then $x = \frac{1}{n} \to 0$.

$$\lim_{n\to\infty} 2n \sin\left(\frac{1}{n}\right) = 2 \lim_{x\to 0} \frac{\sin x}{x}$$

That limit is familiar:

$$= 2(1) = 2$$

Then:

$$\lim_{n\to\infty} \log\left(2n\sin\left(\frac{1}{n}\right)\right) = \log 2$$

Note: if you have forgotten that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, you can also evaluate this limit using l'Hôpital's rule:

$$\underbrace{\lim_{x\to 0} \frac{\sin x}{x}}_{\substack{\text{num}\to 0\\ \text{den}\to 0}} = \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1$$

3.1.2.22. Solution. First, although this sequence is not defined for some small values of n, it is defined as long as $n \geq 5$, so it's not a problem to take the limit as $n \to \infty$. Second, we notice that our limit has the indeterminate form $\infty - \infty$. Since this form is indeterminate, more work is needed to find our limit, if it exists. A standard trick we saw last semester with functions of this form was to multiply and divide by the conjugate of the expression, $\sqrt{n^2+5n} + \sqrt{n^2-5n}$. Then the denominator will be the *sum* of two similar things, rather than their difference. See the work below to find out why that is helpful.

$$\begin{aligned}
&\sqrt{n^2+5n} - \sqrt{n^2-5n} \\
&\quad = \left(\sqrt{n^2+5n} - \sqrt{n^2-5n}\right)\left(\frac{\sqrt{n^2+5n} + \sqrt{n^2-5n}}{\sqrt{n^2+5n} + \sqrt{n^2-5n}}\right) \\
&\quad = \frac{(n^2+5n) - (n^2-5n)}{\sqrt{n^2+5n} + \sqrt{n^2-5n}} \\
&\quad = \frac{10n}{\sqrt{n^2+5n} + \sqrt{n^2-5n}}
\end{aligned}$$

Now, we'll cancel out n from the top and the bottom. Note $n = \sqrt{n^2}$.

$$\begin{aligned}
&= \frac{10n}{\sqrt{n^2+5n} + \sqrt{n^2-5n}}\left(\frac{\frac{1}{n}}{\frac{1}{n}}\right) \\
&= \frac{10n}{\sqrt{n^2+5n} + \sqrt{n^2-5n}}\left(\frac{\frac{1}{n}}{\frac{1}{\sqrt{n^2}}}\right) \\
&= \frac{10}{\sqrt{1+\frac{5}{n}} + \sqrt{1-\frac{5}{n}}}
\end{aligned}$$

Now, the limit is clear.

$$\lim_{n\to\infty} \frac{10}{\sqrt{1+\frac{5}{n}}+\sqrt{1-\frac{5}{n}}} = \frac{10}{\sqrt{1+0}+\sqrt{1+0}} = \frac{10}{1+1} = 5$$

3.1.2.23. Solution. First, although this sequence is not defined for some small values of n, it is defined as long as $n \geq \sqrt{2.5}$, so it's not a problem to take the limit as $n \to \infty$. Second, we notice that our limit has the indeterminate form $\infty - \infty$. Since this form is indeterminate, more work is needed to find our limit, if it exists. In Question 22, we saw a similar limit, and made use of the conjugate. However, in this case, there's an easier path: let's factor out n from each term.

$$\begin{aligned}\sqrt{n^2+5n}-\sqrt{2n^2-5} &= \sqrt{n^2\left(1+\frac{5}{n}\right)}-\sqrt{n^2\left(2-\frac{5}{n^2}\right)} \\ &= n\sqrt{1+\frac{5}{n}}-n\sqrt{2-\frac{5}{n^2}} \\ &= n\left(\sqrt{1+\frac{5}{n}}-\sqrt{2-\frac{5}{n^2}}\right)\end{aligned}$$

Now, the limit is clear.

$$\begin{aligned}\lim_{n\to\infty}\left[\sqrt{n^2+5n}-\sqrt{2n^2-5}\right] &= \lim_{n\to\infty}\left[n\left(\sqrt{1+\frac{5}{n}}-\sqrt{2-\frac{5}{n^2}}\right)\right] \\ &= \lim_{n\to\infty}\left[n\left(\sqrt{1+0}-\sqrt{2-0}\right)\right] \\ &= \lim_{n\to\infty}\left[n\,(-1)\right] = -\infty\end{aligned}$$

Remark: check Question 22 to see whether a similar trick would work there. Why or why not?

3.1.2.24. Solution. First, we note that we have in indeterminate form: as n grows, $2+\frac{1}{n} \to 2$, so $n\left[\left(2+\frac{1}{n}\right)^{100}-2^{100}\right]$ has the form $\infty \cdot 0$. To overcome this difficulty, we could use some algebra and l'Hôpital's rule, but there's a slicker way. If we let $h = \frac{1}{n}$, then $h \to 0$ as $n \to \infty$, and our limit looks like:

$$\lim_{n\to\infty} n\left[\left(2+\frac{1}{n}\right)^{100}-2^{100}\right] = \lim_{h\to 0}\frac{(2+h)^{100}-2^{100}}{h}$$

This reminds us of the definition of a derivative.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^{100}\right\} = \lim_{h\to 0}\frac{(x+h)^{100}-x^{100}}{h}$$

So, if we set $f(x) = x^{100}$, our limit is simply $f'(2)$. That is, $\left[100x^{99}\right]_{x=2} = 100\cdot 2^{99}$.

3.1.2.25. Solution. Using the definition of a derivative,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

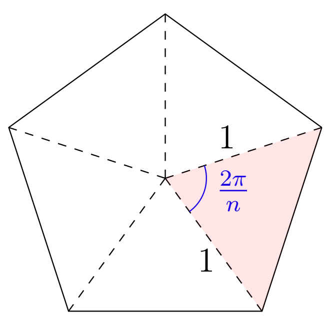
We want $n \to \infty$, so we set $h = \frac{1}{n}$.

$$\begin{aligned} &= \lim_{\frac{1}{n} \to 0} \frac{f\left(a + \frac{1}{n}\right) - f(a)}{\frac{1}{n}} \\ &= \lim_{n \to \infty} n \left[f\left(a + \frac{1}{n}\right) - f(a) \right] \end{aligned}$$

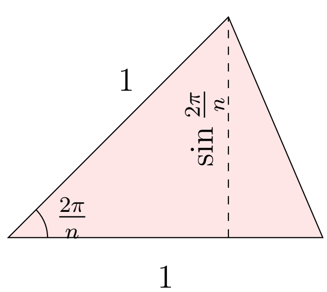
We also could have chosen $h = -\frac{1}{n}$, which leads to the following:

$$\begin{aligned} \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{-\frac{1}{n} \to 0} \frac{f\left(a - \frac{1}{n}\right) - f(a)}{-1/n} \\ &= \lim_{n \to \infty} -n \left(f\left(a - \frac{1}{n}\right) - f(a) \right) \\ &= \lim_{n \to \infty} n \left(f(a) - f\left(a - \frac{1}{n}\right) \right) \end{aligned}$$

3.1.2.26. Solution. (a) To find the area A_n, note that the figure with n sides can be divided up into n isosceles triangles, each with two sides of length 1 and angle between them of $\frac{2\pi}{n}$:



Each of these triangles has area $\frac{1}{2} \sin\left(\frac{2\pi}{n}\right)$:



All together, the area of the n-sided figure is $A_n = \frac{n}{2}\sin\left(\frac{2\pi}{n}\right)$.

(b) We will discuss two ways to find $\lim_{n\to\infty} A_n$, which has the indeterminate form $\infty \times 0$.

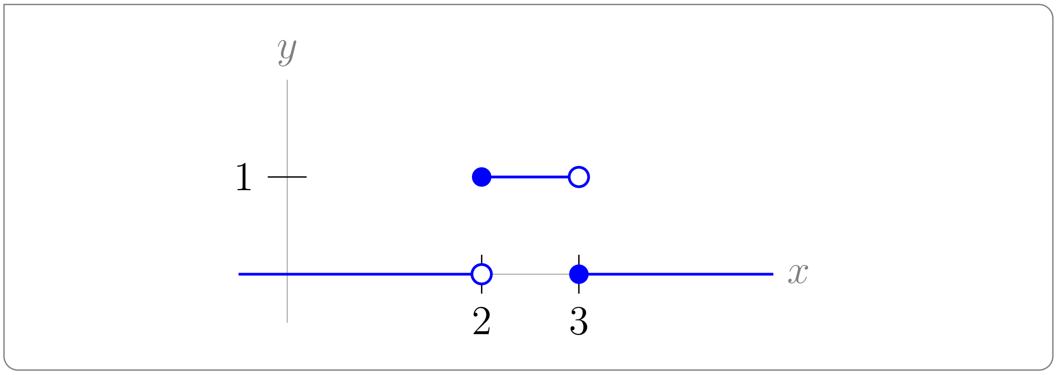
First, note that as $n \to \infty$, our figures look more and more like a circle of radius 1. So, we see A_n is approaching the area of a circle of radius 1. That is, $\lim_{n\to\infty} A_n = \pi$.

Alternately, we can make use of the limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Let $x = \frac{2\pi}{n}$. Note if $n \to \infty$, then $x \to 0$.

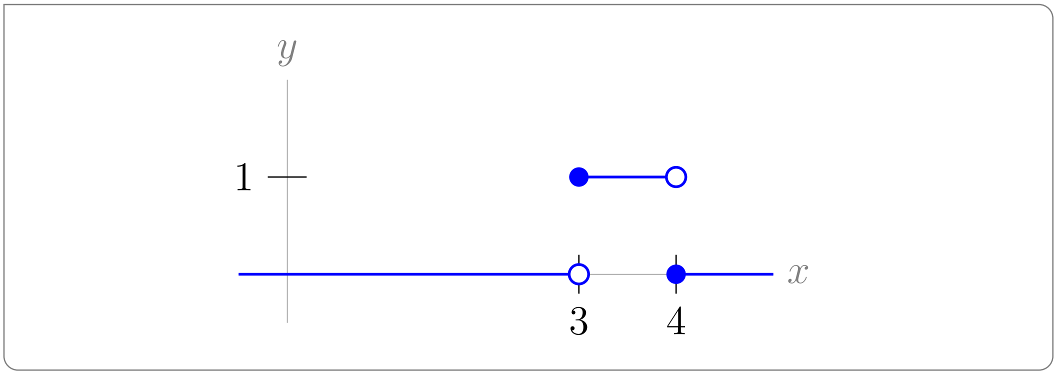
$$\begin{aligned}\lim_{n\to\infty} A_n &= \lim_{n\to\infty} \frac{n}{2}\sin\left(\frac{2\pi}{n}\right) = \lim_{n\to\infty} \frac{\pi}{\frac{2\pi}{n}}\sin\left(\frac{2\pi}{n}\right)\\ &= \lim_{x\to 0} \pi\frac{\sin x}{x} = \pi \times 1 = \pi\end{aligned}$$

3.1.2.27. Solution.

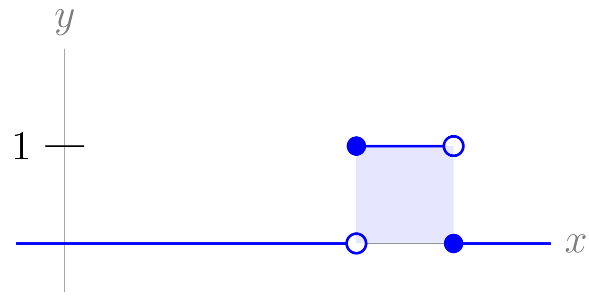
a $f_2(x) = \begin{cases} 1 & 2 \le x < 3 \\ 0 & \text{else} \end{cases}$



b $f_3(x) = \begin{cases} 1 & 3 \le x < 4 \\ 0 & \text{else} \end{cases}$



c For any n, $f_n(x) = 1$ for an interval of length 1, and $f_n(x) = 0$ for all other x. So, the area under the curve is a square of side length one.



Then $A_n = \int_0^\infty f_n(x)\,\mathrm{d}x = 1$ for all n. That is, the sequence $\{A_n\}$ is simply $\{1, 1, \ldots, 1\}$, a sequence of all 1s.

d Given the description above, $\lim\limits_{n\to\infty} A_n = 1$.

e For any fixed x, recall $\{f_n(x)\} = \{0, \ldots, 0, 1, 0, \ldots 0, 0, 0, 0, 0, \ldots\}$. In particular, there are infinitely many zeroes at its end. So, $\lim\limits_{n\to\infty} f_n(x) = 0$. Then $g(x) = 0$ for every x.

f Given the description above, $\displaystyle\int_0^\infty g(x)\,\mathrm{d}x = \int_0^\infty 0\,\mathrm{d}x = 0$.

Remark: what we've shown here is that, for this particular $f_n(x)$,

$$\lim_{n\to\infty}\int_0^\infty f_n(x)\,\mathrm{d}x \neq \int_0^\infty \lim_{n\to\infty} f_n(x)\,\mathrm{d}x$$

That is, we can't necessarily swap a limit with an integral (which is, in this case, another limit, since the integral is improper). The interested reader can look up "uniform convergence" to learn about the conditions under which these *can* be swapped.

3.1.2.28. Solution. If we naively try to find the limit, we run up against the indeterminate form 1^∞. We'd like to use l'Hôpital's rule, but we don't have the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$ — we'll need to use a logarithm. Additionally, l'Hôpital's rule applies to differentiable functions defined for real numbers — so we'll consider a function, rather than the sequence.
Note the terms of the sequence are all positive.

- Solution 1: Define $x = \frac{1}{n}$, and $f(x) = (1 + 3x + 5x^2)^{1/x}$. Then $b_n = f\left(\frac{1}{n}\right) = f(x)$, and

$$\lim_{n\to\infty} f\left(\frac{1}{n}\right) = \lim_{x\to0^+} f(x).$$

If this limit exists, it is equal to $\lim\limits_{n\to\infty} b_n$.

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+}\left(1 + 3x + 5x^2\right)^{1/x}$$
$$\lim_{x\to0^+}\log[f(x)] = \lim_{x\to0^+}\log\left[\left(1 + 3x + 5x^2\right)^{1/x}\right]$$

$$= \underbrace{\lim_{x\to 0^+} \frac{\log\left[1+3x+5x^2\right]}{x}}_{\substack{\text{num}\to 0\\ \text{den}\to 0}}$$

$$= \lim_{x\to 0^+} \frac{\frac{3+10x}{1+3x+5x^2}}{1} = 3$$

$$\lim_{x\to 0^+} f(x) = e^3$$

Since the limit exists, $\lim\limits_{n\to\infty} b_n = e^3$.

- Solution 2: If we didn't see the nice simplifying trick of letting $x = \frac{1}{n}$, we can still solve the problem using $g(x) = \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x$:

$$g(x) = \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x$$

$$\log[g(x)] = x\log\left[1 + \frac{3}{x} + \frac{5}{x^2}\right] = \underbrace{\frac{\log\left[1+\frac{3}{x}+\frac{5}{x^2}\right]}{1/x}}_{\substack{\text{num}\to 0\\ \text{den}\to 0}}$$

$$\lim_{x\to\infty} \log[g(x)] = \lim_{x\to\infty} \frac{\frac{-\frac{3}{x^2}-\frac{10}{x^3}}{1+\frac{3}{x}+\frac{5}{x^2}}}{\frac{-1}{x^2}} = \lim_{x\to\infty} x^2 \frac{\frac{3}{x^2}+\frac{10}{x^3}}{1+\frac{3}{x}+\frac{5}{x^2}}$$

$$= \lim_{x\to\infty} \frac{3+\frac{10}{x}}{1+\frac{3}{x}+\frac{5}{x^2}} = \frac{3+0}{1+0+0}$$

$$= 3$$

$$\lim_{x\to\infty} f(x) = e^3$$

Since the limit exists, $\lim\limits_{n\to\infty} b_n = e^3$

3.1.2.29. Solution.

a When $a_1 = 4$, we see $a_2 = \dfrac{4+8}{3} = 4$, and so on. That is, $a_n = 4$ for every n. So, $\lim\limits_{n\to\infty} a_n = 4$.

b Cross-multiplying, we see $3x = x + 8$, hence $x = 4$.

c In order for our sequence to converge to 4, the terms should be getting infinitely close to 4. So, we find the relationship between $a_{n+1} - 4$ and $a_n - 4$.

$$a_{n+1} = \frac{a_n + 8}{3}$$

$$a_{n+1} - 4 = \frac{a_n + 8}{3} - 4 = \frac{a_n - 4}{3}$$

So, the distance between our sequence terms and the number 4 is decreasing by a factor of 3 each term. This implies that the terms get infinitely close to 4 as n grows. That is, $\lim\limits_{n\to\infty} a_n = 4$.

3.1.2.30. Solution.

a Since w_1 has the highest frequency, w_2 has the next-highest frequency, and so on, we know f_1 is larger than the other members of its sequence, f_2 is the next largest, etc. So, $\{f_n\}$ is a decreasing sequence.

b The most-used word in a language is w_1, while the n-th most used word in a language is w_n. So, we re-state the law as:

$$f_1 = nf_n$$

Then we can rewrite this fomula a little more naturally as $f_n = \frac{1}{n}f_1$.

c Then $f_3 = \frac{1}{3}f_1$. In this case, we expect the third-most used word to account for $\frac{1}{3}(6\%) = 2\%$ of all words.

d From (b), we know $f_{10} = \frac{1}{10}f_1$. Note $f_1 = 6f_6 = 6(0.3\%)$. Then:

$$f_{10} = \frac{1}{10}f_1 = \frac{1}{10}6f_6 = \frac{1}{10}(6)(0.3\%) = \frac{1.8}{10}\% = 0.18\%$$

So, f_{10} should be 0.18% of all words.

e The use of the word "frequency" in the statement of Zipf's law implies $f_n = \frac{\text{uses of } w_n}{\text{total number of words}}$. The question asks for the total uses of w_n. If we call this quantity t_n, and the total number of all words is T, then Zipf's law tells us $\frac{t_n}{T} = \frac{1}{n}\frac{t_1}{T}$, hence $t_n = \frac{1}{n}t_1$.

With this notation, the problem states $t_1 = 22,038,615$, $w_1 =$ `the`, $w_2 =$ `be`, and $w_3 =$ `and`.

Following Zipf's law, $t_n = \frac{1}{n}t_1$. So, we expect $t_2 = \frac{t_1}{2} = 11,019,307.5$; since this isn't an integer, let's say we expect $t_2 \approx 11,019,308$. Similarly, we expect $t_3 = \frac{t_1}{3} = 7,346,205$.

Remark: The 450-million-word source material that used "the" 22,038,615 times also contained 12,545,825 instances of "be," and 10,741,073 instances of "and." While Zipf's Law might be a nice model for our data overall, in these few instances it does not appear to be extremely accurate.

3.2 · Series

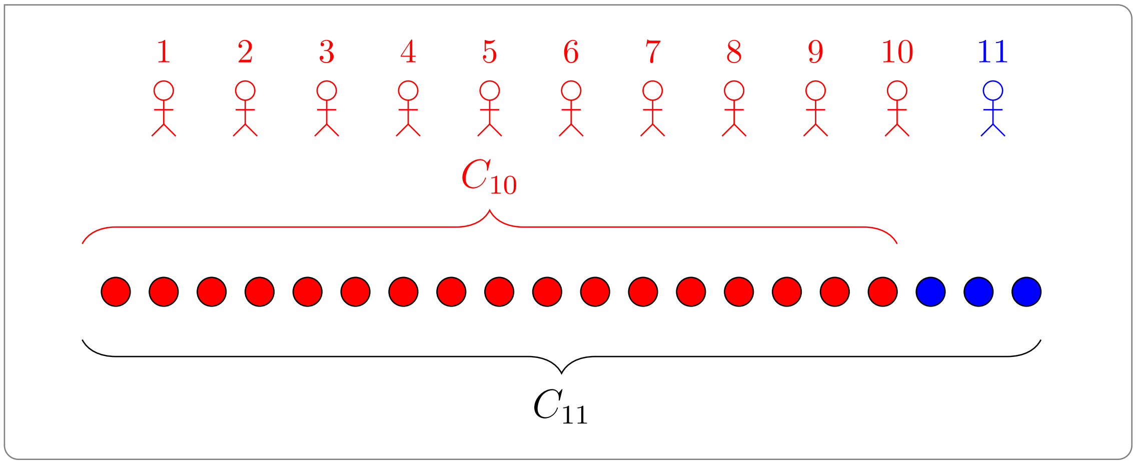
3.2.2 · Exercises

Exercises — Stage 1

3.2.2.1. Solution. The Nth term of the sequence of partial sums, S_N, is the sum of the first N terms of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

$\mathbf{N}$	$\mathbf{S_N}$
1	1
2	$1+\frac{1}{2}$
3	$1+\frac{1}{2}+\frac{1}{3}$
4	$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$
5	$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$

3.2.2.2. Solution. If there were a total of 17 cookies before Student 11 came, and 20 cookies after, then Student 11 brought 3 cookies.



3.2.2.3. Solution.

a We find $\{a_n\}$ from $\{S_N\}$ using the same logic as Question 2. S_N is the sum of the first N terms of $\{a_n\}$, and S_{N-1} is the sum of all the same terms *except* a_N. So, $a_N = S_N - S_{N-1}$ when $N \geq 2$. Written another way:

$$S_N = a_1 + a_2 + a_3 + \cdots + a_{N-2} + a_{N-1} + a_N$$
$$S_{N-1} = a_1 + a_2 + a_3 + \cdots + a_{N-2} + a_{N-1}$$

So,

$$\begin{aligned} S_N - S_{N-1} &= \left[a_1 + a_2 + a_3 + \cdots + a_{N-2} + a_{N-1} + a_N\right] \\ &\quad - \left[a_1 + a_2 + a_3 + \cdots + a_{N-2} + a_{N-1}\right] \\ &= a_N \end{aligned}$$

So, we calculate

$$a_N = S_N - S_{N-1} = \left(\frac{N}{N+1}\right) - \left(\frac{N-1}{N-1+1}\right)$$

$$= \frac{N^2}{N(N+1)} - \frac{N^2-1}{N(N+1)}$$
$$= \frac{1}{N(N+1)}$$

Therefore,

$$a_n = \frac{1}{n(n+1)}$$

Remark: the formula given for S_N has $S_0 = 0$, which makes sense: the sum of no terms at all should be 0. However, it is common for a sequence of partial sums to start at $N = 1$. (This fits our definition of a partial sum–we don't really define the "sum of no terms.") In this case, a_1 must be calculated separately from the other terms of $\{a_n\}$. To find a_1, we simply set $a_1 = S_1$, which (to reiterate) might not be the same as $S_1 - S_0$.

b

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n(n+1)} = 0.$$

That is, the terms we're adding up are getting very, very small as we go along.

c By Definition 3.2.3,

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{N}{N+1} = 1$$

That is, as we add more and more terms of our series, our cumulative sum gets very, very close to 1.

3.2.2.4. Solution. As in Question 3,

$$\begin{aligned}
a_N = S_N - S_{N-1} &= \left[(-1)^N + \frac{1}{N}\right] - \left[(-1)^{N-1} + \frac{1}{N-1}\right] \\
&= (-1)^N - (-1)^{N-1} + \frac{1}{N} - \frac{1}{N-1} \\
&= (-1)^N + (-1)^N + \frac{N-1}{N(N-1)} - \frac{N}{N(N-1)} \\
&= 2(-1)^N - \frac{1}{N(N-1)}
\end{aligned}$$

Note, however, that a_N is only the same as $S_N - S_{N-1}$ when $N \geq 2$: otherwise, we're trying to calculate $S_1 - S_0$, but S_0 is not defined. So, we find a_1 separately:

$$a_1 = S_1 = (-1)^1 + \frac{1}{1} = 0$$

All together:

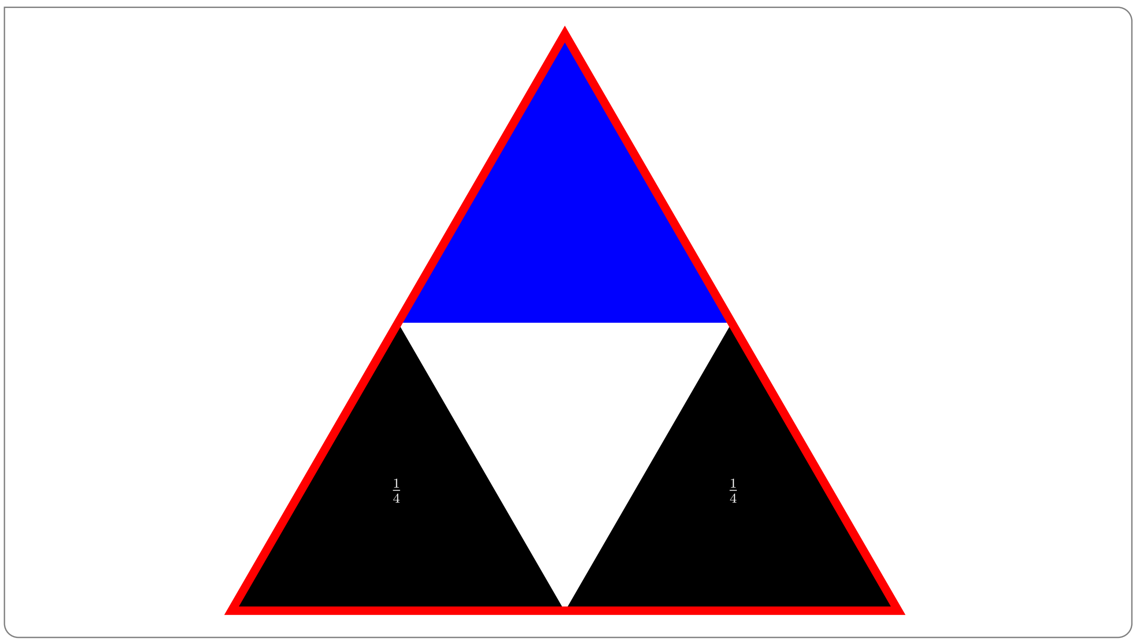
$$a_n = \begin{cases} 0 & \text{if } n = 1 \\ 2(-1)^n - \frac{1}{n(n-1)} & \text{else} \end{cases}$$

3.2.2.5. Solution. If $f'(N) < 0$, that means $f(N)$ is decreasing. So, adding more terms makes for a smaller sum. That means the terms we're adding are negative. That is, $a_n < 0$ for all $n \geq 2$.

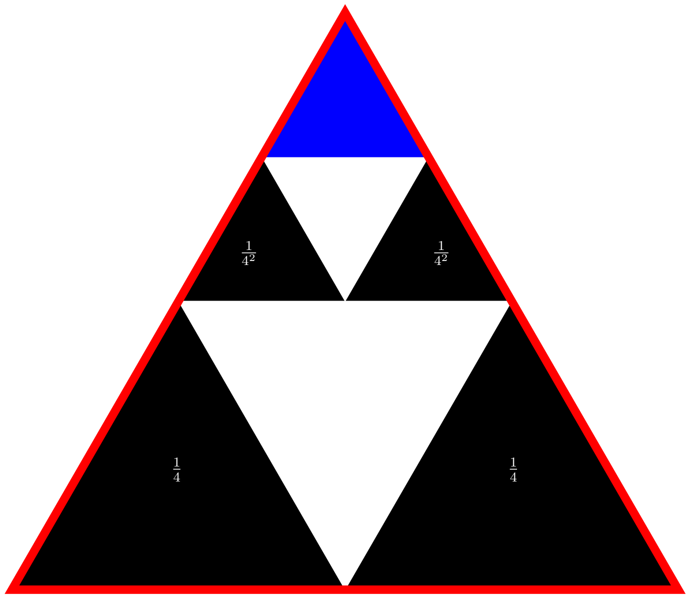
3.2.2.6. Solution. (a) To generate the pattern, we repeat the following steps:

- divide the top triangle into four triangles of equal area,
- colour the bottom two of them black, and
- leave the middle one white.

Every time we repeat this sequence, we divide up a triangle with an area one-quarter the size of our previous triangle, and take two of the four resulting pieces. So, our area should end up as a geometric sum with common ratio $r = \dfrac{1}{4}$, and coefficient $a = 2$. This is shown more explicitly below.
Since the entire triangle (outlined in red) has area 1, the four smaller triangles below each have area $\dfrac{1}{4}$. The two black triangles will be added to our total black area; the blue triangle will be subdivided.



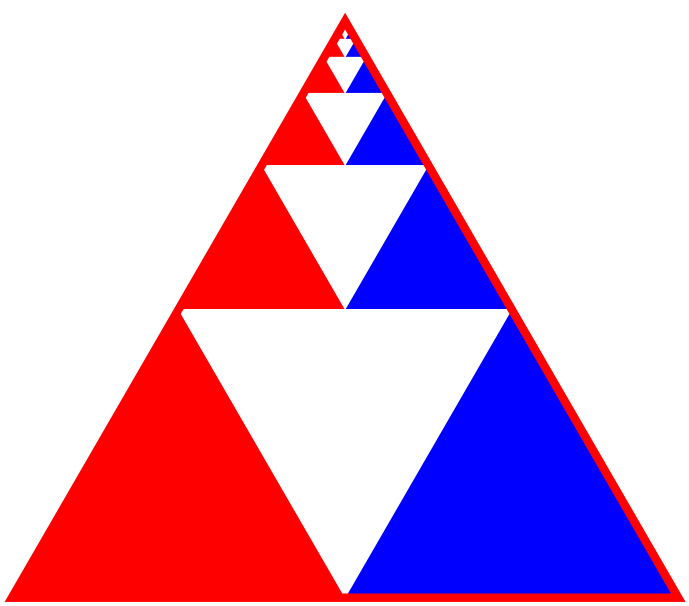
The blue triangle had area $\dfrac{1}{4}$, so each of the small black triangles below has area $\left(\dfrac{1}{4}\right)\left(\dfrac{1}{4}\right) = \dfrac{1}{4^2}$.



Each time we make another subdivision, we add two black triangles, each with $\frac{1}{4}$ the area of the previous black triangles. So, our total black area is:

$$2\left(\frac{1}{4}\right)+2\left(\frac{1}{4^2}\right)+2\left(\frac{1}{4^3}\right)+2\left(\frac{1}{4^4}\right)+\cdots=\sum_{n=1}^{\infty}\frac{2}{4^n}$$

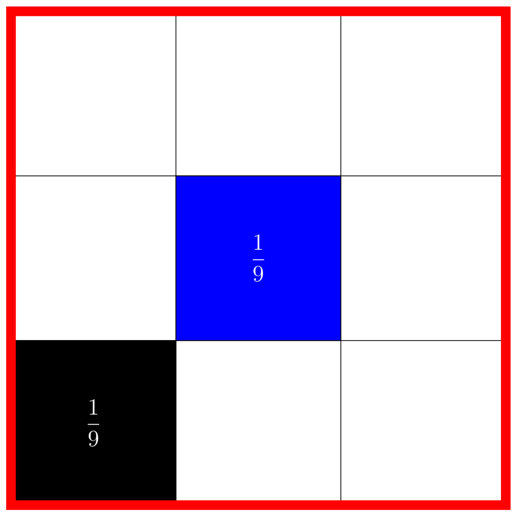
(b) To evalutate the series, we imagine gathering up all our little triangles and sorting them into three identical piles: the bottom three triangles go in three different piles, the three triangles directly above them go in three different piles, etc. (In the picture below, different colours correspond to different piles.)



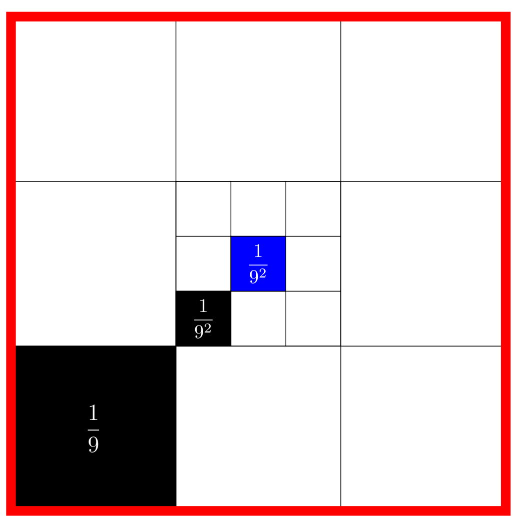
Since the piles all have equal area, each pile has a total area of $\frac{1}{3}$. The black area shaded in the problem corresponds to *two* piles (red and blue above), so

$$\sum_{n=1}^{\infty} \frac{2}{4^n} = \frac{2}{3}$$

3.2.2.7. Solution. (a) The pattern can be described as follows: divide the innermost square into 9 equal parts (a 3×3 grid), choose one square to be black, and another square to subdivide.
The area of the red (outermost) square is 1, so the area of the largest black square is $\frac{1}{9}$. The area of the central, blue square below is also $\frac{1}{9}$.



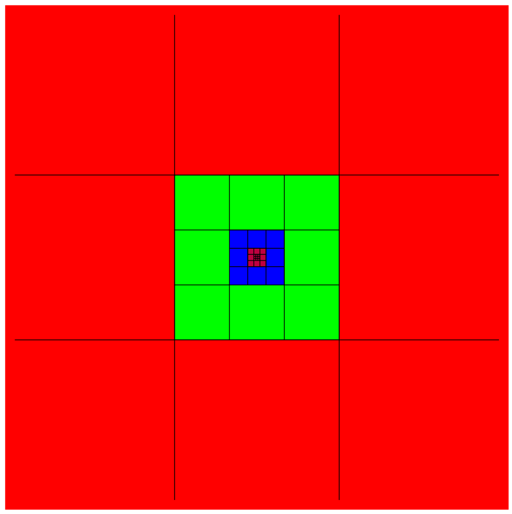
When we subdivide the blue square, the subdivisions each have one-ninth its area, or $\frac{1}{9^2}$.



We continue taking squares that are one-ninth the area of the previous square. So, our total black area is

$$\frac{1}{9}+\frac{1}{9^2}+\frac{1}{9^3}+\cdots=\sum_{n=1}^{\infty}\frac{1}{9^n}$$

(b) If we cut up this square along the marks, we can easily share it equally among 8 friends: there are eight squares of area $\frac{1}{9}$ along the outer ring, eight squares of area $\frac{1}{9^2}$ along the next ring in, and so on.

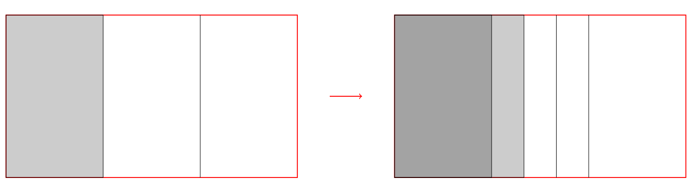


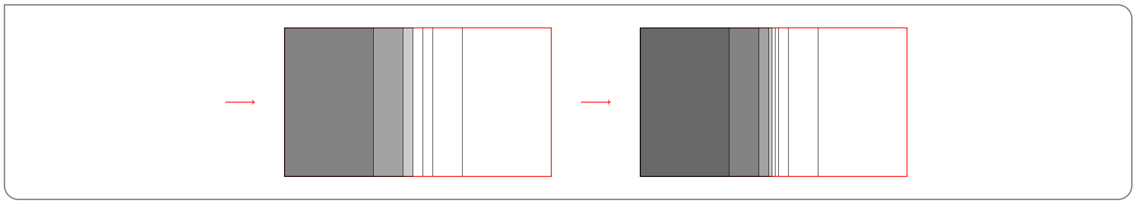
Since the eight friends all get the same total area, the area each friend gets is $\frac{1}{8}$. The area shaded in black in the question corresponds to the pile given to one friend. So,

$$\sum_{n=1}^{\infty}\frac{1}{9^n}=\frac{1}{8}$$

3.2.2.8. Solution. If we start with a shape of area 1, and iteratively divide it into thirds, taking one of the three newly created pieces each time, then the area we take will be equal to the desired series, $\sum_{n=1}^{\infty}\frac{1}{3^n}$.

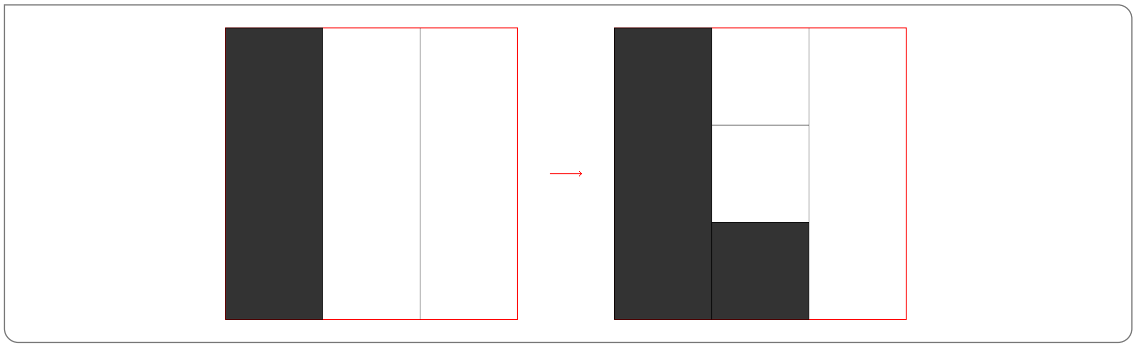
One way to do this is to start with a rectangle, make three vertical strips, then keep the left strip and subdivide the middle strip.

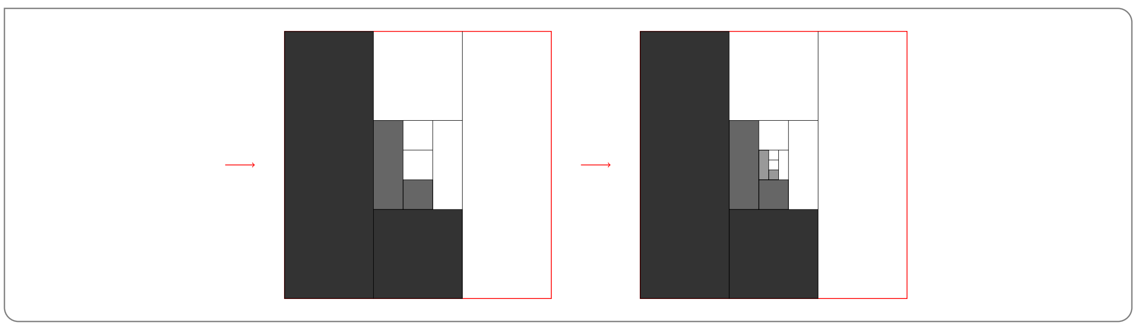




We see that the total area we take approaches one-half the total area of the figure, so $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}$.

Alternately, instead of always taking vertical strips, we could alternate vertical and horizontal slices.





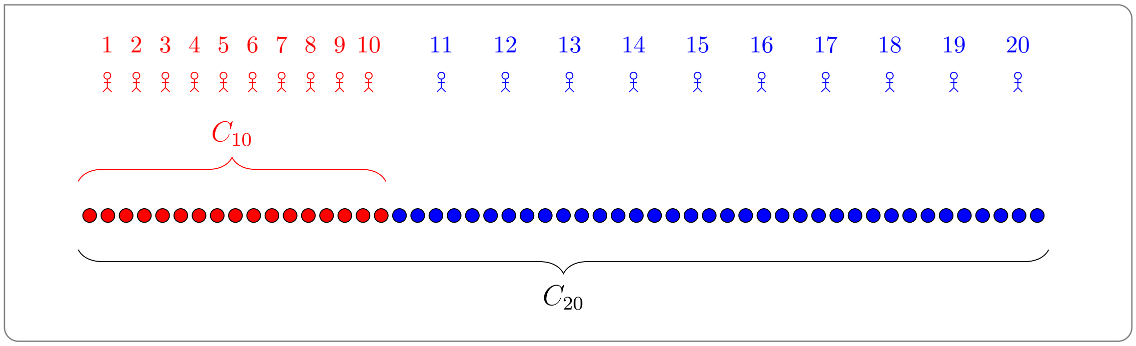
In this setup, we notice that our strips come in pairs: two large vertical strips, two smaller horizontal strips, two smaller vertical strips, etc. We shaed exactly one of each, so the shaded area is one-half the total area: $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}$.

Other solutions are possible, as well.

3.2.2.9. Solution. Lemma 3.2.5 tells us $\sum_{n=0}^{N} ar^n = a\frac{1-r^{N+1}}{1-r}$, for $r \neq 1$. Our geometric sum has $a = 1$, $r = \frac{1}{5}$, and $N = 100$. So:

$$\sum_{n=0}^{100} \frac{1}{5^n} = \frac{1 - \frac{1}{5^{101}}}{1 - \frac{1}{5}} = \frac{5^{101} - 1}{4 \cdot 5^{100}}$$

3.2.2.10. Solution. After twenty students have brought their cookies, the pile numbers 53 cookies. 17 of these cookies were brought by students one through ten. So, the remainder $(53 - 17 = 36)$ is the number of cookies brought by students 11, 12, 13, 14, 15, 16, 17, 18, 19, and 20, together.



3.2.2.11. Solution.

- Solution 1: Using the ideas of Question 10, we see:

$$\sum_{n=50}^{100} \frac{1}{5^n} = \sum_{n=0}^{100} \frac{1}{5^n} - \sum_{n=0}^{49} \frac{1}{5^n}$$

That is, we want start with the sum of all the terms up to $\frac{1}{5^{100}}$, and then subtract off the ones we actually don't want, which is everything up to $\frac{1}{5^{49}}$. Now, both series are in a form appropriate for Lemma 3.2.5.

$$\begin{aligned}
\sum_{n=0}^{100} \frac{1}{5^n} - \sum_{n=0}^{49} \frac{1}{5^n} &= \frac{1-\frac{1}{5^{101}}}{1-\frac{1}{5}} - \frac{1-\frac{1}{5^{50}}}{1-\frac{1}{5}} \\
&= \frac{5^{101}-1}{4\cdot 5^{100}} - \frac{5^{50}-1}{4\cdot 5^{49}}\left(\frac{5^{51}}{5^{51}}\right) \\
&= \frac{5^{101}-1}{4\cdot 5^{100}} - \frac{5^{101}-5^{51}}{4\cdot 5^{100}} \\
&= \frac{5^{51}-1}{4\cdot 5^{100}}
\end{aligned}$$

- Solution 2: If we write out the first few terms of our series, we see we can factor out a constant to change the starting index.

$$\begin{aligned}
\sum_{n=50}^{100} \frac{1}{5^n} &= \frac{1}{5^{50}} + \frac{1}{5^{51}} + \frac{1}{5^{52}} + \frac{1}{5^{53}} + \cdots + \frac{1}{5^{100}} \\
&= \frac{1}{5^{50}}\left(\frac{1}{5^0} + \frac{1}{5^1} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots + \frac{1}{5^{50}}\right) \\
&= \sum_{n=0}^{50} \frac{1}{5^{50}} \cdot \frac{1}{5^n}
\end{aligned}$$

Now, our sum is in the form of Lemma 3.2.5 with $a = \frac{1}{5^{50}}$, $r = \frac{1}{5}$, and $N = 50$.

$$\begin{aligned}
\sum_{n=0}^{50} \frac{1}{5^{50}} \cdot \frac{1}{5^n} = \frac{1}{5^{50}} \cdot \frac{1-\frac{1}{5^{51}}}{1-\frac{1}{5}} &= \frac{1-\frac{1}{5^{51}}}{4\cdot 5^{49}}\left(\frac{5^{51}}{5^{51}}\right) \\
&= \frac{5^{51}-1}{4\cdot 5^{100}}
\end{aligned}$$

3.2.2.12. Solution. (a) The table below is a record of our account, with black entries representing the money your friend gives you, and red entries representing the money you give them (which is why the red entries are negative).

$\mathbf{d}$	$-\frac{\mathbf{1}}{\mathbf{d+1}}$	$\frac{\mathbf{1}}{\mathbf{d}}$	
1	$-\frac{1}{2}$	1	$\frac{1}{2}$
2	$-\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$
3	$-\frac{1}{4}$	$\frac{1}{3}$	$\frac{3}{4}$
4	$-\frac{1}{5}$	$\frac{1}{4}$	$\frac{4}{5}$
5	$-\frac{1}{6}$	$\frac{1}{5}$	$\frac{5}{6}$
6	$-\frac{1}{7}$	$\frac{1}{6}$	$\frac{6}{7}$

After the exchange of day n, the amount you're left with is $\left(1-\frac{1}{n+1}\right)$. We see this by the cancellation in the table: the $\$\frac{1}{2}$ you gave your friend on day 1 was returned on day 2; the $\$\frac{1}{3}$ you gave your friend on day 2 was returned on day 3, etc.
So, after a long time, you'll have gained close to (but always slightly less than) one dollar.
(b) The series $\displaystyle\sum_{d=1}^{\infty}\left(\frac{1}{d}-\frac{1}{(d+1)}\right)$ describes the scenario in (a), so by our reasoning there,

$$\sum_{d=1}^{\infty}\left(\frac{1}{d}-\frac{1}{(d+1)}\right)=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1$$

(c) Again, let's set up an account book.

$\mathbf{d}$	$\mathbf{d+1}$	$-(\mathbf{d+2})$	
1	2	-3	-1
2	3	-4	-2
3	4	-5	-3
4	5	-6	-4
5	6	-7	-5
6	7	-8	-6

By day d, you've *lost* \$ d to your so-called friend. As time goes on, you lose more and more.
(d) The series $\displaystyle\sum_{d=1}^{\infty}((d+1)-(d+2))$ exactly describes the scenario in part (c), so it diverges to $-\infty$. You can also see this by writing $\displaystyle\sum_{d=1}^{\infty}((d+1)-(d+2))=\sum_{d=1}^{\infty}(-1)=-1-1-1-1-1-\cdots$.
Be careful to avoid a common mistake with telescoping series: if we look back at our account book, we see that every negative term will cancel with a positive term, with the initial $+2$ as the only term that never cancels. Your friend takes \$3, which they return the next day; then they take \$4, which they return the next day; then they take \$5, which they return the next day, and so on. It's extremely tempting to say that the series adds up to \$2, since every other term cancels out eventually. This

is where we lean on Definition 3.2.3: we evaluate the *partial sums*, which always leave your friend's last withdrawal unreturned. This definition makes sense: saying "I gained two bucks from this exchange" doesn't really capture the reality of your increasing debt.

3.2.2.13. Solution. Using arithmetic of series, Theorem 3.2.9, we see

$$\sum_{n=1}^{\infty}(a_n+b_n+c_{n+1}) = A+B+\sum_{n=1}^{\infty}c_{n+1}$$

The question remaining is what do to with the last series. If we write out the terms, we see the difference between $\displaystyle\sum_{n=1}^{\infty}c_n$ and $\displaystyle\sum_{n=1}^{\infty}c_{n+1}$ is simply that the latter is missing c_1:

$$\begin{aligned}\sum_{n=1}^{\infty}c_{n+1} &= c_2+c_3+c_4+c_5+\cdots\\ &= -c_1+c_1+c_2+c_3+c_4+c_5+\cdots\\ &= -c_1+\sum_{n=1}^{\infty}c_n\end{aligned}$$

So,

$$\sum_{n=1}^{\infty}(a_n+b_n+c_{n+1}) = A+B+C-c_1$$

3.2.2.14. Solution. Theorem 3.2.9, arithmetic of series, doesn't mention division, because in general it doesn't work the way the question suggests. For example, let $\{a_n\}=\{b_n\}=\frac{1}{2^n}$. Then:

- $\sum_{n=0}^{\infty}a_n=\sum_{n=0}^{\infty}b_n=\frac{1}{1-\frac{1}{2}}=2$, while
- $\sum_{n=0}^{\infty}\dfrac{a_n}{b_n}=\sum_{n=0}^{\infty}1=\infty$.

For the statement in the question, we can take $\{a_n\}=\{b_n\}=\frac{1}{2^n}$, $A=B=2$, $\{c_n\}=\{0,0,0,\ldots\}$, and $C=0$. We see the statement is false in this case.
So, in general, the statement given is false.

Exercises — Stage 2

3.2.2.15. *. Solution. We recognize that this is a geometric series:

$$\begin{aligned}&1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\cdots\\ &\qquad=\frac{1}{3^0}+\frac{1}{3^1}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\frac{1}{3^5}+\cdot\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{1}{3^n}$$

Using Lemma 3.2.5 with $r = \frac{1}{3}$ and $a = 1$,

$$= \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}.$$

3.2.2.16. $*$. Solution. This is a geometric series, with ratio $r = \frac{1}{8}$. However, it doesn't start at $k = 0$, which is what we're used to.

- Solution 1: We write out the first few terms of the series to figure out a convenient constant to factor out.

$$\begin{aligned}
\sum_{k=7}^{\infty} \frac{1}{8^k} &= \frac{1}{8^7} + \frac{1}{8^8} + \frac{1}{8^9} + \cdots \\
&= \frac{1}{8^7}\left(\frac{1}{8^0} + \frac{1}{8^1} + \frac{1}{8^2} + \cdots\right) \\
&= \sum_{k=0}^{\infty} \frac{1}{8^7} \cdot \frac{1}{8^n}
\end{aligned}$$

  We now evaluate the series using Lemma 3.2.5 with $r = \frac{1}{8}$, $a = \frac{1}{8^7}$.

$$= \frac{1}{8^7} \cdot \frac{1}{1 - \frac{1}{8}} = \frac{1}{7 \times 8^6}$$

- Solution 2: Using the idea of Question 10, we express the series we're interested in as the difference of two series that we can easily evaluate.

$$\sum_{k=7}^{\infty} \frac{1}{8^k} = \sum_{k=0}^{\infty} \frac{1}{8^k} - \sum_{k=0}^{6} \frac{1}{8^k}$$

  Using Lemma 3.2.5,

$$\begin{aligned}
&= \frac{1}{1 - \frac{1}{8}} - \frac{1 - \frac{1}{8^7}}{1 - \frac{1}{8}} \\
&= \frac{1}{7 \times 8^6}
\end{aligned}$$

3.2.2.17. *. Solution. We recognize this as a telescoping series.

$\mathbf{k}$	$\frac{\mathbf{6}}{\mathbf{k^2}}$	$-\frac{\mathbf{6}}{\mathbf{(k+1)^2}}$	$\mathbf{s_k}$
1	6	$-\frac{6}{4}$	$6-\frac{6}{4}$
2	$\frac{6}{4}$	$-\frac{6}{9}$	$6-\frac{6}{9}$
3	$\frac{6}{9}$	$-\frac{6}{16}$	$6-\frac{6}{16}$
4	$\frac{6}{16}$	$-\frac{6}{25}$	$6-\frac{6}{25}$
5	$\frac{6}{25}$	$-\frac{6}{36}$	$6-\frac{6}{36}$
6	$\frac{6}{36}$	$-\frac{6}{47}$	$6-\frac{6}{47}$
$\vdots$			

When we compute the n^{th} partial sum, i.e. the sum of of the first n terms, successive terms cancel and only the first half of the first term, $\left(\frac{6}{k^2}-\frac{6}{(k+1)^2}\right)\Big|_{k=1}$, and the second half of the n^{th} term, $\left(\frac{6}{k^2}-\frac{6}{(k+1)^2}\right)\Big|_{k=n}$, survive. That is:

$$s_n = \sum_{k=1}^{n}\left(\frac{6}{k^2}-\frac{6}{(k+1)^2}\right) = \frac{6}{1^2}-\frac{6}{(n+1)^2}$$

Therefore, we can see directly that the sequence of partial sums $\{s_n\}$ is convergent:

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(6-\frac{6}{(n+1)^2}\right) = 6$$

By Definition 3.2.3 the series is also convergent, with limit 6.

3.2.2.18. *. Solution. We recognize that this is a telescoping series, and set up a table to find the sequence of partial sums.

$\mathbf{n}$	$\cos\left(\frac{\pi}{\mathbf{n}}\right)$	$-\cos\left(\frac{\pi}{\mathbf{n+1}}\right)$	$\mathbf{s_n}$
3	$\cos\left(\frac{\pi}{3}\right)$	$-\cos\left(\frac{\pi}{4}\right)$	$\frac{1}{2}-\cos\left(\frac{\pi}{4}\right)$
4	$\cos\left(\frac{\pi}{4}\right)$	$-\cos\left(\frac{\pi}{5}\right)$	$\frac{1}{2}-\cos\left(\frac{\pi}{5}\right)$
5	$\cos\left(\frac{\pi}{5}\right)$	$-\cos\left(\frac{\pi}{6}\right)$	$\frac{1}{2}-\cos\left(\frac{\pi}{6}\right)$
6	$\cos\left(\frac{\pi}{6}\right)$	$-\cos\left(\frac{\pi}{7}\right)$	$\frac{1}{2}-\cos\left(\frac{\pi}{7}\right)$
7	$\cos\left(\frac{\pi}{7}\right)$	$-\cos\left(\frac{\pi}{8}\right)$	$\frac{1}{2}-\cos\left(\frac{\pi}{8}\right)$
8	$\cos\left(\frac{\pi}{8}\right)$	$-\cos\left(\frac{\pi}{9}\right)$	$\frac{1}{2}-\cos\left(\frac{\pi}{9}\right)$
$\vdots$			

The Nth partial sum sees every term cancel except the first part of the first term $\left(\frac{1}{2}\right)$ and the second part of the last term $\left(-\cos\left(\frac{\pi}{n+1}\right)\right)$.

$$\begin{aligned} s_N &= \sum_{n=3}^{N}\left(\cos\left(\frac{\pi}{n}\right)-\cos\left(\frac{\pi}{n+1}\right)\right) \\ &= \cos\left(\frac{\pi}{3}\right)-\cos\left(\frac{\pi}{N+1}\right) \\ &= \frac{1}{2}-\cos\left(\frac{\pi}{N+1}\right). \end{aligned}$$

As $N \to \infty$, the argument $\frac{\pi}{N+1}$ converges to 0, and $\cos x$ is continuous at $x = 0$. By Definition 3.2.3, the value of the series is

$$\begin{aligned}\lim_{N\to\infty} s_N &= \lim_{N\to\infty}\left[\frac{1}{2} - \cos\left(\frac{\pi}{N+1}\right)\right] \\ &= \frac{1}{2} - \cos(0) = -\frac{1}{2}\end{aligned}$$

3.2.2.19. *. Solution. (a) As in Question 2, since

$$\begin{aligned}s_{n-1} &= a_1 + a_2 + \cdots + a_{n-1} \\ s_n &= a_1 + a_2 + \cdots + a_{n-1} + a_n\end{aligned}$$

we can find a_n by subtracting:

$$\begin{aligned}a_n &= s_n - s_{n-1} \\ &= \frac{1+3n}{5+4n} - \frac{1+3(n-1)}{5+4(n-1)} = \frac{3n+1}{4n+5} - \frac{3n-2}{4n+1} \\ &= \frac{(3n+1)(4n+1)-(3n-2)(4n+5)}{(4n+1)(4n+5)} \\ &= \frac{11}{16n^2+24n+5}\end{aligned}$$

(b) Using Definition 3.2.3,

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\frac{1+3n}{5+4n} = \lim_{n\to\infty}\frac{1/n+3}{5/n+4} = \frac{0+3}{0+4} = \frac{3}{4}$$

The series converges to $\dfrac{3}{4}$.

3.2.2.20. *. Solution. What we have is a geometric series, but we need to get it into the proper form before we can evaluate it.

$$\sum_{n=2}^{\infty}\frac{3\cdot 4^{n+1}}{8\cdot 5^n} = \sum_{n=2}^{\infty}\frac{3\cdot 4\cdot 4^n}{8\cdot 5^n} = \frac{3}{2}\sum_{n=2}^{\infty}\left(\frac{4}{5}\right)^n$$

- Solution 1: If we factor our $\left(\frac{4}{5}\right)^2$, we can change our index to something more convenient.

$$\begin{aligned}\frac{3}{2}\sum_{n=2}^{\infty}\left(\frac{4}{5}\right)^n &= \frac{3}{2}\sum_{n=2}^{\infty}\left(\frac{4}{5}\right)^2\left(\frac{4}{5}\right)^{n-2} \\ &= \frac{3}{2}\sum_{n=0}^{\infty}\left(\frac{4}{5}\right)^2\left(\frac{4}{5}\right)^n\end{aligned}$$

We use Lemma 3.2.5 with $r = \frac{4}{5}$.

$$= \frac{3}{2}\left(\frac{4}{5}\right)^2 \cdot \frac{1}{1-\frac{4}{5}} = \frac{24}{5}$$

- Solution 2: Using the idea of Question 10, we view our series as a more convenient series, minus a few initial terms.

$$\begin{aligned}\frac{3}{2}\sum_{n=2}^{\infty}\left(\frac{4}{5}\right)^n &= \frac{3}{2}\left(\left[\sum_{n=0}^{\infty}\left(\frac{4}{5}\right)^n\right] - \left(\frac{4}{5}\right)^1 - \left(\frac{4}{5}\right)^0\right)\\ &= \frac{3}{2}\left(\sum_{n=0}^{\infty}\left(\frac{4}{5}\right)^n - \frac{9}{5}\right)\end{aligned}$$

We use Lemma 3.2.5 with $r = \frac{4}{5}$.

$$= \frac{3}{2}\left(\frac{1}{1-\frac{4}{5}} - \frac{9}{5}\right) = \frac{24}{5}$$

3.2.2.21. $*$. Solution. The number is:

$$\begin{aligned}0.2 + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots &= \frac{1}{5} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots\\ &= \frac{1}{5} + \frac{3}{10^2}\left(\frac{1}{10^0} + \frac{1}{10^1} + \frac{1}{10^2} + \cdots\right)\\ &= \frac{1}{5} + \frac{3}{10^2}\sum_{n=0}^{\infty}\frac{1}{10^n}\end{aligned}$$

We use Lemma 3.2.5 with $r = \frac{1}{10}$.

$$\begin{aligned}&= \frac{1}{5} + \frac{3}{10^2}\cdot\frac{1}{1-\frac{1}{10}}\\ &= \frac{1}{5} + \frac{1}{30} = \frac{7}{30}\end{aligned}$$

3.2.2.22. $*$. Solution. The number is:

$$\begin{aligned}2 + \frac{65}{100} + \frac{65}{10000} + \frac{65}{1000000} + \cdots &= 2 + \frac{65}{100} + \frac{65}{100^2} + \frac{65}{100^3} + \cdots\\ &= 2 + \frac{65}{100}\sum_{n=0}^{\infty}\frac{1}{100^n}\end{aligned}$$

We use Lemma 3.2.5 with $r = \frac{1}{100}$.

$$= 2 + \frac{65}{100}\cdot\frac{1}{1-\frac{1}{100}}$$

$$= 2 + \frac{65}{99} = \frac{263}{99}$$

3.2.2.23. $*$. Solution. The number is:

$$\begin{aligned} 0.\overline{321} &= 0.321321321\ldots \\ &= \frac{321}{1000} + \frac{321}{10^6} + \frac{321}{10^9} + \cdots \\ &= \frac{321}{1000}\left(\left(\frac{1}{10^3}\right)^0 + \left(\frac{1}{10^3}\right)^1 + \left(\frac{1}{10^3}\right)^2 + \cdots\right) \\ &= \frac{321}{1000}\sum_{n=0}^{\infty}\left(\frac{1}{10^3}\right)^n \end{aligned}$$

We use Lemma 3.2.5 with $r = \frac{1}{10^3}$.

$$= \frac{321}{1000} \cdot \frac{1}{1 - \frac{1}{10^3}} = \frac{321}{999} = \frac{107}{333}$$

3.2.2.24. $*$. Solution. We split the sum into two parts.

$$\begin{aligned} &\sum_{n=2}^{\infty}\left(\frac{2^{n+1}}{3^n} + \frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &\qquad = \sum_{n=2}^{\infty}\frac{2^{n+1}}{3^n} \quad + \quad \sum_{n=2}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \end{aligned}$$

The first part is a geometric series.

$$\sum_{n=2}^{\infty}\frac{2^{n+1}}{3^n} = \sum_{n=0}^{\infty}\frac{2^{n+3}}{3^{n+2}} = \sum_{n=0}^{\infty}\frac{2^3}{3^2}\cdot\left(\frac{2}{3}\right)^n$$

We use Lemma 3.2.5 with $r = \frac{2}{3}$ and $a = \frac{8}{9}$.

$$= \frac{8}{9} \cdot \frac{1}{1 - \frac{2}{3}} = \frac{8}{3}$$

The second part is a telescoping series. Let's make a table to see how it cancels.

$\mathbf{n}$	$\frac{1}{2n-1}$	$-\frac{1}{2n+1}$	S_n
2	$\frac{1}{3}$	$-\frac{1}{5}$	$\frac{1}{3} - \frac{1}{5}$
3	$\frac{1}{5}$	$-\frac{1}{7}$	$\frac{1}{3} - \frac{1}{7}$
4	$\frac{1}{7}$	$-\frac{1}{9}$	$\frac{1}{3} - \frac{1}{9}$
5	$\frac{1}{9}$	$-\frac{1}{11}$	$\frac{1}{3} - \frac{1}{11}$
6	$\frac{1}{11}$	$-\frac{1}{13}$	$\frac{1}{3} - \frac{1}{13}$
7	$\frac{1}{13}$	$-\frac{1}{15}$	$\frac{1}{3} - \frac{1}{15}$
$\vdots$			

After adding terms $n = 2$ through $n = N$, the partial sum is

$$s_N = \frac{1}{3} - \frac{1}{2N+1}$$

because all the terms except the first part of the $n = 2$ term, and the last part of the $n = N$ term, cancel. Then:

$$\begin{aligned}\sum_{n=2}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) &= \lim_{N\to\infty} s_N = \lim_{N\to\infty} \frac{1}{3} - \frac{1}{2N+1} \\ &= \frac{1}{3}\end{aligned}$$

All together,

$$\begin{aligned}&\sum_{n=2}^{\infty}\left(\frac{2^{n+1}}{3^n} + \frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &\qquad = \sum_{n=2}^{\infty}\frac{2^{n+1}}{3^n} \quad + \quad \sum_{n=2}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &\qquad = \frac{8}{3} + \frac{1}{3} = 3\end{aligned}$$

3.2.2.25. *. Solution. We split the sum into two parts.

$$\sum_{n=1}^{\infty}\left[\left(\frac{1}{3}\right)^n + \left(-\frac{2}{5}\right)^{n-1}\right] = \sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty}\left(-\frac{2}{5}\right)^{n-1}$$

Both are geometric series.

$$\begin{aligned}&= \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{n+1} + \sum_{n=0}^{\infty}\left(-\frac{2}{5}\right)^n \\ &= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n + \sum_{n=0}^{\infty}\left(-\frac{2}{5}\right)^n\end{aligned}$$

We use Lemma 3.2.5 with $a_1 = \frac{1}{3}$ and $r_1 = \frac{1}{3}$, then with $a_2 = 1$ and $r_2 = -\frac{2}{5}$.

$$\begin{aligned}&= \frac{1}{3}\cdot\frac{1}{1-\frac{1}{3}} + \frac{1}{1+\frac{2}{5}} \\ &= \frac{1}{2} + \frac{5}{7} = \frac{17}{14}\end{aligned}$$

3.2.2.26. *. Solution. We split the sum into two parts.

$$\sum_{n=0}^{\infty}\frac{1+3^{n+1}}{4^n} = \sum_{n=0}^{\infty}\frac{1}{4^n} + \sum_{n=0}^{\infty}\frac{3^{n+1}}{4^n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{4^n} + 3\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n$$

Using Lemma 3.2.5,

$$\begin{aligned} &= \frac{1}{1-\frac{1}{4}} + \frac{3}{1-\frac{3}{4}} \\ &= \frac{4}{3} + 12 = \frac{40}{3} \end{aligned}$$

3.2.2.27. Solution. Using logarithm rules, we see

$$\sum_{n=5}^{\infty} \log\left(\frac{n-3}{n}\right) = \sum_{n=5}^{\infty}\big[\log(n-3) - \log n\big]$$

which looks like a telescoping series. Let's make a table to figure out the partial sums.

$\mathbf{n}$	$\log(\mathbf{n-3})$	$-\log \mathbf{n}$	$\mathbf{s_n}$
5	$\log 2$	$-\log 5$	$\log 2 - \log 5$
6	$\log 3$	$-\log 6$	$\log 2 + \log 3 - \log 5 - \log 6$
7	$\log 4$	$-\log 7$	$\log 2 + \log 3 + \log 4 - \log 5 - \log 6 - \log 7$
8	$\log 5$	$-\log 8$	$\log 2 + \log 3 + \log 4 - \log 6 - \log 7 - \log 8$
9	$\log 6$	$-\log 9$	$\log 2 + \log 3 + \log 4 - \log 7 - \log 8 - \log 9$
10	$\log 7$	$-\log 10$	$\log 2 + \log 3 + \log 4 - \log 8 - \log 9 - \log 10$
11	$\log 8$	$-\log 11$	$\log 2 + \log 3 + \log 4 - \log 9 - \log 10 - \log 11$
$\vdots$			

There is a "lag" before the terms cancel, which is why they "build up" more than we saw in past examples. Still, we can clearly see the Nth partial sum:

$$\begin{aligned} &\sum_{n=5}^{N}\Big(\log(n-3) - \log(n)\Big) \\ &\qquad = \log 2 + \log 3 + \log 4 - \log(N-2) - \log(N-1) - \log(N) \\ &\qquad = \log\left(\frac{24}{N(N-1)(N-2)}\right) \end{aligned}$$

when $N \geq 7$. So,

$$\begin{aligned} \sum_{n=5}^{\infty}\Big(\log(n-3) - \log(n)\Big) &= \lim_{N\to\infty} s_N \\ &= \lim_{N\to\infty} \log\left(\frac{24}{N(N-1)(N-2)}\right) \\ &= -\infty \end{aligned}$$

3.2.2.28. Solution. This is a telescoping series. Let's investigate it in the usual way. To make the pattern of cancellation clearer, we express $\frac{2}{n} - \frac{1}{n+1} - \frac{1}{n-1} = \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n-1}$, and leave the fractions in the middle of the table unsimplified. Then every fraction has numerator one and two terms with the same denominator and opposite sign cancel.

$\mathbf{n}$	$\frac{\mathbf{1}}{\mathbf{n}}$	$\frac{\mathbf{1}}{\mathbf{n}}$	$-\frac{\mathbf{1}}{\mathbf{n+1}}$	$-\frac{\mathbf{1}}{\mathbf{n-1}}$	$\mathbf{s_n}$
2	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{1}$	$\frac{1}{2}+\frac{1}{2}-\frac{1}{3}-\frac{1}{1}$
3	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{2}+\frac{1}{3}-\frac{1}{4}-\frac{1}{1}$
4	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{5}$	$-\frac{1}{3}$	$\frac{1}{2}+\frac{1}{4}-\frac{1}{5}-\frac{1}{1}$
5	$\frac{1}{5}$	$\frac{1}{5}$	$-\frac{1}{6}$	$-\frac{1}{4}$	$\frac{1}{2}+\frac{1}{5}-\frac{1}{6}-\frac{1}{1}$
6	$\frac{1}{6}$	$\frac{1}{6}$	$-\frac{1}{7}$	$-\frac{1}{5}$	$\frac{1}{2}+\frac{1}{6}-\frac{1}{7}-\frac{1}{1}$
7	$\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{8}$	$-\frac{1}{6}$	$\frac{1}{2}+\frac{1}{7}-\frac{1}{8}-\frac{1}{1}$
8	$\frac{1}{8}$	$\frac{1}{8}$	$-\frac{1}{9}$	$-\frac{1}{7}$	$\frac{1}{2}+\frac{1}{8}-\frac{1}{9}-\frac{1}{1}$
$\vdots$					

Concentrate on any row n, except the very first row and the very last row. The first $\frac{1}{n}$ in that row cancels the $-\frac{1}{n}$ in the middle of the row above it, and the second $\frac{1}{n}$ in that row cancels the $-\frac{1}{n}$ at the end of the row below it. As far as the first $(n = 2)$ row is concerned, the first $\frac{1}{2}$ and the last $-\frac{1}{1}$ never get cancelled out because there is no row above the first one. And as far as the very last row is concerned, the two middle terms never get cancelled out because there is no row after the last one. So the partial sum is

$$s_N = \sum_{n=2}^{N} \left(\frac{2}{n} - \frac{1}{n+1} - \frac{1}{n-1} \right) = \overbrace{\frac{1}{2} - \frac{1}{1}}^{\text{from the first row}} + \overbrace{\frac{1}{N} - \frac{1}{N+1}}^{\text{from the last row}}$$

There is another purely algebraic way to find the same s_N, motivated by the above discussion.

$$\begin{aligned} s_N &= \sum_{n=2}^{N} \left(\frac{2}{n} - \frac{1}{n+1} - \frac{1}{n-1} \right) \\ &= \sum_{n=2}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) + \sum_{n=2}^{N} \left(\frac{1}{n} - \frac{1}{n-1} \right) \end{aligned}$$

The first half

$$\sum_{n=2}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) + \cdots$$

$$+\left(\frac{1}{N}-\frac{1}{N+1}\right)$$
$$=\left[\frac{1}{2}-\frac{1}{N+1}\right]$$

and the second half

$$\sum_{n=2}^{N}\left(\frac{1}{n}-\frac{1}{n-1}\right)=\left(\frac{1}{2}-\frac{1}{1}\right)+\left(\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{4}-\frac{1}{3}\right)+\cdots$$
$$+\left(\frac{1}{N}-\frac{1}{N-1}\right)$$
$$=\left\{-\frac{1}{1}+\frac{1}{N}\right\}$$

So

$$\begin{aligned} s_N &= \left[\frac{1}{2}-\frac{1}{N+1}\right]+\left\{-\frac{1}{1}+\frac{1}{N}\right\} \\ &= -\frac{1}{2}+\frac{1}{N}-\frac{1}{N+1} \end{aligned}$$

and the limit

$$\begin{aligned} \sum_{n=2}^{\infty}\left(\frac{2}{n}-\frac{1}{n+1}-\frac{1}{n-1}\right) &= \lim_{N\to\infty} s_N \\ &= \lim_{N\to\infty}\left[-\frac{1}{2}+\frac{1}{N}-\frac{1}{N+1}\right] \\ &= -\frac{1}{2} \end{aligned}$$

Exercises — Stage 3

3.2.2.29. Solution. The stone at position x has mass $\frac{1}{4^x}$ kg, and we have to pull it a distance of 2^x metres, so the work involved in moving that one stone is

$$\left(\frac{1}{4^x}\text{ kg}\right)\left(2^x\text{ m}\right)\left(9.8\frac{\text{m}}{\text{sec}^2}\right)=\frac{9.8}{2^x}\text{ J}$$

Therefore, the work to move *all* the stones is:

$$\begin{aligned} \sum_{x=1}^{\infty}\frac{9.8}{2^x} &= \sum_{x=0}^{\infty}\frac{9.8}{2^{x+1}} \\ &= \sum_{x=0}^{\infty}\frac{9.8}{2}\cdot\frac{1}{2^x}=\frac{9.8}{2}\cdot\frac{1}{1-\frac{1}{2}}=9.8\text{ J} \end{aligned}$$

3.2.2.30. Solution. The volume of a sphere of radius $\frac{1}{\pi^n}$ is

$$v_n = \frac{4}{3}\pi\left(\frac{1}{\pi^n}\right)^3 = \frac{4\pi}{3}\left(\frac{1}{\pi^3}\right)^n$$

So, the volume of all the spheres together is:

$$\begin{aligned}\sum_{n=1}^{\infty} v_n &= \sum_{n=1}^{\infty}\frac{4\pi}{3}\left(\frac{1}{\pi^3}\right)^n\\ &= \sum_{n=0}^{\infty}\frac{4\pi}{3}\left(\frac{1}{\pi^3}\right)^{n+1}\\ &= \sum_{n=0}^{\infty}\frac{4}{3\pi^2}\left(\frac{1}{\pi^3}\right)^n\end{aligned}$$

We use Lemma 3.2.5 with $a = \frac{4}{3\pi^2}$ and $r = \frac{1}{\pi^3}$.

$$= \frac{4}{3\pi^2}\cdot\frac{1}{1-\frac{1}{\pi^3}} = \frac{4\pi}{3\left(\pi^3-1\right)}$$

3.2.2.31. Solution. Let's make a table. Keep in mind $\cos^2\theta + \sin^2\theta = 1$.

$\mathbf{n}$	$\frac{\sin^2 n}{2^n}$	$\frac{\cos^2(n+1)}{2^{n+1}}$	s_n
3	$\frac{\sin^2 3}{2^3}$	$\frac{\cos^2 4}{2^4}$	$\frac{\sin^2 3}{2^3}+\frac{\cos^2 4}{2^4}$
4	$\frac{\sin^2 4}{2^4}$	$\frac{\cos^2 5}{2^5}$	$\frac{\sin^2 3}{2^3}+\frac{1}{2^4}+\frac{\cos^2 5}{2^5}$
5	$\frac{\sin^2 5}{2^5}$	$\frac{\cos^2 6}{2^6}$	$\frac{\sin^2 3}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{\cos^2 6}{2^6}$
6	$\frac{\sin^2 6}{2^6}$	$\frac{\cos^2 7}{2^7}$	$\frac{\sin^2 3}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}+\frac{\cos^2 7}{2^7}$
7	$\frac{\sin^2 7}{2^7}$	$\frac{\cos^2 8}{2^8}$	$\frac{\sin^2 3}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}+\frac{1}{2^7}+\frac{\cos^2 8}{2^8}$
$\vdots$			

This gives us an equation for the partial sum s_N, when $N \geq 4$:

$$\begin{aligned}s_N &= \sum_{n=3}^{N}\left(\frac{\sin^2 n}{2^n}+\frac{\cos^2(n+1)}{2^{n+1}}\right)\\ &= \frac{\sin^2 3}{2^3}+\left(\sum_{n=4}^{N}\frac{1}{2^n}\right)+\frac{\cos^2(N+1)}{2^{N+1}}\end{aligned}$$

Using Definition 3.2.3, our series evaluates to:

$$\lim_{N\to\infty} s_N = \lim_{N\to\infty}\left[\frac{\sin^2 3}{2^3}+\left(\sum_{n=4}^{N}\frac{1}{2^n}\right)+\frac{\cos^2(N+1)}{2^{N+1}}\right]$$

$$= \frac{\sin^2 3}{8} + \left[\lim_{N\to\infty} \frac{\cos^2(N+1)}{2^{N+1}}\right] + \sum_{n=4}^{\infty} \frac{1}{2^n}$$

We evaluate the limit using the squeeze theorem; the series is geometric.

$$\begin{aligned}
&= \frac{\sin^2 3}{8} + 0 + \sum_{n=0}^{\infty} \frac{1}{2^{n+4}} \\
&= \frac{\sin^2 3}{8} + \frac{1}{2^4}\sum_{n=0}^{\infty} \frac{1}{2^n}
\end{aligned}$$

Using Lemma 3.2.5,

$$\begin{aligned}
&= \frac{\sin^2 3}{8} + \frac{1}{2^4}\frac{1}{1-\frac{1}{2}} \\
&= \frac{\sin^2 3}{8} + \frac{1}{8} \approx 0.1275
\end{aligned}$$

3.2.2.32. Solution. Since $\{\mathbb{S}_M\}$ is the sequence of partial sums of $\displaystyle\sum_{N=1}^{\infty} S_N$, we can find $\{S_N\}$ from $\{\mathbb{S}_M\}$ as in Question 3:

$$\begin{aligned}
S_N &= \mathbb{S}_N - \mathbb{S}_{N-1} = \frac{N+1}{N} - \frac{N}{N-1} = -\frac{1}{N(N-1)} && \text{if } N \geq 2, \\
S_1 &= \mathbb{S}_1 = 2
\end{aligned}$$

Similarly, we find $\{a_n\}$ from $\{S_N\}$. Do be careful: S_N only follows the formula we found above when $N \geq 2$. In the next line, we use an expression containing S_{n-1}; in order for the subscript to be at least two (so the formula fits), we need $n \geq 3$.

$$\begin{aligned}
a_n &= S_n - S_{n-1} = -\frac{1}{n(n-1)} - \frac{-1}{(n-1)(n-2)} \\
&= \frac{2}{n(n-1)(n-2)} && \text{if } n \geq 3, \\
a_2 &= S_2 - S_1 = -\frac{1}{2(2-1)} - 2 = -\frac{5}{2}, \\
a_1 &= S_1 = 2
\end{aligned}$$

All together,

$$a_n = \begin{cases} \frac{2}{n(n-1)(n-2)} & \text{if } n \geq 3, \\ -\frac{5}{2} & \text{if } n = 2, \\ 2 & \text{if } n = 1 \end{cases}$$

3.2.2.33. Solution. We consider a circle of radius R, with an “inner ring” from

$\frac{R}{3}$ to $\frac{2R}{3}$ and an "outer ring" from $\frac{2R}{3}$ to R.
The area of the outer ring is:

$$\pi R^2 - \pi\left(\frac{2R}{3}\right)^2 = \frac{5}{9}\pi R^2$$

The area of the inner ring is:

$$\pi\left(\frac{2R}{3}\right)^2 - \pi\left(\frac{R}{3}\right)^2 = \frac{3}{9}\pi R^2$$

So, the ratio of the inner ring's area to the outer ring's area is $\frac{3}{5}$.
In our bullseye diagram, if we pair up any red ring with the blue ring just inside it, the blue ring has $\frac{3}{5}$ the area of the red ring. So, the blue portion of the bullseye has $\frac{3}{5}$ the area of the red portion.
Since the circle has area 1, if we let the red portion have area A, then

$$1 = A + \frac{3}{5}A = \frac{8}{5}A$$

So, the red portion has area $\frac{5}{8}$.

3.3 · Convergence Tests

3.3.11 · Exercises

Exercises — Stage 1

3.3.11.1. Solution.

A $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, so the divergence test is inconclusive. It's true that this series diverges, but we can't show it using the divergence test.

B $\lim\limits_{n\to\infty} \dfrac{n^2}{n+1} = \infty$, which is not zero, so the divergence test tells us this series diverges.

C We'll show below that $\lim\limits_{n\to\infty} \sin n$ does not exist. In particular, it is not zero. Therefore, the divergence test tells us this series diverges.

Now we'll show that $\lim\limits_{n\to\infty} \sin n$ does not exist. Suppose that it does exist and takes the value S. We will now see that this assumption leads to a contradiction. Add together the two trig identities (see Appendix A.8)

$$\sin(n+1) = \sin(n)\cos(1) + \cos(n)\sin(1)$$
$$\sin(n-1) = \sin(n)\cos(1) - \cos(n)\sin(1)$$

This gives

$$\sin(n+1) + \sin(n-1) = 2\sin(n)\cos(1)$$

Taking the limit $n \to \infty$ gives $2S = 2S\cos(1)$. Since $\cos(1) \neq 1$, this forces $S = 0$. Now the first trig identity above gives

$$\cos(n) = \frac{\sin(n+1) - \sin(n)\cos(1)}{\sin(1)}$$

Taking the limit as $n \to \infty$ of that gives

$$\lim_{n\to\infty} \cos(n) = \frac{S - S\cos(1)}{\sin(1)} = 0$$

But that provides the contradiction. Because $\sin^2(n) + \cos^2(n) = 1$, we can't have both $\sin(n)$ and $\cos(n)$ converging to zero. So $\lim\limits_{n\to\infty} \sin n$ does not exist.

D For all whole numbers n, $\sin(\pi n) = 0$, so $\lim\limits_{n\to\infty} \sin(\pi n) = 0$ and the divergence test is inconclusive.

3.3.11.2. Solution. Let $f(x)$ be a function with $f(n) = a_n$ for all whole numbers n. In order to apply the integral test (Theorem 3.3.5) we need $f(x)$ to be positive and decreasing for all sufficiently large values of n.

A $f(x) = \frac{1}{x}$, which is positive and decreasing for all $x \geq 1$, so the integral test does apply here.

B $f(x) = \frac{x^2}{x+1}$, which is not decreasing — in fact, it goes to infinity. So, the integral test does not apply here. (The divergence test tells us the series diverges, though.)

C $f(x) = \sin x$, which is neither consistently positive nor consistently decreasing, so the integral test does not apply. (The divergence test tells us the series diverges, though.)

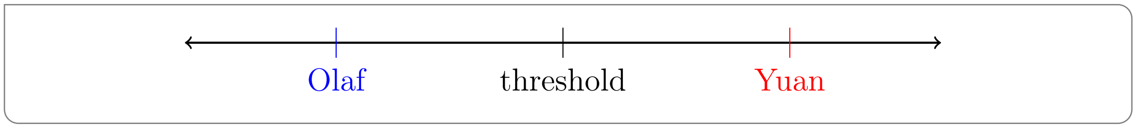
D $f(x) = \frac{\sin x + 1}{x^2}$ is positive for all whole numbers n. To determine whether it is decreasing, we consider its derivative.

$$f'(x) = \frac{x^2(\cos x) - (\sin x + 1)(2x)}{x^4} = \frac{x\cos x - 2\sin x - 2}{x^3}$$

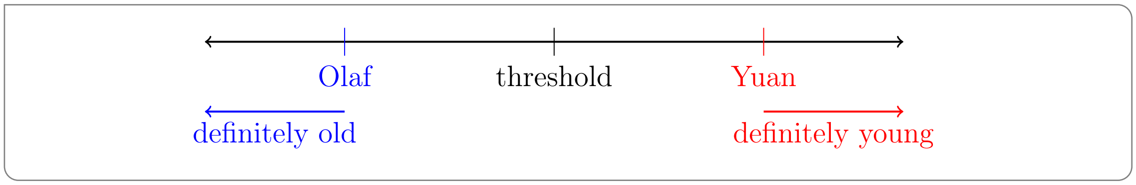
This is sometimes positive, and sometimes negative. (For example, if $x = 100\pi$, $f'(x) = \frac{100\pi - 0 - 2}{(100\pi)^3} > 0$, but if $x = 101\pi$ then $f'(x) = \frac{101\pi(-1) - 0 - 2}{(101\pi)^3} < 0$.) Then $f(x)$ is not a decreasing function, so the integral test does not apply.

3.3.11.3. Solution. (a) If Olaf is old, and I am *even older*, then I am old as well.
(b) If Olaf is old, and I am {not as old}, then perhaps I am old as well (just slightly less so), or perhaps I am young. There is not enough information to tell.
(c) If Yuan is young, and I am older, then perhaps I am much older and I am old, or perhaps I am only a little older, and I am young. There is not enough information

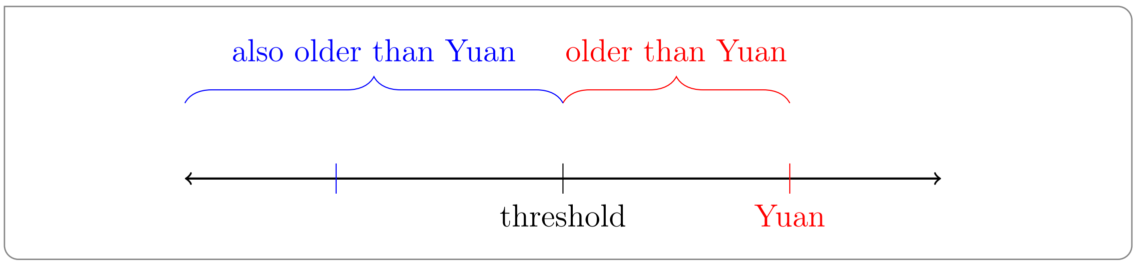
to tell.
(d) If Yuan is young, and I am *even younger*, then I must also be young.
Another way to think about this is with a timeline of birthdates. People born before the threshold are old, and people born after it are young.



If I'm born before (older than) Olaf, I'm born before the threshold, so I'm old. If I'm born after (younger than) Yuan I'm born after the threshold, so I'm young.



If I'm born after Olaf or before Yuan, I don't know which side of the threshold I'm on. I could be old or I could be young.



3.3.11.4. Solution. The comparison test is Theorem 3.3.8. However, rather than trying to memorize which way the inequalities go in all cases, we use the same reasoning as Question 3.
If a sequence has positive terms, it either converges, or it diverges to infinity, with the partial sums increasing and increasing without bound. If one sequence diverges, and the other sequence is larger, then the other sequence diverges — just like being older than an old person makes you old.
If $\sum a_n$ converges, and $\{a_n\}$ is the red (larger) series, then $\sum b_n$ converges: it's smaller than a sequence that doesn't add up to infinity, so it too does not add up to infinity.
If $\sum a_n$ diverges, and $\{a_n\}$ is the blue (smaller) series, then $\sum b_n$ diverges: it's larger than a sequence that adds up to infinity, so it too adds up to infinity.
In the other cases, we can't say anything. If $\{a_n\}$ is the red (larger) series, and $\sum a_n$ diverges, then perhaps $\{b_n\}$ behaves similarly to $\{a_n\}$ and $\sum b_n$ diverges, or perhaps $\{b_n\}$ is much, much smaller than $\{a_n\}$ and $\sum b_n$ converges.
Similarly, if $\{a_n\}$ is the blue (smaller) series, and $\sum a_n$ converges, then perhaps $\{b_n\}$ behaves similarly to $\{a_n\}$ and $\sum b_n$ converges, or perhaps $\{b_n\}$ is much, much bigger than $\{a_n\}$ and $\sum b_n$ diverges.

	if $\sum a_n$ converges	if $\sum a_n$ diverges
and if $\{a_n\}$ is the red series	then $\sum b_n$ CONVERGES	inconclusive
and if $\{a_n\}$ is the blue series	inconclusive	then $\sum b_n$ DIVERGES

3.3.11.5. Solution. (a) Since $\sum \frac{1}{n}$ is divergent, we can only use it to prove series with *larger* terms are divergent. This is the case here, since $\frac{1}{n-1} > \frac{1}{n}$. So, the direct comparison test is valid.
For the limit comparison test, we calculate:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n-1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1-\frac{1}{n}} = 1$$

Since the limit exists and is not zero, the limit comparison test is also valid.
(b) Since the series $\sum \frac{1}{n^2}$ converges, we can only use the direct comparison test to show the convergence of a series if its terms have *smaller* absolute values. Indeed,

$$\left|\frac{\sin n}{n^2+1}\right| = \frac{|\sin n|}{n^2+1} < \frac{1}{n^2}$$

so the series are set for a direct comparison.
To check whether a limit comparison will work, we compute:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sin n}{n^2+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} \sin n = \lim_{n\to\infty} (1) \sin n$$

The limit does not exist, so the limit comparison test is not a valid test to compare these two series.
(c) Since the series $\sum \frac{1}{n^3}$ converges, we can only use the direct comparison test to conclude something about a series with *smaller* terms. However,

$$\frac{n^3+5n+1}{n^6-2} > \frac{n^3}{n^6} = \frac{1}{n^3}.$$

Therefore the direct comparison test does not apply to this pair of series.
For the limit comparison test, we calculate:

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{n^3+5n+1}{n^6-2}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3+5n+1}{n^3-\frac{2}{n^3}} \left(\frac{\frac{1}{n^3}}{\frac{1}{n^3}}\right) \\ &= \lim_{n\to\infty} \frac{1+\frac{5}{n^2}+\frac{1}{n^3}}{1-\frac{2}{n^6}} = 1\end{aligned}$$

Since the limit is a nonzero real number, we can use the limit comparison test to compare this pair of series.
(d) Since the series $\sum \frac{1}{\sqrt[4]{n}}$ diverges, we can only use the direct comparison test to show that a series with *larger* terms diverges. However,

$$\frac{1}{\sqrt{n}} < \frac{1}{\sqrt[4]{n}}$$

so the direct comparison test isn't valid with this pair of series.
For the limit comparison test, we calculate:

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt[4]{n}}}=\lim_{n\to\infty}\frac{1}{\sqrt[4]{n}}=0$$

Since the limit is zero, the limit comparison test doesn't apply.

3.3.11.6. Solution. It diverges by the divergence test, because $\lim_{n\to\infty} a_n \neq 0$.

3.3.11.7. Solution. The divergence test (Theorem 3.3.1) is inconclusive when $\lim_{n\to\infty} a_n = 0$. We cannot use the divergence test to show that a series converges.

3.3.11.8. Solution. The integral test does not apply because $f(x)$ is not decreasing.

3.3.11.9. Solution. The inequality goes the wrong way, so the direct comparison test (with this comparison series) is inconclusive.

3.3.11.10. Solution. Although the terms of (A) are sometimes negative and sometimes positive, they are not strictly alternating in a positive-negative-positive-negative pattern. For instance, $\sin 1$ and $\sin 2$ are both postive. So, (A) is not an alternating series.
When n is a whole number, $\cos(\pi n) = (-1)^n$, so (B) is alternating.
Since the exponent of $(-n)$ in (C) is even, the terms are always positive. Therefore (C) is not alternating.
(D) is an alternating series.

3.3.11.11. Solution. One possible answer: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$. This series converges (it's a p-series with $p = 2 > 1$), but if we take the ratio of consecutive terms:

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{n^2}{(n+1)^2}=1$$

The limit of the ratio is 1, so the ratio test is inconclusive.

3.3.11.12. Solution. By the divergence test, for a series $\sum a_n$ to *converge*, we need $\lim_{n\to\infty} a_n = 0$. That is, the magnitude (absolute value) of the terms needs to be getting smaller. If $\displaystyle\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| < 1$ or (equivalently) $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, then $|a_{n+1}| > |a_n|$ for sufficiently large n, so the terms are actually *growing* in magnitude. That means the series diverges, by the divergence test.

3.3.11.13. Solution. The terms of the series only see a small portion of the domain of the integral. We can try to think of a function $f(x)$ that behaves "nicely" when x is a whole number (that is, it produces a sequence whose sum converges), but is more unruly when x is not a whole number.
For example, suppose $f(x) = \sin(\pi x)$. Then $f(x) = 0$ for every integer x, but this

is not representative of the function as a whole. Indeed, our corresponding series has terms $\{a_n\} = \{0, 0, 0, \ldots\}$.

- $\displaystyle\int_1^\infty \sin(\pi x)\,\mathrm{d}x = \lim_{R\to\infty}\left[-\frac{1}{\pi}\cos(\pi x)\right]_1^R = \lim_{R\to\infty}\Big[-\cos(\pi R)\Big] - \frac{1}{\pi}$ Since the limit does not exist, the integral diverges.

- $\displaystyle\sum_{n=1}^{\infty}\sin(\pi n) = \sum_{n=1}^{\infty} 0 = 0$. The series converges.

3.3.11.14. *. Solution. When n is very large, the term 2^n dominates the numerator, and the term 3^n dominates the denominator. So when n is very large $a_n \approx \frac{2^n}{3^n}$. Therefore we should take $b_n = \dfrac{2^n}{3^n}$. Note that, with this choice of b_n,

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2^n+n}{3^n+1}\,\frac{3^n}{2^n} = \lim_{n\to\infty}\frac{1+n/2^n}{1+1/3^n} = 1$$

as desired.

3.3.11.15. *. Solution. (a) In general false. The harmonic series $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ diverges by the p-test with $p = 1$.
(b) Be careful. You were not told that the a_n's are positive. So this is false in general. If $a_n = (-1)^n\frac{1}{n}$, then $\sum\limits_{n=1}^{\infty}(-1)^n a_n$ is again the harmonic series $\sum\limits_{n=1}^{\infty}\frac{1}{n}$, which diverges.
(c) In general false. Take, for example, $a_n = 0$ and $b_n = 1$.

Exercises — Stage 2

3.3.11.16. *. Solution. First, we'll check the divergence test. It doesn't always work, but if it does, it's likely the easiest path.

$$\lim_{n\to\infty}\frac{n^2}{3n^2+\sqrt{n}}\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right) = \lim_{n\to\infty}\frac{1}{3+\frac{1}{n\sqrt{n}}} = \frac{1}{3} \neq 0$$

Since the limit of the terms being added is not zero, the series diverges by the divergence test.

3.3.11.17. *. Solution. This precise question was asked on a 2014 final exam. Note that the n^{th} term in the series is $a_n = \frac{5^k}{4^k+3^k}$ and does *not* depend on n! There are two possibilities. Either this was intentional (and the instructor was being particularly nasty) or it was a typo and the intention was to have $a_n = \frac{5^n}{4^n+3^n}$. In both cases, the limit

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{5^k}{4^k+3^k} = \frac{5^k}{4^k+3^k} \neq 0$$
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{5^n}{4^n+3^n} = \lim_{n\to\infty}\frac{(5/4)^n}{1+(3/4)^n} = +\infty \neq 0$$

is nonzero, so the series diverges by the divergence test.

3.3.11.18. *. Solution. We usually check the divergence test first, to look for low-hanging fruit. The limit of the terms being added is zero:

$$\lim_{n\to\infty} \frac{1}{n+\frac{1}{2}} = 0$$

so the divergence test is inconclusive. That is, we need to look harder.
Next, we might consider a comparison test — these can also provide us (if we're lucky) with an easy path. The terms we're adding look somewhat like $\frac{1}{n}$, but our terms are *smaller* than these terms, which form the terms of the *divergent* harmonic series. So, a direct comparison seems unlikely. Now we search for more exotic tests.
Let $f(x) = \dfrac{1}{x+\frac{1}{2}}$. Note $f(x)$ is positive and decreases as x increases. So, by the integral test, which is Theorem 3.3.5, the given series converges if and only if the integral $\int_0^\infty \frac{1}{x+\frac{1}{2}}\,\mathrm{d}x$ converges. Since

$$\begin{aligned}
\int_0^\infty \frac{1}{x+\frac{1}{2}}\,\mathrm{d}x = \lim_{R\to\infty}\int_0^R \frac{1}{x+\frac{1}{2}}\,\mathrm{d}x &= \lim_{R\to\infty}\left[\log\left(x+\frac{1}{2}\right)\right]_{x=0}^{x=R}\\
&= \lim_{R\to\infty}\left[\log\left(R+\frac{1}{2}\right) - \log\frac{1}{2}\right]
\end{aligned}$$

diverges, the series diverges.

3.3.11.19. Solution. The terms of the series tend to 0, so we can't use the divergence test.
To generate a guess about its convergence, we do the following:

$$\sum \frac{1}{\sqrt{k}\sqrt{k+1}} = \sum \frac{1}{\sqrt{k^2+k}} \approx \sum \frac{1}{\sqrt{k^2}} = \sum \frac{1}{k}$$

We guess that our series behaves like the harmonic series, and the harmonic series diverges (which can be demonstrated by p-test or integral test). So, we guess that our series diverges. However, in order to directly compare our series to the harmonic series and show our series diverges, our terms would have to be *bigger* than the terms in the harmonic series, and this is not the case. So, we use limit comparison.

$$\begin{aligned}
\frac{\frac{1}{k}}{\frac{1}{\sqrt{k^2+k}}} &= \frac{\sqrt{k^2+k}}{k} = \frac{\sqrt{k^2+k}}{\sqrt{k^2}} = \sqrt{\frac{k^2+k}{k^2}} = \sqrt{1+\frac{1}{k}}, \text{ so}\\
\lim_{k\to\infty}\frac{\frac{1}{k}}{\frac{1}{\sqrt{k^2+k}}} &= \lim_{k\to\infty}\sqrt{1+\frac{1}{k}} = 1
\end{aligned}$$

Since 1 is a real number greater than 0, by the Limit Comparison Test, $\sum \frac{1}{\sqrt{k}\sqrt{k+1}}$ diverges, like $\sum \frac{1}{k}$.

3.3.11.20. Solution. This is a geometric series with $r = 1.001$. Since $|r| > 1$, it is divergent.

3.3.11.21. Solution. This is a geometric series with $r = \frac{-1}{5}$. Since $|r| < 1$, it is convergent.
We want to use the formula $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$, but our series does not start at 0, so we re-write it:

$$\begin{aligned}\sum_{n=3}^{\infty}\left(\frac{-1}{5}\right)^n &= \sum_{n=0}^{\infty}\left(\frac{-1}{5}\right)^n - \sum_{n=0}^{2}\left(\frac{-1}{5}\right)^n \\ &= \frac{1}{1-(-1/5)} - \left(1 - \frac{1}{5} + \frac{1}{25}\right) \\ &= \frac{1}{6/5} - 1 + \frac{1}{5} - \frac{1}{25} = \frac{5}{6} + \frac{-25+5-1}{25} \\ &= -\frac{1}{150}\end{aligned}$$

3.3.11.22. Solution. For any integer n, $\sin(\pi n) = 0$, so $\sum \sin(\pi n) = \sum 0 = 0$. So, this series converges.

3.3.11.23. Solution. For any integer n, $\cos(\pi n) = \pm 1$, so $\lim\limits_{n\to\infty} \cos(\pi n) \neq 0$. By the divergence test, this series diverges.

3.3.11.24. Solution. Factorials grow super fast. Like, wow, really fast. Even faster than exponentials. So the terms are going to zero, and the divergence test won't help us. Let's use ratio — it's a good go-to test with factorials.

$$\begin{aligned}\frac{a_{k+1}}{a_k} &= \frac{\frac{e^{k+1}}{(k+1)!}}{\frac{e^k}{k!}} = \frac{e^{k+1}}{e^k}\cdot\frac{k!}{(k+1)!} \\ &= e\cdot\frac{k(k-1)\cdots(1)}{(k+1)(k)(k-1)\cdots(1)} = e\cdot\frac{1}{k+1} = \frac{e}{k+1}\end{aligned}$$

Since e is a constant,

$$\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\frac{e}{k+1} = 0$$

Since $0 < 1$, by the ratio test, the series converges.

3.3.11.25. Solution. This is close to being in the form of a geometric series. First, we should have our powers be k, not $k+2$, but we notice $3^{k+2} = 3^k 3^2 = 9\cdot 3^2$, so:

$$\sum_{k=0}^{\infty}\frac{2^k}{3^{k+2}} = \sum_{k=0}^{\infty}\frac{2^k}{9\cdot 3^k} = \frac{1}{9}\sum_{k=0}^{\infty}\frac{2^k}{3^k} = \frac{1}{9}\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k$$

Now it looks like a geometric series with $r = \frac{2}{3}$

$$= \frac{1}{9}\left(\frac{1}{1-(2/3)}\right) = \frac{1}{3}$$

In conclusion: this (geometric) series is convergent, and its sum is $\dfrac{1}{3}$.

3.3.11.26. Solution. Usually with factorials, we want to use the divergence test or the ratio test. Since the terms are indeed tending towards zero, we are left with the ratio test.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\frac{(n+1)!(n+1)!}{(2n+2)!}}{\frac{n!n!}{(2n)!}} = \frac{(n+1)!(n+1)!}{n!n!}\cdot\frac{(2n)!}{(2n+2)!}\\ &= \frac{(n+1)(n)(n-1)\cdots(1)}{n(n-1)\cdots(1)}\cdot\frac{(n+1)(n)(n-1)\cdots(1)}{n(n-1)\cdots(1)}\cdot\frac{(2n)(2n-1)(2n-2)\cdots(1)}{(2n+2)(2n+1)(2n)(2n-1)(2n-2)\cdots(1)}\\ &= (n+1)(n+1)\cdot\frac{1}{(2n+2)(2n+1)}\end{aligned}$$

So

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \lim_{n\to\infty}\frac{(n+1)(n+1)}{2(n+1)(2n+1)}\\ &= \lim_{n\to\infty}\frac{n+1}{4n+2} = \frac{1}{4}\end{aligned}$$

Since the limit is a number less than 1, the series converges by the ratio test.

3.3.11.27. Solution. We want to make an estimation, when n gets big:

$$\frac{n^2+1}{2n^4+n}\approx\frac{n^2}{2n^4}=\frac{1}{2n^2}$$

Since $\sum\frac{1}{2n^2}$ is a convergent series (by p-test, or integral test), we guess that our series is convergent as well. If we wanted to use comparison test, we should have to show $\frac{n^2+1}{2n^4+n}<\frac{1}{2n^2}$, which seems unpleasant, so let's use limit comparison.

$$\begin{aligned}\lim_{n\to\infty}\frac{\frac{n^2+1}{2n^4+n}}{\frac{1}{2n^2}} &= \lim_{n\to\infty}\frac{(n^2+1)2n^2}{2n^4+n} = \lim_{n\to\infty}\frac{2n^4+2n^2}{2n^4+n}\left(\frac{1/n^4}{1/n^4}\right)\\ &= \lim_{n\to\infty}\frac{2+\frac{2}{n^2}}{2+\frac{1}{n^3}} = 1\end{aligned}$$

Since the limit is a positive finite number, by the Limit Comparison Test, $\sum\frac{n^1+1}{2n^4+n}$ does the same thing $\sum\frac{1}{2n^2}$ does: it converges.

3.3.11.28. $*$. Solution. First, we rule out some of the easier tests. The limit of the terms being added is zero, so the divergence test is inconclusive. The terms being added are smaller than the terms of the (divergent) harmonic series, $\sum\frac{1}{n}$, so we can't directly compare these two series, and there isn't another obvious series to compare ours to. However, the terms being added seem like a function we could integrate.

Let $f(x)=\dfrac{5}{x(\log x)^{3/2}}$. Then $f(x)$ is positive and decreases as x increases. So the sum $\sum_{3}^{\infty} f(n)$ and the integral $\int_3^\infty f(x)\,\mathrm{d}x$ either both converge or both diverge, by the integral test, which is Theorem 3.3.5. For the integral, we use the substitution $u = \log x$, $\mathrm{d}u = \frac{\mathrm{d}x}{x}$ to get

$$\int_3^\infty \frac{5\,\mathrm{d}x}{x(\log x)^{3/2}} = \int_{\log 3}^\infty \frac{5\,\mathrm{d}u}{u^{3/2}}$$

which converges by the p-test (which is Example 1.12.8) with $p = \frac{3}{2} > 1$.

3.3.11.29. *. Solution. Let $f(x) = \frac{1}{x(\log x)^p}$. Then $f(x)$ is positive for $n \geq 3$, and $f(x)$ decreases as x increases. So, we can use the integral test, Theorem 3.3.5.

$$\int_2^\infty \frac{1}{x(\log x)^p}\mathrm{d}x = \lim_{R\to\infty}\int_2^R \frac{1}{(\log x)^p}\frac{\mathrm{d}x}{x} = \lim_{R\to\infty}\int_{\log 2}^{\log R}\frac{1}{u^p}\mathrm{d}u \quad \text{with },$$

Using the results about p-series, Example 3.3.6, we know this integral converges if and only if $p > 1$, so the same is true for the series by the integral test.

3.3.11.30. *. Solution. As usual, let's see whether the "easy" tests work. The terms we're adding converge to zero:

$$\lim_{n\to\infty}\frac{e^{-\sqrt{n}}}{\sqrt{n}} = \lim_{n\to\infty}\frac{1}{\sqrt{n}e^{\sqrt{n}}} = 0$$

so the divergence test is inconclusive. Our series isn't geometric, and it doesn't seem obvious how to compare it to a geometric series. However, the terms we're adding seem like they would make an integrable function.

Set $f(x) = \frac{e^{-\sqrt{x}}}{\sqrt{x}}$. For $x \geq 1$, this function is positive and decreasing (since it is the product of the two positive decreasing functions $e^{-\sqrt{x}}$ and $\frac{1}{\sqrt{x}}$). We use the integral test with this function. Using the substitution $u = \sqrt{x}$, so that $\mathrm{d}u = \frac{1}{2\sqrt{x}}\,\mathrm{d}x$, we see that

$$\begin{aligned}
\int_1^\infty f(x)\,\mathrm{d}x &= \lim_{R\to\infty}\int_1^R \left(\frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x\right) \\
&= \lim_{R\to\infty}\left(\int_1^{\sqrt{R}} e^{-u}\cdot 2\,\mathrm{d}u\right) \\
&= \lim_{R\to\infty}\left(-2e^{-u}\Big|_1^{\sqrt{R}}\right) \\
&= \lim_{R\to\infty}\left(-2e^{-\sqrt{R}} + 2e^{-\sqrt{1}}\right) = 0 + 2e^{-1},
\end{aligned}$$

and so this improper integral converges. By the integral test, the given series also converges.

3.3.11.31. $*$. **Solution**. We first develop some intuition. For very large n, $3n^2$ dominates 7 so that

$$\frac{\sqrt{3n^2-7}}{n^3} \approx \frac{\sqrt{3n^2}}{n^3} = \frac{\sqrt{3}}{n^2}$$

The series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges by the p-test with $p = 2$, so we expect the given series to converge too.
To verify that our intuition is correct, it suffices to observe that

$$0 < a_n = \frac{\sqrt{3n^2-7}}{n^3} < \frac{\sqrt{3n^2}}{n^3} = \frac{\sqrt{3}}{n^2} = c_n$$

for all $n \geq 2$. As the series $\sum_{n=2}^{\infty} c_n$ converges, the comparison test says that $\sum_{n=2}^{\infty} a_n$ converges too.

3.3.11.32. $*$. **Solution**. We first develop some intuition. For very large k, k^4 dominates 1 so that the numerator $\sqrt[3]{k^4+1} \approx \sqrt[3]{k^4} = k^{4/3}$, and k^5 dominates 9 so that the denominator $\sqrt{k^5+9} \approx \sqrt{k^5} = k^{5/2}$ and the summand

$$\frac{\sqrt[3]{k^4+1}}{\sqrt{k^5+9}} \approx \frac{k^{4/3}}{k^{5/2}} = \frac{1}{k^{7/6}}$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{k^{7/6}}$ converges by the p-test with $p = \frac{7}{6} > 1$, so we expect the given series to converge too.
To verify that our intuition is correct, we apply the limit comparison test with

$$a_k = \frac{\sqrt[3]{k^4+1}}{\sqrt{k^5+9}} \quad \text{and} \quad b_k = \frac{1}{k^{7/6}} = \frac{k^{4/3}}{k^{5/2}}$$

which is valid since

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sqrt[3]{k^4+1}/k^{4/3}}{\sqrt{k^5+9}/k^{5/2}} = \lim_{k\to\infty} \frac{\sqrt[3]{1+1/k^4}}{\sqrt{1+9/k^5}} = 1$$

exists. Since the series $\sum_{k=1}^{\infty} b_k$ is a convergent p-series (with ratio $p = \frac{7}{6} > 1$), the given series converges.
Note: to apply the direct comparison test with our chosen comparison series, we would need to show that

$$\frac{\sqrt[3]{k^4+1}}{\sqrt{k^5+9}} \leq \frac{1}{k^{7/6}}$$

for all k sufficiently large. However, this is not true: the opposite inequality holds when k is large.

3.3.11.33. $*$. **Solution**.

- Solution 1: Let's see whether the divergence test works here.

$$\lim_{n\to\infty}\frac{n^4 2^{n/3}}{(2n+7)^4}\left(\frac{\frac{1}{n^4}}{\frac{1}{n^4}}\right)=\lim_{n\to\infty}\frac{2^{n/3}}{(2+7/n)^4}=\lim_{n\to\infty}\frac{2^{n/3}}{(2+0)^4}=\infty$$

The summands of our series do not converge to zero. By the divergence test, the series diverges.

- Solution 2: Let's develop some intuition for a comparison. For very large n, $2n$ dominates 7 so that

$$\frac{n^4 2^{n/3}}{(2n+7)^4}\approx\frac{n^4 2^{n/3}}{(2n)^4}=\frac{1}{16}2^{n/3}$$

The series $\sum_{n=1}^{\infty}2^{n/3}$ is a geometric series with ratio $r=2^{1/3}>1$ and so diverges. (It also fails the divergence test.) We expect the given series to diverge too.

To verify that our intuition is correct, we apply the limit comparison test with

$$a_n=\frac{n^4 2^{n/3}}{(2n+7)^4}\quad\text{and}\quad b_n=2^{n/3}$$

which is valid since

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{n^4}{(2n+7)^4}=\lim_{n\to\infty}\frac{1}{(2+7/n)^4}=\frac{1}{2^4}$$

exists and is nonzero. Since the series $\sum_{n=1}^{\infty}b_n$ is a divergent geometric series (with ratio $r=2^{1/3}>1$), the given series diverges.

(It is possible to use the plain comparison test as well. One needs to show something like $a_n=\frac{n^4 2^{n/3}}{(2n+7)^4}\geq\frac{n^4 2^{n/3}}{(2n+7n)^4}=\frac{1}{9^4}b_n$.)

- Solution 3: Alternately, one can apply the ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|&=\lim_{n\to\infty}\left|\frac{(n+1)^4 2^{(n+1)/3}/(2(n+1)+7)^4}{n^4 2^{n/3}/(2n+7)^4}\right|\\&=\lim_{n\to\infty}\frac{(n+1)^4(2n+7)^4}{n^4(2n+9)^4}\ \frac{2^{(n+1)/3}}{2^{n/3}}\\&=\lim_{n\to\infty}\frac{(1+1/n)^4(2+7/n)^4}{(2+9/n)^4}\cdot 2^{1/3}=1\cdot 2^{1/3}>1.\end{aligned}$$

Since the ratio of consecutive terms is greater than one, by the ratio test, the series diverges.

3.3.11.34. *. Solution. (a) For large n, $n^2 \gg 1$ and so $\sqrt{n^2+1} \approx \sqrt{n^2} = n$. This suggests that we apply the limit comparison test with $a_n = \frac{1}{\sqrt{n^2+1}}$ and $b_n = \frac{1}{n}$. Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/\sqrt{n^2+1}}{1/n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n^2}} = 1$$

and since $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges, the given series diverges.

(b) Since $\cos(n\pi) = (-1)^n$, the given series converges by the alternating series test. To check that $a_n = \frac{n}{2^n}$ decreases to 0 as n tends to infinity, note that

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)2^{-(n+1)}}{n2^{-n}} = \left(1+\frac{1}{n}\right)\frac{1}{2}$$

is smaller than 1 (so that $a_{n+1} \le a_n$) for all $n \ge 1$, and is smaller than $\frac{3}{4}$ (so $a_{n+1} \le \frac{3}{4}a_n$) for all $n \ge 2$.

3.3.11.35. *. Solution. For large k, $k^4 \gg 2k^3 - 2$ and $k^5 \gg k^2 + k$ so

$$\frac{k^4 - 2k^3 + 2}{k^5 + k^2 + k} \approx \frac{k^4}{k^5} = \frac{1}{k}.$$

This suggests that we apply the limit comparison test with $a_k = \frac{k^4-2k^3+2}{k^5+k^2+k}$ and $b_k = \frac{1}{k}$. Since

$$\begin{aligned}\lim_{k\to\infty} \frac{a_k}{b_k} &= \lim_{k\to\infty} \frac{k^4 - 2k^3 + 2}{k^5 + k^2 + k} \cdot \frac{k}{1} = \lim_{k\to\infty} \frac{k^5 - 2k^4 + k^2}{k^5 + k^2 + k} \\ &= \lim_{k\to\infty} \frac{1 - 2/k + 1/k^3}{1 + 1/k^3 + 1/k^4} \\ &= 1\end{aligned}$$

and since $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ diverges (by the p-test with $p = 1$), the given series diverges.

3.3.11.36. *. Solution. (a) For large n, $n^2 \gg n + 1$ and so the numerator $n^2 + n + 1 \approx n^2$. For large n, $n^5 \gg n$ and so the denominator $n^5 - n \approx n^5$. So, for large n,

$$\frac{n^2 + n + 1}{n^5 - n} \approx \frac{n^2}{n^5} = \frac{1}{n^3}.$$

This suggests that we apply the limit comparison test with $a_n = \frac{n^2+n+1}{n^5-n}$ and $b_n = \frac{1}{n^3}$. Since

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{(n^2 + n + 1)/(n^5 - n)}{1/n^3} = \lim_{n\to\infty} \frac{n^5 + n^4 + n^3}{n^5 - n} \\ &= \lim_{n\to\infty} \frac{1 + 1/n + 1/n^2}{1 - 1/n^4} \\ &= 1\end{aligned}$$

exists and is nonzero, and since $\sum\limits_{n=1}^{\infty} \frac{1}{n^3}$ converges (by the p-test with $p = 3 > 1$), the given series converges.

(b) For large m, $3m \gg |\sin\sqrt{m}|$ and so

$$\frac{3m + \sin\sqrt{m}}{m^2} \approx \frac{3m}{m^2} = \frac{3}{m}.$$

This suggests that we apply the limit comparison test with $a_m = \frac{3m+\sin\sqrt{m}}{m^2}$ and $b_m = \frac{1}{m}$. (We could also use $b_m = \frac{3}{m}$.) Since

$$\begin{aligned}\lim_{m\to\infty} \frac{a_m}{b_m} &= \lim_{m\to\infty} \frac{(3m + \sin\sqrt{m})/m^2}{1/m} = \lim_{m\to\infty} \frac{3m + \sin\sqrt{m}}{m} \\ &= \lim_{m\to\infty} 3 + \frac{\sin\sqrt{m}}{m} \\ &= 3\end{aligned}$$

exists and is nonzero, and since $\sum\limits_{m=1}^{\infty} \frac{1}{m}$ diverges (by the p-test with $p = 1$), the given series diverges.

3.3.11.37. Solution.

$$\begin{aligned}\sum_{n=5}^{\infty} \frac{1}{e^n} &= \sum_{n=5}^{\infty} \left(\frac{1}{e}\right)^n \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n - \sum_{n=0}^{4} \left(\frac{1}{e}\right)^n \\ &= \frac{1}{1-\frac{1}{e}} - \frac{1-\left(\frac{1}{e}\right)^5}{1-\frac{1}{e}} \\ &= \frac{\left(\frac{1}{e}\right)^5}{1-\frac{1}{e}} = \frac{1}{e^5\left(1-\frac{1}{e}\right)} \\ &= \frac{1}{e^5 - e^4}\end{aligned}$$

3.3.11.38. *. Solution. This is a geometric series.

$$\sum_{n=2}^{\infty} \frac{6}{7^n} = \sum_{n=0}^{\infty} \frac{6}{7^{n+2}} = \sum_{n=0}^{\infty} \frac{6}{7^2} \cdot \frac{1}{7^n}$$

We use Lemma 3.2.5 with $a = \frac{6}{7^2}$ and $r = \frac{1}{7}$.

$$= \frac{6}{7^2} \cdot \frac{1}{1-\frac{1}{7}} = \frac{6}{42} = \frac{1}{7}$$

3.3.11.39. *. Solution. (a)

- Solution 1: The given series is

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} a_n \text{ with}$$

First we'll develop some intuition by observing that, for very large n, $a_n \approx \frac{1}{2n}$. We know that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-test with $p = 1$. So let's apply the limit comparison test with $b_n = \frac{1}{n}$. Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{1}{2 - \frac{1}{n}} = \frac{1}{2}$$

the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} b_n$ converges. So the given series diverges.

- Solution 2: The series

$$\begin{aligned} 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots &\geq \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \cdots \\ &= \frac{1}{2}\Big(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots\Big) \end{aligned}$$

The series in the brackets is the harmonic series which we know diverges, by the p-test with $p = 1$. So the series on the right hand side diverges. By the direct comparison test, the series on the left hand side diverges too.

(b) We'll use the ratio test with $a_n = \dfrac{(2n+1)}{2^{2n+1}}$. Since

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(2n+3)}{2^{2n+3}} \frac{2^{2n+1}}{(2n+1)} = \frac{1}{4}\frac{(2n+3)}{(2n+1)} = \frac{1}{4}\frac{(2+3/n)}{(2+1/n)} \\ &\to \frac{1}{4} < 1 \text{ as } n \to \infty \end{aligned}$$

the series converges.

3.3.11.40. *. Solution. (a) For very large k, $k \ll k^2$ so that

$$a_n = \frac{\sqrt[3]{k}}{k^2 - k} \approx \frac{\sqrt[3]{k}}{k^2} = \frac{1}{k^{5/3}}.$$

We apply the limit comparison test with $b_k = \frac{1}{k^{5/3}}$. Since

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{\sqrt[3]{k}/(k^2-k)}{1/k^{5/3}} = \lim_{k\to\infty} \frac{k^2}{k^2 - k} = \lim_{k\to\infty} \frac{1}{1 - 1/k} = 1$$

exists and is nonzero, and $\sum_{k=1}^{\infty} \frac{1}{k^{5/3}}$ converges (by the p-test with $p = \frac{5}{3} > 1$), the

given series converges by the limit comparison test.
(b) The k^{th} term in this series is $a_k = \frac{k^{10}10^k(k!)^2}{(2k)!}$. Factorials often work well with the ratio test, because they simplify so nicely in quotients.

$$\begin{aligned}\frac{a_{k+1}}{a_k} &= \frac{(k+1)^{10}10^{k+1}((k+1)!)^2}{(2k+2)!} \cdot \frac{(2k)!}{k^{10}10^k(k!)^2} \\ &= 10\Big(\frac{k+1}{k}\Big)^{10} \frac{(k+1)^2}{(2k+2)(2k+1)} \\ &= 10\Big(1+\frac{1}{k}\Big)^{10} \frac{(1+1/k)^2}{(2+2/k)(2+1/k)}\end{aligned}$$

As k tends to ∞, this converges to $10 \times 1 \times \frac{1}{2\times 2} > 1$. So the series diverges by the ratio test.
(c) We'll use the integal test. The k^{th} term in the series is $a_k = \frac{1}{k(\log k)(\log\log k)} = f(k)$ with $f(x) = \frac{1}{x(\log x)(\log\log x)}$, which is continuous, positive and decreasing for $x \ge 3$.

$$\begin{aligned}\int_3^\infty f(x)\ \mathrm{d}x = \int_3^\infty \frac{\mathrm{d}x}{x(\log x)(\log\log x)} &= \lim_{R\to\infty}\int_3^R \frac{\mathrm{d}x}{x(\log x)(\log\log x)} \\ &= \lim_{R\to\infty}\int_{\log 3}^{\log R} \frac{\mathrm{d}y}{y\log y} \qquad \text{with } y=\log x,\ \mathrm{d}y = \frac{\mathrm{d}x}{x} \\ &= \lim_{R\to\infty}\int_{\log\log 3}^{\log\log R} \frac{\mathrm{d}t}{t} \qquad \text{with } t=\log y,\ \mathrm{d}t = \frac{\mathrm{d}y}{y} \\ &= \lim_{R\to\infty}\Big[\log t\Big]_{\log\log 3}^{\log\log R} = \infty\end{aligned}$$

Since the integral is divergent, the series is divergent as well by the integral test.

3.3.11.41. $*$. Solution. For large n, the numerator $n^3 - 4 \approx n^3$ and the denominator $2n^5 - 6n \approx 2n^5$, so the nth term is approximately $\frac{n^3}{2n^5} = \frac{1}{2n^2}$. So we apply the limit comparison test with $a_n = \frac{n^3-4}{2n^5-6n}$ and $b_n = \frac{1}{n^2}$. Since

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{(n^3-4)/(2n^5-6n)}{1/n^2} = \lim_{n\to\infty}\frac{1-\frac{4}{n^3}}{2-\frac{6}{n^4}} = \frac{1}{2}$$

exists and is nonzero, the given series $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if the series $\sum\limits_{n=1}^{\infty} b_n$ converges. Since the series $\sum\limits_{n=1}^{\infty} b_n = \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series (with $p = 2$), both series converge.

3.3.11.42. $*$. Solution. By the alternating series test, the error introduced when we approximate the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n\cdot 10^n}$ by $\displaystyle\sum_{n=1}^{N}\frac{(-1)^n}{n\cdot 10^n}$ is at most the magnitude of the first omitted term, $\dfrac{1}{(N+1)10^{(N+1)}}$. By trial and error, we find that this expression becomes smaller than 10^{-6} when $N+1 \geq 6$. So the smallest allowable value is $N=5$.

3.3.11.43. $*$. **Solution**. The sequence $\{\frac{1}{n^2}\}$ decreases to zero as n increases to infinity. So, by the alternating series error bound, which is given in Theorem 3.3.14, $\frac{\pi^2}{12} - S_N$ lies between zero and the first omitted term, $\frac{(-1)^N}{(N+1)^2}$. We therefore need $\frac{1}{(N+1)^2} \leq 10^{-6}$, which is equivalent to $N+1 \geq 10^3$ and $N \geq 999$.

3.3.11.44. $*$. **Solution**. The error introduced when we approximate S by the N^{th} partial sum $S_N = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{(2n+1)^2}$ lies between 0 and the first term dropped, which is $\frac{(-1)^{n+1}}{(2n+1)^2}\Big|_{n=N+1} = \frac{(-1)^{N+2}}{(2N+3)^2}$. So we need the smallest positive integer N obeying

$$\begin{aligned} \frac{1}{(2N+3)^2} &\leq \frac{1}{100} \\ (2N+3)^2 &\geq 100 \\ 2N+3 &\geq 10 \\ N &\geq \frac{7}{2} \end{aligned}$$

So we need $N=4$ and then

$$S_4 = \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{9^2}$$

Exercises — Stage 3

3.3.11.45. $*$. **Solution**. (a) There are plenty of powers/factorials. So let's try the ratio test with $a_n = \frac{n^n}{9^n n!}$.

$$\begin{aligned} \lim_{n\to\infty} \frac{a_{n+1}}{a_n} &= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{9^{n+1}(n+1)!} \frac{9^n n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n\, 9\,(n+1)} \\ &= \lim_{n\to\infty} \frac{(1+1/n)^n}{9} = \frac{e}{9} \end{aligned}$$

Here we have used that $\lim\limits_{n\to\infty}(1+1/n)^n = e$. See Example 3.7.20 in the CLP-1 text, with $x = \frac{1}{n}$ and $a=1$. As $e < 9$, our series converges.
(b) We know that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by the p-test with $p=2$, and also that $\log n \geq 2$ for all $n \geq e^2$. So let's use the limit comparison test with $a_n = \frac{1}{n^{\log n}}$ and $b_n = \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n^{\log n}} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{1}{n^{\log n - 2}} = 0$$

So our series converges, by the limit comparison test.

3.3.11.46. $*$. **Solution**. (a)

- Solution 1:
    - Our first task is to identify the potential sources of impropriety for this

integral.

- The domain of integration extends to $+\infty$. On the domain of integration the denominator is never zero so the integrand is continuous. Thus the only problem is at $+\infty$.
- Our second task is to develop some intuition about the behavior of the integrand for very large x. When x is very large:
  - $|\sin x| \leq 1 \ll x$, so that the numerator $x + \sin x \approx x$, and
  - $1 \ll x^2$, so that denominator $1 + x^2 \approx x^2$, and
  - the integrand $\dfrac{x+\sin x}{1+x^2} \approx \dfrac{x}{x^2} = \dfrac{1}{x}$
- Now, since $\displaystyle\int_2^\infty \frac{\mathrm{d}x}{x}$ diverges, we would expect $\displaystyle\int_2^\infty \frac{x+\sin x}{1+x^2}\,\mathrm{d}x$ to diverge too.
- Our final task is to verify that our intuition is correct. To do so, we set

$$f(x) = \frac{x+\sin x}{1+x^2} \qquad\qquad g(x) = \frac{1}{x}$$

and compute

$$\begin{aligned}
\lim_{x\to\infty} \frac{f(x)}{g(x)} &= \lim_{x\to\infty} \frac{x+\sin x}{1+x^2} \div \frac{1}{x} \\
&= \lim_{x\to\infty} \frac{(1+\sin x/x)x}{(1/x^2+1)x^2} \times x \\
&= \lim_{x\to\infty} \frac{1+\sin x/x}{1/x^2+1} \\
&= 1
\end{aligned}$$

- Since $\displaystyle\int_2^\infty g(x)\,\mathrm{d}x = \int_2^\infty \frac{\mathrm{d}x}{x}$ diverges, by Example 1.12.8 [a], with $p = 1$, Theorem 1.12.22(b) now tells us that $\displaystyle\int_2^\infty f(x)\,\mathrm{d}x = \int_2^\infty \frac{x+\sin x}{1+x^2}\,\mathrm{d}x$ diverges too.

- Solution 2: Let's break up the integrand as $\dfrac{x+\sin x}{1+x^2} = \dfrac{x}{1+x^2} + \dfrac{\sin x}{1+x^2}$. First, we consider the integral $\displaystyle\int_2^\infty \frac{\sin x}{1+x^2}\,\mathrm{d}x$.
  - $\dfrac{|\sin x|}{1+x^2} \leq \dfrac{1}{1+x^2}$, so if we can show $\displaystyle\int \frac{1}{1+x^2}\,\mathrm{d}x$ converges, we can conclude that $\displaystyle\int \frac{|\sin x|}{1+x^2}\,\mathrm{d}x$ converges as well by the comparison test.
  - $\displaystyle\int_2^\infty \frac{1}{1+x^2}\,\mathrm{d}x \leq \int_2^\infty \frac{1}{x^2}\,\mathrm{d}x$

- $\displaystyle\int_2^\infty \frac{1}{x^2}\,\mathrm{d}x$ converges (by the p-test with $p = 2$)
- So the integral $\displaystyle\int_2^\infty \frac{\sin x}{1+x^2}\,\mathrm{d}x$ converges by the comparison test, and hence
- $\displaystyle\int_2^\infty \frac{\sin x}{1+x^2}\,\mathrm{d}x$ converges as well.

Therefore, $\displaystyle\int_2^\infty \frac{x+\sin x}{1+x^2}\,\mathrm{d}x$ converges if and only if $\displaystyle\int_2^\infty \frac{x}{1+x^2}\,\mathrm{d}x$ converges. But

$$\int_2^\infty \frac{x}{1+x^2}\,\mathrm{d}x = \lim_{r\to\infty}\int_2^r \frac{x}{1+x^2}\,\mathrm{d}x = \lim_{r\to\infty}\left[\frac{1}{2}\log(1+x^2)\right]_2^r = \infty$$

diverges, so $\displaystyle\int_2^\infty \frac{x+\sin x}{1+x^2}\,\mathrm{d}x$ diverges.

(b) The problem is that $f(x) = \dfrac{x+\sin x}{1+x^2}$ is not a decreasing function. To see this, compute the derivative:

$$\begin{aligned} f'(x) &= \frac{(1+\cos x)(1+x^2)-(x+\sin x)(2x)}{(1+x^2)^2} \\ &= \frac{(\cos x - 1)x^2 - 2x\sin x + 1 + \cos x}{(1+x^2)^2} \end{aligned}$$

If $x = 2m\pi$, the numerator is $0 - 0 + 1 + 1 > 0$.
Therefore, the integral test does not apply.
(c)

- Solution 1: Set $a_n = \frac{n+\sin n}{1+n^2}$. We first try to develop some intuition about the behaviour of a_n for large n and then we confirm that our intuition was correct.
  - *Step 1: Develop intuition.* When $n \gg 1$, the numerator $n + \sin n \approx n$, and the denominator $1 + n^2 \approx n^2$ so that $a_n \approx \frac{n}{n^2} = \frac{1}{n}$ and it looks like our series should diverge by the p-test (Example 3.3.6) with $p = 1$.
  - *Step 2: Verify intuition.* To confirm our intuition we set $b_n = \frac{1}{n}$ and compute the limit

$$\begin{aligned} \lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{\frac{n+\sin n}{1+n^2}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n[n+\sin n]}{1+n^2} \\ &= \lim_{n\to\infty}\frac{1+\frac{\sin n}{n}}{\frac{1}{n^2}+1} = 1 \end{aligned}$$

  We already know that the series $\sum_{n=1}^\infty b_n = \sum_{n=1}^\infty \frac{1}{n}$ diverges by the p-test with $p = 1$. So our series diverges by the limit comparison test, Theorem 3.3.11.

- Solution 2: Since $\left|\frac{\sin n}{1+n^2}\right| \le \frac{1}{n^2}$ and the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test with $p = 2$, the series $\sum\limits_{n=1}^{\infty} \frac{\sin n}{1+n^2}$ converges. Hence $\sum\limits_{n=1}^{\infty} \frac{n+\sin n}{1+n^2}$ converges if and only if the series $\sum\limits_{n=1}^{\infty} \frac{n}{1+n^2}$ converges. Now $f(x) = \frac{x}{1+x^2}$ is a continuous, positive, decreasing function on $[1, \infty)$ since

$$f'(x) = \frac{(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

is negative for all $x > 1$. We saw in part (a) that the integral $\int_2^\infty \frac{x}{1+x^2}\,\mathrm{d}x$ diverges. So the integral $\int_1^\infty \frac{x}{1+x^2}\,\mathrm{d}x$ diverges too and the sum $\sum\limits_{n=1}^{\infty} \frac{n}{1+n^2}$ diverges by the integral test. So $\sum\limits_{n=1}^{\infty} \frac{n+\sin n}{1+n^2}$ diverges.

[a] To change the lower limit of integration from 1 to 2, just apply Theorem 1.12.20.

3.3.11.47. *. **Solution**. Note that $\dfrac{e^{-\sqrt{x}}}{\sqrt{x}} = \dfrac{1}{\sqrt{x}e^{\sqrt{x}}}$ decreases as x increases. Hence, for every $n \ge 1$,

$$\frac{e^{-\sqrt{x}}}{\sqrt{x}} \ge \frac{e^{\sqrt{n}}}{\sqrt{n}} \qquad \text{for } x \text{ in the interval } [n-1, n]$$

$$\begin{aligned}
\text{So,} \qquad \int_{n-1}^{n} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x &\ge \int_{n-1}^{n} \frac{e^{-\sqrt{n}}}{\sqrt{n}}\,\mathrm{d}x \\
&= \left[\frac{e^{-\sqrt{n}}}{\sqrt{n}}x\right]_{x=n-1}^{x=n} \\
&= \frac{e^{-\sqrt{n}}}{\sqrt{n}}
\end{aligned}$$

Then, for every $N \ge 1$,

$$\begin{aligned}
E_N = \sum_{n=N+1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}} &\le \sum_{n=N+1}^{\infty} \int_{n-1}^{n} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x \\
&= \int_{N}^{N+1} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x + \int_{N+1}^{N+2} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x + \cdots \\
&= \int_{N}^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x
\end{aligned}$$

Substituting $y = \sqrt{x}$, $\mathrm{d}y = \frac{1}{2}\frac{\mathrm{d}x}{\sqrt{x}}$,

$$\int_N^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x = 2\int_{\sqrt{N}}^\infty e^{-y}\,\mathrm{d}y = -2e^{-y}\Big|_{\sqrt{N}}^\infty = 2e^{-\sqrt{N}}$$

This shows that $\sum_{n=N+1}^{\infty}\frac{e^{-\sqrt{n}}}{\sqrt{n}}$ converges and is between 0 and $2e^{-\sqrt{N}}$. Since $E_{14} = 2e^{-\sqrt{14}} = 0.047$, we may truncate the series at $n = 14$.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{e^{-\sqrt{n}}}{\sqrt{n}} &= \sum_{n=1}^{14}\frac{e^{-\sqrt{n}}}{\sqrt{n}} + E_{14} \\ &= \quad 0.3679 + 0.1719 + 0.1021 + 0.0677 + 0.0478 \\ &\quad + 0.0352 + 0.0268 + 0.0209 + 0.0166 + 0.0134 \\ &\quad + 0.0109 + 0.0090 + 0.0075 + 0.0063 + E_{14} \\ &= 0.9042 + E_{14}\end{aligned}$$

The sum is between 0.9035 and 0.9535. (This even allows for a roundoff error of 0.00005 in each term as we were calculating the partial sum.)

3.3.11.48. *. Solution. Let's get some intuition to guide us through a proof. Since $\sum\limits_{n=1}^{\infty} a_n$, converges a_n must converge to zero as $n \to \infty$. So, when n is quite large, $\frac{a_n}{1-a_n} \approx \frac{a_n}{1-0} = \frac{a_n}{1}$, and we know $\sum a_n$ converges. So, we want to separate the "large" indices from a finite number of smaller ones.
Since $\lim\limits_{n\to\infty} a_n = 0$, there must be [a] some integer N such that $\frac{1}{2} > a_n \geq 0$ for all $n > N$. Then, for $n > N$,

$$\frac{a_n}{1-a_n} \leq \frac{a_n}{1-1/2} = 2a_n$$

From the information in the problem statement, we know

$$\sum_{n=N+1}^{\infty} 2a_n = 2\sum_{n=N+1}^{\infty} a_n \qquad \text{converges.}$$

So, by the direct comparison test,

$$\sum_{n=N+1}^{\infty}\frac{a_n}{1-a_n} \qquad \text{converges as well.}$$

Since the convergence of a series is not affected by its first N terms, as long as N is finite, we conclude

$$\sum_{n=1}^{\infty}\frac{a_n}{1-a_n} \qquad \text{converges.}$$

a We could have chosen any positive number strictly less than 1, not only $\frac{1}{2}$.

3.3.11.49. *. Solution. By the divergence test, the fact that $\sum\limits_{n=0}^{\infty}(1-a_n)$ converges guarantees that $\lim\limits_{n\to\infty}(1-a_n) = 0$, or equivalently, that $\lim\limits_{n\to\infty} a_n = 1$. So, by the divergence test, a second time, the fact that

$$\lim_{n\to\infty} 2^n a_n = +\infty$$

guarantees that $\sum\limits_{n=0}^{\infty} 2^n a_n$ diverges too.

3.3.11.50. *. Solution. By the divergence test, the fact that $\sum\limits_{n=1}^{\infty} \dfrac{na_n - 2n+1}{n+1}$ converges guarantees that $\lim\limits_{n\to\infty} \dfrac{na_n - 2n+1}{n+1} = 0$, or equivalently, that

$$0 = \lim_{n\to\infty} \frac{n}{n+1} a_n - \lim_{n\to\infty} \frac{2n-1}{n+1} = \lim_{n\to\infty} a_n - 2 \iff \lim_{n\to\infty} a_n = 2$$

The series of interest can be written $-\log a_1 + \sum\limits_{n=1}^{\infty} \big[\log(a_n) - \log(a_{n+1})\big]$ which looks like a telescoping series. So we'll compute the partial sum

$$\begin{aligned} S_N &= -\log a_1 + \sum_{n=1}^{N} \big[\log(a_n) - \log(a_{n+1})\big] \\ &= -\log a_1 + \big[\log(a_1) - \log(a_2)\big] + \big[\log(a_2) - \log(a_3)\big] + \cdots \\ &\qquad + \big[\log(a_N) - \log(a_{N+1})\big] \\ &= -\log(a_{N+1}) \end{aligned}$$

and then take the limit $N \to \infty$

$$\begin{aligned} -\log a_1 + \sum_{n=1}^{\infty} \big[\log(a_n) - \log(a_{n+1})\big] &= \lim_{N\to\infty} S_N = -\lim_{N\to\infty} \log(a_{N+1}) \\ &= -\log 2 = \log \frac{1}{2} \end{aligned}$$

3.3.11.51. *. Solution. We are told that $\sum_{n=1}^{\infty} a_n$ converges. Thus we must have that $\lim\limits_{n\to\infty} a_n = 0$. In particular, there is an index N such that $0 \le a_n \le 1$ for all $n \ge N$. Then:

$$0 \le a_n^2 \le a_n \qquad \text{for}$$

By the direct comparison test,

$$\sum_{n=N+1}^{\infty} a_n^2 \qquad \text{converges.}$$

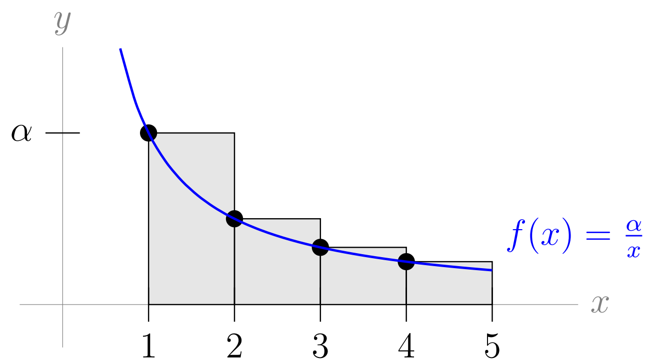
Since convergence doesn't depend on the first N terms of a series for any finite N,

$$\sum_{n=1}^{\infty} a_n^2 \qquad \text{converges as well.}$$

3.3.11.52. Solution. The most-commonly used word makes up α percent of all the words. So, we want to find α.
If we add together the frequencies of all the words, they should amount to 100%. That is,

$$\sum_{n=1}^{20{,}000} \frac{\alpha}{n} = 100$$

We can approximate the sum (with α left as a parameter) using the ideas behind the integral test. (See Example 3.3.4.)



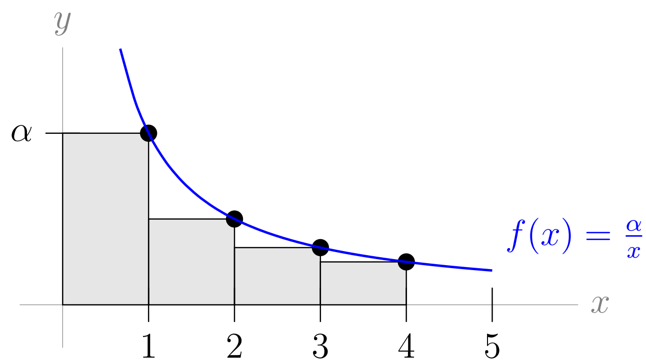
As we see in the diagram above, $\displaystyle\sum_{n=1}^{N} \frac{\alpha}{n}$ (which is the sum of the areas of the rectangles) is *greater* than $\displaystyle\int_1^{N+1} \frac{\alpha}{x}\mathrm{d}x$ (the area under the curve). That is,

$$\int_1^{N+1} \frac{\alpha}{x}\mathrm{d}x < \sum_{n=1}^{N} \frac{\alpha}{n}.$$

Using the fact that our language's 20,000 words make up 100% of the words used, we can find a lower bound for α.

$$\begin{aligned} 100 &= \sum_{n=1}^{20{,}000} \frac{\alpha}{n} > \int_1^{20{,}001} \frac{\alpha}{x}\mathrm{d}x = \Big[\alpha\log(x)\Big]_1^{20{,}001} = \alpha\log(20,001) \\ \alpha &< \frac{100}{\log(20,001)} \end{aligned}$$

We can find an upper bound for α in a similar manner.



From the diagram, we see $\sum_{n=2}^{N}\frac{\alpha}{n}$ (which is the sum of the areas of the rectangles, excluding the first) is *less* than $\int_1^N \frac{\alpha}{x}\mathrm{d}x$. (The reason for excluding the first rectangle is to avoid comparing our series to an integral that diverges.) That is,

$$\sum_{n=2}^{N}\frac{\alpha}{n} < \int_1^N \frac{\alpha}{x}\mathrm{d}x\,.$$

Therefore,

$$\begin{aligned} 100 &= \sum_{n=1}^{20{,}000}\frac{\alpha}{n} = \alpha + \sum_{n=2}^{20{,}000}\frac{\alpha}{n} \\ &< \alpha + \int_1^{20{,}00}\frac{\alpha}{x}\mathrm{d}x = \alpha + \alpha\log(20,000) = \alpha\big[1+\log(20,000)\big] \\ \alpha &> \frac{100}{1+\log(20,000)} \end{aligned}$$

Using a calculator, we see

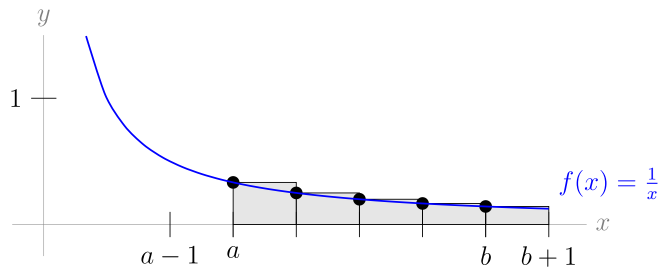
$$9.17 < \alpha < 10.01$$

So, the most-commonly used word makes up about 9-10 percent of the total words.

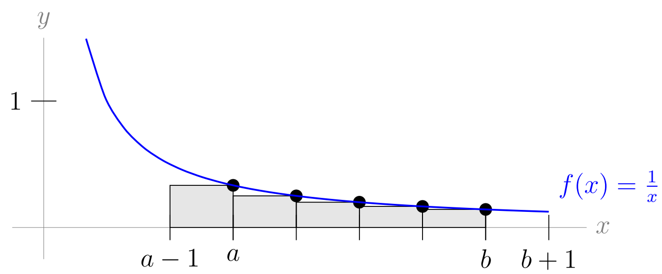
3.3.11.53. Solution. Generalizing our work in Question 52, we find the approximations:

$$\int_a^{b+1}\frac{1}{x}\mathrm{d}x < \sum_{n=a}^{b}\frac{1}{n} < \int_{a-1}^{b}\frac{1}{x}\mathrm{d}x$$

when $a \geq 2$. The inequality $\int\limits_a^{b+1}\frac{1}{x}\mathrm{d}x < \sum\limits_{n=a}^{b}\frac{1}{n}$ can be read off of the sketch



and the inequality $\sum\limits_{n=a}^{b} \frac{1}{n} < \int\limits_{a-1}^{b} \frac{1}{x} \mathrm{d}x$ can be read off of the sketch



We will evaluate the total population by writing

$$\sum_{n=1}^{2\times 10^6} \frac{2\times 10^6}{n} = \sum_{n=1}^{a-1} \frac{2\times 10^6}{n} + \sum_{n=a}^{2\times 10^6} \frac{2\times 10^6}{n}$$

and applying the above integral approximations to the second sum. We want our error to be less than one million, so we need to choose a value of a such that:

$$\begin{aligned}
\underbrace{2\times 10^6 \int_{a-1}^{2\times 10^6} \frac{1}{x}\mathrm{d}x}_{\text{upper bound}} - \underbrace{2\times 10^6 \int_{a}^{2\times 10^6+1} \frac{1}{x}\mathrm{d}x}_{\text{lower bound}} &< 10^6 \\
\int_{a-1}^{2\times 10^6} \frac{1}{x}\mathrm{d}x - \int_{a}^{2\times 10^6+1} \frac{1}{x}\mathrm{d}x &< \frac{1}{2} \\
\left[\log\left(2\times 10^6\right) - \log(a-1)\right] - \left[\log\left(2\times 10^6+1\right) - \log(a)\right] &< \frac{1}{2} \\
\left[\log\left(2\times 10^6\right) - \log\left(2\times 10^6+1\right)\right] + \left[\log(a) - \log(a-1)\right] &< \frac{1}{2} \\
\log\left(\frac{2\times 10^6}{2\times 10^6+1}\right) + \log\left(\frac{a}{a-1}\right) &< \frac{1}{2}
\end{aligned}$$

The first term is extremely close to 0, so we ignore it.

$$\begin{aligned}
\log\left(\frac{a}{a-1}\right) &< \frac{1}{2} \\
\frac{a}{a-1} &< e^{1/2} = \sqrt{e}
\end{aligned}$$

$$\begin{aligned}a &< a\sqrt{e} - \sqrt{e}\\ \sqrt{e} &< a(\sqrt{e} - 1)\\ \frac{\sqrt{e}}{\sqrt{e}-1} &< a\end{aligned}$$

Since $\dfrac{\sqrt{e}}{\sqrt{e}-1} \approx 2.5$, we use $a = 3$. That is, we will approximate the value of $\displaystyle\sum_{n=3}^{2\times 10^6} \frac{1}{n}$ using an integral. Then, we will use that approximation to estimate our total population.

$$\begin{aligned}\int_3^{2\times10^6+1} \frac{1}{x}\mathrm{d}x < \sum_{n=3}^{2\times10^6} \frac{1}{n} &< \int_2^{2\times10^6} \frac{1}{x}\mathrm{d}x\\ \log\left(2\times10^6+1\right) - \log(3) < \sum_{n=3}^{2\times10^6} \frac{1}{n} &< \log\left(2\times10^6\right) - \log(2)\\ 1 + \frac{1}{2} + \log\left(2\times10^6+1\right) - \log(3) < \sum_{n=1}^{2\times10^6} \frac{1}{n} &< 1 + \frac{1}{2} + \log\left(2\times10^6\right)\\ &\qquad - \log(2)\\ \frac{3}{2} + \log\left(\frac{2\times10^6+1}{3}\right) < \sum_{n=1}^{2\times10^6} \frac{1}{n} &< \frac{3}{2} + 6\log(10)\end{aligned}$$

So the population, namely $\sum_{n=1}^{2\times10^6} \frac{2\times10^6}{n}$, is between

$$2\times10^6\left(\frac{3}{2} + \log\left(\tfrac{2}{3}\times10^6+\tfrac{1}{3}\right)\right) = 29,820,091$$

and

$$2\times10^6\left(\frac{3}{2} + 6\log(10)\right) = 30,631,021$$

3.4 · Absolute and Conditional Convergence

3.4.3 · Exercises

Exercises — Stage 1

3.4.3.1. ∗. **Solution**. False. For example if $b_n = \frac{1}{n}$, then $\sum\limits_{n=1}^{\infty}(-1)^{n+1}b_n = \sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges by the alternating series test, but $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ diverges by the p–test.

Remark: if we had added that $\{b_n\}$ is a sequence of *alternating terms*, then by

Theorem 3.4.2, the statement would have been true. This is because $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ would either be equal to $\sum_{n=1}^{\infty}|b_n|$ or $-\sum_{n=1}^{\infty}|b_n|$.

3.4.3.2. Solution. Absolute convergence describes the situation where $\sum|a_n|$ converges (see Definition 3.4.1). By Theorem 3.4.2, this guarantees that also $\sum a_n$ converges.
Conditional convergence describes the situation where $\sum|a_n|$ diverges but $\sum a_n$ converges (see again Definition 3.4.1).
If $\sum a_n$ diverges, we just say it diverges. The reason is that if $\sum a_n$ diverges, we automatically know $\sum|a_n|$ diverges as well, so there's no need for a special name.

	$\sum a_n$ converges	$\sum a_n$ diverges
$\sum\|a_n\|$ converges	converges absolutely	not possible
$\sum\|a_n\|$ diverges	converges conditionally	diverges

Exercises — Stage 2

3.4.3.3. *. Solution. The series $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{9n+5}$ converges by the alternating series test. On the other hand the series $\sum\limits_{n=1}^{\infty}\left|\frac{(-1)^n}{9n+5}\right| = \sum_{n=1}^{\infty}\frac{1}{9n+5}$ diverges by the limit comparison test with $b_n=\frac{1}{n}$. So the given series is conditionally convergent.

3.4.3.4. *. Solution. Note that $(-1)^{2n+1} = (-1)\cdot(-1)^{2n} = -1$. So we can simplify

$$\sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{1+n} = -\sum_{n=1}^{\infty}\frac{1}{1+n}$$

Since $\dfrac{1}{1+n} \geq \dfrac{1}{n+n} = \dfrac{1}{2n}$, $\quad\sum\limits_{n=1}^{\infty}\dfrac{1}{1+n}$ diverges by the comparison test with the divergent harmonic series $\sum\limits_{n=1}^{\infty}\frac{1}{n}$. The extra overall factor of -1 in the original series does not change the conclusion of divergence.

3.4.3.5. *. Solution. Since

$$\lim_{n\to\infty}\frac{1+4^n}{3+2^{2n}} = \lim_{n\to\infty}\frac{1+4^n}{3+4^n} = 1$$

the alternating series test cannot be used. Indeed, $\lim\limits_{n\to\infty}(-1)^{n-1}\dfrac{1+4^n}{3+2^{2n}}$ does not exist (for very large n, $(-1)^{n-1}\frac{1+4^n}{3+2^{2n}}$ alternates between a number close to $+1$ and a number close to -1) so the divergence test says that the series diverges. (Note that "none of the above" cannot possibly be the correct answer — *every* series either converges absolutely, converges conditionally, or diverges.)

3.4.3.6. $*$. Solution. First, we'll develop some intuition. For very large n

$$\left|\frac{\sqrt{n}\cos(n)}{n^2-1}\right| \approx \left|\frac{\sqrt{n}\cos(n)}{n^2}\right| = \left|\frac{\cos(n)}{n^{3/2}}\right| \le \frac{1}{n^{3/2}}$$

since $|\cos(n)| \le 1$ for all n. By the p-test, which is in Example 3.3.6, the series $\sum_{n=5}^{\infty}\frac{1}{n^p}$ converges for all $p > 1$. So we would expect the given series to converge absolutely.

Now, to confirm that our intuition is correct, we'll first try the limit comparison theorem, which is Theorem 3.3.11, with $a_n = \left|\frac{\sqrt{n}\cos(n)}{n^2-1}\right|$ and $b_n = \frac{1}{n^{3/2}}$.

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{\left|\frac{\sqrt{n}\cos(n)}{n^2-1}\right|}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty}\frac{\sqrt{n}\cdot\sqrt{n}^3|\cos n|}{n^2-1}\\ &= \lim_{n\to\infty}\frac{n^2|\cos n|}{n^2-1} = \lim_{n\to\infty}\left(\frac{1}{1-1/n^2}\right)|\cos n|\\ &= \lim_{n\to\infty} 1\cdot|\cos n|\end{aligned}$$

Unfortunately, this limit doesn't exist, so this attempt to use the limit comparison theorem has failed. Fortunately, having seen that the $\cos n$ caused the failure, it is not hard to adjust our strategy to get a successful proof of absolute convergence. First, in step 1 below, we use the comparison test to eliminate the $\cos n$ and then, in step 2 below, we apply the limit comparison test.

- Step 1: Since $|\cos n| \le 1$, we have

  $$\left|\frac{\sqrt{n}\cos(n)}{n^2-1}\right| \le \frac{\sqrt{n}}{n^2-1}$$

  for all $n > 1$. So, by part (a) of the comparison test, which is Theorem 3.3.8, if the series $\sum_{n=5}^{\infty}\frac{\sqrt{n}}{n^2-1}$ converges, then we will have that the series $\sum_{n=5}^{\infty}\left|\frac{\sqrt{n}\cos(n)}{n^2-1}\right|$ also converges, and hence that the series $\sum_{n=5}^{\infty}\frac{\sqrt{n}\cos(n)}{n^2-1}$ converges absolutely.

- Step 2: Now, to prove that the series $\sum_{n=5}^{\infty}\frac{\sqrt{n}}{n^2-1}$ converges, we apply the limit comparison test with $a_n = \frac{\sqrt{n}}{n^2-1}$ and $b_n = \frac{1}{n^{3/2}}$ (for $n \ge 5$). Since

  $$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{\sqrt{n}}{n^2-1}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty}\frac{\sqrt{n}\cdot\sqrt{n}^3}{n^2-1}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{n^2}{n^2-1} = \lim_{n\to\infty} \frac{1}{1-1/n^2} \\
&= 1
\end{aligned}
$$

and since $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n^{3/2}}$ converges by the p-test, the limit comparison test tells us that the series $\displaystyle\sum_{n=5}^{\infty} \frac{\sqrt{n}}{n^2-1}$ converges. So, by step 1, $\displaystyle\sum_{n=5}^{\infty} \frac{\sqrt{n}\cos(n)}{n^2-1}$ converges absolutely.

3.4.3.7. ∗. Solution. We first develop some intuition about $\displaystyle\sum_{n=1}^{\infty} \left|\frac{n^2-\sin n}{n^6+n^2}\right|$, where we take the absolute value of the summands to consider whether the series converges *absolutely*. For very large n, n^2 dominates $\sin n$ and n^6 dominates n^2 so that

$$
\left|\frac{n^2-\sin n}{n^6+n^2}\right| \approx \frac{n^2}{n^6} = \frac{1}{n^4}
$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges by the p-test with $p = 4 > 1$. We expect the given series to converge too.
To verify that our intuition is correct, we apply the limit comparison test with

$$
a_n = \frac{n^2-\sin n}{n^6+n^2} \quad \text{and} \quad b_n = \frac{1}{n^4}
$$

which is valid since

$$
\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \left|\frac{(n^2-\sin n)}{n^6+n^2}\right| \cdot \frac{n^4}{1} = \lim_{n\to\infty} \frac{|n^6-n^4\sin n|}{n^6+n^2} \\
&= \lim_{n\to\infty} \frac{1-n^{-2}\sin n}{1+n^{-4}} = 1
\end{aligned}
$$

exists and is nonzero. Since the series $\sum_{n=1}^{\infty} b_n$ converges, the series $\sum_{n=1}^{\infty} \frac{|n^2-\sin n|}{n^6+n^2}$ converges too. Therefore, the series $\sum_{n=1}^{\infty} \frac{n^2-\sin n}{n^6+n^2}$ converges absolutely.

3.4.3.8. ∗. Solution. You might think that this series converges by the alternating series test. But you would be wrong. The problem is that $\{a_n\}$ does not converge to zero as $n \to \infty$, so that the series actually diverges by the divergence test. To verify that the n^{th} term does not converge to zero as $n \to \infty$ let's write $a_n = \frac{(2n)!}{(n^2+1)(n!)^2}$ (i.e. a_n is the n^{th} term without the sign) and check to see whether

a_{n+1} is bigger than or smaller than a_n.

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(2n+2)!}{((n+1)^2+1)((n+1)!)^2}\frac{(n^2+1)(n!)^2}{(2n)!} \\
&= \frac{(2n+2)(2n+1)}{(n+1)^2}\frac{n^2+1}{(n+1)^2+1} \\
&= \frac{2(2n+1)}{(n+1)}\frac{1+1/n^2}{(1+1/n)^2+1/n^2} \\
&= 4\frac{1+1/2n}{1+1/n}\frac{1+1/n^2}{(1+1/n)^2+1/n^2}
\end{aligned}$$

So

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 4$$

and, in particular, for large n, $a_{n+1} > a_n$. Thus, for large n, a_n increases with n and so cannot converge to 0. So the series diverges by the divergence test.

3.4.3.9. *. Solution. This series converges by the alternating series test. We want to know whether it converges absolutely, so we consider the seris$\displaystyle\sum_{n=2}^{\infty}\left|\frac{(-1)^n}{n(\log n)^{101}}\right| = \sum_{n=2}^{\infty}\frac{1}{n(\log n)^{101}}$.
We've seen similar function before (e.g. Example 3.3.7, with $p = 101 > 1$) and it yields nicely to the integral test. Let $f(x) = \frac{1}{x(\log x)^{101}}$. Note $f(x)$ is positive and decreasing for $n \geq 3$. Then by the integral test, the series $\sum_{n=2}^{\infty}\frac{1}{n(\log n)^{101}}$ converges if and only if the integral $\int_2^{\infty}\frac{1}{x(\log x)^{101}}\mathrm{d}x$ does. We evaluate the integral using the substitution $u = \log x$, $\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x$.

$$\begin{aligned}
\int_2^{\infty}\frac{1}{x(\log x)^{101}}\mathrm{d}x &= \lim_{b\to\infty}\int_2^b\frac{1}{x(\log x)^{101}}\mathrm{d}x \\
&= \lim_{b\to\infty}\int_{\log 2}^{\log b}\frac{1}{u^{101}}\,\mathrm{d}u \\
&= \lim_{b\to\infty}\left[\frac{-1}{100u^{100}}\right]_{\log 2}^{\log b} \\
&= \frac{1}{100(\log 2)^{100}}
\end{aligned}$$

Since the integral converges, the series $\sum\limits_{n=2}^{\infty}\frac{1}{n(\log n)^{101}}$ converges, and therefore the series $\sum\limits_{n=2}^{\infty}\frac{(-1)^n}{n(\log n)^{101}}$ converges absolutely.

3.4.3.10. Solution. The sequence has some positive terms and some negative terms, which limits the tests we can use. However, if we consider the series

$\displaystyle\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^2}\right|$, we can use the direct comparison test.

For every n, $|\sin n| < 1$, so $0 \le \left|\dfrac{\sin n}{n^2}\right| < \dfrac{1}{n^2}$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, then by the direct comparison test, $\displaystyle\sum_{n=1}^{\infty}\left|\frac{\sin n}{n^2}\right|$ converges as well. Then $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ converges absolutely — in particular, it converges.

3.4.3.11. Solution. The terms of this series are sometimes negative (for odd values of n where $\sin n < \frac{1}{2}$) and sometimes positive. But, they are not strictly alternating, so we can't use the alternating series test. Instead, we use a direct comparison test to show the series converges absolutely.

$$\begin{aligned}
& -\frac{1}{4} \le \frac{\sin n}{4} \le \frac{1}{4} \\
\Rightarrow\qquad & \left(-\frac{1}{4}-\frac{1}{8}\right) \le \left(\frac{\sin x}{4}-\frac{1}{8}\right) < \left(\frac{1}{4}-\frac{1}{8}\right) \\
\Rightarrow\qquad & -\frac{3}{8} \le \left(\frac{\sin x}{4}-\frac{1}{8}\right) < \frac{1}{8} \\
\Rightarrow\qquad & 0 \le \left|\frac{\sin x}{4}-\frac{1}{8}\right| < \frac{3}{8} \\
\Rightarrow\qquad & 0 \le \left|\left(\frac{\sin x}{4}-\frac{1}{8}\right)^n\right| < \left(\frac{3}{8}\right)^n
\end{aligned}$$

Since $\displaystyle\sum_{n=1}^{\infty}\left(\frac{3}{8}\right)^n$ converges (it's a geometric sum with $|r| < 1$), by the direct comparison test, $\displaystyle\sum_{n=1}^{\infty}\left|\left(\frac{\sin x}{4}-\frac{1}{8}\right)^n\right|$ converges as well.

Then $\displaystyle\sum_{n=1}^{\infty}\left(\frac{\sin x}{4}-\frac{1}{8}\right)^n$ converges absolutely — and so it converges.

3.4.3.12. Solution. The terms of this series are sometimes negative and sometimes positive. But, they are not strictly alternating, so we can't use the alternating series test. Instead, we use a direct comparison test to show the series converges absolutely.

$$\left|\frac{\sin^2 n - \cos^2 n + \frac{1}{2}}{2^n}\right| \le \frac{1+1+\frac{1}{2}}{2^n} = \frac{5}{2^{n+1}}$$

The series $\displaystyle\sum_{n=1}^{\infty}\frac{5}{2^{n+1}}$ converges, because it's a geometric series with $r = \frac{1}{2}$. By the direct comparison test, $\displaystyle\sum_{n=1}^{\infty}\left|\frac{\sin^2 n - \cos^2 n + \frac{1}{2}}{2^n}\right|$ converges as well. Then

$$\sum_{n=1}^{\infty} \frac{\sin^2 n - \cos^2 n + \frac{1}{2}}{2^n} \text{ converges absolutely, so it converges.}$$

Exercises — Stage 3

3.4.3.13. *. Solution. (a)

- Solution 1: We need to show that $\sum_{n=1}^{\infty} 24n^2 e^{-n^3}$ converges. If we replace n by x in the summand, we get $f(x) = 24x^2 e^{-x^3}$, which we can integate. (Just substitute $u = x^3$.) So let's try the integral test. First, we have to check that $f(x)$ is positive and decreasing. It is certainly positive. To determine if it is decreasing, we compute

  $$\frac{\mathrm{d}f}{\mathrm{d}x} = 48xe^{-x^3} - 24 \times 3x^4 e^{-x^3} = 24x(2 - 3x^3)e^{-x^3}$$

  which is negative for $x \geq 1$. Therefore $f(x)$ is decreasing for $x \geq 1$, and the integral test applies. The substitution $u = x^3$, $\mathrm{d}u = 3x^2\,\mathrm{d}x$, yields

  $$\begin{aligned}\int f(x)\,\mathrm{d}x &= \int 24x^2 e^{-x^3}\,\mathrm{d}x = \int 8e^{-u}\,\mathrm{d}u = -8e^{-u} + C \\ &= -8e^{-x^3} + C.\end{aligned}$$

  Therefore

  $$\begin{aligned}\int_1^{\infty} f(x)\,\mathrm{d}x &= \lim_{R\to\infty} \int_1^R f(x)\,\mathrm{d}x = \lim_{R\to\infty} \Big[-8e^{-x^3}\Big]_1^R \\ &= \lim_{R\to\infty} (-8e^{-R^3} + 8e^{-1}) = 8e^{-1}\end{aligned}$$

  Since the integral is convergent, the series $\sum_{n=1}^{\infty} 24n^2 e^{-n^3}$ converges and the series $\sum_{n=1}^{\infty} (-1)^{n-1} 24n^2 e^{-n^3}$ converges absolutely.

- Solution 2: Alternatively, we can use the ratio test with $a_n = 24n^2 e^{-n^3}$. We calculate

  $$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{24(n+1)^2 e^{-(n+1)^3}}{24n^2 e^{-n^3}}\right| \\ &= \lim_{n\to\infty} \left(\frac{(n+1)^2}{n^2} \frac{e^{n^3}}{e^{(n+1)^3}}\right) \\ &= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 e^{-(3n^2+3n+1)} = 1 \cdot 0 = 0 < 1,\end{aligned}$$

  and therefore the series converges absolutely.

- Solution 3: Alternatively, alternatively, we can use the limiting comparison test. First a little intuition building. Recall that we need to show that $\sum\limits_{n=1}^{\infty} 24n^2e^{-n^3}$ converges. The n^{th} term in this series is

$$a_n = 24n^2e^{-n^3} = \frac{24n^2}{e^{n^3}}$$

It is a ratio with both the numerator and denominator growing with n. A good rule of thumb is that exponentials grow a lot faster than powers. For example, if $n = 10$ the numerator is $2400 = 2.4 \times 10^3$ and the denominator is about 2×10^{434}. So we would guess that a_n tends to zero as $n \to \infty$. The question is "does a_n tend to zero fast enough with n that our series converges?". For example, we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (by the p-test with $p = 2$). So if a_n tends to zero faster than $\frac{1}{n^2}$ does, our series will converge. So let's try the limiting convergence test with $a_n = 24n^2e^{-n^3} = \frac{24n^2}{e^{n^3}}$ and $b_n = \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{24n^2e^{-n^3}}{1/n^2} = \lim_{n\to\infty} \frac{24n^4}{e^{n^3}}$$

By l'Hôpital's rule, twice,

$$\begin{aligned} \lim_{x\to\infty} \frac{24x^4}{e^{x^3}} &= \lim_{x\to\infty} \frac{4 \times 24x^3}{3x^2e^{x^3}} && \text{by l'Hôpital} \\ &= \lim_{x\to\infty} \frac{32x}{e^{x^3}} && \text{just cleaning up} \\ &= \lim_{x\to\infty} \frac{32}{3x^2e^{x^3}} && \text{by l'Hôpital, again} \\ &= 0 \end{aligned}$$

That's it. The limit comparison test now tells us that $\sum_{n=1}^{\infty} a_n$ converges.

(b) In part (a) we saw that $24n^2e^{-n^3}$ is positive and decreasing. The limit of this sequence equals 0 (as can be shown with l'Hôpital's Rule, just as we did at the end of the third solution of part (a)). Therefore, we can use the alternating series test, so that the error made in approximating the infinite sum $S = \sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} (-1)^{n-1}24n^2e^{-n^3}$ by the sum of its first N terms, $S_N = \sum\limits_{n=1}^{N} a_n$, lies between 0 and the first omitted term, a_{N+1}. If we use 5 terms, the error satisfies

$$|S - S_5| \le |a_6| = 24 \times 36e^{-6^3} \approx 1.3 \times 10^{-91}$$

3.4.3.14. Solution. The error in our approximation using through term N is at most $\frac{1}{(2(N+1))!}$. We want $\frac{1}{(2(N+1))!} < \frac{1}{1000}$. By checking small values of N, we see that $8! = 40320 > 1000$, so if $N = 3$, then $\frac{1}{2(N+1)!} = \frac{1}{40320} < \frac{1}{1000}$. So, for our approximation, it suffices to consider the first four terms of our series.

$$\begin{aligned}\cos(1) &\approx \sum_{N=0}^{3} \frac{(-1)^n}{(2n)!} = \frac{1}{0!} - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} \\ &= 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} \\ &= \frac{720 - 360 + 30 - 1}{720} = \frac{379}{720}\end{aligned}$$

When we use a calculator, we see

$$\begin{aligned}\frac{389}{720} &= 0.540\overline{2}\overline{77} \\ \cos(1) &\approx 0.540302 \\ \cos(1) - \frac{389}{720} &\approx 0.000024528 \approx \frac{1}{40770}\end{aligned}$$

So, our error is reasonably close to our bound of $\frac{1}{40320}$, and far smaller than $\frac{1}{1000}$.

3.4.3.15. Solution. The terms of this series are sometimes negative and sometimes positive. But, they are not strictly alternating, so we can't use the alternating series test. Instead, we use a direct comparison test to show the series converges absolutely.
If n is prime, then

$$\left|\frac{a_n}{e^n}\right| = \left|-\frac{e^{n/2}}{e^n}\right| = \frac{1}{e^{n/2}} = \left(\frac{1}{\sqrt{e}}\right)^n$$

If n is not prime, then

$$\left|\frac{a_n}{e^n}\right| = \left|-\frac{n^2}{e^n}\right| = \frac{n^2}{e^n}$$

For n sufficiently large, $n^2 < e^{n/2}$, so for n sufficiently large,

$$\frac{n^2}{e^n} \le \left(\frac{1}{\sqrt{e}}\right)^n.$$

Since $e > 1$, then $\sqrt{e} > 1$, so the geometric series $\sum \left(\frac{1}{\sqrt{e}}\right)^n$ has $|r| = r = \frac{1}{\sqrt{e}} < 1$, so it converges. By the direct comparison test, $\sum_{n=1}^{\infty} \left|\frac{a_n}{e^n}\right|$ converges as well. Then $\sum_{n=1}^{\infty} \frac{a_n}{e^n}$ converges absolutely, so it converges.

3.5 · Power Series

3.5.3 · Exercises

Exercises — Stage 1

3.5.3.1. Solution.

$$\begin{aligned} f(1) &= \sum_{n=0}^{\infty} \left(\frac{3-1}{4}\right)^n \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \end{aligned}$$

This is a geometric series with $r = \frac{1}{2}$.

$$= \frac{1}{1-\frac{1}{2}} = 2$$

3.5.3.2. Solution. Following Theorem 3.5.13, we differentiate our function term-by-term.

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{(x-5)^n}{n!+2} \\ f'(x) &= \sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{(x-5)^n}{n!+2}\right\} \\ &= \sum_{n=1}^{\infty} \frac{n(x-5)^{n-1}}{n!+2} \end{aligned}$$

Keep in mind that x is our variable, and *for each term*, n is constant.

3.5.3.3. Solution. If $x = c$, then

$$\begin{aligned} f(x) &= A_a(c-c)^a + A_{a+1}(c-c)^{a+1} + A_{a+2}(c-c)^{a+2} + \cdots \\ &= A_a \cdot 0 + A_{a+1} \cdot 0 + A_{a+2} \cdot 0 + \cdots \\ &= 0 \end{aligned}$$

So, $f(x)$ converges (to the constant 0) when $x = c$. (Had we allowed $a = 0$, it would be possible for $f(x)$ to converge to a nonzero number A_0, because we use the convention $0^0 = 1$.)
Depending on the sequence $\{A_n\}$, it's possible that $f(x)$ diverges for all $x \neq c$. For example, suppose $A_n = n!$, so $f(x) = \sum_{n=0}^{\infty} n!(x-c)^n$. If $x \neq c$, then the limit $\lim_{n\to\infty} \left|\frac{(n+1)!(x-c)^{n+1}}{n!(x-c)^n}\right| = \lim_{n\to\infty}(n+1)|x-c|$ is infinity, since $x - c \neq 0$. So, the series diverges.
We've now shown that the series definitely converges at $x = c$, but at any other point, it may fail to converge.

3.5.3.4. Solution. According to Theorem 3.5.9, because $f(x)$ diverges somewhere, and because it converges at a point other than its centre, $f(x)$ has a positive

radius of convergence R. That is, $f(x)$ converges whenever $|x-5|<R$, and it diverges whenever $|x-5|>R$.
If $R>6$, then $|11-5|<R$, so $f(x)$ converges at $x=11$; since we are told $f(x)$ *diverges* at $x=11$, we see $R\le 6$.
If $R<6$, then $|-1-5|>R$, so $f(x)$ diverges at $x=-1$; since we are told $f(x)$ *converges* at $x=-1$, we see $R\ge 6$.
Therefore, $R=6$.

Exercises — Stage 2

3.5.3.5. $*$. Solution. (a) We apply the ratio test for the series whose k^{th} term is $a_k=(-1)^k 2^{k+1}x^k$. Then

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{(-1)^{k+1}2^{k+2}x^{k+1}}{(-1)^k 2^{k+1}x^k}\right|\\ &= \lim_{k\to\infty}|2x| = |2x|\end{aligned}$$

Therefore, by the ratio test, the series converges for all x obeying $|2x|<1$, i.e. $|x|<\frac{1}{2}$, and diverges for all x obeying $|2x|>1$, i.e. $|x|>\frac{1}{2}$. So the radius of convergence is $R=\frac{1}{2}$.
Alternatively, one can set $A_k=(-1)^k 2^{k+1}$ and compute

$$A=\lim_{k\to\infty}\left|\frac{A_{k+1}}{A_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+1}2^{k+2}}{(-1)^k 2^{k+1}}\right| = \lim_{k\to\infty} 2 = 2$$

so that $R=\frac{1}{A}=\frac{1}{2}$, again.
(b) The series is

$$\begin{aligned}\sum_{k=0}^{\infty}(-1)^k 2^{k+1}x^k = 2\sum_{k=0}^{\infty}(-2x)^k &= 2\sum_{k=0}^{\infty} r^k\Big|_{r=-2x} = 2\times\frac{1}{1-r}\\ &= \frac{2}{1+2x}\end{aligned}$$

for all $|r|=|2x|<1$, i.e. all $|x|<\frac{1}{2}$.

3.5.3.6. $*$. Solution. We apply the ratio test for the series whose k^{th} term is $a_k=\frac{x^k}{10^{k+1}(k+1)!}$. Then

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{x^{k+1}}{10^{k+2}(k+2)!}\cdot\frac{10^{k+1}(k+1)!}{x^k}\right|\\ &= \lim_{k\to\infty}\left|\frac{10^{k+1}}{10^{k+2}}\right|\cdot\left|\frac{(k+1)!}{(k+2)!}\right|\cdot\left|\frac{x^{k+1}}{x^k}\right|\\ &= \lim_{k\to\infty}\frac{1}{10(k+2)}|x| = 0<1\end{aligned}$$

for all x. Therefore, by the ratio test, the series converges for all x and the radius of convergence is $R = \infty$.

Alternatively, one can set $A_k = \dfrac{1}{10^{k+1}(k+1)!}$ and compute $A = \lim_{k\to\infty}\left|\dfrac{A_{k+1}}{A_k}\right| = 0$, so that R is again $+\infty$.

3.5.3.7. *. Solution. We apply the ratio test with $a_n = \frac{(x-2)^n}{n^2+1}$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^2+1}\cdot\frac{n^2+1}{(x-2)^n}\right| \\
&= \lim_{n\to\infty}\frac{n^2+1}{(n+1)^2+1}|x-2| \\
&= \lim_{n\to\infty}\frac{1+1/n^2}{(1+1/n)^2+1/n^2}|x-2| \\
&= |x-2|
\end{aligned}$$

So, the series converges if $|x-2| < 1$ and diverges if $|x-2| > 1$. That is, the radius of convergence is 1.

3.5.3.8. *. Solution. We apply the ratio test for the series whose n^{th} term is $a_n = \dfrac{(-1)^n(x+2)^n}{\sqrt{n}}$. Then

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{\sqrt{n+1}}\ \frac{\sqrt{n}}{(x+2)^n}\right| \\
&= \lim_{n\to\infty}|x+2|\ \frac{\sqrt{n}}{\sqrt{n+1}} \\
&= \lim_{n\to\infty}|x+2|\ \frac{1}{\sqrt{1+1/n}} \\
&= |x+2|
\end{aligned}$$

So the series must converge when $|x+2| < 1$ and must diverge when $|x+2| > 1$. When $x+2 = 1$, the series reduces to

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$$

which converges by the alternating series test. When $x+2 = -1$, the series reduces to

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$$

which diverges by the p–series test with $p = \frac{1}{2}$. So the interval of convergence is $-1 < x+2 \le 1$ or $(-3, -1]$.

3.5.3.9. $*$. Solution. We apply the ratio test for the series whose n^{th} term is $a_n = \frac{(-1)^n}{n+1}\left(\frac{x+1}{3}\right)^n$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{n+2}\left(\frac{x+1}{3}\right)^{n+1}}{\frac{(-1)^n}{n+1}\left(\frac{x+1}{3}\right)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\right|\cdot\left|\frac{n+1}{n+2}\right|\cdot\left|\frac{(x+1)^{n+1}}{(x+1)^n}\right|\cdot\left|\frac{3^n}{3^{n+1}}\right| \\
&= \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)\cdot\left|\frac{x+1}{3}\right| = \frac{|x+1|}{3}
\end{aligned}$$

Therefore, by the ratio test, the series converges when $\frac{|x+1|}{3} < 1$ and diverges when $\frac{|x+1|}{3} > 1$. In particular, it converges when

$$|x+1| < 3 \iff -3 < x+1 < 3 \iff -4 < x < 2$$

and the radius of convergence is $R = 3$. (Alternatively, one can set $A_n = \frac{(-1)^n}{(n+1)3^n}$ and compute $A = \lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \frac{1}{3}$, so that $R = \frac{1}{A} = 3$.)
Next, we consider the endpoints 2 and -4. At $x = 2$, i.e. $x+1 = 3$, the series is simply $\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$, which is an alternating series: the signs alternate, and the unsigned terms decrease to zero. Therefore the series converges at $x = 2$ by the alternating series test.
At $x = -4$ the series is

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\left(\frac{-4+1}{3}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(-1)^n = \sum_{n=0}^{\infty}\frac{1}{n+1},$$

since $(-1)^n\cdot(-1)^n = (-1)^{2n} = \left((-1)^2\right)^n = 1$. This series diverges, either by comparison or limit comparison with the harmonic series (the p-series with $p = 1$). (For that matter, it is exactly equal to the standard harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$, re-indexed to start at $n = 0$.)
In summary, the interval of convergence is $-4 < x \le 2$, or simply $(-4, 2]$.

3.5.3.10. $*$. Solution. We first apply the ratio test with $a_n = \frac{(x-2)^n}{n^{4/5}(5^n-4)}$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^{4/5}(5^{n+1}-4)}\cdot\frac{n^{4/5}(5^n-4)}{(x-2)^n}\right| \\
&= \lim_{n\to\infty}\frac{n^{4/5}(5^n-4)}{(n+1)^{4/5}(5^{n+1}-4)}|x-2| \\
&= \lim_{n\to\infty}\frac{(1-4/5^n)}{(1+1/n)^{4/5}(5-4/5^n)}|x-2| \\
&= \frac{|x-2|}{5}
\end{aligned}$$

Therefore the series converges if $|x-2| < 5$ and diverges if $|x-2| > 5$. When

$x-2=+5$, i.e. $x=7$, the series reduces to $\sum\limits_{n=1}^{\infty}\frac{5^n}{n^{4/5}(5^n-4)}=\sum\limits_{n=1}^{\infty}\frac{1}{n^{4/5}(1-4/5^n)}$ which diverges by the limit comparison test with $b_n=\frac{1}{n^{4/5}}$. When $x-2=-5$, i.e. $x=-3$, the series reduces to $\sum\limits_{n=1}^{\infty}\frac{(-5)^n}{n^{4/5}(5^n-4)}=\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n^{4/5}(1-4/5^n)}$ which converges by the alternating series test. So the interval of convergence is $-3\le x<7$ or $[-3,7)$.

3.5.3.11. $*$. Solution. We apply the ratio test with $a_n=\frac{(x+2)^n}{n^2}$. Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{(x+2)^{n+1}}{(n+1)^2}}{\frac{(x+2)^n}{n^2}}\right|=\lim_{n\to\infty}\frac{n^2}{(n+1)^2}|x+2|$$
$$=\lim_{n\to\infty}\frac{1}{(1+1/n)^2}|x+2|=|x+2|$$

we have convergence for

$$|x+2|<1\iff -1<x+2<1\iff -3<x<-1$$

and divergence for $|x+2|>1$. For $|x+2|=1$, i.e. for $x+2=\pm1$, i.e. for $x=-3,-1$, the series reduces to $\sum\limits_{n=1}^{\infty}\frac{(\pm1)^n}{n^2}$, which converges absolutely, because $\sum\limits_{n=1}^{\infty}\frac{1}{n^p}$ converges for $p=2>1$. So the given series converges if and only if $-3\le x\le -1$.

3.5.3.12. $*$. Solution. We apply the ratio test with $a_n=\frac{4^n}{n}(x-1)^n$. Since

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|&=\lim_{n\to\infty}\left|\frac{4^{n+1}(x-1)^{n+1}/(n+1)}{4^n(x-1)^n/n}\right|\\&=\lim_{n\to\infty}4|x-1|\frac{n}{n+1}\\&=4|x-1|\lim_{n\to\infty}\frac{n}{n+1}=4|x-1|\cdot 1.\end{aligned}$$

the series converges if

$$\begin{aligned}4|x-1|<1&\iff -1<4(x-1)<1\iff -\frac{1}{4}<x-1<\frac{1}{4}\\&\iff \frac{3}{4}<x<\frac{5}{4}\end{aligned}$$

and diverges if $4|x-1|>1$. Checking the right endpoint $x=\frac{5}{4}$, we see that

$$\sum_{n=1}^{\infty}\frac{4^n}{n}\left(\frac{5}{4}-1\right)^n=\sum_{n=1}^{\infty}\frac{1}{n}$$

is the divergent harmonic series. At the left endpoint $x=\frac{3}{4}$,

$$\sum_{n=1}^{\infty}\frac{4^n}{n}\left(\frac{3}{4}-1\right)^n=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

converges by the alternating series test. Therefore the interval of convergence of the original series is $\frac{3}{4}\le x<\frac{5}{4}$, or $\left[\frac{3}{4},\frac{5}{4}\right)$.

3.5.3.13. *. Solution. We apply the ratio test with $a_n = (-1)^n \frac{(x-1)^n}{2^n(n+2)}$. Since

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{2^{n+1}(n+3)}\ \frac{2^n(n+2)}{(x-1)^n}\right| \\ &= \lim_{n\to\infty}\frac{|x-1|}{2}\frac{n+2}{n+3} \\ &= \frac{|x-1|}{2}\lim_{n\to\infty}\frac{1+2/n}{1+3/n} = \frac{|x-1|}{2}\end{aligned}$$

the series converges if

$$\begin{aligned}\frac{|x-1|}{2} < 1 &\iff |x-1| < 2 \iff -2 < (x-1) < 2 \\ &\iff -1 < x < 3\end{aligned}$$

and diverges if $|x-1| > 2$. So the series has radius of convergence 2. Checking the left endpoint $x = -1$, so that $\frac{x-1}{2} = -1$, we see that

$$\sum_{n=0}^{\infty}(-1)^n\frac{(-1-1)^n}{2^n(n+2)} = \sum_{n=0}^{\infty}\frac{1}{n+2}$$

is the divergent harmonic series. At the right endpoint $x = 3$, so that $\frac{x-1}{2} = +1$ and

$$\sum_{n=0}^{\infty}(-1)^n\frac{(3-1)^n}{2^n(n+2)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+2}$$

converges by the alternating series test. Therefore the interval of convergence of the original series is $-1 < x \le 3$, or $\big(-1, 3\big]$.

3.5.3.14. *. Solution. We apply the ratio test with $a_n = (-1)^n n^2 (x-a)^{2n}$. Since

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)^2(x-a)^{2(n+1)}}{(-1)^n n^2(x-a)^{2n}}\right| \\ &= \lim_{n\to\infty}|x-a|^2\frac{(n+1)^2}{n^2} \\ &= |x-a|^2\lim_{n\to\infty}\big(1+\frac{1}{n}\big)^2 = |x-a|^2\cdot 1.\end{aligned}$$

the series converges if

$$\begin{aligned}|x-a|^2 < 1 &\iff |x-a| < 1 \iff -1 < x-a < 1 \\ &\iff a-1 < x < a+1\end{aligned}$$

and diverges if $|x-a| > 1$. Checking both endpoints $x - a = \pm 1$, we see that

$$\sum_{n=1}^{\infty}(-1)^n n^2 (x-a)^{2n}\Big|_{x-a=\pm 1} = \sum_{n=1}^{\infty}(-1)^n n^2$$

fails the divergence test — the n^{th} term does not converge to zero as $n \to \infty$. Therefore the interval of convergence of the original series is $a - 1 < x < a + 1$, or $\big(a - 1, a + 1\big)$.

3.5.3.15. *. Solution. (a) We apply the ratio test for the series whose k^{th} term is $A_k = \frac{(x+1)^k}{k^2 9^k}$. Then

$$\begin{aligned}
\lim_{k\to\infty} \left|\frac{A_{k+1}}{A_k}\right| &= \lim_{k\to\infty} \left|\frac{(x+1)^{k+1}}{(k+1)^2 9^{k+1}} \ \frac{k^2 9^k}{(x+1)^k}\right| \\
&= \lim_{k\to\infty} |x+1| \ \frac{1}{9} \ \frac{k^2}{(k+1)^2} \\
&= \lim_{k\to\infty} |x+1| \ \frac{1}{9} \ \frac{1}{(1+1/k)^2} \\
&= \frac{|x+1|}{9}
\end{aligned}$$

So the series must converge when $|x + 1| < 9$ and must diverge when $|x + 1| > 9$. When $x + 1 = \pm 9$, the series reduces to

$$\sum_{k=1}^{\infty} \frac{(\pm 9)^k}{k^2 9^k} = \sum_{k=1}^{\infty} \frac{(\pm 1)^k}{k^2}$$

which converges (since, by the p–test, $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for any $p > 1$). So the interval of covnergence is $|x + 1| \le 9$ or $-10 \le x \le 8$ or $[-10, 8]$.
(b) The partial sum

$$\begin{aligned}
&\sum_{k=1}^{N} \Big(\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}}\Big) \\
&\qquad = \Big(\frac{a_1}{a_2} - \frac{a_2}{a_3}\Big) + \Big(\frac{a_2}{a_3} - \frac{a_3}{a_4}\Big) + \cdots + \Big(\frac{a_N}{a_{N+1}} - \frac{a_{N+1}}{a_{N+2}}\Big) \\
&\qquad = \frac{a_1}{a_2} - \frac{a_{N+1}}{a_{N+2}}
\end{aligned}$$

We are told that $\displaystyle\sum_{k=1}^{\infty} \Big(\frac{a_k}{a_{k+1}} - \frac{a_{k+1}}{a_{k+2}}\Big) = \frac{a_1}{a_2}$. This means that the above partial sum converges to $\frac{a_1}{a_2}$ as $N \to \infty$, or equivalently, that

$$\lim_{N\to\infty} \frac{a_{N+1}}{a_{N+2}} = 0$$

and hence that

$$\lim_{k\to\infty} \frac{|a_{k+1}(x-1)^{k+1}|}{|a_k(x-1)^k|} = |x-1| \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$$

is infinite for any $x \ne 1$. So, by the ratio test, this series converges only for $x = 1$.

3.5.3.16. *. Solution. Using the geometric series $\sum\limits_{n=0}^{\infty} x^n = \frac{1}{1-x}$,

$$\frac{x^3}{1-x} = x^3 \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+3} = \sum_{n=3}^{\infty} x^n$$

3.5.3.17. Solution. We can find $f(x)$ by differentiating its integral, or antidifferentiating its derivative. In the latter case, we'll have to solve for the arbitrary constant of integration; in the former case, we do not. (Remember that many different functions have the same derivative, but a single function has only one derivative.) To avoid the necessity of finding the arbitrary constant, we can ignore the given equation for $f'(x)$, which makes the problem much simpler. This is the method used in Solution 1.

- Solution 1: Using the Fundamental Theorem of Calculus Part 1:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_5^x f(t)\mathrm{d}t\right\} = f(x)$$

$$\begin{aligned} \text{So,} \qquad f(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{3x + \sum_{n=0}^{\infty}\frac{(x-1)^{n+1}}{n(n+1)^2}\right\} \\ &= 3 + \sum_{n=1}^{\infty}\frac{(n+1)(x-1)^n}{n(n+1)^2} \\ &= 3 + \sum_{n=1}^{\infty}\frac{(x-1)^n}{n(n+1)} \end{aligned}$$

- Solution 2: Suppose we had used $f'(x)$ instead. We would antidifferentiate to find:

$$\begin{aligned} f(x) &= \int\left(\sum_{n=0}^{\infty}\frac{(x-1)^n}{n+2}\right)\mathrm{d}x \\ &= \left(\sum_{n=0}^{\infty}\frac{(x-1)^{n+1}}{(n+1)(n+2)}\right) + C \\ &= \left(\sum_{n=1}^{\infty}\frac{(x-1)^n}{n(n+1)}\right) + C \end{aligned}$$

Notice $f(1) = 0 + C$. So, to find C, we must find $f(1)$. We can't get that information from $f'(x)$, so our only option is to consider the given formula for $\int_5^x f(t)\mathrm{d}t$. Using the Fundamental Theorem of Calculus Part 1:

$$\begin{aligned} f(1) &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_5^x f(t)\mathrm{d}t\right\}\Bigg|_{x=1} \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{3x + \sum_{n=1}^{\infty}\frac{(x-1)^{n+1}}{n(n+1)^2}\right\}\Bigg|_{x=1} \end{aligned}$$

$$= \left[3 + \sum_{n=1}^{\infty} \frac{(n+1)(x-1)^n}{n(n+1)^2} \right]_{x=1}$$
$$= \left[3 + \sum_{n=1}^{\infty} \frac{(x-1)^n}{n(n+1)} \right]_{x=1}$$
$$= 3 + \sum_{n=1}^{\infty} \frac{0^n}{n(n+1)}$$
$$= 3$$

So, $f(x) = 3 + \sum_{n=1}^{\infty} \frac{(x-1)^n}{n(n+1)}$.

Note that in Solution 2, we did the same calculation as Solution 1, and more.

Exercises — Stage 3

3.5.3.18. *. Solution. We apply the ratio test for the series whose n^{th} term is either $a_n = \frac{x^n}{3^{2n}\log n}$ or $a_n = \left|\frac{x^n}{3^{2n}\log n}\right|$. For both series

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{x^{n+1}}{3^{2(n+1)}\log(n+1)} \; \frac{3^{2n}\log n}{x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{x\log n}{3^2\log(n+1)} \right| = \lim_{n\to\infty} \left| \frac{x\log n}{3^2[\log(n)+\log(1+1/n)]} \right| \\
&= \lim_{n\to\infty} \left| \frac{x}{3^2[1+\log(1+1/n)/\log(n)]} \right| \\
&= \frac{|x|}{9}
\end{aligned}$$

Therefore, by the ratio test, our series converges absolutely when $|x| < 9$ and diverges when $|x| > 9$.

For $x = -9$, $\sum_{n=2}^{\infty} \frac{x^n}{3^{2n}\log n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\log n}$ which converges by the alternating series test.

For $x = +9$, $\sum_{n=2}^{\infty} \frac{x^n}{3^{2n}\log n} = \sum_{n=2}^{\infty} \frac{1}{\log n}$ which is the same series as $\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\log n}\right|$. We shall shortly show that $n \ge \log n$, and hence $\frac{1}{\log n} \ge \frac{1}{n}$ for all $n \ge 1$. This implies that the series $\sum_{n=2}^{\infty} \frac{1}{\log n}$ diverges by comparison with the divergent series $\left.\sum_{n=2}^{\infty} \frac{1}{n^p}\right|_{p=1}$.

This yelds both divergence for $x = 9$ and also the failure of absolute convergence for $x = -9$.

Finally, we show that $n - \log n > 0$, for all $n \ge 1$. Set $f(x) = x - \log x$. Then $f(1) = 1 > 0$ and

$$f'(x) = 1 - \frac{1}{x} \ge 0 \qquad \text{for all } x \ge 1$$

So $f(x)$ is (strictly) positive when $x = 1$ and is increasing for all $x \geq 1$. So $f(x)$ is (strictly) positive for all $x \geq 1$.

3.5.3.19. *. Solution. (a) Applying $\dfrac{1}{1+r} = \sum_{n=0}^{\infty}(-1)^n r^n$ with $r = x^3$ gives

$$\int \frac{1}{1+x^3}\,\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n \int x^{3n}\,\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{3n+1}}{3n+1} + C$$

(b) By part (a),

$$\int_0^{1/4} \frac{1}{1+x^3}\,\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{3n+1}}{3n+1}\bigg|_0^{1/4} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(3n+1)4^{3n+1}}$$

This is an alternating series with successively smaller terms that converge to zero as $n \to \infty$. So truncating it introduces an error no larger than the magnitude of the first dropped term. We want that first dropped term to obey

$$\frac{1}{(3n+1)4^{3n+1}} < 10^{-5} = \frac{1}{10^5}$$

So let's check the first few terms.

$$\begin{aligned}
\frac{1}{(3n+1)4^{3n+1}}\bigg|_{n=0} &= \frac{1}{4} > \frac{1}{10^5}\\
\frac{1}{(3n+1)4^{3n+1}}\bigg|_{n=1} &= \frac{1}{4^5} > \frac{1}{10^5}\\
\frac{1}{(3n+1)4^{3n+1}}\bigg|_{n=2} &= \frac{1}{7\times 4^7} = \frac{1}{7\times 2^{14}} = \frac{1}{7\times 16\times 1024}\\
&= \frac{1}{112\times 1024} < \frac{1}{10^5}
\end{aligned}$$

So we need to keep two terms (the $n = 0$ and $n = 1$ terms).

3.5.3.20. *. Solution. (a) Differentiating both sides of

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

gives

$$\sum_{n=0}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$$

Now multiplying both sides by x gives

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

as desired.
(b) Differentiating both sides of the conclusion of part (a) gives

$$\sum_{n=0}^{\infty} n^2 x^{n-1} = \frac{(1-x)^2 - 2x(x-1)}{(1-x)^4} = \frac{(1-x)(1-x+2x)}{(1-x)^4}$$
$$= \frac{1+x}{(1-x)^3}$$

Now multiplying both sides by x gives

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$$

We know that differentiation preserves the radius of convergence of power series. So this series has radius of convergence 1 (the radius of convergence of the original geometric series). At $x = \pm 1$ the series diverges by the divergence test. So the series converges for $-1 < x < 1$.

3.5.3.21. *. Solution. By the divergence test, the fact that $\sum_{n=0}^{\infty}(1-b_n)$ converges guarantees that $\lim_{n\to\infty}(1-b_n) = 0$, or equivalently, that $\lim_{n\to\infty} b_n = 1$. So, by equation (3.5.2), the radius of convergence is

$$R = \left[\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right|\right]^{-1} = \left[\frac{1}{1}\right]^{-1} = 1$$

3.5.3.22. *. Solution. (a) We know that the radius of convergence R obeys

$$\frac{1}{R} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n}{n+1} \frac{(n+1)a_{n+1}}{na_n} = 1\frac{C}{C} = 1$$

because we are told that $\lim_{n\to\infty} na_n = C$. So $R = 1$.
(b) Just knowing that the radius of convergence is 1, we know that the series converges for $|x| < 1$ and diverges for $|x| > 1$. That leaves $x \pm 1$.
When $x = +1$, the series reduces to $\sum_{n=1}^{\infty} a_n$. We are told that na_n *decreases* to $C > 0$. So $a_n \geq \frac{C}{n}$. By the comparison test with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges by the p–test with $p = 1$, our series diverges when $x = 1$.
When $x = -1$, the series reduces to $\sum_{n=1}^{\infty}(-1)^n a_n$. We are told that na_n *decreases* to $C > 0$. So $a_n > 0$ and a_n converges to 0 as $n \to \infty$. Consequently $\sum_{n=1}^{\infty}(-1)^n a_n$ converges by the alternating series test.
In conclusion $\sum_{n=1}^{\infty} a_n x^n$ converges when $-1 \leq x < 1$.

3.5.3.23. Solution. Equation 2.3.1 tells us the centre of mass of a rod with weights $\{m_n\}$ at positions $\{x_n\}$ is $\bar{x} = \dfrac{\sum m_n x_n}{\sum m_n}$.

We find the combined mass of our weights using Lemma 3.2.5 with $r = \frac{1}{2}$ and $r = \frac{1}{3}$, respectively.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{\infty} \frac{1}{3^n} &= \sum_{n=0}^{\infty} \frac{1}{2} \cdot \frac{1}{2^n} + \sum_{n=0}^{\infty} \frac{1}{3} \cdot \frac{1}{3^n} \\ &= \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} + \frac{1}{3} \cdot \frac{1}{1 - \frac{1}{3}} \\ &= 1 + \frac{1}{2} = \frac{3}{2}\end{aligned}$$

Now, we want to calculate the sum of the products of the masses and their positions.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \cdot n + \sum_{n=1}^{\infty} \frac{1}{3^n} \cdot (-n)$$

We don't have such a nice formula for this, but we can make one by differentiating. The following formula is true for any x with $|x| < 1$:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Differentiating both sides with respect to x:

$$\begin{aligned}\sum_{n=0}^{\infty} n x^{n-1} &= \frac{1}{(1-x)^2} \\ \sum_{n=1}^{\infty} n x^{n-1} &= \frac{1}{(1-x)^2}\end{aligned}$$

Multiplying both sides by x:

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$$

This allows us to evaluate our series.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n}{2^n} - \sum_{n=1}^{\infty} \frac{n}{3^n} &= \frac{\frac{1}{2}}{\left(1 - \frac{1}{2}\right)^2} - \frac{\frac{1}{3}}{\left(1 - \frac{1}{3}\right)^2} \\ &= 2 - \frac{3}{4} = \frac{5}{4}\end{aligned}$$

Therefore,

$$\bar{x} = \frac{5/4}{3/2} = \frac{5}{6} = 0.8\overline{33}$$

Remark: we can check that this makes some sense. Since the weights to the right of $x = 0$ are heavier than those to the left, but spaced the same, we would expect our rod to balance to the right of $x = 0$.

3.5.3.24. Solution. First, we differentiate.

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} A_n(x-c)^n \\
f'(x) &= \sum_{n=0}^{\infty} nA_n(x-c)^{n-1} \\
&= \sum_{n=1}^{\infty} nA_n(x-c)^{n-1} \\
f'(c) &= \sum_{n=1}^{\infty} nA_n \cdot 0^{n-1} \\
&= A_1 \cdot 1 + 2A_2 \cdot 0 + 3A_3 \cdot 0 + \cdots \\
&= A_1
\end{aligned}$$

So, if $A_1 = 0$, then $f'(c) = 0$. That is, $f(x)$ has a critical point at $x = c$.
To determine the behaviour of this critical point, we use the second derivative test.

$$\begin{aligned}
f'(x) &= \sum_{n=1}^{\infty} nA_n(x-c)^{n-1} \\
f''(x) &= \sum_{n=1}^{\infty} n(n-1)A_n(x-c)^{n-2} \\
&= \sum_{n=2}^{\infty} n(n-1)A_n(x-c)^{n-2} \\
f''(c) &= \sum_{n=2}^{\infty} n(n-1)A_n \cdot 0^{n-2} \\
&= 2(1)A_2 \cdot 0^0 + 3(2)A_3 \cdot 0^1 + 4(3)A_4 \cdot 0^2 + \cdots \\
&= 2A_2
\end{aligned}$$

Following the second derivative test, $x = c$ is the location of a local maximum if $A_2 < 0$, and it is the location of a local minimum if $A_2 > 0$. (If $A_2 = 0$, the critical point may or may not be a local extremum.)

3.5.3.25. Solution. We recognize $\displaystyle\sum_{n=3}^{\infty} \frac{n}{5^{n-1}}$ as $f(x) = \displaystyle\sum_{n=3}^{\infty} n \cdot x^{n-1}$, evaluated at $x = \dfrac{1}{5}$. We should figure out what $f(x)$ is in equation form (as opposed to power series form). Notice that this looks similar to the derivative of the geometric series $\sum x^n$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad \text{when}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{1-x}\right\} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n=0}^{\infty} x^n\right\}\\
\frac{1}{(1-x)^2} &= \sum_{n=1}^{\infty} nx^{n-1}\\
&= 1x^0 + 2x^1 + \sum_{n=3}^{\infty} nx^{n-1}\\
&= 1 + 2x + \sum_{n=3}^{\infty} nx^{n-1}\\
\text{So,}\qquad \frac{1}{(1-x)^2} - 1 - 2x &= \sum_{n=3}^{\infty} nx^{n-1}\\
\text{Setting :}\qquad \frac{1}{(1-1/5)^2} - 1 - \frac{2}{5} &= \sum_{n=3}^{\infty} n\left(\frac{1}{5}\right)^{n-1}\\
\left(\frac{5}{4}\right)^2 - 1 - \frac{2}{5} &= \sum_{n=3}^{\infty} \frac{n}{5^{n-1}}
\end{aligned}$$

So, our series evaluates to $\dfrac{25}{16} - 1 - \dfrac{2}{5} = \dfrac{13}{80}$.

3.5.3.26. Solution. As we saw in in Example 3.5.20,

$$\log(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$$

which is an alternating series when x is positive. If we use its partial sum S_N to approximate $\log(1+x)$, the absolute error involved is no more than

$$\frac{x^{(N+1)+1}}{(N+1)+1} = \frac{x^{N+2}}{N+2}$$

We want this error to be at most 10^{-5} whenever $0 < x < \frac{1}{10}$. For this range of x values, $\dfrac{x^{N+2}}{N+2} < \dfrac{1}{(N+2)10^{N+2}}$, so we want N that satisfies the inequality:

$$\begin{aligned}
& \frac{1}{(N+2)10^{N+2}} \le \frac{1}{10^5}\\
\Rightarrow \qquad & (N+2)10^{N+2} \ge 10^5
\end{aligned}$$

We see $N = 3$ suffices.
So, the partial sum

$$\sum_{n=0}^{3}(-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

approximates $\log(1+x)$ to within an error of $\dfrac{x^5}{5}$.

When x is between 0 and $\frac{1}{10}$, that error is at most $\dfrac{1}{5\cdot 10^5} < 10^{-5}$, as desired.

Now we can approximate $\log(1.05)$.

$$\begin{aligned}
\log(1.05) &= \log\left(1+\frac{1}{20}\right)\\
&\approx \left(\tfrac{1}{20}\right) - \frac{\left(\frac{1}{20}\right)^2}{2} + \frac{\left(\frac{1}{20}\right)^3}{3} - \frac{\left(\frac{1}{20}\right)^4}{4}\\
&= \frac{12\times 20^3 - 6\times 20^2 + 4\times 20 - 3}{12\times 20^4} = \frac{93677}{1920000}
\end{aligned}$$

We note that a computer approximates $\frac{93677}{1920000} \approx 0.04879010$ and $\log(1.05) \approx 0.04879016$. So, our actual error is around 6×10^{-8}.

3.5.3.27. Solution. As we saw in in Example 3.5.21,

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

which is an alternating series when x is nonzero. If we use its partial sum S_N to approximate $\arctan x$, the absolute error involved is no more than

$$\frac{|x|^{2(N+1)+1}}{N(n+1)+1} = \frac{|x|^{2N+3}}{2N+3}$$

We want this error to be at most 10^{-6} whenever $-\frac{1}{4} < x < \frac{1}{4}$. For this range of x values, $\dfrac{|x|^{2N+3}}{2N+3} < \dfrac{1}{(2N+3)4^{2N+3}}$, so we want N that satisfies the inequality:

$$\frac{1}{(2N+3)4^{2N+3}} \le \frac{1}{10^5}$$

A quick check with a calculator shows that $N=2$ suffices.

So, the partial sum

$$\sum_{n=0}^{2} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5}$$

approximates $\arctan x$ to within an error of $\dfrac{x^7}{7}$.

When x is between $-\frac{1}{4}$ and $\frac{1}{4}$, that error is at most $\dfrac{1}{7\cdot 4^7} = \dfrac{1}{114688} < \dfrac{1}{100000} = 10^{-5}$, as desired. (When $x=0$, our approximation is 0, the exact value of $\arctan 0$.)

3.6 · Taylor Series

3.6.8 · Exercises

Exercises — Stage 1

3.6.8.1. Solution. All functions A, B, and C intersect the function $y = f(x)$ when $x = 2$. B is a constant function, so this is the constant approximation. A is the tangent line, so A is the linear approximation. C is a tangent parabola, so C is the quadratic approximation.

3.6.8.2. Solution. Following how a Taylor series is constructed, the Taylor series and the function agree at the point chosen as the centre. So, $T(5) = \arctan^3(e^5 + 7)$. If we were evaluating a Taylor series at a point *other than its centre*, we would generally need to check that (a) the series converges, and (b) it converges to the same value as the function we used to create it.

3.6.8.3. Solution. These are listed in Theorem 3.6.7. However, it's possible to figure out many of them without a lot of memorization. For example, $e^0 = \cos(0) = \frac{1}{1-0} = 1$, while $\sin(0) = \log(1 + 0) = \arctan(0) = 0$. So by plugging in $x = 0$ to the series listed, we can divide them into these two categories.
The derivative of sine is cosine, so we can also look for one series that is the derivative of another. The derivative of e^x is e^x, so we can look for a series that is its own derivative.
Furthermore, sine and arctangent are odd functions and only II and IV are odd. Cosine is an even function and only III is even.
Alternately, we can find the first few terms of each series using the definition of a Taylor series, and match them up.
All together, the functions correspond to the following series:
A - V
B - I
C - IV
D - VI
E - II
F - III

3.6.8.4. Solution.

a Using the definition of a Taylor series, we know

$$\sum_{n=0}^{\infty} \frac{n^2}{(n!+1)}(x-3)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n$$

So, the coefficient of $(x-3)^{20}$ is $\frac{f^{(20)}(3)}{20!}$ (using the definition). Using the given series, the coefficient of $(x-3)^{20}$ is $\frac{20^2}{20!+1}$. So,

$$\begin{aligned} \frac{f^{(20)}(3)}{20!} &= \frac{20^2}{20!+1} \\ \Rightarrow \qquad f^{(20)}(3) &= 20^2\left(\frac{20!}{20!+1}\right) \end{aligned}$$

(which is extremely close to 20^2).

b Using the definition of a Taylor series, we know

$$\sum_{n=0}^{\infty} \frac{n^2}{(n!+1)}(x-3)^{2n} = \sum_{k=0}^{\infty} \frac{g^{(k)}(3)}{k!}(x-3)^k$$

So, the coefficient of $(x-3)^{20}$ is $\frac{g^{(20)}(3)}{20!}$ (using the definition). Looking at the given series, the coefficient of $(x-3)^{20}$ occurs when $n=10$, so it is $\frac{10^2}{10!+1}$. So,

$$\begin{aligned} \frac{g^{(20)}(3)}{20!} &= \frac{10^2}{10!+1} \\ \Rightarrow \qquad g^{(20)}(3) &= 10^2\left(\frac{20!}{10!+1}\right) \end{aligned}$$

c With the previous two examples in mind, we find the Maclaurin series for $h(x)$. (Using the series representation will be much easier than differentiating $h(x)$ directly twenty times.) Recall from the text that we know the Maclaurin series for $\arctan x$.

$$\begin{aligned} \arctan(x) &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} \\ \arctan(5x^2) &= \sum_{n=0}^{\infty}(-1)^n\frac{(5x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{5^{2n+1}}{2n+1}x^{4n+2} \\ \frac{\arctan(5x^2)}{x^4} &= \sum_{n=0}^{\infty}(-1)^n\frac{5^{2n+1}}{2n+1}x^{4n-2} \\ \sum_{k=0}^{\infty}\frac{h^{(k)}(0)}{k!}x^k &= \sum_{n=0}^{\infty}(-1)^n\frac{5^{2n+1}}{2n+1}x^{4n-2} \end{aligned}$$

Using the definition of a Maclaurin series, the coefficient of x^{22} is $\dfrac{h^{(22)}(0)}{22!}$. This occurs in the given series when $n=6$, so

$$\begin{aligned} \frac{h^{(22)}(0)}{22!} &= (-1)^6\frac{5^{2\times 6+1}}{2\times 6+1} = \frac{5^{13}}{13} \\ \Rightarrow \qquad h^{(22)}(0) &= \frac{22!\cdot 5^{13}}{13} \end{aligned}$$

Similarly, the coefficient of x^{20} in the Maclaurin series is $\dfrac{h^{(20)(0)}}{20!}$. Since no term x^{20} occurs in our series, that coefficient is 0, so $h^{(20)}(0) = 0$.

Exercises — Stage 2

3.6.8.5. Solution. The definition of a Taylor series tells us we will be computing the coefficients in the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

That is, we need a general description of $f^{(n)}(1)$. To find this, we take a few derivatives, and look for a pattern.

$$\begin{aligned}
f(x) &= \log(x) & f(1) &= 0\\
f'(x) &= x^{-1} & f'(1) &= 1\\
f''(x) &= (-1)x^{-2} & f''(1) &= -1\\
f^{(3)}(x) &= (-2)(-1)x^{-3} & f^{(3)}(1) &= 2!\\
f^{(4)}(x) &= (-3)(-2)(-1)x^{-4} & f^{(4)}(1) &= -3!\\
f^{(5)}(x) &= (-4)(-3)(-2)(-1)x^{-5} & f^{(5)}(1) &= 4!\\
f^{(6)}(x) &= (-5)(-4)(-3)(-2)(-1)x^{-6} & f^{(6)}(1) &= -5!\\
&\vdots & &\vdots\\
f^{(n)}(x) &= (-1)^{n-1}(n-1)!\,x^{-n} & f^{(n)}(1) &= (-1)^{n-1}(n-1)!
\end{aligned}$$

Using the convention $0! = 1$, our pattern for $f^{(n)}(1)$ begins when $n = 1$.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n &= 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n\\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n
\end{aligned}$$

3.6.8.6. Solution. To find the Taylor series for sine, centred at $a = \pi$, we'll need to know the various derivatives of sine at π.

$$\begin{aligned}
f(x) &= \sin x & f(\pi) &= 0\\
f'(x) &= \cos x & f'(\pi) &= -1\\
f''(x) &= -\sin x & f''(\pi) &= 0\\
f'''(x) &= -\cos x & f'''(\pi) &= 1\\
f^{(4)}(x) &= \sin x = f(x) & f^{(4)}(\pi) &= 0
\end{aligned}$$

Even derivatives are 0; odd derivatives alternate between -1 and $+1$. (If you're following along with the derivation of the Maclaurin series for sine in the text, note $f^{(n)}(\pi) = -f^{(n)}(0)$.)
In our Taylor series, every even-indexed term will be zero, and we will be left with only odd-indexed terms. If we let n be our index, then the term $2n+1$ will capture all the odd numbers. Since the signs alternate, $f^{(2n+1)}(\pi) = (-1)^{n+1}$. So, our Taylor

series is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(\pi)}{k!}(x-\pi)^k = \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(\pi)}{(2n+1)!}(x-\pi)^{2n+1} \qquad \text{(since the even terms are all zero)}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}(x-\pi)^{2n+1}$$

3.6.8.7. Solution. The definition of a Taylor series tells us we will be computing the coefficients in the series

$$\sum_{n=0}^{\infty} \frac{g^{(n)}(10)}{n!}(x-10)^n$$

That is, we need a general description of $g^{(n)}(10)$. To find this, we take a few derivatives, and look for a pattern.

$$\begin{aligned}
g(x) &= x^{-1} & g(10) &= \frac{1}{10}\\
g'(x) &= (-1)x^{-2} & g'(10) &= \frac{-1}{10^2}\\
g''(x) &= (-2)(-1)x^{-3} & g''(10) &= \frac{(-1)^2 2!}{10^3}\\
g^{(3)}(x) &= (-3)(-2)(-1)x^{-4} & g^{(3)}(10) &= \frac{(-1)^3 3!}{10^4}\\
g^{(4)}(x) &= (-4)(-3)(-2)(-1)x^{-5} & g^{(4)}(10) &= \frac{(-1)^4 4!}{10^5}\\
g^{(5)}(x) &= (-5)(-4)(-3)(-2)(-1)x^{-6} & g^{(5)}(10) &= \frac{(-1)^5 5!}{10^6}\\
&\vdots & &\vdots\\
g^{(n)}(x) &= (-1)^n n! x^{-(n+1)} & g^{(n)}(10) &= \frac{(-1)^n n!}{10^{n+1}}
\end{aligned}$$

Using the convention $0! = 1$, our pattern for $g^{(n)}(10)$ begins when $n = 0$.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{g^{(n)}(1)}{n!}(x-10)^n &= \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!10^{n+1}}(x-10)^n\\
&= -\sum_{n=0}^{\infty} \frac{(x-10)^n}{(-10)^{n+1}}\\
&= \frac{1}{10}\sum_{n=0}^{\infty} \left(\frac{10-x}{10}\right)^n
\end{aligned}$$

For fixed x, we recognize this as a geometric series with $r = \frac{10-x}{10}$. So it converges precisely when $|r| < 1$, i.e.

$$\begin{aligned}
\left|\frac{10-x}{10}\right| &< 1\\
|10-x| &< 10\\
-10 < x - 10 &< 10\\
0 < x &< 20
\end{aligned}$$

So, its interval of convergence is $(0, 20)$.

3.6.8.8. Solution. The definition of a Taylor series tells us we will be computing the coefficients in the series

$$\sum_{n=0}^{\infty} \frac{h^{(n)}(a)}{n!}(x-a)^n$$

That is, we need a general description of $h^{(n)}(a)$. To find this, we take a few derivatives, and look for a pattern.

$$\begin{aligned}
h(x) &= e^{3x} & h(a) &= e^{3a}\\
h'(x) &= 3e^{3x} & h'(a) &= 3e^{3a}\\
h''(x) &= 3^2e^{3x} & h''(a) &= 3^2e^{3a}\\
h'''(x) &= 3^3e^{3x} & h'''(a) &= 3^3e^{3a}\\
&\vdots & &\vdots\\
h^{(n)}(x) &= 3^ne^{3x} & h^{(n)}(a) &= 3^ne^{3a}
\end{aligned}$$

The pattern for $h^{(n)}(a)$ holds for all (whole numbers) $n \geq 0$. So, our Taylor series for $h(x)$ is

$$\sum_{n=0}^{\infty} \frac{3^ne^{3a}}{n!}(x-a)^n$$

To find its radius of convergence, we use the ratio test.

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{3^{n+1}e^{3a}(x-a)^{n+1}}{(n+1)!}\cdot\frac{n!}{3^ne^{3a}(x-a)^n}\right|\\
&= \left|\frac{3^{n+1}}{3^n}\cdot\frac{e^{3a}}{e^{3a}}\cdot\frac{n!}{(n+1)!}\cdot\frac{(x-a)^{n+1}}{(x-a)^n}\right|\\
&= 3\cdot\frac{1}{n+1}\cdot|x-a|\\
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left[\frac{3}{n+1}\cdot|x-a|\right] = 0
\end{aligned}$$

Our series converges for every value of x, so its radius of convergence is ∞.

3.6.8.9. *. Solution. Substituting $y = 2x$ into $\dfrac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ (which is valid for all $-1 < y < 1$) gives

$$f(x) = \frac{1}{2x-1} = -\frac{1}{1-2x} = -\sum_{n=0}^{\infty}(2x)^n = -\sum_{n=0}^{\infty} 2^n x^n$$

for all $-\frac{1}{2} < x < \frac{1}{2}$.

3.6.8.10. *. Solution. Substituting first $y = -x$ and then $y = 2x$ into $\dfrac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ (which is valid for all $-1 < y < 1$) gives

$$\begin{aligned} \frac{1}{1-(-x)} &= \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n \\ \frac{1}{1-(2x)} &= \sum_{n=0}^{\infty}(2x)^n = \sum_{n=0}^{\infty} 2^n x^n \end{aligned}$$

for all $-\frac{1}{2} < x < \frac{1}{2}$. Hence, for all $-\frac{1}{2} < x < \frac{1}{2}$,

$$\begin{aligned} f(x) &= \frac{3}{x+1} - \frac{1}{2x-1} = \frac{3}{1-(-x)} + \frac{1}{1-2x} \\ &= 3\sum_{n=0}^{\infty}(-1)^n x^n + \sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty}\big(3(-1)^n + 2^n\big)x^n \end{aligned}$$

So $b_n = 3(-1)^n + 2^n$.

3.6.8.11. *. Solution. We found the Taylor series for e^{3x} from scratch in Question 8. If we hadn't just done that, we could easily find it by modifying the series for e^x.
Substituting $y = 3x$ into the exponential series

$$e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$$

gives

$$e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n\, x^n}{n!}$$

so that c_5, the coefficient of x^5, which appears only in the $n = 5$ term, is $c_5 = \dfrac{3^5}{5!}$

3.6.8.12. $*$. Solution. Since

$$f'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\log(1+2t) = \frac{2}{1+2t} = 2\sum_{n=0}^{\infty}(-2t)^n \quad \text{if } |2t|<1 \text{ i.e. } |t|<\frac{1}{2}$$

and $f(0)=0$, we have

$$f(x) = \int_0^x f'(t)\,\mathrm{d}t = 2\sum_{n=0}^{\infty}\int_0^x (-1)^n 2^n t^n\,\mathrm{d}t = \sum_{n=0}^{\infty}(-1)^n 2^{n+1}\frac{x^{n+1}}{n+1}$$

for all $|x| < \frac{1}{2}$.

3.6.8.13. $*$. Solution. We just need to substitute $y = x^3$ into the known Maclaurin series for $\sin y$, to get the Maclaurin series for $\sin(x^3)$, and then multiply the result by x^2.

$$\begin{aligned}\sin y &= y - \frac{y^3}{3!} + \cdots \\ \sin(x^3) &= x^3 - \frac{x^9}{3!} + \cdots \\ x^2\sin(x^3) &= x^5 - \frac{x^{11}}{3!} + \cdots\end{aligned}$$

so $a=1$ and $b = -\frac{1}{3!} = -\frac{1}{6}$.

3.6.8.14. $*$. Solution. Recall that

$$e^y = \sum_{n=0}^{\infty}\frac{y^n}{n!} = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + \cdots$$

Setting $y=-x^2$, we have

$$\begin{aligned}e^{-x^2} &= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots \\ e^{-x^2} - 1 &= -x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots \\ \frac{e^{-x^2}-1}{x} &= -x + \frac{x^3}{2} - \frac{x^5}{6} + \cdots \\ \int \frac{e^{-x^2}-1}{x}\,\mathrm{d}x &= C - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{36} + \cdots\end{aligned}$$

3.6.8.15. $*$. Solution. Recall that

$$\arctan(y) = \sum_{n=0}^{\infty}(-1)^n\frac{y^{2n+1}}{2n+1}$$

Setting $y = 2x$, we have

$$\begin{aligned}\int x^4 \arctan(2x)\,\mathrm{d}x &= \int \left(x^4 \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n+1}}{2n+1} \right) \mathrm{d}x \\ &= \int \left(\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+5}}{2n+1} \right) \mathrm{d}x \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1} x^{2n+6}}{(2n+1)(2n+6)} + C \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n+6}}{(2n+1)(n+3)} + C\end{aligned}$$

3.6.8.16. *. Solution. Substituting $y = -3x^3$ into $\dfrac{1}{1-y} = \sum\limits_{n=0}^{\infty} y^n$ gives

$$\frac{\mathrm{d}f}{\mathrm{d}x} = x \cdot \frac{1}{1+3x^3} = x \sum_{n=0}^{\infty} \big(-3x^3 \big)^n = \sum_{n=0}^{\infty} (-1)^n 3^n x^{3n+1}$$

Now integrating,

$$f(x) = \sum_{n=0}^{\infty} (-1)^n 3^n \frac{x^{3n+2}}{3n+2} + C$$

To have $f(0) = 1$, we need $C = 1$. So, finally

$$f(x) = 1 + \sum_{n=0}^{\infty} (-1)^n \frac{3^n}{3n+2} x^{3n+2}$$

3.6.8.17. *. Solution. We're given a big hint: that our series resembles the Taylor series for arctangent.

The terms of arctangent are $(-1)^n \dfrac{x^{2n+1}}{2n+1}$. Our terms resemble those terms, with x^{2n+1} replaced by $\dfrac{1}{3^n}$.

Since $3^n = \big(\sqrt{3}\big)^{2n} = \frac{1}{\sqrt{3}}\big(\sqrt{3}\big)^{2n+1}$:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} &= \sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\big(\sqrt{3}\big)^{2n+1}} \\ &= \sqrt{3} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \bigg|_{x=\frac{1}{\sqrt{3}}} = \sqrt{3} \arctan \frac{1}{\sqrt{3}} \\ &= \sqrt{3}\,\frac{\pi}{6} = \frac{\pi}{2\sqrt{3}}\end{aligned}$$

3.6.8.18. $*$. **Solution**. Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. So

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \Big[\sum_{n=0}^{\infty} \frac{x^n}{n!}\Big]_{x=-1} = \Big[e^x\Big]_{x=-1} = e^{-1}$$

3.6.8.19. $*$. **Solution**. Recall that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. So

$$\sum_{k=0}^{\infty} \frac{1}{e^k k!} = \Big[\sum_{k=0}^{\infty} \frac{x^k}{k!}\Big]_{x=1/e} = \Big[e^x\Big]_{x=1/e} = e^{1/e}$$

3.6.8.20. $*$. **Solution**. Recall that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. So

$$\sum_{k=0}^{\infty} \frac{1}{\pi^k k!} = \Big[\sum_{k=0}^{\infty} \frac{x^k}{k!}\Big]_{x=1/\pi} = \Big[e^x\Big]_{x=1/\pi} = e^{1/\pi}$$

This series differs from the given one only in that it starts with $k = 0$ while the given series starts with $k = 1$. So

$$\sum_{k=1}^{\infty} \frac{1}{\pi^k k!} = \sum_{k=0}^{\infty} \frac{1}{\pi^k k!} - \underbrace{1}_{k=0} = e^{1/\pi} - 1$$

3.6.8.21. $*$. **Solution**. Recall, from Theorem 3.6.7, that, for all $-1 < x \le 1$,

$$\log(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

(To get from the first sum to the second sum we substituted $n = k+1$. If you don't see why the two sums are equal, write out the first few terms of each.) So

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\, 2^n} = \Big[\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}\Big]_{x=1/2} = \Big[\log(1+x)\Big]_{x=1/2} = \log(3/2)$$

3.6.8.22. $*$. **Solution**. Write

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n+2}{n!} e^n &= \sum_{n=1}^{\infty} \frac{n}{n!} e^n + \sum_{n=1}^{\infty} \frac{2}{n!} e^n \\
&= \sum_{n=1}^{\infty} \frac{e^n}{(n-1)!} + 2 \sum_{n=1}^{\infty} \frac{e^n}{n!} \\
&= e \sum_{n=1}^{\infty} \frac{e^{n-1}}{(n-1)!} + 2 \sum_{n=1}^{\infty} \frac{e^n}{n!}
\end{aligned}$$

$$= e\sum_{n=0}^{\infty}\frac{e^n}{n!} + 2\sum_{n=1}^{\infty}\frac{e^n}{n!}$$

Recall that $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$. So

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n+2}{n!}e^n &= e\Big[\sum_{n=0}^{\infty}\frac{x^n}{n!}\Big]_{x=e} + 2\Big[\sum_{n=1}^{\infty}\frac{x^n}{n!}\Big]_{x=e}\\ &= e\Big[e^x\Big]_{x=e} + 2\Big[e^x-1\Big]_{x=e} = e^{e+1} + 2(e^e-1)\\ &= (e+2)e^e - 2\end{aligned}$$

3.6.8.23. Solution. Let's use the ratio test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{n+1}}{\frac{2^n}{n}}\right|\\ &= \lim_{n\to\infty} 2\frac{n}{n+1} = 2 > 1\end{aligned}$$

So, the series diverges.

Remark: it's tempting to note that $\log(1+y) = \displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{y^{n+1}}{n+1} = -\sum_{n=1}^{\infty}\frac{(-y)^n}{n}$, and try to substitute in $y=-2$. But, the Maclaurin series for $\log(1+y)$ has radius of convergence $R=1$, so it doesn't converge at $y=-2$. Furthermore, $\log(1+(-2)) = \log(-1)$, but this is undefined.

3.6.8.24. Solution. Our series looks something like the Taylor series for sine, $\sin x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$.

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{4}\right)^{2n+1}\left(1+2^{2n+1}\right)\\ &\qquad= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left[\left(\frac{\pi}{4}\right)^{2n+1} + \left(\frac{\pi}{2}\right)^{2n+1}\right]\\ &\qquad= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{4}\right)^{2n+1} + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n+1}\\ &\qquad= \sin\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{2}\right)\\ &\qquad= \frac{1}{\sqrt{2}} + 1 = \frac{1+\sqrt{2}}{\sqrt{2}}\end{aligned}$$

3.6.8.25. *. Solution. (a)

- Solution 1: The naive strategy is to set $a_n = \dfrac{x^{2n}}{(2n)!}$ and apply the ratio test.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{x^{2n+2}}{(2n+2)!}}{\frac{x^{2n}}{(2n)!}}\right| = \left|\frac{x^{2n+2}}{x^{2n}}\cdot\frac{(2n)!}{(2n+2)(2n+1)(2n)!}\right| \\ &= \lim_{n\to\infty}\frac{x^2}{(2n+2)(2n+1)} \\ &= 0\end{aligned}$$

  This is smaller than 1 no matter what x is. So the series converges for all x.

- Solution 2: Alternatively, the sneaky way is to observe that both $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ and $e^{-x} = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}$ are known to converge for all x. So

$$\frac{1}{2}\left(e^x + e^{-x}\right) = \sum_{n\text{ even}}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

  also converges for all x.

(b) Recall that $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$. Then:

$$\begin{aligned}e &= \sum_{n=0}^{\infty}\frac{1}{n!} \\ e^{-1} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!} \\ e + e^{-1} &= \sum_{n=0}^{\infty}\frac{1+(-1)^n}{n!} = 2\sum_{n\text{ even}}\frac{1}{n!} = 2\sum_{n=0}^{\infty}\frac{1}{(2n)!}\end{aligned}$$

Hence $\sum_{n=0}^{\infty}\frac{1}{(2n)!} = \frac{1}{2}\left(e + \frac{1}{e}\right)$.

3.6.8.26. Solution. All three series we're adding up are alternating, so we can bound the absolute error in the approximation S_N (the N-th partial sum) by $|a_{N+1}|$. The Taylor series for arctangent is

$$\arctan(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

for every real x.

a Using the Taylor series for arctangent when $x = 1$, we see

$$\frac{\pi}{4} = \arctan(1) = \sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}$$
$$\pi = \sum_{n=0}^{\infty}(-1)^n\frac{4}{2n+1}$$

The error involved in approximating π with the partial sum S_N is at most $|a_{N+1}| = \frac{4}{2N+3}$. In order for this to be at most 4×10^{-5}, we need:

$$\begin{aligned}\frac{4}{2N+3} &\le 4 \times 10^{-5}\\ 2N+3 &\ge 10^5\\ N &\ge \frac{10^5-3}{2} = 5 \times 10^4 - \frac{3}{2} = 50,000 - 1.5\end{aligned}$$

Since n must be an integer, we need to add up the terms from $n = 0$ to $n = 49,999$. That is, we add up the first 50,000 terms.

b Using the Taylor series for arctangent:

$$\begin{aligned}\pi &= 16\arctan\frac{1}{5} - 4\arctan\frac{1}{239}\\ &= 16\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)5^{2n+1}} - 4\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)\cdot 239^{2n+1}}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{16}{5^{2n+1}} - \frac{4}{239^{2n+1}}\right)\end{aligned}$$

This is an alternating sum, so the absolute error in using the partial sum S_N is at most:

$$|a_{N+1}| = \frac{1}{2N+3}\left(\frac{16}{5^{2N+3}} - \frac{4}{239^{2N+3}}\right)$$

So, we want to find a value of N that makes this at most 4×10^{-5}. Several values of N are given below.

N	$\|a_{N+1}\|$
1	$\frac{1}{5}\left(\frac{16}{5^5} - \frac{4}{239^5}\right) \approx 0.001$
2	$\frac{1}{7}\left(\frac{16}{5^7} - \frac{4}{239^7}\right) \approx 0.000029 < 4 \times 10^{-5}$

So, it suffices to add up the first three terms ($n = 0$, $n = 1$, and $n = 2$) of the series.

c Again, we use the Taylor series for arctangent.

$$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\left(\frac{3+2}{2\cdot 3 - 1}\right) = \arctan(1) = \frac{\pi}{4}$$

so that

$$\begin{aligned}
\pi &= 4\left(\arctan\frac{1}{2} + \arctan\frac{1}{3}\right) \\
&= 4\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)2^{2n+1}} + 4\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)3^{2n+1}} \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{4}{2n+1}\left(\frac{1}{2^{2n+1}} + \frac{1}{3^{2n+1}}\right)
\end{aligned}$$

If we use the partial sum S_N, our absolute error is at most

$$|a_{N+1}| = \frac{4}{2N+3}\left(\frac{1}{2^{2N+3}} + \frac{1}{3^{2N+3}}\right).$$

Several of these values are given below.

N	$\lvert a_{N+1}\rvert$
1	$\frac{4}{5}\left(\frac{1}{2^5}+\frac{1}{3^5}\right) \approx 0.028$
2	$\frac{4}{7}\left(\frac{1}{2^7}+\frac{1}{3^7}\right) \approx 0.0047$
3	$\frac{4}{9}\left(\frac{1}{2^9}+\frac{1}{3^9}\right) \approx 0.00089$
4	$\frac{4}{11}\left(\frac{1}{2^{11}}+\frac{1}{3^{11}}\right) \approx 0.00018$
5	$\frac{4}{13}\left(\frac{1}{2^{13}}+\frac{1}{3^{13}}\right) \approx 0.000038 < 4\times 10^{-5}$

So, it suffices to add the first six terms ($n = 0$ to $n = 5$) of the series.

Remark: if we actually wanted to approximate π this way, the series from part (a) is probably not ideal — adding 50,000 terms sounds rough. The series from (b) and (c) seem much more practical.

3.6.8.27. Solution. Using the Taylor series for $\log(1+x)$:

$$\log(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

$$\log(1.5) = \log\left(1+\frac{1}{2}\right) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n2^n}$$

Since this is an alternating series, the error involved in using the partial sum S_N is at most

$$|a_{N+1}| = \frac{1}{(N+1)2^{N+1}}.$$

We want this to be at most 5×10^{-11}.

N	$\|a_{N+1}\|$
10	$\frac{1}{11 \cdot 2^{11}} \approx 4 \times 10^{-5}$
15	$\frac{1}{16 \cdot 2^{16}} \approx 9.5 \times 10^{-7}$
20	$\frac{1}{21 \cdot 2^{21}} \approx 2 \times 10^{-8}$
25	$\frac{1}{26 \cdot 2^{26}} \approx 6 \times 10^{-10}$
26	$\frac{1}{27 \cdot 2^{27}} \approx 3 \times 10^{-10}$
27	$\frac{1}{28 \cdot 2^{28}} \approx 1 \times 10^{-10}$
28	$\frac{1}{29 \cdot 2^{29}} \approx 6 \times 10^{-11}$
29	$\frac{1}{30 \cdot 2^{30}} \approx 3 \times 10^{-11}$

So, it suffices to add up the first 29 terms.

3.6.8.28. Solution. The Taylor Series for e^x is not alternating, so we'll use Theorem 3.6.3 to bound the error in a partial-sum approximation. The error in the partial-sum approximation S_N is

$$E_N = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$

for some c strictly between a and x. In our case, $a = 0$ and $x = 1$. So, we want to find a value of N such that

$$\left| \frac{f^{(N+1)}(c)}{(N+1)!}(1-0)^{N+1} \right| = \frac{e^c}{(N+1)!} < 5 \times 10^{-11}$$

for *all* c in $(0, 1)$.

If c is between 0 and 1, then e^c is between 1 and e. However, since the purpose of this problem is to approximate e precisely, it doesn't make much sense to use e in our bound. Since e is less than 3, then $e^c < 3$ for all c in $(0, 1)$. Now we can search for an appropriate value of N.

N	$\dfrac{3}{(N+1)!}$
10	$\dfrac{3}{11!} = \dfrac{1}{9^{10}} \approx 8 \times 10^{-8}$
11	$\dfrac{3}{12!} \approx 6 \times 10^{-9}$
12	$\dfrac{3}{13!} \approx 5 \times 10^{-10}$
13	$\dfrac{3}{14!} \approx 3 \times 10^{-11}$

So, it suffices to use the partial sum S_{13}.

3.6.8.29. Solution. The Taylor Series for $\log(1-x)$ is not alternating, so we'll use Theorem 3.6.3 to bound the error in a partial-sum approximation. The error in the partial-sum approximation S_N is

$$E_N = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$

for some c strictly between a and x. In our case, $a = 0$ and $x = \dfrac{1}{10}$. So, we want to find a value of N such that

$$\left|\frac{f^{(N+1)}(c)}{(N+1)!}\left(\frac{1}{10}\right)^{N+1}\right| < 5 \times 10^{-11}$$

for *all* c in $(0, \frac{1}{10})$.
To find this N, we to know $f^{(N+1)}(x)$. Just like when we create a Taylor polynomial from scratch, we'll differentiate $f(x)$ several times, and look for a pattern.

$$\begin{aligned}
f(x) &= \log(1-x) & f^{(6)}(x) &= \frac{-2(3)(4)(5)}{(1-x)^6}\\
f'(x) &= \frac{-1}{1-x} & f^{(7)}(x) &= \frac{-2(3)(4)(5)(6)}{(1-x)^7}\\
f''(x) &= \frac{-1}{(1-x)^2} & &\vdots\\
f'''(x) &= \frac{-2}{(1-x)^3} & f^{(N+1)}(x) &= \frac{-N!}{(1-x)^{N+1}}\\
f^{(4)}(x) &= \frac{-2(3)}{(1-x)^4} & &\\
f^{(5)}(x) &= \frac{-2(3)(4)}{(1-x)^5} & &
\end{aligned}$$

Now we want a reasonable bound on $f^{(N+1)}(c)$, when c is in $(0, \frac{1}{10})$. Note that in this range, $1 - c > 0$.

$$0 < c < \frac{1}{10}$$

$$\begin{aligned}
\Rightarrow \qquad & \frac{9}{10} < 1-c < 1 \\
\Rightarrow \qquad & \left(\frac{9}{10}\right)^{N+1} < (1-c)^{N+1} < 1 \\
\Rightarrow \qquad & 1 < \frac{1}{(1-c)^{N+1}} < \left(\frac{10}{9}\right)^{N+1} \\
\Rightarrow \qquad & N! < \frac{N!}{(1-c)^{N+1}} < N!\left(\frac{10}{9}\right)^{N+1}
\end{aligned}$$

This bound provides us with a "worst-case scenario" error. We don't know exactly what c is, but we don't need to — the bound above holds for *all* c between 0 and $\frac{1}{10}$.

Now we're ready to choose an N that results in a sufficiently small error bound.

$$\begin{aligned}
\left|\frac{f^{(N+1)}(c)}{(N+1)!}\left(\frac{1}{10}\right)^{N+1}\right| &< \frac{N!\left(\frac{10}{9}\right)^{N+1}}{(N+1)!}\left(\frac{1}{10}\right)^{N+1} \\
&= \frac{1}{9^{N+1}\cdot(N+1)}
\end{aligned}$$

So, we want: $$\frac{1}{9^{N+1}\cdot(N+1)} < 5\times10^{-11}$$

To find an appropriate N, we test several values.

N	$\frac{1}{9^{N+1}\cdot(N+1)}$
8	$\frac{1}{9\cdot9^9}=\frac{1}{9^{10}}\approx 3\times10^{-10}$
9	$\frac{1}{10\cdot9^{10}}\approx 3\times10^{-11}$

So, it suffices to use the partial sum S_9.

3.6.8.30. Solution. We'll use Theorem 3.6.3 to bound the error of a partial-sum approximation. The error in the partial-sum approximation S_N is

$$E_N = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$

for some c strictly between a and x. In our case, $a=0$ and x is in $(-2,1)$. So, we want to find a value of N such that

$$\left|\frac{f^{(N+1)}(c)}{(N+1)!}(x)^{N+1}\right| < 5\times10^{-11}$$

for *all* x in $(-2,1)$, and *all* c in $(-2,1)$.

To find this N, we to know $f^{(N+1)}(x)$. Just like when we create a Taylor polynomial from scratch, we'll differentiate $f(x)$ several times, and look for a pattern.

$$\begin{aligned} f(x) = \sinh(x) &= \frac{e^x - e^{-x}}{2} \\ f'(x) &= \frac{e^x + e^{-x}}{2} \\ f''(x) &= \frac{e^x - e^{-x}}{2} \\ f'''(x) &= \frac{e^x + e^{-x}}{2} \end{aligned}$$

That is, even derivatives of $f(x)$ are $f(x)$, and odd derivatives of $f(x)$ are $\frac{e^x+e^{-x}}{2}$ (which, incidentally, is the function called $\cosh x$).

Now we want a reasonable bound on $f^{(N+1)}(c)$, when c is in $(-2, 1)$. Since powers of e are always positive, we begin by noting that $0 < \frac{e^x-e^{-x}}{2} < \frac{e^x+e^{-x}}{2}$. So, all derivatives of $f(x)$ are bounded above by $\frac{e^x+e^{-x}}{2}$.

$$\begin{aligned} & -2 < c < 1 \\ \Rightarrow \quad & e^{-2} < e^c < e \quad \text{and } e^{-1} < e^{-c} < e^2 \\ \Rightarrow \quad & f^{(N+1)}(c) < \frac{e^c + e^{-c}}{2} \quad < \frac{e^2+e^2}{2} = e^2 < 9 \end{aligned}$$

This bound provides us with a "worst-case scenario" error. We don't know exactly what c is, but we don't need to — the bound above holds for *all* c between -2 and 1.

We also don't know exactly what x will be, only that it's between -2 and 1. So, we note $|x|^{N+1} < 2^{N+1}$.

Now we're ready to choose an N that results in a sufficiently small error bound.

$$\left| \frac{f^{(N+1)}(c)}{(N+1)!} (x)^{N+1} \right| < \frac{9 \cdot 2^{N+1}}{(N+1)!}$$

$$\text{So, we want:} \quad \frac{9 \cdot 2^{N+1}}{(N+1)!} < 5 \times 10^{-11}$$

To find an appropriate N, we test several values.

N	$\dfrac{9 \cdot 2^{N+1}}{(N+1)!}$
10	$\dfrac{9 \cdot 2^{11}}{(11)!} \approx 5 \times 10^{-4}$
15	$\dfrac{9 \cdot 2^{16}}{(16)!} \approx 3 \times 10^{-8}$
17	$\dfrac{9 \cdot 2^{18}}{(18)!} \approx 4 \times 10^{-10}$
18	$\dfrac{9 \cdot 2^{19}}{(19)!} \approx 4 \times 10^{-11} < 5 \times 10^{-11}$

3.6.8.31. Solution. We'll use Theorem 3.6.3 to bound the error in a partial-sum

So, it suffices to use the partial sum S_{18}.

approximation. The error in the partial-sum approximation S_N is

$$E_N = \frac{f^{(N+1)}(c)}{(N+1)!}(x-a)^{N+1}$$

for some c strictly between a and x. In our case, $a = \frac{1}{2}$, $x = -\frac{1}{3}$, and we are given the nth derivative of $f(x)$:

$$\begin{aligned}
E_6 &= \frac{f^{(7)}(c)}{7!}\left(-\frac{1}{3}-\frac{1}{2}\right)^7 \\
&= \frac{1}{7!}\cdot\frac{6!}{2}\left[(1-c)^{-7} + (-1)^6(1+c)^{-7}\right]\left(-\frac{5}{6}\right)^7 \\
&= \frac{-5^7}{14\cdot 6^7}\cdot\left[(1-c)^{-7}+(1+c)^{-7}\right]
\end{aligned}$$

for some c in $(-\frac{1}{3}, \frac{1}{2})$.
We want to provide actual numeric bounds for this expression. That is, we want to find the absolute max and min of

$$E(c) = \frac{-5^7}{14\cdot 6^7}\cdot\left[(1-c)^{-7}+(1+c)^{-7}\right]$$

over the interval $\left(-\frac{1}{3}, \frac{1}{2}\right)$. Absolute extrema occur at endpoints and critical points. So, we'll start by differentiating $E(c)$, and finding its critical points (if any) in the interval $\left(-\frac{1}{3}, \frac{1}{2}\right)$.

$$\begin{aligned}
E(c) &= \frac{-5^7}{14\cdot 6^7}\cdot\left[(1-c)^{-7}+(1+c)^{-7}\right] \\
E'(c) &= \frac{-5^7}{14\cdot 6^7}\cdot\left[7(1-c)^{-8}-7(1+c)^{-8}\right] = 0 \\
(1-c)^{-8} &= (1+c)^{-8} \\
1-c &= 1+c \\
c &= 0
\end{aligned}$$

Since $E'(c)$ is defined over our entire interval, its only critical point is $c = 0$.

- $E(0) = \dfrac{-5^7}{14\cdot 6^7}[2]$
- $E\left(-\frac{1}{3}\right) = \dfrac{-5^7}{14\cdot 6^7}\left[\left(\frac{4}{3}\right)^{-7}+\left(\frac{2}{3}\right)^{-7}\right] = \dfrac{-5^7}{14\cdot 6^7}\left[\left(\frac{3}{4}\right)^{7}+\left(\frac{3}{2}\right)^{7}\right]$
- $E\left(\frac{1}{2}\right) = \dfrac{-5^7}{14\cdot 6^7}\left[\left(\frac{1}{2}\right)^{-7}+\left(\frac{3}{2}\right)^{-7}\right] = \dfrac{-5^7}{14\cdot 6^7}\left[2^7+\left(\frac{2}{3}\right)^{7}\right]$

We want to decide which of these numbers is biggest, and which smallest. Note that 2^7 is much, much bigger than $(3/2)^7$, and both $(3/4)^7$ and $(2/3)^7$ are less than one. Furthermore, $(3/2)^7$ is much larger than 2. So: $[2^7 + (2/3)^7] > [(3/2)^7 + (3/4)^7] > 2$. Therefore,

$$\frac{-5^7}{14 \cdot 6^7}\left[2^7 + \left(\frac{2}{3}\right)^7\right] < \frac{-5^7}{14 \cdot 6^7}\left[\left(\frac{3}{4}\right)^7 + \left(\frac{3}{2}\right)^7\right] < \frac{-5^7}{14 \cdot 6^7}[2]$$

We conclude that the error E_6 is in the interval

$$\left(\frac{-5^7}{14 \cdot 6^7}\left[2^7 + \left(\frac{2}{3}\right)^7\right] \quad , \quad \frac{-5^7}{14 \cdot 6^7}[2]\right)$$

or, equivalently,

$$\left(\frac{-5^7}{14 \cdot 3^7}\left[1 + \frac{1}{3^7}\right] \quad , \quad \frac{-5^7}{7 \cdot 6^7}\right)$$

which is approximately $(-0.199, -0.040)$.

Exercises — Stage 3

3.6.8.32. *. Solution. Using the Maclaurin series expansions of $\cos x$ and e^x,

$$\begin{aligned}
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \\
1 - \cos x &= \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
1 + x - e^x &= -\frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \\
\frac{1 - \cos x}{1 + x - e^x} &= \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots}{-\frac{x^2}{2!} - \frac{x^3}{3!} + \cdots} = \frac{\frac{1}{2!} - \frac{x^2}{4!} + \cdots}{-\frac{1}{2!} - \frac{x}{3!} + \cdots}
\end{aligned}$$

we have

$$\lim_{x\to 0}\frac{1 - \cos x}{1 + x - e^x} = \lim_{x\to 0}\frac{\frac{1}{2!} - \frac{x^2}{4!} + \cdots}{-\frac{1}{2!} - \frac{x}{3!} + \cdots} = \frac{\frac{1}{2!}}{-\frac{1}{2!}} = -1$$

3.6.8.33. *. Solution. Using the Maclaurin series expansion of $\sin x$,

$$\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\sin x - x + \frac{x^3}{6} &= \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\frac{\sin x - x + \frac{x^3}{6}}{x^5} &= \frac{1}{5!} - \frac{x^2}{7!} + \cdots
\end{aligned}$$

we have
$$\lim_{x\to 0}\frac{\sin x - x + \frac{x^3}{6}}{x^5} = \lim_{x\to 0}\left(\frac{1}{5!}-\frac{x^2}{7!}+\cdots\right) = \frac{1}{5!} = \frac{1}{120}$$
Remark: to solve this using l'Hôpital's rule we would differentiate five times, making series a practical alternative.

3.6.8.34. Solution. Our limit has the indeterminate form 1^∞; as with l'Hôpital's rule, we can change it to a friendlier form using the natural logarithm.

$$\begin{aligned} f(x) &= \left(1+x+x^2\right)^{2/x} \\ \log(f(x)) &= \log\left[\left(1+x+x^2\right)^{2/x}\right] = \frac{2}{x}\log\left(1+x+x^2\right) \end{aligned}$$

Recall $\log(1+y) = \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}y^n}{n}$, and set $y = x+x^2$. The series converges when $|y| < 1$, and since we only consider values of x that are very close to 0, we can assume $|x+x^2| < 1$.

$$\begin{aligned} \log(f(x)) &= \frac{2}{x}\log\left(1+(x+x^2)\right) = \frac{2}{x}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x+x^2)^n}{n} \\ &= \frac{2}{x}\left[(x+x^2) - \frac{(x+x^2)^2}{2} + \frac{(x+x^2)^3}{3} - \cdots\right] \\ &= 2+2x - \frac{(x+x^2)^2}{2x} + \frac{(x+x^2)^3}{3x} - \cdots \\ &= 2+2x - \frac{(x^2+x)(1+x)}{2} + \frac{(x^2+x)^2(1+x)}{3} - \cdots \end{aligned}$$

so that

$$\begin{aligned} &\lim_{x\to 0}\log(f(x)) \\ &\quad= \lim_{x\to 0}\left[2+2x - \frac{(x^2+x)(1+x)}{2} + \frac{(x^2+x)^2(1+x)}{3} - \cdots\right] \\ &\quad= 2+0+0\cdots = 2 \end{aligned}$$

and

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{\log f(x)} = e^2$$

3.6.8.35. Solution. We have an indeterminate form 1^∞. We can use a natural logarithm to change this to a friendlier form. Furthermore, to avoid negative powers, we substitute $y = \dfrac{1}{2x}$. As x grows larger and larger, y gets closer and closer to zero, while staying positive.

$$\log\left[\left(1+\frac{1}{2x}\right)^x\right] = x\log\left(1+\frac{1}{2x}\right) = \frac{1}{2y}\log(1+y)$$

$$
\begin{aligned}
&= \frac{1}{2y}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}y^n \\
&= \frac{1}{2y}\left[y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots\right] \\
&= \left[\frac{1}{2} - \frac{y}{4} + \frac{y^2}{6} - \frac{y^3}{8} + \cdots\right] \\
\lim_{x\to\infty}\log\left[\left(1+\frac{1}{2x}\right)^x\right] &= \lim_{y\to 0^+}\left[\frac{1}{2} - \frac{y}{4} + \frac{y^2}{6} - \frac{y^3}{8} + \cdots\right] = \frac{1}{2} \\
\lim_{x\to\infty}\left[\left(1+\frac{1}{2x}\right)^x\right] &= e^{1/2} = \sqrt{e}
\end{aligned}
$$

3.6.8.36. Solution. The factor $(n+1)(n+2)$ reminds us of a derivative. Start with the geometric series.

$$
\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{1-x}\right\} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n=0}^{\infty} x^n\right\} \\
\frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} nx^{n-1} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{(1-x)^2}\right\} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n=1}^{\infty} nx^{n-1}\right\} \\
\frac{2}{(1-x)^3} &= \sum_{n=1}^{\infty} n(n-1)x^{n-2} = \sum_{n=2}^{\infty} n(n-1)x^{n-2} \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)x^n
\end{aligned}
$$

Let $x = \frac{1}{7}$. Then $|x| < 1$, so our series converges.

$$
\begin{aligned}
\frac{2}{(1-1/7)^3} &= \sum_{n=0}^{\infty}(n+2)(n+1)\left(\frac{1}{7}\right)^n \\
\frac{2}{(6/7)^3} &= \sum_{n=0}^{\infty}\frac{(n+2)(n+1)}{7^n}
\end{aligned}
$$

3.6.8.37. Solution. Recall the Taylor series for arctangent is:

$$
\arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}
$$

There are similarities between this and our given series: skipping powers of x, and a denominator that's not factorial. We'll try to manipulate it to look like our series.

First, we antidifferentiate, to get a factor of $(2n+2)$ on the bottom.

$$\int \arctan x \, \mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{(2n+1)(2n+2)} + C$$

We can find the antiderivative of arctangent using integration by parts. Let $u = \arctan x$ and $\mathrm{d}v = \mathrm{d}x$; then $\mathrm{d}u = \frac{1}{1+x^2}\mathrm{d}x$ and $v = x$.

$$\int \arctan x \, \mathrm{d}x = x \arctan x - \int \frac{x}{1+x^2} \, \mathrm{d}x + C$$

Now, we use the substitution $w = 1 + x^2$, $\mathrm{d}w = 2x\mathrm{d}x$.

$$= x \arctan x - \frac{1}{2} \log(1+x^2) + C$$

$$\text{So, } \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{(2n+1)(2n+2)} = x \arctan x - \frac{1}{2} \log(1+x^2) + C$$

To find C, we evaluate both sides of the equation at $x = 0$.

$$0 = 0 \arctan 0 - \frac{1}{2} \log(1) + C = C$$

$$\text{Therefore, } \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{(2n+1)(2n+2)} = x \arctan x - \frac{1}{2} \log(1+x^2)$$

Multiplying both sides by x^2,

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+4}}{(2n+1)(2n+2)} = x^3 \arctan x - \frac{x^2}{2} \log(1+x^2)$$

3.6.8.38. Solution.

(a) We'll start, as we usually do, by finding a pattern for $f^{(n)}(0)$.

$$\begin{aligned}
f(x) &= (1-x)^{-1/2} \\
f'(x) &= \frac{1}{2}(1-x)^{-3/2} \\
f''(x) &= \frac{1 \cdot 3}{2^2}(1-x)^{-5/2} \\
f'''(x) &= \frac{1 \cdot 3 \cdot 5}{2^3}(1-x)^{-7/2} \\
f^{(4)}(x) &= \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^4}(1-x)^{-9/2} \\
&\vdots \\
f^{(n)}(x) &= \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2^n}(1-x)^{-(2n+1)/2} \\
f^{(n)}(0) &= \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2^n}
\end{aligned}$$

We could leave it like this, but we simplify, to make our work cleaner later on.

$$\begin{aligned}
&= \frac{1}{2^n} \cdot \frac{(2n)!}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)} \\
&= \frac{1}{2^n} \cdot \frac{(2n)!}{2^n \cdot n!} \\
&= \frac{(2n)!}{2^{2n}\, n!}
\end{aligned}$$

This pattern holds for $n \geq 0$. Now, we can write our Maclaurin series for $f(x)$.

$$(1-x)^{-1/2} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}\,(n!)^2} x^n$$

To find the radius of convergence, we use the ratio test.

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(2n+2)!}{2^{2n+2}((n+1)!)^2} \cdot \frac{2^{2n}\,(n!)^2}{(2n)!} \cdot |x| \\
&= \frac{(2n+2)!}{(2n)!} \left(\frac{n!}{(n+1)!}\right)^2 \cdot \frac{2^{2n}}{2^{2n+2}} |x| \\
&= (2n+2)(2n+1) \left(\frac{1}{n+1}\right)^2 \cdot \frac{1}{4} |x| \\
&= \frac{4n^2+4n+2}{4n^2+8n+4} |x| \\
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left[\frac{4n^2+4n+2}{4n^2+8n+4}|x|\right] = |x|
\end{aligned}$$

So, the radius of convergence is $R = 1$.

(b) We note the derivative of the arcsine function is $\dfrac{1}{\sqrt{1-x^2}} = f(x^2)$. With this insight, we can manipulate our Taylor series for $f(x)$ into a Taylor series for arcsine.

$$\begin{aligned}
\frac{1}{\sqrt{1-x}} &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}\,(n!)^2} x^n \\
\frac{1}{\sqrt{1-x^2}} &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}\,(n!)^2} x^{2n} \\
\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x &= \int \left(\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}\,(n!)^2} x^{2n}\right) \mathrm{d}x \\
\arcsin x &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}\,(n!)^2(2n+1)} x^{2n+1} + C \\
\arcsin x &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}\,(n!)^2(2n+1)} x^{2n+1}
\end{aligned}$$

where we found the value of C by setting $x = 0$. Its radius of convergence is also 1, by Theorem 3.5.13.

3.6.8.39. *. Solution. We use that

$$\log(1+y) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{y^n}{n} \qquad \text{for all } -1 < y \le 1$$

with $y = \frac{x-2}{2}$ to give

$$\begin{aligned} \log(x) &= \log(2+x-2) = \log\left[2\left(1+\frac{x-2}{2}\right)\right] \\ &= \log 2 + \log\left(1+\frac{x-2}{2}\right) = \log 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\,2^n}(x-2)^n \end{aligned}$$

It converges when $-1 < y \le 1$, or equivalently, $0 < x \le 4$.

3.6.8.40. *. Solution. (a) Using the geometric series expansion with $r = -t^4$,

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \implies \frac{1}{1+t^4} = \sum_{n=0}^{\infty} (-t^4)^n = \sum_{n=0}^{\infty} (-1)^n t^{4n}$$

Substituting this into our integral,

$$\begin{aligned} I(x) &= \int_0^x \frac{1}{1+t^4}\,\mathrm{d}t \\ &= \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^{4n}\right)\mathrm{d}t \\ &= \left[\sum_{n=0}^{\infty} (-1)^n \frac{t^{4n+1}}{4n+1}\right]_{t=0}^{t=x} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{4n+1} \end{aligned}$$

(b) Substituting in $x = \frac{1}{2}$,

$$\begin{aligned} I(1/2) &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(4n+1)2^{4n+1}} \\ &= \frac{1}{2} - \frac{1}{5\times 2^5} + \frac{1}{9\times 2^9} - \frac{1}{13\times 2^{13}} + \cdots \\ &= 0.5 - 0.00625 + 0.000217 - 0.0000094 + \cdots \\ &= \boxed{0.493967} - 0.0000094 + \cdots \end{aligned}$$

See part (c) for the error analysis.

(c) The series for $I(x)$ is an alternating series (that is, the sign alternates) with successively smaller terms that converge to zero. So the error introduced by truncating the series is between zero and the first omitted term. In this case, the first omitted term was negative (-0.0000094). So the exact value of $I(1/2)$ is the approximate value found in part (b) plus a negative number whose magnitude is smaller than $0.00001 = 10^{-5}$. So the approximate value of part (b) is *larger* than the true value of $I(1/2)$.

3.6.8.41. *. Solution. Expanding the exponential using its Maclaurin series,

$$\begin{aligned} I &= \int_0^1 x^4 e^{-x^2}\,\mathrm{d}x = \sum_{n=0}^{\infty} \int_0^1 x^4 \frac{(-x^2)^n}{n!}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n+4}\,\mathrm{d}x \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+5)} = \underbrace{\frac{1}{5}}_{n=0} - \underbrace{\frac{1}{7}}_{n=1} + \underbrace{\frac{1}{18}}_{n=2} - \underbrace{\frac{1}{3!(11)}}_{n=3} + \cdots \end{aligned}$$

The signs of successive terms in this series alternate. Futhermore the magnitude of the n^{th} term decreases with n. Hence, by the alternating series test, I lies between $\frac{1}{5} - \frac{1}{7} + \frac{1}{18}$ and $\frac{1}{5} - \frac{1}{7} + \frac{1}{18} - \frac{1}{3!(11)}$. So

$$|I - a| \le \frac{1}{3!(11)} = \frac{1}{66}$$

3.6.8.42. *. Solution. Expanding the exponential using its Taylor series,

$$\begin{aligned} I &= \int_0^{\frac{1}{2}} x^2 e^{-x^2}\,\mathrm{d}x = \sum_{n=0}^{\infty} \int_0^{\frac{1}{2}} x^2 \frac{(-x^2)^n}{n!}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^{\frac{1}{2}} x^{2n+2}\,\mathrm{d}x \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+3)} \frac{1}{2^{2n+3}} \end{aligned}$$

The signs of successive terms in this series alternate. Futhermore the magnitude of the n^{th} term decreases with n. Hence, by the alternating series test, I lies between $\sum_{n=0}^{N} \frac{(-1)^n}{n!(2n+3)} \frac{1}{2^{2n+3}}$ and $\sum_{n=0}^{N+1} \frac{(-1)^n}{n!(2n+3)} \frac{1}{2^{2n+3}}$, for every N. The first few terms are, to five decimal places,

n	0	1	2	3
$\frac{(-1)^n}{n!(2n+3)} \frac{1}{2^{2n+3}}$	0.04167	-0.00625	0.00056	-0.00004

Allowing for a roundoff error of 0.000005 in each of these, I must be between

$$0.04167 - 0.00625 + 0.00056 + 0.000005 \times 3 = 0.035995$$

and

$$0.04167 - 0.00625 + 0.00056 - 0.00004 - 0.000005 \times 4 = 0.035920$$

where the multiples of 0.000005 are the maximum possible accumulated roundoff errors in the added terms.

3.6.8.43. $*$. **Solution**. (a) Using the Taylor series expansion of e^x with $x = -t$,

$$e^{-t} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \implies e^{-t} - 1 = \sum_{n=1}^{\infty} (-1)^n \frac{t^n}{n!}$$
$$\implies \frac{e^{-t} - 1}{t} = \sum_{n=1}^{\infty} (-1)^n \frac{t^{n-1}}{n!}$$

Substituting this into our integral,

$$I(x) = \int_0^x \frac{e^{-t} - 1}{t} \, \mathrm{d}t = \sum_{n=1}^{\infty} (-1)^n \int_0^x \frac{t^{n-1}}{n!} \, \mathrm{d}t = \sum_{n=1}^{\infty} (-1)^n \, \frac{x^n}{n \cdot n!}$$

(b) Substituting in $x = 1$,

$$\begin{aligned} I(1) &= \sum_{n=1}^{\infty} (-1)^n \, \frac{1}{n \cdot n!} \\ &= -1 + \frac{1}{2 \cdot 2!} - \frac{1}{3 \cdot 3!} + \frac{1}{4 \cdot 4!} - \frac{1}{5 \cdot 5!} + \cdots \\ &= -1 + 0.25 - 0.0556 + 0.0104 - 0.0017 + \cdots = -0.80 \end{aligned}$$

See part (c) for the error analysis.
(c) The series for $I(x)$ is an alternating series (that is, the sign alternates) with successively smaller terms that converge to zero. So the error introduced by truncating the series is no larger than the first omitted term. So the magnitude of $-\frac{1}{5\,5!} + \cdots$ is no larger than 0.0017. Allowing for a roundoff error of at most 0.0001 in each of the two terms $-0.0556 + 0.0104$

$$I(1) = -1 + 0.25 - 0.0556 + 0.0104 \pm 0.0019 = -0.7952 \pm 0.0019$$

3.6.8.44. $*$. **Solution**. (a) Using the Taylor series expansion of $\sin x$ with $x = t$,

$$\sin t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} \implies \frac{\sin t}{t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}$$

So

$$\begin{aligned} \Sigma(x) &= \int_0^x \frac{\sin t}{t} \, \mathrm{d}t = \sum_{n=0}^{\infty} (-1)^n \int_0^x \frac{t^{2n}}{(2n+1)!} \, \mathrm{d}t \\ &= \sum_{n=0}^{\infty} (-1)^n \, \frac{x^{2n+1}}{(2n+1)(2n+1)!} \end{aligned}$$

(b) The critical points of $\Sigma(x)$ are the solutions of $\Sigma'(x) = 0$. By the fundamental theorem of calculus $\Sigma'(x) = \frac{\sin x}{x}$, so the critical points of $\Sigma(x)$ are $x = \pm\pi, \pm 2\pi, \cdots$.

The absolute maximum occurs at $x = \pi$.
(c) Substituting in $x = \pi$,

$$\begin{aligned}\Sigma(\pi) &= \sum_{n=0}^{\infty}(-1)^n \ \frac{\pi^{2n+1}}{(2n+1)(2n+1)!} \\ &= \pi - \frac{\pi^3}{3\cdot 3!} + \frac{\pi^5}{5\cdot 5!} - \frac{\pi^7}{7\cdot 7!} + \cdots \\ &= 3.1416 - 1.7226 + 0.5100 - 0.0856 + 0.0091 - 0.0007 + \cdots\end{aligned}$$

The series for $\Sigma(\pi)$ is an alternating series (that is, the sign alternates) with successively smaller terms that converge to zero. So the error introduced by truncating the series is no larger than the first omitted term. So

$$\Sigma(\pi) = 3.1416 - 1.7226 + 0.5100 - 0.0856 + 0.0091 = 1.8525$$

with an error of magnitude at most $0.0007 + 0.0005$ (the 0.0005 is the maximum possible accumulated roundoff error in all five retained terms).

3.6.8.45. $*$. Solution. (a) Using the Taylor series expansion of $\cos t$,

$$\begin{aligned}\cos t &= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{(2n)!} \\ \frac{\cos t - 1}{t^2} &= -\frac{1}{2!} + \frac{t^2}{4!} - \frac{t^4}{6!} + \cdots \\ &= \sum_{n=1}^{\infty}(-1)^n \frac{t^{2n-2}}{(2n)!} \\ I(x) = \int_0^x \frac{\cos t - 1}{t^2}\, \mathrm{d}t &= -\frac{x}{2!} + \frac{x^3}{4!3} - \frac{x^5}{6!5} + \cdots \\ &= \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n-1}}{(2n)!(2n-1)}\end{aligned}$$

(b), (c) Substituting in $x = 1$,

$$\begin{aligned}I(1) &= -\frac{1}{2} + \frac{1}{4!3} - \frac{1}{6!5} + \cdots \\ &= -0.5 + 0.0139 - 0.0003 - \cdots \\ &= -0.486 \pm 0.001\end{aligned}$$

The series for $I(1)$ is an alternating series with decreasing successive terms that converge to zero. So approximating $I(1)$ by $-\frac{1}{2} + \frac{1}{4!3}$ introduces an error between 0 and $-\frac{1}{6!5}$. Hence $I(1) < -\frac{1}{2} + \frac{1}{4!3}$.

3.6.8.46. ∗. Solution. (a) Using the Taylor series expansions of $\sin x$ and $\cos x$ with $x = t$,

$$\begin{aligned}
\sin t &= \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n+1}}{(2n+1)!} \\
&= t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots \\
t\sin t &= \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n+2}}{(2n+1)!} \\
&= \qquad t^2 - \frac{t^4}{3!} + \frac{t^6}{5!} - \frac{t^8}{7!} + \cdots \\
&= -\sum_{n=1}^{\infty}(-1)^n\frac{t^{2n}}{(2n-1)!} \\
\cos t &= \sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{(2n)!} \\
&= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} + \cdots \\
\cos t - 1 &= \sum_{n=1}^{\infty}(-1)^n\frac{t^{2n}}{(2n)!} \\
&= -\frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} + \cdots \\
\cos t + t\sin t - 1 &= \sum_{n=1}^{\infty}(-1)^n\frac{t^{2n}}{(2n)!} - \sum_{n=1}^{\infty}(-1)^n\frac{t^{2n}}{(2n-1)!} \\
&= \sum_{n=1}^{\infty}(-1)^n t^{2n}\left(\frac{1}{(2n)!} - \frac{1}{(2n-1)!}\right) \\
&= \left(1 - \frac{1}{2!}\right)t^2 - \left(\frac{1}{3!} - \frac{1}{4!}\right)t^4 + \cdots \\
&= \sum_{n=1}^{\infty}(-1)^n t^{2n}\left(\frac{1}{(2n)!} - \frac{2n}{(2n)!}\right) \\
&= \sum_{n=1}^{\infty}(-1)^n t^{2n}\left(\frac{1-2n}{(2n)!}\right) \\
&= \left(\frac{2}{2!} - \frac{1}{2!}\right)t^2 - \left(\frac{4}{4!} - \frac{1}{4!}\right)t^4 + \cdots \\
&= \sum_{n=1}^{\infty}(-1)^{n+1} t^{2n}\left(\frac{2n-1}{(2n)!}\right) \\
&= \frac{1}{2!}t^2 - \frac{3}{4!}t^4 + \frac{5}{6!}t^6 - \frac{7}{8!}t^8 + \cdots \\
\frac{\cos t + t\sin t - 1}{t^2} &= \sum_{n=1}^{\infty}(-1)^{n+1} t^{2n-2}\left(\frac{2n-1}{(2n)!}\right)
\end{aligned}$$

$$= \frac{1}{2!}t - \frac{3}{4!}t^2 + \frac{5}{6!}t^4 - \frac{7}{8!}t^6 + \cdots$$

Now, we're ready to integrate.

$$\begin{aligned}
I(x) &= \int_0^x \left(\frac{\cos t + t\sin t - 1}{t^2}\right) \\
&= \int_0^x \left(\sum_{n=1}^{\infty}(-1)^{n+1}t^{2n-2}\left(\frac{2n-1}{(2n)!}\right)\right)\mathrm{d}t \\
&= \left[\sum_{n=1}^{\infty}(-1)^{n+1}\frac{t^{2n-1}}{(2n)!}\right]_0^x \\
&= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{(2n)!}
\end{aligned}$$

(b)

$$\begin{aligned}
I(1) &= \frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!} - \frac{1}{8!} + \cdots \\
&= 0.5 - 0.041\dot{6} + 0.00139 - 0.000024 + \cdots \\
&= \boxed{0.460}
\end{aligned}$$

The error analysis is in part (c).
(c) The series for $I(1)$ is an alternating series with decreasing successive terms that convege to zero. So approximating $I(1)$ by $\frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!}$ introduces an error between 0 and $-\frac{1}{8!}$. So $I(1) < \frac{1}{2!} - \frac{1}{4!} + \frac{1}{6!} < 0.460$.

3.6.8.47. *. Solution. (a) Substituting $x = -t$ into the known power series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$, we see that:

$$\begin{aligned}
e^{-t} &= 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \cdots \\
1 - e^{-t} &= t - \frac{t^2}{2!} + \frac{t^3}{3!} - \frac{t^4}{4!} + \cdots \\
\frac{1 - e^{-t}}{t} &= 1 - \frac{t}{2!} + \frac{t^2}{3!} - \frac{t^3}{4!} + \cdots \\
\int \frac{1 - e^{-t}}{t}\,\mathrm{d}t &= C + x - \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!} - \frac{x^4}{4\cdot 4!} + \cdots
\end{aligned}$$

Finally, $f(0) = 0$ (since $f(0)$ is an integral from 0 to 0) and so $C = 0$. Therefore

$$f(x) = \int_0^x \frac{1 - e^{-t}}{t}\,\mathrm{d}t = x - \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!} - \frac{x^4}{4\cdot 4!} + \cdots .$$

We can also do this calculation entirely in summation notation: $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$, and so

$$\begin{aligned}
e^{-t} &= \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n!} \\
1 - e^{-t} &= -\sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^n}{n!} \\
\frac{1 - e^{-t}}{t} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{n-1}}{n!} \\
f(x) = \int_0^x \frac{1 - e^{-t}}{t}\, \mathrm{d}t &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n \cdot n!}
\end{aligned}$$

(b) We set $a_n = A_n x^n = \frac{(-1)^{n-1}}{n \cdot n!} x^n$ and apply the ratio test.

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(-1)^n x^{n+1}/((n+1)\cdot(n+1)!)}{(-1)^{n-1} x^n/(n \cdot n!)}\right| \\
&= \lim_{n\to\infty} \left(\frac{|x|^{n+1}}{|x|^n} \frac{n \cdot n!}{(n+1)\cdot(n+1)!}\right) \\
&= \lim_{n\to\infty} \left(|x| \frac{n}{(n+1)^2}\right) \qquad \text{since } (n+1)! = (n+1)\, n! \\
&= 0
\end{aligned}$$

This is smaller than 1 no matter what x is. So the series converges for all x.

3.6.8.48. $*$. Solution.

$$\begin{aligned}
& e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \ge 1 + x \qquad \text{for all } x \ge 0 \\
\implies\ & e^x - 1 \ge x \\
\implies\ & \frac{x^3}{e^x - 1} \le \frac{x^3}{x} = x^2 \\
\implies\ & \int_0^1 \frac{x^3}{e^x - 1}\, \mathrm{d}x \le \int_0^1 x^2\, \mathrm{d}x = \frac{1}{3}
\end{aligned}$$

3.6.8.49. $*$. Solution. (a) We know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x. Replacing x by $-x$, we also have $e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$ for all x and hence

$$\begin{aligned}
\cosh(x) &= \frac{1}{2}\big[e^x + e^{-x}\big] = \frac{1}{2}\Big[\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\Big] = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}
\end{aligned}$$

for all x. In particular, the interval of convergence is all real numbers.
(b) Using the power series expansion of part (a),

$$\cosh(2) = 1 + \frac{2^2}{2!} + \frac{2^4}{4!} + \sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!} = 3\frac{2}{3} + \sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!}$$

So it suffices to show that $\sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!} \le 0.1$. Let's write $b_n = \frac{2^{2n}}{(2n)!}$. The first term in $\sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!}$ is

$$b_3 = \frac{2^6}{6!} = \frac{2^6}{6 \times 5 \times 4 \times 3 \times 2} = \frac{4}{45}$$

The ratio between successive terms in $\sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!}$ is

$$\begin{aligned} \frac{b_{n+1}}{b_n} &= \frac{2^{2n+2}/2^{2n}}{(2n+2)!/(2n)!} = \frac{4}{(2n+2)(2n+1)} \\ &\le \frac{4}{8 \times 7} = \frac{1}{14} \qquad \text{for all } n \ge 3 \end{aligned}$$

Hence

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!} &\le \overbrace{\frac{4}{45}}^{b_3} + \overbrace{\frac{4}{45} \times \frac{1}{14}}^{b_4 \le} + \overbrace{\frac{4}{45} \times \frac{1}{14^2}}^{b_5 \le} + \overbrace{\frac{4}{45} \times \frac{1}{14^3}}^{b_6 \le} + \cdots \\ &= \frac{4}{45} \frac{1}{1 - \frac{1}{14}} = \frac{4}{45} \frac{14}{13} = \frac{56}{585} < \frac{1}{10} \end{aligned}$$

(c) Comparing

$$\begin{aligned} \cosh(t) &= \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(t^2)^n}{(2n)!} \\ \text{and} \qquad e^{\frac{1}{2}t^2} &= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}t^2\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{(t^2)^n}{2^n n!} \end{aligned}$$

we see that it suffices to show that $(2n)! \ge 2^n n!$. Now. for all $n \ge 1$,

$$\begin{aligned} (2n)! &= \overbrace{1 \times 2 \times \cdots \times n}^{n \text{ factors}} \overbrace{(n+1) \times (n+2) \times \cdots \times 2n}^{n \text{ factors}} \\ &\ge \overbrace{1 \times 2 \times \cdots \times n}^{n \text{ factors}} \overbrace{2 \times 2 \times \cdots \times 2}^{n \text{ factors}} \\ &= 2^n \, n! \end{aligned}$$

3.6.8.50. Solution.

(a) For Newton's method, recall we approximate a root of the function $g(x)$ in iterations: given an approximation x_n, our next approximation is $x_{n+1} =$

$x_n - \dfrac{g(x_n)}{g'(x_n)}$. In our case,

$$x_{n+1} = x_n - \frac{x_n^3 - 2}{3x_n^2} = \frac{2}{3}\left(x_n + \frac{1}{x_n^2}\right).$$

We want to start somewhere reasonably close to the actual root we want, so let's set $x_0 = 1$. (Your starting point may vary.)

$$\begin{aligned}
x_0 &= 1 & \implies x_1 &= \frac{2}{3}\left(1 + \frac{1}{1}\right) = \frac{4}{3} \\
& & &\approx 1.3333 \\
x_1 &= \frac{4}{3} & \implies x_2 &= \frac{2}{3}\left(\frac{4}{3} + \frac{9}{16}\right) = \frac{91}{72} \\
& & &\approx 1.2639 \\
x_2 &= \frac{91}{72} & \implies x_3 &= \frac{2}{3}\left(\frac{91}{72} + \frac{72^2}{91^2}\right) = \frac{1126819}{894348} \\
& & &\approx 1.2599 \\
x_3 &= \frac{1126819}{894348} & \implies x_4 &= \frac{2}{3}\left(\frac{1126819}{894348} + \frac{894348^2}{1126819^2}\right) \\
& & &\approx 1.2599
\end{aligned}$$

So, $\sqrt[3]{2} \approx 1.26$.

(b) We'll evaluate the given series at $x = 2$. This yields the series

$$\sqrt[3]{2} = 1 + \frac{1}{6} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{(2)(5)(8)\cdots(3n-4)}{3^n\, n!}.$$

This series is alternating, so if we use the partial sum S_N, our absolute error is at most

$$|a_{N+1}| = \frac{(2)(5)(8)\cdots(3N-1)}{3^{N+1}\,(N+1)!}$$

(if $N \geq 2$). We want to know which value of N makes this at most 0.01. We test several values.

N	$\lvert a_{N+1}\rvert$
3	$\dfrac{(2)(5)(8)}{3^4 \cdot 4!} \approx 0.04$
4	$\dfrac{(2)(5)(8)(11)}{3^5\, 5!} \approx 0.03$
5	$\dfrac{(2)(5)(8)(11)(14)}{3^6\, 6!} \approx 0.023$
6	$\dfrac{(2)(5)(8)(11)(14)(17)}{3^7\, 7!} \approx 0.019$
7	$\dfrac{(2)(5)(8)(11)(14)(17)(20)}{3^8\, 8!} \approx 0.016$
8	$\dfrac{(2)(5)(8)(11)(14)(17)(20)(23)}{3^9\, 9!} \approx 0.013$
9	$\dfrac{(2)(5)(8)(11)(14)(17)(20)(23)(26)}{3^{10}\, 10!} \approx 0.012$
10	$\dfrac{(2)(5)(8)(11)(14)(17)(20)(23)(26)(29)}{3^{11}\, 11!} \approx 0.0103$
11	$\dfrac{(2)(5)(8)(11)(14)(17)(20)(23)(26)(29)(32)}{3^{12}\, 12!} \approx 0.009$

So, the approximation S_{11} has a sufficiently small error. That is, we would add up the first twelve terms.

3.6.8.51. Solution. Our plan is as follows:

- Make a Taylor series for $f(x)$
- Calculate the tenth derivative of the Taylor series of $f(x)$.
- Decide how many terms we need to add to achieve the desired accuracy.
- Approximate $f^{(10)}\left(\frac{1}{5}\right)$ with a partial sum.

We know that the Taylor series for $\arctan x$ is $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$, which converges for $-1 \le x \le 1$. So, the Taylor series for $\arctan(x^3)$ is

$$f(x) = \arctan(x^3) = \sum_{n=0}^{\infty}(-1)^n\frac{(x^3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+3}}{2n+1}$$

It is much easier to differentiate this series many times than it is to differentiate $\arctan(x^3)$ directly many times.

$$f'(x) = \sum_{n=0}^{\infty}(-1)^n\frac{(6n+3)x^{6n+2}}{2n+1}$$
$$f''(x) = \sum_{n=0}^{\infty}(-1)^n\frac{(6n+3)(6n+2)x^{6n+1}}{2n+1}$$

$$
\begin{aligned}
f'''(x) &= \sum_{n=0}^{\infty}(-1)^n\frac{(6n+3)(6n+2)(6n+1)x^{6n}}{2n+1}\\
&\vdots\\
f^{(10)}(x) &= \sum_{n=0}^{\infty}(-1)^n\frac{(6n+3)(6n+2)(6n+1)\cdots(6n-6)x^{6n-7}}{2n+1}\\
&= \sum_{n=2}^{\infty}(-1)^n\frac{(6n+3)(6n+2)(6n+1)\cdots(6n-6)x^{6n-7}}{2n+1}\\
&= \sum_{n=2}^{\infty}(-1)^n\frac{(6n+3)!}{(2n+1)(6n-7)!}x^{6n-7}\\
f^{(10)}\left(\frac{1}{5}\right) &= \sum_{n=2}^{\infty}(-1)^n\frac{(6n+3)!}{(2n+1)(6n-7)!\cdot 5^{6n-7}}
\end{aligned}
$$

(Notice, after ten differentiations, the terms a_0 and a_1 are both zero.)
Since this is an alternating series, the absolute error involved in using the approximation S_N is at most

$$|a_{N+1}| = \frac{(6N+9)!}{(2N+3)(6N-1)!\cdot 5^{6N-1}}$$

By testing a few values of N, we find

$$|a_6| = |a_{5+1}| = \frac{39!}{(13)(29!)\cdot 5^{29}} \approx 0.00000095 < 10^{-6}$$

So, S_5 is a sufficient approximation. That is,

$$
\begin{aligned}
f^{(10)}\left(\frac{1}{5}\right) &\approx \sum_{n=2}^{5}(-1)^n\frac{(6n+3)!}{(2n+1)(6n-7)!\cdot 5^{6n-7}}\\
&= (-1)^2\frac{15!}{5\cdot 5!\cdot 5^5} + (-1)^3\frac{21!}{7!\cdot 11!\cdot 5^{11}} + (-1)^4\frac{27!}{9!\cdot 17!\cdot 5^{17}}\\
&\qquad + (-1)^5\frac{33!}{11!\cdot 23!\cdot 5^{23}}\\
&= \frac{15!}{5!\cdot 5^6} - \frac{21!}{7!\cdot 11!\cdot 5^{11}} + \frac{27!}{9!\cdot 17!\cdot 5^{17}} - \frac{33!}{11!\cdot 23!\cdot 5^{23}}
\end{aligned}
$$

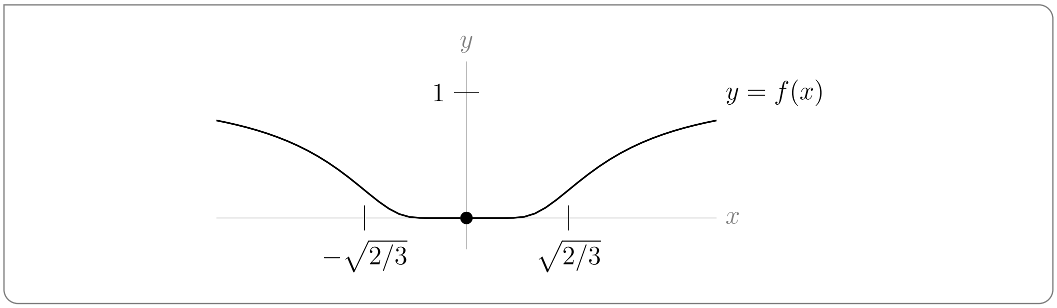
Remark: if we had calculated $f^{(10)}(1/5)$ directly, using derivative rules instead of series, we would have found an exact value; however, our value here is easier to find, and is highly accurate (if not exact).

3.6.8.52. Solution.

(a) To sketch $y = f(x)$, we note the following:

- $f(x)$ is never negative.

- $\lim_{x\to\pm\infty} f(x) = e^0 = 1$, so the curve has horizontal asymptotes in both directions at $y = 1$.
- $\lim_{x\to\pm 0} f(x) = \lim_{x\to\pm 0} \frac{1}{e^{1/x^2}} = \lim_{u\to+\infty} \frac{1}{e^u} = 0 = f(0)$, so the curve is continuous at $x = 0$.
- For $x \neq 0$, $f'(x) = \frac{2}{x^3}e^{-1/x^2}$, so our curve is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$
- For $x \neq 0$, $f''(x) = 2x^{-6}(2 - 3x^2)e^{-1/x^2}$, so our curve is concave up on $(-\sqrt{2/3}, \sqrt{2/3})$, and concave down elsewhere.



(b) Since $f^{(n)}(0) = 0$ for all whole n (that is, the graph is really quite flat at the origin), and since $f(0) = 0$, the Maclaurin series for $f(x)$ is $\displaystyle\sum_{n=0}^{\infty} \frac{0}{n!}x^n = 0$.

(c) The Maclaurin series converges for all real values of x (to the constant 0).

(d) Since $e^y > 0$ for any real y, we see $f(x) = 0$ only when $x = 0$. So, $f(x)$ is only equal to its Maclaurin series at the single point $x = 0$.

Remark: the function $f(x)$ is an example of a function whose Maclaurin series converges, but not to $f(x)$! To describe this behaviour, we say $f(x)$ is *non-analytic*.

3.6.8.53. Solution.

- Solution 1: Since $f(x)$ is odd, $f(-x) = -f(x)$ for all x in its domain. We plug this into our power series, then consider the even-indexed terms and the odd-indexed terms separately.

$$f(-x) = -f(x)$$
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(-x)^n = -\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Now separating the even powered and odd powered terms

$$\sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!}(-x)^{2n+1} + \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!}(-x)^{2n}$$
$$= -\sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{n!}x^{2n+1} - \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{n!}x^{2n}$$

For any integer n, we have that $(-1)^{2n} = 1$ and $(-1)^{2n+1} = -1$ so that

$$\begin{aligned}
-\sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} + \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} \\
= -\sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{n!} x^{2n+1} - \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{n!} x^{2n}
\end{aligned}$$

and

$$\sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} = -\sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{n!} x^{2n}$$

and

$$\begin{aligned}
2\sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} &= 0 \\
\sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} &= 0
\end{aligned}$$

- Solution 2: Alternately, we could note the following:
  - Since all derivative of $f(x)$ exist, all its derivatives are continuous.
  - The derivative of an odd function is even, and the derivative of an even function is odd.
  - So, the even-indexed derivatives of $f(x)$ are continuous, odd functions.
  - Every continuous, odd function passes through the origin. That is, $f^{(2n)}(0) = 0$.
  - So, every term in the series is 0.